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3-valued problem and reduction of some integer programming problems

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1. Introduction

The so-called *integer linear programming problem* is described as follows; Given integers

(1.1) $$a_{ij},\ b_i \text{ and } c_j,\ 1 \leqq i \leqq m,\ 1 \leqq j \leqq n,$$

find non-negative integers $x_1, \dots, x_n$ such that $\sum_{j=1}^{n} c_j x_j$ takes the maximum value under the constraints

(1.2) $$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i,\ i = 1, \dots, m.$$

Various methods for solving this problem have been discussed in [3, 5, 11, 18, 20, 22, 24]. Especially, useful methods are exploited in [9, 10, 13, 14, 17, 35] for 2-valued problems in which a_{ij}, x_j are supposed to belong to $\{0, 1\}$. The 2-valued problems can be applied to various problems concerning graphs, networks, and so on.

In this paper we are concerned with the following four types of integer problems:

Integer Selection Problem, or shortly ISP: Let n, k, $m_1, \dots, m_k$ be positive integers and let a_{ij}^r, b_i^r, c_j, $1 \leqq i \leqq m_r$, $1 \leqq r \leqq k$, $1 \leqq j \leqq n$ and z be given integers. Find an integer r with $1 \leqq r \leqq k$ and non-negative integers $x_1, \dots, x_n$ satisfying

$$\sum_{j=1}^{n} c_j x_j = z \text{ and } \sum_{j=1}^{n} a_{ij}^r x_j \leqq b_i^r \text{ for } 1 \leqq i \leqq m_r.$$

3-valued Problem: Given integers $a_{ij} \in \{-1, 0, 1\}$ and b_i stated in (1.1), find $x_1, \dots, x_n \in \{0, 1\}$ satisfying (1.2).

Indeterminate Coefficient Problem, or shortly IDCP: Let m, n, p and q be positive integers, a_{ij}, b_i integers given in (1.1), and let g_{si}, $d_{jt} \in \{-1, 0, 1\}$ and ℓ_{st}, $1 \leqq s \leqq p$, $1 \leqq t \leqq q$ be given integers. Find non-negative integers x_j and $y_{ij} \in \{0, 1\}$ satisfying

$$\sum_{j=1}^{n} a_{ij} y_{ij} x_j \leqq b_i,\ i = 1, \dots, m \text{ and}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} g_{si} y_{ij} d_{jt} \leqq \ell_{st},\ s = 1, \dots, p,\ t = 1, \dots, q.$$

IDCP with boundedness conditions: Given integers $u_j \geqq 0$, $1 \leqq j \leqq n$, solve the IDCP under the boundedness conditions $x_j \leqq u_j$, $j = 1, \dots, n$.

The first problem is a modification of the problem of "selection from several regions" due to Dantzig [11].

The main objective in this paper is to show that the two problems ISP and IDCP are equivalent, and that any IDCP with boundedness conditions is reduced to a 3-valued problem, as stated below:

THEOREM A. *Any solution of a given ISP (resp. IDCP and IDCP with boundedness conditions) is derived from solutions of the associated IDCP (resp. ISP and* 3*-valued problem) and vice versa.*

This is proved by combining Theorems 1 and 2 given in Section 2 and Theorem 3 stated in Section 3.

In order to state the second result on the 3-valued problem for given integers $a_{ij} \in \{-1, 0, 1\}$ and b_i, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, we introduce two notions: We say that a subset J' of $J = \{1, \dots, n\}$ is *weakly* (resp. *strongly*) *removable*, if for each $j \in J'$ there exists $i \in \{1, \dots, m\}$ satisfying

$$a_{ij} \geqq 0 \text{ (resp. } a_{ij} = 1) \text{ and } \textstyle\sum_{h \in J - J'} a_{ih} > b_i - a_{ij};$$

and we say that J' is *maximal* if $J' \cup \{k\}$ for any $k \in J - J'$ is not weakly (resp. strongly) removable. Using the above terminologies, we may state the following result which provides useful necessary conditions for the existence of solutions of 3-valued problems:

THEOREM B. *Let $x_1, \dots, x_n$ be a solution of the* 3*-valued problem formulated for a_{ij} and b_i.*

(i) *If there are no solutions $y_1, \dots, y_n \in \{0, 1\}$ such that $\{j | y_j = 1\} = \{j | x_j = 1\} \cup \{k\}$ for k with $x_k = 0$, then the subset $\{j | x_j = 0\}$ is maximal as a weakly removable set and also maximal as a strongly removable set.*

(ii) *If $b_i < 0$ for any i and the inequality*

$$\textstyle\sum_{j \in J_- - J'} \sum_{i=1}^m a_{ij} > \sum_{i=1}^m b_i, \quad J_- = \{j | \sum_{i=1}^m a_{ij} < 0\}$$

holds for a subset J' of $\{1, \dots, n\}$, then $x_j = 1$ for some $j \in J'$.

We note that we may assume $b_i < 0$ in (ii) without loss of generality. See the first part of Section 4.

An indeterminate coefficient problem with boundedness conditions arose in the study of the problems of inventory controls, production planning problems and so on. Theorem A shows that these problems can be treated as special cases of the 3-valued problem. In the forthcoming paper we shall discuss these problems by applying Theorem B.

This paper is organized as follows. In Section 2 the relationship between the integer selection problem and the indeterminate coefficient problem is discussed. In Section 3 indeterminate coefficient problems and 3-valued problems are treated. Finally, various conditions for the existence of solutions of the 3-valued problems are investigated in Section 4.

2. Integer Programming Problems

In this section, we consider a problem of finding an integer vector satisfying various integer constraints which we call IEP (see Definition 1 below). We show that any integer selection problem ISP introduced in Section 1 can be regarded as a special case of the IEP. We also introduce a kind of non-linear integer problem called an indeterminate coefficient problem, or shortly IDCP, and consider the relationship between IDCP and ISP.

We introduce some notation which we employ throughout this paper. We denote by $[p_1, \dots, p_m]$ the m-dimensional row vector $\boldsymbol{x}$, and $\boldsymbol{x}^t$ stands for the transposed vector of $\boldsymbol{x}$. For the same dimensional vector $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the usual inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$.

For a given integer p and two positive integers m and n, $[p]_{m,n}$ and $[p]_m$ denote the $m \times n$ matrix whose components are all equal to p and m-dimensional row vector such that all the components are equal to p, respectively. For an $m \times n$ matrix A and an $m \times k$ matrix B, $[A\ B]$ denotes the $m \times (n+k)$ block matrix. Similarly, for an $m \times n$ matrix A and a $k \times n$ matrix B, we denote by $\begin{bmatrix} A \\ B \end{bmatrix}$ the $(m+k) \times n$ block matrix: the first m rows form the matrix A and the last k rows form the matrix B.

We denote by Z and Z_+ the set of all integers and the set of all non-negative integers, respectively.

Definition 1. Let X be a subspace of n-dimensional space Z^n, $f = [f_1, \dots, f_m]$ a Z^m-valued function on X and let g be an integer-valued function on X. The *integer existence problem* IEP(n, m, X, f, g), or shortly IEP, is a problem of finding a vector $\boldsymbol{x}$ of X such that $g(\boldsymbol{x}) = 0$ and $f_i(\boldsymbol{x}) \leqq 0$ for $i = 1, \dots, m$. The function g is called an objective function.

In what follows, we denote the class of all the integer existence problems introduced above by $\mathscr{P}$. For IEP(n, m, X, f, g), we write $\mathscr{S}$(IEP(n, m, X, f, g)), or shortly $\mathscr{S}$(IEP), for the set of all solutions of IEP(n, m, X, f, g).

Remark 1. An integer existence problem represented by IEP(n, m, X, f, g) is equivalent to a problem of finding a feasible solution of an integer programming problem such that the constraints are given by $f_i(\boldsymbol{x}) \leqq 0$,

$i = 1, \dots, m$, where f_i is the i-th component function of f, and such that the associated objective function is $g_c(x) = g(x) - c$ for some integer c.

We here give two typical examples of the integer existence problems.

EXAMPLE 1. Consider IEP(n, m, X, f, g) in which

$$X = Z_+^n,\ f = [f_1, \dots, f_m],\ f_i(x) = \sum_{j=1}^n a_{ij}x_j - b_i \quad \text{for} \quad i = 1, \dots, m,$$

and

$$g(x) = \sum_{j=1}^n c_j x_j - c,$$

where a_{ij}, b_i, c_j, $i = 1, \dots, m$, $j = 1, \dots, n$ and c are given integers and x_j denotes the j-th component of x. Then any element $x \in \mathscr{S}(\text{IEP}(n, m, X, f, g))$ is a feasible solution of the linear integer programming problem to find an element $x \in Z_+^n$ at which $g(x)$ is maximized subject to the constraints $f_i(x) \leqq b_i$, $i = 1, \dots, m$.

EXAMPLE 2. Take the same region X and functions $f_1, \dots, f_m$ as in Example 1, and define a new objective function $g(x)$ given by

$$g(x) = \sum_{j=1}^n c_j x_j - (1/2) \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j - c,$$

where q_{ij} are integers. In this case the associated problem IEP(n, m, X, f, g) corresponds to so-called quadratic programming problem.

DEFINITION 2. Consider two integer existence problems P and P' in the class $\mathscr{P}$. We say that P *is stronger than* P', and write $P \prec P'$, if the existence of a solution of P implies that of a solution of P'.

In particular, $P' \prec P$ and $P \prec P'$ are valid for a pair of problems P and P' in the class $\mathscr{P}$, then P and P' are said to be *equivalent* and, in this case, we write $P \sim P'$.

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two subclasses of $\mathscr{P}$. If there is a mapping φ from $\mathscr{P}_1$ into $\mathscr{P}_2$ such that $P \sim \varphi(P)$ for any problem P in the class $\mathscr{P}_1$, we write $\mathscr{P}_1 \underset{\varphi}{\vdash} \mathscr{P}_2$.

The following is a modification of the problem of "selection from several regions" due to Dantzig [11, 12].

DEFINITION 3. Let n, k, $m_1, \dots, m_k$ be positive integers. Given integers

$$a_{ij}^r,\ b_i^r,\ c_j,\ 1 \leqq i \leqq m_r,\ 1 \leqq r \leqq k,\ 1 \leqq j \leqq n, \ \text{and}\ z, \tag{2.1}$$

the *integer selection problem*, or shortly ISP, is a problem of finding an integer r with $1 \leqq r \leqq k$ and non-negative integers $x_1, \dots, x_n$ satisfying

$$\sum_{j=1}^{n} c_j x_j = z \text{ and } \sum_{j=1}^{n} a_{ij}^r x_j \leqq b_i^r \quad \text{for} \quad i = 1, \dots, m_r.$$

We write $\mathrm{ISP}(n, k, \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z)$ for the integer selection problem associated with

$$\begin{aligned} &\mathfrak{m} = [m_1, \dots, m_k], \quad \mathfrak{A} = [A_1, \dots, A_k], \quad A_r = [a_{ij}^r], \\ (2.2) \qquad &\mathfrak{b} = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_k], \quad \boldsymbol{b}_r = [b_1^r, \dots, b_{m_r}^r], \quad \boldsymbol{c} = [c_1, \dots, c_n], \end{aligned}$$

where m_r, a_{ij}^r, b_i^r and c_j are integers appearing in (2.1). By means of these symbols the ISP is rewritten as the problem of finding an integer r and a vector $\boldsymbol{x} \in Z_+^n$ satisfying

$$\boldsymbol{c} \cdot \boldsymbol{x} = z \text{ and } A_r \boldsymbol{x}^t \leqq \boldsymbol{b}_r^t.$$

In what follows, the symbol $\mathscr{P}_S$ stands for the class of all integer selection problems. For $\mathrm{ISP}(n, k, \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z)$ we write $\mathscr{S}(\mathrm{ISP}(n, k, \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z))$, or shortly $\mathscr{S}(\mathrm{ISP})$, for the set of all pairs of an integer r and an n-dimensional vector $\boldsymbol{x}$ which provides a solution of the ISP.

We here give a simple example of the integer selection problem and its solutions.

Example 3. Let A_1, A_2, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ and $\boldsymbol{c}$ be defined by

$$A_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

$$\boldsymbol{b}_1 = \boldsymbol{b}_2 = [0, 0, 0], \qquad \boldsymbol{c} = [1, 1, 0].$$

Then

$$\begin{aligned} &\mathscr{S}(\mathrm{ISP}(3, 2, [3, 3], [A_1, A_2], [\boldsymbol{b}_1, \boldsymbol{b}_2], \boldsymbol{c}, z)) \\ &= \{[u, z-u, |z-2u|] \mid u \in Z_+, u \leqq z\}. \end{aligned}$$

Proposition 1. *An integer selection problem is an integer existence problem.*

Proof. Consider an ISP for the integers given in (2.1). One can define a subspace D_r of n-dimensional space Z^n by

$$D_r = \{[x_1, \dots, x_n] \in Z_+^n \mid \textstyle\sum_{j=1}^{n} a_{ij}^r x_j \leqq b_i^r, \ 1 \leqq i \leqq m_r\}, \quad r = 1, \dots, k,$$

and formulate a problem $\mathrm{IEP}(n, 1, \bigcup_{r=1}^{k} D_r, [f], g)$ for the function $f(\boldsymbol{x}) = 0$ and $g(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} - z$. Then we have

$$\mathscr{S}(\mathrm{ISP}) = \{(r, \boldsymbol{x}) \mid \boldsymbol{x} \in \mathscr{S}(\mathrm{IEP}), \boldsymbol{x} \in D_r\} \quad \text{and}$$

$$\mathscr{S}(\mathrm{IEP}) = \{x \mid (r, x) \in \mathscr{S}(\mathrm{ISP})\}.$$

This shows that the ISP with respect to (2.1) is regarded as the IEP$(n, 1, \bigcup_{r=1}^{k} D_r, [f], g)$. This proves the proposition.

DEFINITION 4. Let m, n, p and q be positive integers. Take the integers

$$a_{ij},\ b_i,\ g_{si} \in \{-1, 0, 1\},\ d_{jt} \in \{-1, 0, 1\} \text{ and } \ell_{st} \in Z \tag{2.3}$$

for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, $1 \leqq s \leqq p$ and $1 \leqq t \leqq q$. The *indeterminate coefficient problem*, or shortly IDCP, is a problem of finding integers $x_j \geqq 0$ and $y_{ij} \in \{0, 1\}$ satisfying

$$\sum_{j=1}^{n} a_{ij} y_{ij} x_j \leqq b_i \text{ and } \sum_{i=1}^{m} \sum_{j=1}^{n} g_{si} y_{ij} d_{jt} \leqq \ell_{st}$$

for $1 \leqq i \leqq m$, $1 \leqq s \leqq p$ and $1 \leqq t \leqq q$.

The IDCP for the integers given in (2.3) may be formulated in terms of integer matrices in the following way: Find an n-dimensional vector x of non-negative integers and an $m \times n$ 0–1 matrix $Y = [y_{ij}]$ satisfying

$$(A \circ Y)x^t \leqq b^t \text{ and } GYD \leqq L,$$

where

$$A = [a_{ij}],\ b = [b_1, \ldots, b_m],\ G = [g_{si}],\ D = [d_{jt}],\ L = [\ell_{st}] \tag{2.4}$$

and $A \circ Y$ denotes the $m \times n$ matrix $[a_{ij} y_{ij}]$.

In what follows, we write IDCP$(n, m, p, q, A, b, G, D, L)$ for the IDCP with respect to (2.4). We denote by $\mathscr{Y}(G, D, L)$ the set of all 0–1 matrices which satisfy the inequality $GYD \leqq L$, and by $\mathscr{S}(\mathrm{IDCP}(n, m, p, q, A, b, G, D, L))$, or shortly $\mathscr{S}(\mathrm{IDCP})$, the set of all pairs x and Y which give solutions of the IDCP. The symbol $\mathscr{P}_{ID}$ stands for the class of all indeterminate coefficient problems.

The following is an example of the indeterminate coefficient problem and its solutions.

EXAMPLE 4. Consider the problem IDCP$(3, 4, 5, 8, A, b, G, D, L)$ for the matrices and the vector

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad b = [0, 0, 0, 0], \quad G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 4 & -4 & -2 & 2 & 2 & 2 & 2 & 0 \\ 3 & -3 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$\mathscr{Y}(G, D, L) = \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

and

$$\mathscr{S}(\mathrm{IDCP}(3,4,5,8,A,\boldsymbol{b},G,D,L))$$
$$= \{(\boldsymbol{x}, Y) \mid \boldsymbol{x} = [u, v, \max\{u, v\}],\ u, v \in Z_+,\ Y \in \mathscr{Y}(G, D, L)\}.$$

PROPOSITION 2. *An indeterminate coefficient problem is an integer existence problem.*

PROOF. Consider an IDCP for the integers given in (2.3). One can formulate the integer existence problem IEP(n', m', W, f, g) for the integers $n' = n + mn$ and $m' = m + pq$, the subspace $W = Z^n_+ \times \{0, 1\}^{mn}$ of $Z^{n'}$, the $Z^{m'}$-valued function $f = [f_{01}, \ldots, f_{0m}, f_{11}, \ldots, f_{1q}, \ldots, f_{p1}, \ldots, f_{pq}]$ and the function $g(\boldsymbol{w}) \equiv 0$, where

$$f_{0i}(\boldsymbol{w}) = \textstyle\sum_{j=1}^{n} a_{ij} w_{in+j} w_j - b_i \qquad \text{for} \quad 1 \leqq i \leqq m,$$

$$f_{st}(\boldsymbol{w}) = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} g_{si} w_{in+j} d_{jt} - \ell_{st} \qquad \text{for} \quad 1 \leqq s \leqq p \text{ and } 1 \leqq t \leqq q.$$

Suppose that a pair of a vector $[x_1, \ldots, x_n]$ and an $m \times n$ matrix $[y_{ij}]$ is a solution of the IDCP. Then the vector $[x_1, \ldots, x_n, y_{11}, \ldots, y_{1n}, \ldots, y_{m1}, \ldots, y_{mn}]$ is a solution of the IEP. Conversely, for a solution $[w_1, \ldots, w_{n+mn}]$ of the IEP, the pair of the vector $[w_1, \ldots, w_n]$ and the matrix $[y_{ij}]$ with $y_{ij} = w_{in+j}$ is a solution of the IDCP. This shows that the IDCP with respect to (2.3) is regarded as the IEP(n', m', W, f, g). The proof is now complete.

In order to state the relationship between the class of ISPs and the class of IDCPs, we introduce the modified problem called IDCPs associated with ISPs. To formulate the problem, we introduce some notation: We write I_n for the $n \times n$ identity matrix. We also write $J_n = [j_{uv}]$ for the $n \times n$ matrix such that

$$j_{uv} = \begin{cases} -1 & \text{if } u = v \quad \text{for } v = 1, \dots, n, \\ 1 & \text{if } u = v + 1 \text{ for } v = 1, \dots, n-1, \\ 1 & \text{if } u = 1 \text{ and } v = n, \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer n, $\Xi(n)$ denote the vector $[-1, [0]_{n-1}]$. For k positive integers $m_1, \dots, m_k$ and $m = \sum_{i=1}^{k} m_i$, $\Gamma(m_1, \dots, m_k)$ denotes the $(2m+5) \times (m+4)$ matrix of the form

$$\begin{bmatrix} \Gamma_1(m_1, \dots, m_k) & [0]_{m_1,4} \\ \vdots & \vdots \\ \Gamma_k(m_1, \dots, m_k) & [0]_{m_k,4} \\ \multicolumn{2}{c}{-I_{m+4}} \\ \Xi(m_1) \dots \Xi(m_k) & [0]_4 \end{bmatrix},$$

where $\Gamma_i(m_1, \dots, m_k) = [\Gamma_{i1}(m_1, \dots, m_k) \dots \Gamma_{ik}(m_1, \dots, m_k)]$ and

$$\Gamma_{ij}(m_1, \dots, m_k) = \begin{cases} J_{m_i} & \text{if } i = j, \\ [0]_{m_i, m_j} & \text{if } i \neq j. \end{cases} \tag{2.5}$$

Given an integer n we write $\Delta(n)$ for the matrix $[I_n \ J_n]$. Furthermore, for any pair of positive integers m and n, $\Lambda(m, n)$ denotes the $(2m+5) \times 2n$ matrix of the form

$$\begin{bmatrix} [0]_{2m,n} & [0]_{2m,n} \\ [-1]_{4,n} & [0]_{4,n} \\ \Xi(n) & [0]_n \end{bmatrix}.$$

We now state the definition of the IDCP associated with an integer selection problem.

DEFINITION 5. Consider the problem ISP$(n, k, \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z)$ for $\mathfrak{m} = [m_1, \dots, m_k]$, $\mathfrak{A} = [A_1, \dots, A_k]$, $\mathfrak{b} = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_k]$, $A_r = [a_{ij}^r]$, $\boldsymbol{b}_r = [b_1^r, \dots, b_{m_r}^r]$ and $\boldsymbol{c} = [c_1, \dots, c_n]$. The *indeterminate coefficient problem associated with the* ISP is the problem IDCP$(n', m', p, q, A, \boldsymbol{b}, G, D, L)$ in which

(1) $m = \sum_{r=1}^{k} m_r$, $n' = n + 1$, $m' = m + 4$, $p = 2m + 5$, $q = 2n + 2$,

(2) A is the $m' \times n'$ matrix whose i-th row $\boldsymbol{a}_{i*}$ is given by

$$\boldsymbol{a}_{i*} = \begin{cases} [a_{u1}^r, \dots, a_{un}^r, -b_u^r] & \text{if } i = \sum_{h=1}^{r-1} m_h + u,\ 1 \leqq r \leqq k,\ 1 \leqq u \leqq m_r, \\ [c_1, \dots, c_n, 0] & \text{if } i = m + 1, \\ [-c_1, \dots, -c_n, 0] & \text{if } i = m + 2, \\ [[0]_n, 1] & \text{if } i = m + 3, \\ [[0]]_n, -1] & \text{if } i = m + 4, \end{cases}$$

(3) $\boldsymbol{b} = [[0]_m, z, -z, 1, -1]$,

(4) $G = \Gamma(m_1, \dots, m_k)$,

(5) $D = \Delta(n + 1)$,

(6) $L = \Lambda(m, n + 1)$.

The IDCP associated with an ISP is an indeterminate coefficient problem in the sence that A can be taken as the $m' \times n'$ matrix of integers, $\boldsymbol{b}$ as the n'-dimensional vector of integers, G as the $p \times m'$ matrix $\Gamma(m_1, \dots, m_k)$ of $\{-1, 0, 1\}$, D as the $n' \times q$ matrix $\Delta(n + 1)$ of $\{-1, 0, 1\}$ and L can be taken as the $p \times q$ matrix $\Lambda(m, n + 1)$ of integers.

The following lemma is particularly important for proving Lemmas 2 and 3.

LEMMA 1. *Let $m_1, \dots, m_k$ and n be positive integers, $m = \sum_{h=1}^{k} m_h$ and let Y be an $(m + 4) \times (n + 1)$ matrix. Then $Y \in \mathscr{Y}(\Gamma(m_1, \dots, m_k), \Delta(n + 1), \Lambda(m, n + 1))$ if and only if Y is of the form*

$$Y = \begin{bmatrix} [p_1]_{m_1, n+1} \\ \vdots \\ [p_k]_{m_k, n+1} \\ [1]_{4, n+1} \end{bmatrix}, \tag{2.6}$$

where $p_h \in \{0, 1\}$, $1 \leqq h \leqq k$, and $p_r = 1$ for some r with $1 \leqq r \leqq k$.

PROOF. Since Y is a $(\sum_{h=1}^{k} m_h + 4) \times (n + 1)$ matrix, we can write it as the block matrix

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_{k+1} \end{bmatrix}, \tag{2.7}$$

where Y_h is an $m_h \times (n+1)$ matrix for $h = 1, \dots, k$ and Y_{k+1} is a $4 \times (n+1)$ matrix. In view of the definitions of $\Gamma(m_1, \dots, m_k)$ and $\Delta(n+1)$, we have

$$\Gamma(m_1, \dots, m_k)\, Y \Delta(n+1)$$

$$= \begin{bmatrix} J_{m_1} Y_1 & J_{m_1} Y_1 J_{n+1} \\ \vdots & \vdots \\ J_{m_k} Y_k & J_{m_k} Y_k J_{n+1} \\ -Y_1 & -Y_1 J_{n+1} \\ \vdots & \vdots \\ -Y_k & -Y_k J_{n+1} \\ -Y_{k+1} & -Y_{k+1} J_{n+1} \\ \sum_{h=1}^k \Xi(m_h) Y_h & \sum_{h=1}^k \Xi(m_h) Y_h J_{n+1} \end{bmatrix}.$$

Hence, Y belongs to $\mathscr{Y}(\Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$ if and only if the submatrices $Y_1, \dots, Y_{k+1}$ satisfy the following inequalities;

(2.8) $\quad J_{m_h} Y_h \leqq [0]_{m_h, n+1} \quad$ for $h = 1, \dots, k$,

(2.9) $\quad J_{m_h} Y_h J_{n+1} \leqq [0]_{m_h, n+1} \quad$ for $h = 1, \dots, k$,

(2.10) $\quad -Y_h \leqq [0]_{m_h, n+1} \quad$ for $h = 1, \dots, k$,

(2.11) $\quad -Y_h J_{n+1} \leqq [0]_{m_h, n+1} \quad$ for $h = 1, \dots, k$,

(2.12) $\quad -Y_{k+1} \leqq [-1]_{4, n+1}$,

(2.13) $\quad -Y_{k+1} J_{n+1} \leqq [0]_{4, n+1}$,

(2.14) $\quad \sum_{h=1}^k \Xi(m_h) Y_h \leqq \Xi(n+1)$,

(2.15) $\quad \sum_{h=1}^k \Xi(m_h) Y_h J_{n+1} \leqq [0]_{n+1}$.

Let y_{ij}^h denote the (i, j)-component of the matrix Y_h. We first suppose that $Y_h = [p_h]_{m_h, n+1}$ for $1 \leqq h \leqq k$ and $Y_{k+1} = [1]_{4, n+1}$, where $p_h \in \{0, 1\}$ and $p_r = 1$ for some r with $1 \leqq r \leqq k$. From the simple calculation for $J_{m_h} Y_h$, we have

$$-y_{1j}^h + y_{m_h j}^h = -p_h + p_h = 0 \qquad \text{for } j = 1, \dots, n+1, \text{ and}$$

$$y_{i-1,j}^h - y_{ij}^h = p_h - p_h = 0 \qquad \text{for } i = 2, 3, \dots, m_h, \ j = 1, \dots, n+1.$$

This shows that $J_{m_h} Y_h = [0]_{m_h, n+1}$ for $h = 1, \dots, k$. Similarly, we have $Y_h J_{n+1} = [0]_{m_h, n+1}$ for $h = 1, \dots, k+1$. Therefore, the inequalities (2.8), (2.9), (2.11), (2.13) and (2.15) hold. Since any element of Y is 1 or 0, it follows that the inequality (2.10) is valid. From $Y_{k+1} = [1]_{4, n+1}$, we immediately obtain the inequality (2.12). Recalling that $p_r = 1$ for some r, we have

$$- \sum_{h=1}^{k} y_{11}^{h} \leqq - p_r = - 1.$$

This implies that the inequality (2.14) is valid. Thus, $Y \in \mathscr{Y}(\Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$.

Conversely, we suppose that a matrix Y is an element of $\mathscr{Y}(\Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$. Then, $Y_1, \dots, Y_{k+1}$ satisfy the inequalities (2.8) through (2.15). From (2.8), we see that

$$y_{m_h j}^{h} \leqq y_{1j}^{h} \leqq \cdots \leqq y_{m_h-1,j}^{h} \leqq y_{m_h j}^{h}$$

for $1 \leqq j \leqq n+1$ and $1 \leqq h \leqq k$. These inequalities together imply that any column of Y_h is equal to $[0]_{m_h}^{t}$ or $[1]_{m_h}^{t}$. On the other hand, by (2.11), we have

$$y_{i1}^{h} \leqq y_{in}^{h} \leqq \cdots \leqq y_{i2}^{h} \leqq y_{i1}^{h}$$

for $i = 1, \dots, m_h$ and $h = 1, \dots, k+1$, i.e., any row of Y_h is equal to $[0]_{n+1}$ or $[1]_{n+1}$. Therefore, it follows that either $Y_h = [0]_{m_h, n+1}$ or $[1]_{m_h, n+1}$ for $h = 1, \dots, k$. Moreover, the inequality (2.14) implies that at least one element in $\{y_{11}^{1}, \dots, y_{11}^{k}\}$, say y_{11}^{r}, is equal to 1. Thus, we have $Y_r = [1]_{m_r, n+1}$ for such an r. Finally, $Y_{k+1} = [1]_{4, n+1}$, since Y_{k+1} satisfies the inequality (2.12). This completes the proof of Lemma 1.

LEMMA 2. *Let a pair of an integer r and a vector $\boldsymbol{x}$ be a solution of $ISP(n, k, \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z)$, where $\mathfrak{m}$, $\mathfrak{A}$, $\mathfrak{b}$ and $\boldsymbol{c}$ are given by (2.2). Put $\boldsymbol{w} = [\boldsymbol{x}, 1]$ and let the matrix Y be given by (2.6) in which $p_r = 1$ and $p_i = 0$ for $1 \leqq i \leqq k$ with $i \neq r$. Then the pair of $\boldsymbol{w}$ and Y gives a solution of the IDCP associated with the ISP.*

PROOF. Let the IDCP associated with the ISP be specified by IDCP $(n+1, m+4, 2m+5, 2n+2, A, \boldsymbol{b}, \Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$, where the matrix A is given in Definition 5 and $m = \sum_{h=1}^{k} m_h$. Let $\boldsymbol{a}_{i*}$ and $\boldsymbol{y}_{i*}$ be the i-th row of A and Y, respectively. Then it follows from Lemma 1 that $Y \in \mathscr{Y}(\Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$. Moreover in view of the definition of Y, we have

$$(\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{w} = 0 \qquad \text{for} \quad i < s \text{ or } s + m_r + 1 \leqq i \leqq m, \quad \textit{and}$$

$$(\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{w} = \boldsymbol{a}_{i*} \cdot \boldsymbol{w} = \boldsymbol{a}_{i-s,*}^{r} \cdot \boldsymbol{x} - b_{i-s}^{r} \leqq 0 \qquad \text{for} \quad s+1 \leqq i \leqq s + m_r,$$

where $s = \sum_{h=1}^{r-1} m_h$ and $\boldsymbol{a}_{j*}^{r}$ is the j-th row of A_r. On the other hand, for $i \geqq m+1$, the equations $\boldsymbol{c} \cdot \boldsymbol{x} = z$ and $\boldsymbol{w} = [\boldsymbol{x}, 1]$ imply the four inequalities below:

$$(\boldsymbol{a}_{m+1,*} \circ \boldsymbol{y}_{m+1,*}) \cdot \boldsymbol{w} = \boldsymbol{c} \cdot \boldsymbol{x} \leqq z,$$

$$(\boldsymbol{a}_{m+2,*} \circ \boldsymbol{y}_{m+2,*}) \cdot \boldsymbol{w} = - \boldsymbol{c} \cdot \boldsymbol{x} \leqq - z,$$

$$(\boldsymbol{a}_{m+3,*} \circ \boldsymbol{y}_{m+3,*}) \cdot \boldsymbol{w} \leqq 1,$$

$$(\boldsymbol{a}_{m+4,*} \circ \boldsymbol{y}_{m+4,*}) \cdot \boldsymbol{w} \leqq -1.$$

Since $\boldsymbol{b} = [[0]_m, z, -z, 1, -1]$, it follows that $(A \circ Y)\boldsymbol{w}^t \leqq \boldsymbol{b}^t$. Thus, the proof of Lemma 2 is completed.

LEMMA 3. *Let* $\mathfrak{m}$, $\mathfrak{A}$, $\mathfrak{b}$ *and* $\boldsymbol{c}$ *be given by* (2.2), *and let* $IDCP(n+1, m+4, 2m+5, 2n+2, A, \boldsymbol{b}, \Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$ *be the IDCP associated with* $ISP(n, k, \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z)$. *If a pair of* $\boldsymbol{w} = [w_1, \dots, w_{n+1}]$ *and a matrix* Y *is a solution of the IDCP, then there is an integer* r *such that* $1 \leqq r \leqq k$, r *is determined by* Y, *and a pair of* r *and* $\boldsymbol{x} = [w_1, \dots, w_n]$ *gives a solution of the ISP.*

PROOF. Since $Y \in \mathscr{Y}(\Gamma(m_1, \dots, m_k), \Delta(n+1), \Lambda(m, n+1))$, Lemma 1 implies that there exists an integer r such that Y is of the form (2.6) in which $p_r = 1$. Let $s = \sum_{h=1}^{r-1} m_h$. Since $(A \circ Y)\boldsymbol{w}^t \leqq \boldsymbol{b}^t$, we get

$$0 \geqq (\boldsymbol{a}_{s+j,*} \circ \boldsymbol{y}_{s+j,*}) \cdot \boldsymbol{w} = \boldsymbol{a}_{s+j,*} \cdot \boldsymbol{w} = \boldsymbol{a}^r_{j*} \cdot \boldsymbol{x} - b^r_j w_{n+1}, \tag{2.16}$$

for $1 \leqq j \leqq m_r$, where $\boldsymbol{a}_{j*}$, $\boldsymbol{a}^r_{j*}$ and $\boldsymbol{y}_{j*}$ denote the j-th row of A, A_r and Y, respectively. Comparing the $(m+1)$-th and $(m+2)$-th components of the both sides of $(A \circ Y)\boldsymbol{w}^t \leqq \boldsymbol{b}^t$, we have

$$z \geqq (\boldsymbol{a}_{m+1,*} \circ \boldsymbol{y}_{m+1,*}) \cdot \boldsymbol{w} = \boldsymbol{a}_{m+1,*} \cdot \boldsymbol{w} = \boldsymbol{c} \cdot \boldsymbol{x} \quad \text{and} \tag{2.17}$$

$$-z \geqq (\boldsymbol{a}_{m+2,*} \circ \boldsymbol{y}_{m+2,*}) \cdot \boldsymbol{w} = \boldsymbol{a}_{m+2,*} \cdot \boldsymbol{w} = -\boldsymbol{c} \cdot \boldsymbol{x}. \tag{2.18}$$

Similarly, taking the $(m+3)$-th and $(m+4)$-th components of the both sides of $(A \circ Y)\boldsymbol{x}^t \leqq \boldsymbol{b}^t$ into account, we have

$$w_{n+1} = 1. \tag{2.19}$$

Hence, combining (2.16) through (2.19), we have

$$A_r \boldsymbol{x}^t \leqq \boldsymbol{b}_r \quad \text{and} \quad \boldsymbol{c} \cdot \boldsymbol{x} = z.$$

This completes the proof of Lemma 3.

Combining the above-mentioned results we obtain the first main result.

THEOREM 1. *Any integer selection problem is equivalent to some indeterminate coefficient problem.*

PROOF. For each problem P in the class $\mathscr{P}_S$ we assign $\varphi(P)$ in the class $\mathscr{P}_{ID}$ which is associated with the problem P. Then φ defines a mapping from $\mathscr{P}_S$ into $\mathscr{P}_{ID}$. In view of Definition 5, the mapping φ is well-defined. It then follows from Lemmas 2 and 3 that both $P \prec \varphi(P)$ and $\varphi(P) \prec P$ are valid. This completes the proof of Theorem 1.

Let $(r, \boldsymbol{x})$ be a solution of an ISP. By Lemma 2, we get a solution $(\boldsymbol{w}, Y)$ of the IDCP associated with the ISP. Lemma 3 shows that the solution $(r, \boldsymbol{x})$ can be constructed from $(\boldsymbol{w}, Y)$. This implies that any solution of an ISP is obtained through a solution of the associated IDCP. On the other hand, for any solution $(\boldsymbol{x}, Y)$ of an IDCP, the matrix Y is given by (2.6) (and so Y is of the form (2.7)) and there exist $r_1, \ldots, r_s$ in $\{1, \ldots, k\}$ such that $p_{r_i} = 1, i = 1, \ldots, s$. Hence the solution $(\boldsymbol{x}, Y)$ is constructed from solutions $(r_1, \boldsymbol{x}), \ldots, (r_s, \boldsymbol{x})$ of the ISP associated with the IDCP. Hence, any solution of an IDCP which is associated with an ISP is constructed from solutions of the ISP. This is nothing but the first assertion of Theorem A stated in Section 1.

In order to show the converse of Theorem 1, we formulate a problem ISP associated with an IDCP. The following notation is used to state the problem.

For a positive integer m, we denote by $\mathfrak{P}_m$ the set of all sequences of integers $\Pi = (i_1, \ldots, i_p)$ where $1 \leqq i_1 < \cdots < i_p \leqq m$. The sumbol $|\mathfrak{P}_m|$ means the number of elements of $\mathfrak{P}_m$, $|\Pi|$ is the length of the sequence Π. The length of the sequence which has no elements is defined to be 0. Two sequences $\Pi_1 = (i_1, \ldots, i_p)$ and $\Pi_2 = (j_1, \ldots, j_{m-p})$ of $\mathfrak{P}_m$ are said to be complemented if $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{m-p}\} = \{1, \ldots, m\}$ holds. The complement of Π is denoted by Π^*.

For $k \times m$ matrix A and $\Pi \in \mathfrak{P}_m$, $\Pi(A)$ denotes that

$$\Pi(A) = \begin{cases} [0] & \text{if } |\Pi| = 0, \\ [\boldsymbol{a}_{*i_1} \ldots \boldsymbol{a}_{*i_p}] & \text{if } \Pi = (i_1, \ldots, i_p),\ 1 \leqq p \leqq m, \end{cases}$$

where $\boldsymbol{a}_{*i}$ is the i-th column of A.

For three positive integers j, m and n, we write Θ_{jmn} for the sequence $(j, n + j, \ldots, n(m-1) + j)$ with length m. For $\mathfrak{T} = (\Pi_1, \ldots, \Pi_n) \in (\mathfrak{P}_m)^n$, we denote by $\mathfrak{T}^*$ and $|\mathfrak{T}|$ the sequence $(\Pi_1^*, \ldots, \Pi_n^*)$ and $\sum_{i=1}^n \max\{|\Pi_i|, 1\}$, respectively. The symbols $\Phi(\mathfrak{T})$ and $\Psi(\mathfrak{T})$ denote $|\mathfrak{T}| \times mn$ matrices of the forms

$$\begin{bmatrix} \Phi_1(\mathfrak{T}) \\ \vdots \\ \Phi_n(\mathfrak{T}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Psi_1(\mathfrak{T}) \\ \vdots \\ \Psi_n(\mathfrak{T}) \end{bmatrix},$$

respectively. Here $\mathfrak{T} = (\Pi_1, \ldots, \Pi_n)$, and $\Phi_j(\mathfrak{T})$ and $\Psi_j(\mathfrak{T})$ are defined to be $[0]_{mn}$ if $|\Pi_j| = 0$, otherwise they are defined as follows:

$$\Pi_j(\Theta_{jmn}(\Phi_j(\mathfrak{T}))) = J_{|\Pi_j|},$$

$$\Pi_j^*(\Theta_{jmn}(\Phi_j(\mathfrak{T}))) = [0]_{|\Pi_j|, m-|\Pi_j|} \quad \text{if } |\Pi_j| < m,$$

$$\Theta^*_{jmn}(\Phi_j(\mathfrak{T})) = [0]_{|\Pi_j|, m(n-1)},$$

$$\Pi_j(\Theta_{jmn}(\Psi_j(\mathfrak{T}))) = I_{|\Pi_j|},$$

$$\Pi^*_j(\Theta_{jmn}(\Psi_j(\mathfrak{T}))) = [0]_{|\Pi_j|, m-|\Pi_j|} \quad \text{if } |\Pi_j| < m,$$

$$\Theta^*_{jmn}(\Psi_j(\mathfrak{T})) = [0]_{|\Pi_j|, m(n-1)}.$$

In particular $n = 1$ and $\mathfrak{T} = (\Pi)$, we write $\Phi_1(\Pi)$ and $\Psi_1(\Pi)$ for the matrices $\Phi(\mathfrak{T}) = [\Phi_1(\mathfrak{T})]$ and $\Psi(\mathfrak{T}) = [\Psi_1(\mathfrak{T})]$, respectively.

The following two propositions can be checked easily.

PROPOSITION 3. *For any* $\mathfrak{T} = (\Pi_1, \dots, \Pi_n) \in (\mathfrak{P}_m)^n$ *and* $j = 1, \dots, n$,

$$\Theta_{jmn}(\Phi_j(\mathfrak{T})) = \Phi_1(\Pi_j) \text{ and } \Theta_{jmn}(\Psi_j(\mathfrak{T})) = \Psi_1(\Pi_j).$$

PROPOSITION 4. *Let* $\Pi \in \mathfrak{P}_m$ *and* x *be* m*-dimensional vector. Then*

$$\Phi_1(\Pi)x^t = \Pi(\Phi_1(\Pi))(\Pi(x))^t \quad \text{and}$$

$$\Psi_1(\Pi)x^t = \Pi(\Psi_1(\Pi))(\Pi(x))^t.$$

DEFINITION 6. Consider the problem IDCP$(n, m, p, q, A, b, G, D, L)$, where $A = [a_{ij}]$, $G = [g_{si}]$, $D = [d_{jt}]$ and $L = [\ell_{st}]$ are given in (2.4). The *integer selection problem* ISP *associated with IDCP* is ISP$(2mn, k(m, n), \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, [0]_{2mn}, 0)$ for $k(m, n) = |\mathfrak{P}_m|^n$, $\mathfrak{m} = [m_{\mathfrak{T}_1}, \dots, m_{\mathfrak{T}_{k(m,n)}}]$, $\mathfrak{A} = [A_{\mathfrak{T}_1}, \dots, A_{\mathfrak{T}_{k(m,n)}}]$, $\mathfrak{b} = [b_{\mathfrak{T}_1}, \dots, b_{\mathfrak{T}_{k(m,n)}}]$. Here $m_{\mathfrak{T}_h} = m + pq + 3|\mathfrak{T}_h| + 2|\mathfrak{T}^*_h|$, $A_{\mathfrak{T}_h}$ is the $m_{\mathfrak{T}_h} \times 2mn$ matrix of the form

$$\begin{bmatrix} A' & [0]_{m,mn} \\ & g_{11}D^t \ \cdots \ g_{1m}D^t \\ [0]_{pq,mn} & \vdots \qquad \vdots \\ & g_{p1}D^t \ \cdots \ g_{pm}D^t \\ \Phi(\mathfrak{T}_h) & \\ \Psi(\mathfrak{T}^*_h) & [0]_{|\mathfrak{T}_h| + |\mathfrak{T}^*_h|, mn} \\ & \Psi(\mathfrak{T}^*_h) \\ [0]_{2|\mathfrak{T}_h| + |\mathfrak{T}^*_h|, mn} & \Psi(\mathfrak{T}_h) \\ & -\Psi(\mathfrak{T}_h) \end{bmatrix},$$

A' is the $m \times mn$ matrix whose i-th row is given by

$$[[0]_{ni-n}, a_{i1}, \dots, a_{in}, [0]_{mn-ni}],$$

$b_{\mathfrak{T}_h} = [b, \ell_{1*}, \dots, \ell_{p*}, [0]_{|\mathfrak{T}_h| + 2|\mathfrak{T}^*_h|}, [1]_{|\mathfrak{T}_h|}, [-1]_{|\mathfrak{T}_h|}]$, ℓ_{s*}, $s = 1, \dots, p$, are the s-

th rows of L, and $\{\mathfrak{T}_1, \dots, \mathfrak{T}_{k(m,n)}\} = (\mathfrak{P}_m)^n$.

The ISP associated with an IDCP is an integer selection problem in the sense that n can be taken as the integer $2mn$, k as the integer $k(m, n)$, A_r as the $m_{\mathfrak{T}_r} \times 2mn$ matrix $A_{\mathfrak{T}_r}$, $\boldsymbol{b}_r$ as the $m_{\mathfrak{T}_r}$-dimensional vector $\boldsymbol{b}_{\mathfrak{T}_r}$, $\boldsymbol{c}$ as the $2mn$-dimensional zero vector, $\mathfrak{m}$ as the vector $[m_{\mathfrak{T}_1}, \dots, m_{\mathfrak{T}_{k(m,n)}}]$, $\mathfrak{A}$ as the vector $[A_{\mathfrak{T}_1}, \dots, A_{\mathfrak{T}_{k(m,n)}}]$, $\mathfrak{b}$ as the vector $[\boldsymbol{b}_{\mathfrak{T}_1}, \dots, \boldsymbol{b}_{\mathfrak{T}_{k(m,n)}}]$ and z can be taken as 0. The following lemma is important for proving Lemmas 5 and 6.

LEMMA 4. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be an m-dimensional vector of non-negative integers and an m-dimensional* 0–1 *vector, respectively. Then there exists a non-negative integer w such that $\boldsymbol{x} = w\boldsymbol{y}$, if and only if there exists a sequence Π of $\mathfrak{P}_m$ satisfying*

$$(2.20)\qquad \begin{aligned} &\Phi_1(\Pi)\boldsymbol{x}^t \leqq (\Pi([0]_m))^t, \quad \Psi_1(\Pi^*)\boldsymbol{x}^t \leqq (\Pi^*([0]_m))^t, \\ &-\Psi_1(\Pi)\boldsymbol{y}^t \leqq (\Pi([-1]_m))^t, \quad \Psi_1(\Pi^*)\boldsymbol{y}^t \leqq (\Pi^*([0]_m))^t. \end{aligned}$$

PROOF. We first suppose that $\boldsymbol{x} = w\boldsymbol{y}$ holds for some non-negative integer w. Define a sequence Π belonging to $\mathfrak{P}_m$ by $\Pi(\boldsymbol{y}) = \Pi([1]_m)$ and $\Pi^*(\boldsymbol{y}) = \Pi^*([0]_m)$. Then $\Pi(\boldsymbol{x}) = w\Pi(\boldsymbol{y})$ and $\Pi^*(\boldsymbol{x}) = \Pi^*(\boldsymbol{y}) = \Pi^*([0]_m)$. Let $|\Pi| = k$. If $k = 0$ then $\boldsymbol{x} = \boldsymbol{y} = [0]_m$, and hence the inequalities listed in (2.20) are valid. We next assume that $k > 0$. It follows from Propositions 3 and 4 and the definition of Φ that

$$\begin{aligned} \Phi_1(\Pi)\boldsymbol{x}^t &= \Pi(\Phi_1(\Pi))(\Pi(\boldsymbol{x}))^t = \Pi(\Theta_{1m1}(\Phi_1(\Pi)))(\Pi(\boldsymbol{x}))^t \\ &= wJ_k\Pi(\boldsymbol{y})^t = ([0]_k)^t. \end{aligned}$$

Similarly, we have $\Psi_1(\Pi)\boldsymbol{y}^t = ([1]_k)^t$, $\Psi_1(\Pi^*)\boldsymbol{x}^t = ([0]_{m-k})^t$ and $\Psi_1(\Pi^*)\boldsymbol{y}^t = ([0]_{m-k})^t$. These four equalities together imply that the inequalities listed in (2.20) are valid.

Conversely, we suppose that the inequalities stated in (2.20) hold for some $\Pi \in \mathfrak{P}_m$ with length k. In the case that $k = 0$, it is clear that $\boldsymbol{x} = \boldsymbol{y} = [0]_m$. Hence, $\boldsymbol{x} = w\boldsymbol{y}$ for any non-negative integer w. In the case of $k = m$, we obtain $\Phi_1(\Pi)\boldsymbol{x}^t = J_m\boldsymbol{x}^t \leqq [0]_m^t$ and $-\Psi_1(\Pi)\boldsymbol{y}^t = -I_m\boldsymbol{y}^t \leqq [-1]_m^t$. These two inequalities imply that $\boldsymbol{y} = [1]_m$ and all components of $\boldsymbol{x}$ are equal to some constant integer, say w. So we can take the integer w in this case. Finally, we assume that $0 < k < m$ and $\Pi = (i_1, \dots, i_k)$. Then we have $J_k(\Pi(\boldsymbol{x}))^t \leqq [0]_k^t$ from the inequality in (2.20). This shows that $x_{i_p} = w$, $p = 1, \dots, k$, for some integer w, where x_{i_p} is the i_p-th component of $\boldsymbol{x}$. Also, we see that $\Pi^*(\boldsymbol{x}) \leqq \Pi^*([0]_m)$, $\Pi^*(\boldsymbol{y}) \leqq \Pi^*([0]_m)$ and $-\Pi(\boldsymbol{y}) \leqq \Pi([-1]_m)$. Therefore, we have

$$y_{i_p} = 1,\ p = 1, \dots, k \text{ and } x_j = y_j = 0 \text{ for } j \notin \{i_1, \dots, i_k\}.$$

Hence, $x = wy$ holds for the integer $w = x_{i_1}$. This proves the lemma.

LEMMA 5. *Suppose that a pair of* $w = [w_1, \dots, w_n]$ *and a matrix* $Y = [y_{ij}]$ *gives a solution of* $IDCP(n, m, p, q, A, b, G, D, L)$. *Then there is an integer* r *such that* $1 \leqq r \leqq k(m, n)$, r *is determined by* w *and* Y, *and the pair of* r *and* $x = [u, v]$ *gives a solution of the ISP associated with the IDCP, where* $u = [x_{1*}, \dots, x_{m*}]$, $v = [y_{1*}, \dots, y_{m*}]$, $x_{i*} = [w_1 y_{i1}, \dots, w_n y_{in}]$ *and* $y_{i*} = [y_{i1}, \dots, y_{in}]$, $i = 1, \dots, m$.

PROOF. It follows from the definitions of the vectors u and v that $\Theta_{jmn}(u) = w_j \Theta_{jmn}(v)$, $j = 1, \dots, n$. Therefore, by Lemma 4, there exist sequences Π_j, $j = 1, \dots, n$, belonging to $\mathfrak{P}_m$ such that

$$\Phi_1(\Pi_j)(\Theta_{jmn}(u))^t \leqq (\Pi_j([0]_m))^t, \tag{2.21}$$

$$\Psi_1(\Pi_j^*)(\Theta_{jmn}(u))^t \leqq (\Pi_j^*([0]_m))^t, \tag{2.22}$$

$$\Psi_1(\Pi_j^*)(\Theta_{jmn}(v))^t \leqq (\Pi_j^*([0]_m))^t, \tag{2.23}$$

$$-\Psi_1(\Pi_j)(\Theta_{jmn}(v))^t \leqq (\Pi_j([-1]_m))^t. \tag{2.24}$$

Since $\{\mathfrak{T}_1, \dots, \mathfrak{T}_{k(m,n)}\} = (\mathfrak{P}_m)^n$, there exists r such that $\mathfrak{T}_r = (\Pi_1, \dots, \Pi_n)$. Hence, it is sufficient to show that

$$A_{\mathfrak{T}_r} x^t \leqq b_{\mathfrak{T}_r}^t. \tag{2.25}$$

In view of the definitions of $A_{\mathfrak{T}_r}$ and $b_{\mathfrak{T}_r}$, we can decompose the inequality (2.25) into the following seven inequalities;

$$a_{i*} \cdot x_i \leqq b_i, \qquad i = 1, \dots, m, \tag{2.26}$$

$$\textstyle\sum_{i=1}^m g_{si} d_{*t}^t \cdot y_{i*} \leqq \ell_{st}, \ s = 1, \dots, p, \ t = 1, \dots, q, \tag{2.27}$$

$$\Phi(\mathfrak{T}_r) u^t \leqq [0]_{|\mathfrak{T}_r|}^t, \tag{2.28}$$

$$\Psi(\mathfrak{T}_r^*) u^t \leqq [0]_{|\mathfrak{T}_r^*|}^t, \tag{2.29}$$

$$\Psi(\mathfrak{T}_r^*) v^t \leqq [0]_{|\mathfrak{T}_r^*|}^t, \tag{2.30}$$

$$\Psi(\mathfrak{T}_r) v^t \leqq [0]_{|\mathfrak{T}_r|}^t, \tag{2.31}$$

$$-\Psi(\mathfrak{T}_r) v^t \leqq [-1]_{|\mathfrak{T}_r|}^t, \tag{2.32}$$

where a_{i*}, b_i, g_{ij}, d_{*j} and ℓ_{ij} are the i-th row of A, the i-th component of b, the (i, j)-component of G, the j-th column of D and the (i, j)-component of L, respectively. It follows from Proposition 3 and the definitions of Φ_j and Ψ_j that

$$\Phi_j(\mathfrak{T}_r) u^t = \Theta_{jmn}(\Phi_j(\mathfrak{T}_r))(\Theta_{jmn}(u))^t = \Phi_1(\Pi_j)(\Theta_{jmn}(u))^t \quad \text{and}$$

$$\Psi_j(\mathfrak{T}_r^*)u^t = \Theta_{jmn}(\Psi_j(\mathfrak{T}_r^*))(\Theta_{jmn}(u))^t = \Psi_1(\Pi_j^*)(\Theta_{jmn}(u))^t$$

for $j = 1, \ldots, n$. Therefore, we obtain the inequalities (2.28) and (2.29) from (2.21) and (2.22). Similarly, we have (2.30) and (2.32) from (2.23) and (2.24). Since $\boldsymbol{v}$ is a 0–1 vecotr, the inequality (2.31) is valid. Finally, recalling the assumption that the pair of $\boldsymbol{x}$ and Y is a solution of the IDCP, we obtain (2.26) and (2.27). Thus, we conclude that (2.25) is valid. This completes the proof of Lemma 5.

LEMMA 6. *Suppose that a pair of an integer r and a 2mn-dimensional vector* $\boldsymbol{x}$ *is a solution of* $ISP(2mn, k(m, n), \mathfrak{m}, \mathfrak{A}, \mathfrak{b}, \boldsymbol{c}, z)$ *associated with the problem* $IDCP(n, m, p, q, A, \boldsymbol{b}, G, D, L)$. *Put* $\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]$, $\boldsymbol{u} = [\boldsymbol{x}_{1*}, \ldots, \boldsymbol{x}_{m*}]$, $\boldsymbol{v} = [\boldsymbol{y}_{1*}, \ldots, \boldsymbol{y}_{m*}]$, *where* $\boldsymbol{x}_{i*}$ *and* $\boldsymbol{y}_{i*}$, $i = 1, \ldots, m$, *are the n-dimensional vectors. Then*

(a) *There exists a non-negative integer* w_j *satisfying*

$$\Theta_{jmn}(\boldsymbol{u}) = w_j \Theta_{jmn}(\boldsymbol{v}), \quad j = 1, \ldots, n.$$

(b) *A pair of* $\boldsymbol{w} = [w_1, \ldots, w_n]$ *and the* $m \times n$ *matrix Y whose i-th rows,* $i = 1, \ldots, m$, *are given by* $\boldsymbol{y}_{i*}$, *gives a solution of the IDCP.*

PROOF. By the hypothesis, the inequality (2.25) is valid. Hence, the inequalities (2.26) through (2.32) hold. Using an argument similar to the proof of Lemma 5, we have (2.21) through (2.24) for each component Π_j of $\mathfrak{T}_r$. This shows that $\Theta_{jmn}(\boldsymbol{u})$ and $\Theta_{jmn}(\boldsymbol{v})$ satisfy the hypotheses of Lemma 4. Therefore, we can determine non-negative integers $w_1, \ldots, w_n$ such that $\Theta_{jmn}(\boldsymbol{u}) = w_j \Theta_{jmn}(\boldsymbol{v})$, $j = 1, \ldots, n$. Thus, assertion (a) is verified.

We now show the second assertion (b). Let $\boldsymbol{w} = [w_1, \ldots, w_n]$ be a vector determined by (a). Since $x_{ij} = w_j y_{ij}$ for such w_j, we have $\boldsymbol{a}_{i*} \cdot \boldsymbol{x}_{i*} = (\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{w}$, $i = 1, \ldots, m$, where $\boldsymbol{a}_{i*}$ is the i-th row of A. Therefore, we have $(A \circ Y)\boldsymbol{w}^t \leqq \boldsymbol{b}^t$. The left and right hand sides of (2.27) are the (s, t)-component of GYD and L, respectively. Hence we have $GYD \leqq L$, and the lemma is proved.

Combining Lemmas 5 and 6, and using the same argument as in the proof of Theorem 1, we obtain the second main theorem:

THEOREM 2. *Any indeterminate coefficient problem is equivalent to some integer selection problem.*

Let $(r, \boldsymbol{x})$ be a solution of an ISP which is associated with an IDCP. Since the sequence $\mathfrak{T}_r$ stated in the proof of Lemma 5 is uniquely determined by $\boldsymbol{w}$ and Y, it is easily to show that the solution of the ISP constructed from $(\boldsymbol{w}, Y)$ is equal to $(r, \boldsymbol{x})$. Also, we see easily that any solution $(\boldsymbol{w}, Y)$ of the IDCP can be constructed from $(r, \boldsymbol{x})$ which is obtained by the application of Lemma 6 for

(w, Y). Hence, any solution of an IDCP is obtained through a solution of the associated ISP and vice versa. This is the second assertion of Theorem A stated in Section 1.

3. Indeterminate Coefficient Problem and Three-valued Coefficient Problem

In this section, we consider the relationship between indeterminate coefficient problems and linear 0–1 problems. We begin by giving a precise statement of the linear 0–1 problems. In the Introduction we called them 3-valued problems. However, in order to emphasize that the coefficients take their values in $\{-1, 0, 1\}$, we rephrase the 3-valued problems as follws:

DEFINITION 7. Let

(3.1) $$a_{ij} \in \{-1, 0, 1\},\ b_i,\ i = 1, \dots, m,\ j = 1, \dots, n.$$

The *three-valued coefficient problem*, or shortly TCP, is a problem of finding $x_1, \dots, x_n$ belonging to $\{0, 1\}$ and satisfying

(3.2) $$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i, \quad i = 1, \dots, m.$$

A three-valued coefficient problem is the same as a 3-valued problem introduced in the Introduction. It is sometimes more convenient to formulate the problem in a matrix form. Let $A = [a_{ij}]$ and $\boldsymbol{b} = [b_1, \dots, b_m]$ for the integers a_{ij} and b_i given in (3.1). Then the TCP is rewritten as the problem of finding a 0–1 vector $\boldsymbol{x}$ satisfying $A\boldsymbol{x}^t \leqq \boldsymbol{b}^t$. In what follows, we denote this problem by $\mathrm{TCP}(n, m, A, \boldsymbol{b})$.

A TCP is clearly regarded as an IEP. In what follows, the symbol $\mathscr{P}_T$ stands for the class of all three-valued coefficient problems and $\mathscr{S}(\mathrm{TCP}(n, m, A, \boldsymbol{b}))$, or shortly $\mathscr{S}(\mathrm{TCP})$, for the set of all solutions of the TCP.

We now formulate three types of IDCPs called a three-valued IDCP, an IDCP with boundedness conditions and an IDCP with 0–1 variables.

DEFINITION 8. A *three-valued indeterminate coefficient problem* is an indeterminate coefficient problem for the integers given by (2.3) in which a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$, are supposed to belong to $\{-1, 0, 1\}$.

Let $u_1, \dots, u_n$ be non-negative integers. Then an *indeterminate coefficient problem with boundedness conditions* for $u_1, \dots, u_n$ is an indeterminate coefficient problem with the additional conditions $x_j \leqq u_j$, $j = 1, \dots, n$. The integers $u_1, \dots, u_n$ are called the upper bounds of the IDCP.

An indeterminate coefficient problem with boundedness conditions for $u_1, \dots, u_n$ such that $u_j = 1$, $j = 1, \dots, n$, is called an *indeterminate coefficient problem with* 0–1 *variables*.

Since an IDCP is regarded as an IEP, any problem defined above is also regarded as an IEP. In what follows, we denote by $\mathscr{P}_3$, $\mathscr{P}_b$ and $\mathscr{P}_{01}$ the class of all three-valued indeterminate coefficient problems, the class of all indeterminate coefficient problems with boundedness conditions and the class of all indeterminate coefficient problems with 0–1 variables, respectively.

Let $u_1, \ldots, u_n$ be the upper bounds of an IDCP with boundedness conditions. Then we can determine the minimum integer u satisfying

$$u_j < 2^u, \quad j = 1, \ldots, n. \tag{3.3}$$

We write $u(P)$ for the integer u, where P stands for the IDCP. For an $m \times n$ integer matrix $A = [a_{ij}]$, we denote $\max\{\max\{|a_{ij}| \mid 1 \leqq i \leqq m, 1 \leqq j \leqq n\}, 1\}$ by $r(A)$. The symbol $J^*_{n,r}$ denotes the $nr \times nr$ matrix defined by

$$(\Theta_{jrn}((\Theta_{krn}(J^*_{n,r}))^t))^t = \begin{cases} J_r & \text{if } j = k, j = 1, \ldots, n, \\ [0]_{r,r} & \text{if } j \neq k, 1 \leqq j, k \leqq n. \end{cases}$$

We now state the definitions of a three-valued IDCP associated with an IDCP and an IDCP with 0–1 variables that is associated with an IDCP.

Definition 9. Let $A = [a_{ij}]$. The *three-valued IDCP associated with the problem* IDCP$(n, m, p, q, A, \boldsymbol{b}, G, D, L)$ is IDCP$(n', m', p', q', A', \boldsymbol{b}', G', D', L')$ for

(1) $r = r(A)$, $n' = nr$, $m' = m + nr$, $p' = p + m + nr$, $q' = 2nr + qr$,

(2) A' is the $m' \times n'$ matrix of the form

$$\begin{bmatrix} E_1 & E_2 & \cdots & E_r \\ & & J^*_{n,r} & \end{bmatrix},$$

(3) $\boldsymbol{b}' = [\boldsymbol{b}, [0]_{nr}]$,

(4) G' is the $p' \times m'$ matrix of the form

$$\begin{bmatrix} G & [0]_{p,nr} \\ & I_{m+nr} \end{bmatrix},$$

(5) D' is the $n' \times q'$ matrix of the form

$$[D^*_r \quad J^*_{n,r} \quad -I_{nr}],$$

(6) L' is the $p' \times q'$ matrix of the form

$$\begin{bmatrix} L^*_r & [0]_{p,nr} & [m]_{p,nr} \\ [n]_{m,qr} & [0]_{m,nr} & [0]_{m,nr} \\ [n]_{nr,qr} & [0]_{nr,nr} & [-1]_{nr,nr} \end{bmatrix},$$

where $E_k = [e^k_{ij}]$ is the $m \times n$ matrix whose components are defined by

$$e_{ij}^k = \begin{cases} 1 & \text{if } a_{ij} > 0 \text{ and } k \leqq a_{ij}, \\ 0 & \text{if } a_{ij} = 0 \text{ or } k > |a_{ij}|, \\ -1 & \text{if } a_{ij} < 0 \text{ and } k \leqq |a_{ij}|, \end{cases}$$

D_r^* is the $nr \times qr$ matrix defined by

$$(\Theta_{(n(j-1)+1)n1}((\Theta_{(q(k-1)+1)q1}(D_r^*))^t))^t$$
$$= \begin{cases} D & \text{if } j = k,\ j = 1, \dots, q, \\ [0]_{n,q} & \text{if } j \neq k,\ 1 \leqq j,\ k \leqq q, \end{cases}$$

and L_r^* is the $p \times qr$ matrix of the form $[L \dots L]$.

DEFINITION 10. Consider two problems $P \equiv \mathrm{IDCP}(n, m, p, q, A, \boldsymbol{b}, G, D, L)$ with boundedness conditions for $u_1, \dots, u_n$ and $Q \equiv \mathrm{IDCP}(n'', m'', p'', q'', A'', \boldsymbol{b}'', G'', D'', L'')$. We say that *Q is an IDCP with* 0–1 *variables whch is associated with P* if

(1) $u = u(P)$, $n'' = un + n$, $m'' = m + n$, $p'' = p + m + n$, $q'' = qu + q + 2nu + 2n$,
(2) A'' is the $m'' \times n''$ matrix of the form

$$\begin{bmatrix} 2^0 A & 2^1 A & \cdots & 2^u A \\ 2^0 I_n & 2^1 I_n & \cdots & 2^u I_n \end{bmatrix},$$

(3) $\boldsymbol{b}'' = [\boldsymbol{b}, u_1, u_2, \dots, u_n]$,
(4) G'' is the $p'' \times m''$ matrix of the form

$$\begin{bmatrix} G & [0]_{p,n} \\ & I_{m+n} \end{bmatrix},$$

(5) D'' is the $n'' \times q''$ matrix of the form

$$[D_{u+1}^* \quad J_{n,u+1}^* \quad -I_{nu+n}],$$

(6) L'' is the $p'' \times q''$ matrix of the form

$$\begin{bmatrix} L_{u+1}^* & [0]_{p,nu+n} & [m]_{p,nu+n} \\ [n]_{m,qu+q} & [0]_{m,nu+n} & [0]_{m,nu+n} \\ [n]_{n,qu+q} & [0]_{n,nu+n} & [-1]_{n,nu+n} \end{bmatrix},$$

where D_{u+1}^*, $J_{n,u+1}^*$ and L_{u+1}^* are the matrices stated as in Definition 9.

The following lemma shows a relationship between solutions of an IDCP and its associated three-valued IDCP.

LEMMA 7. *Consider the problem* $P \equiv IDCP(n, m, p, q, A, \boldsymbol{b}, G, D, L)$ *and its associated three-valued problem* $P_3 = IDCP(nr, m+nr, p+m+nr, 2nr + qr, A', \boldsymbol{b}', G', D', L')$. *Then*

(a) *If a pair of a vector* $\boldsymbol{x}$ *and a matrix* Y *is a solution of* P, *then a pair of the nr-dimensional vector* $\boldsymbol{z} = [\boldsymbol{x}, \dots, \boldsymbol{x}]$ *and the* $(m+nr) \times nr$ *matrix* $Y' = \begin{bmatrix} Y \cdots Y \\ W \cdots W \end{bmatrix}$ *gives a solution of* P_3, *where* $W = [1]_{nr,n}$.

(b) *Suppose that a pair of a vector* $\boldsymbol{z} = [\boldsymbol{z}_{1*}, \dots, \boldsymbol{z}_{r*}]$ *and a matrix* $Y' = \begin{bmatrix} Y_1 & \cdots & Y_r \\ W_1 & \cdots & W_r \end{bmatrix}$ *is a solution of* P_3, *where* $\boldsymbol{z}_{k*}$ *is an n-dimensional vector,* Y_k *is an* $m \times n$ *matrix and* W_k *is an* $nr \times n$ *matrix for* $k = 1, \dots, r$. *Then a pair of* $\boldsymbol{z}_{1*}$ *and* Y_1 *gives a solution of* P.

PROOF. First, we show the assertion (a). In view of the definition of E_k, we see that $A = \sum_{k=1}^{r} E_k$. Therefore we get

$$([E_1 \cdots E_r] \circ [Y \cdots Y])\boldsymbol{z}^t = ((\textstyle\sum_{k=1}^{r} E_k) \circ Y)\boldsymbol{x}^t = (A \circ Y)\boldsymbol{x}^t.$$

We have observe that the first m components of $(A' \circ Y')\boldsymbol{z}^t$ form the vector $(A \circ Y)\boldsymbol{x}^t$. Let $\boldsymbol{t}_{i*}^*$ and $\boldsymbol{t}_{i*}$ be the i-th rows of $J_{n,r}^*$ and J_r, respectively. Then we have

$$\begin{aligned} &\Theta_{jrn}(\boldsymbol{t}^*_{n(k-1)+j,*}) = \boldsymbol{t}_{j*}, && k = 1, \dots r, \quad j = 1, \dots, n, \\ &\Theta^*_{jrn}(\boldsymbol{t}^*_{n(k-1)+j,*}) = [0]_{n(r-1)}, && k = 1, \dots, r, \quad j = 1, \dots, n, \end{aligned} \tag{3.4}$$

where Θ_{jrn} is an element of $\mathfrak{P}_{nr}$. The expressions listed in (3.4) together imply that $J^*_{n,r}\boldsymbol{z}^t = [0]^t_{nr}$. Therefore, we obtain $(A' \circ Y')\boldsymbol{z}^t \leqq (\boldsymbol{b}')^t$. On the other hand, in view of the definitions of G', Y' and D', we have

$$\Theta_{(q(k-1)+1)q1}(G'Y'D') = \begin{bmatrix} GYD \\ YD \\ WD \end{bmatrix},$$

$$\Theta_{(qr+n(k-1)+1)n1}(G'Y'D') = \begin{bmatrix} GY - GY \\ Y - Y \\ W - W \end{bmatrix} \text{ and}$$

$$\Theta_{(qr+nr+n(k-1)+1)n1}(G'Y'D') = \begin{bmatrix} -GY \\ -Y \\ -W \end{bmatrix}$$

for $k = 1, \dots, r$. Since G and D are the matrices of the integers $-1, 0, 1$ and Y

is a 0–1 matrix, we see easily the following three inequalities:

$$GY \leqq [1]_{p,m}[1]_{m,n} = [m]_{p,n},$$
$$YD \leqq [1]_{m,n}[1]_{n,q} = [n]_{m,q}, \tag{3.5}$$
$$WD \leqq [1]_{nr,n}[1]_{n,q} = [n]_{nr,q}.$$

Combining the above inequalities and $Y \in \mathscr{Y}(G, D, L)$, we conclude that $Y' \in \mathscr{Y}(G', D', L')$. Thus, the first assertion (a) is established.

Next, we prove the assertion (b). Since z is an nr-dimensional vector, one can decompose it as $[z_{1*}, \ldots, z_{r*}]$, where z_{k*}, $k = 1, \ldots, r$, are n-dimensional vectors. We also write $\begin{bmatrix} Y_1 \cdots Y_r \\ W_1 \cdots W_r \end{bmatrix}$ for the $(m + nr) \times nr$ matrix Y, where Y_k and W_k, $k = 1, \ldots, r$, are $m \times n$ and $nr \times n$ 0–1 matrices, respectively. Since $Y' \in \mathscr{Y}(G', D', L')$, we have

$$\Theta_{(q(k-1)+1)q1}(G'Y'D') = \begin{bmatrix} GY_kD \\ Y_kD \\ W_kD \end{bmatrix} \leqq \begin{bmatrix} L \\ [n]_{m,q} \\ [n]_{nr,q} \end{bmatrix},$$

$$\Theta_{(qr+n(k-1)+1)n1}(G'Y'D') = \begin{bmatrix} GY_k - GY_{k+} \\ Y_k - Y_{k+1} \\ W_k - W_{k+1} \end{bmatrix} \leqq \begin{bmatrix} [0]_{p,n} \\ [0]_{m,n} \\ [0]_{nr,n} \end{bmatrix} \quad \text{and}$$

$$\Theta_{(qr+nr+n(k-1)+1)n1}(G'Y'D') = \begin{bmatrix} -GY_k \\ -Y_k \\ -W_k \end{bmatrix} \leqq \begin{bmatrix} [m]_{p,n} \\ [0]_{m,n} \\ [-1]_{nr,n} \end{bmatrix},$$

for $k = 1, \ldots, r$, where $Y_{r+1} = Y_1$ and $W_{r+1} = W_1$. The above inequalities imply that $Y_k \in \mathscr{Y}(G, D, L)$, $Y_k = Y_1$ and $W_k = [1]_{nr,n}$ for $k = 1, \ldots, r$. Hence we infer that $Y = Y_1 \in \mathscr{Y}(G, D, L)$. On the other hand, it follows from $(A' \circ Y')z^t \leqq (\boldsymbol{b}')^t$ that $J^*_{n,r}z^t \leqq [0]^t_{nr}$. By (3.4), we have

$$\begin{aligned} \boldsymbol{t}^*_{n(k-1)+j,*} \cdot z &= \Theta_{jrn}(\boldsymbol{t}^*_{n(k-1)+j,*}) \cdot \Theta_{jrn}(z) \\ &= \boldsymbol{t}_{j*} \cdot [\Theta_{j11}(z_{1*}), \ldots, \Theta_{j11}(z_{r*})] = \Theta_{j11}(z_{j-1,*}) - \Theta_{j11}(z_{j*}) \leqq 0 \end{aligned}$$

for $1 \leqq j \leqq n$ and $1 \leqq k \leqq r$, where $z_{0*} = z_{r*}$. Therefore, it follows that $z_{k*} = z_{1*}$, $k = 1, \ldots, r$. Put $\boldsymbol{x} = z_{1*}$, then

$$\boldsymbol{b}^t \geqq \textstyle\sum_{k=1}^r (E_k \circ Y_k)z_k^t = \sum_{k=1}^r (E_k \circ Y)\boldsymbol{x}^t = ((\sum_{k=1}^r E_k) \circ Y)\boldsymbol{x}^t = (A \circ Y)\boldsymbol{x}^t$$

holds. Thus, the lemma is proved.

We also give the similar assertion for an IDCP with boundedness conditions and its associated IDCP with 0–1 variables.

LEMMA 8. *Consider two problems* $P_b \equiv IDCP(n, m, p, q, A, \boldsymbol{b}, G, D, L)$ *with boundedness conditions in which upper bounds are* $u_1, \dots, u_n$ *and* $P_0 \equiv IDCP$ $(nu+n, m+n, p+m+n, qu+q+2nu+2n, A'', \boldsymbol{b}'', G'', D'', L'')$ *with* 0–1 *variables. Assume that* P_0 *is associated with* P_b. *Then we have the following.*

(a) *If a pair of a vector* $\boldsymbol{x}$ *and a matrix* Y *gives a solution of* P_b, *then a pair of* $(nu+n)$*-dimensional vector* $\boldsymbol{z} = [\boldsymbol{z}_{0*}, \boldsymbol{z}_{1*}, \dots, \boldsymbol{z}_{u*}]$ *and the* $(m+n) \times (nu+n)$ *matrix* $Y'' = \begin{bmatrix} Y \cdots Y \\ W \cdots W \end{bmatrix}$ *gives a solution of* P_0, *where* $W = [1]_{n,n}$ *and* $\boldsymbol{z}_{0*}, \boldsymbol{z}_{1*}, \dots, \boldsymbol{z}_{u*}$ *are n-dimensional* 0–1 *vectors defined through the binomial decomposition* $\sum_{k=0}^{u} 2^k \boldsymbol{z}_{k*} = \boldsymbol{x}$.

(b) *Assume that a pair of a vector* $\boldsymbol{z} = [\boldsymbol{z}_{0*}, \boldsymbol{z}_{1*}, \dots, \boldsymbol{z}_{u*}]$ *and a matrix* $Y'' = \begin{bmatrix} Y_0\ Y_1 \cdots Y_u \\ W_0\ W_1 \cdots W_u \end{bmatrix}$ *is a solution of* P_0, *where* $\boldsymbol{z}_{k*}$ *is an n-dimensional vector,* Y_k *is an* $m \times n$ 0–1 *matrix and* W_k *is an* $n \times n$ 0–1 *matrix for* $k = 0, 1, \dots, u$. *Then a pair of* $\boldsymbol{x} = \sum_{k=0}^{u} 2^k \boldsymbol{z}_{k*}$ *and* Y_0 *is a solution of* P_b.

PROOF. First, we show that assertion (a) holds. Let $\boldsymbol{a}''_{i*}$, $\boldsymbol{a}_{i*}$, $\boldsymbol{y}''_{i*}$ and $\boldsymbol{y}_{i*}$ be the i-th row of A'', A, Y'' and Y, respectively. Then the i-th component of $(A'' \circ Y'')\boldsymbol{z}^t$ is given by

$$\begin{aligned}(\boldsymbol{a}''_{i*} \circ \boldsymbol{y}''_{i*}) \cdot \boldsymbol{z} &= \textstyle\sum_{k=0}^{u} (\Theta_{(nk+1)n1}(\boldsymbol{a}''_{i*} \circ \boldsymbol{y}''_{i*})) \cdot \boldsymbol{z}_{k*} \\ &= \textstyle\sum_{k=0}^{u} 2^k (\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{z}_{k*} = (\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{x}.\end{aligned}$$

This implies that the first m components of $(A'' \circ Y'')\boldsymbol{z}^t$ form the vector $(A \circ Y)\boldsymbol{x}^t$. Since $\Theta_{(nk+1)n1}(\boldsymbol{a}''_{m+j,*})$ is equal to the product of 2^k and the j-th row of I_n, we have

$$(\boldsymbol{a}''_{m+j,*} \circ \boldsymbol{y}''_{m+j,*}) \cdot \boldsymbol{z} = \textstyle\sum_{k=0}^{u} 2^k\, \Theta_{j11}(\boldsymbol{z}_{k*}) = \Theta_{j11}(\boldsymbol{x}) \leqq u_j,\ j = 1, \dots, n.$$

Therefore, we have $(A'' \circ Y'')\boldsymbol{z}^t \leqq (\boldsymbol{b}'')^t$. In view of the definitions of G'', Y'' and D'', we have

$$\Theta_{(qk+1)q1}(G''Y''D'') = \begin{bmatrix} GYD \\ YD \\ WD \end{bmatrix},$$

$$\Theta_{(q(u+1)+nk+1)n1}(G''Y''D'') = \begin{bmatrix} GY - GY \\ Y - Y \\ W - W \end{bmatrix} \text{ and}$$

$$\Theta_{(q(u+1)+n(u+1)+nk+1)n1}(G''Y''D'') = \begin{bmatrix} -GY \\ -Y \\ -W \end{bmatrix}$$

for $k = 0, 1, \ldots, u$. Hence, we see that $Y'' \in \mathscr{Y}(G'', D'', L')$, because $Y \in \mathscr{Y}(G, D, L)$, $W = [1]_{n,n}$ and three inequalities in (3.5) are valid. This proves the first assertion (a) of the lemma.

Next, we show the second assertion (b). It follows from $Y'' \in \mathscr{Y}(G'', D'', L')$ that

$$\Theta_{(qk+1)q1}(G''Y''D'') = \begin{bmatrix} GY_kD \\ Y_kD \\ W_kD \end{bmatrix} \leqq \begin{bmatrix} L \\ [n]_{m,q} \\ [n]_{n,q} \end{bmatrix},$$

$$\Theta_{(q(u+1)+nk+1)n1}(G''Y''D'') = \begin{bmatrix} GY_k - GY_{k+1} \\ Y_k - Y_{k+1} \\ W_k - W_{k+1} \end{bmatrix} \leqq \begin{bmatrix} [0]_{p,n} \\ [0]_{m,n} \\ [0]_{n,n} \end{bmatrix} \quad \text{and}$$

$$\Theta_{(q(u+1)+n(u+1)+nk+1)n1}(G''Y''D'') = \begin{bmatrix} -GY_k \\ -Y_k \\ -W_k \end{bmatrix} \leqq \begin{bmatrix} [m]_{p,n} \\ [0]_{m,n} \\ [-1]_{n,n} \end{bmatrix}$$

for $k = 0, 1, \ldots, u$, where $Y_{u+1} = Y_0$ and $W_{u+1} = W_0$. The above inequalities together imply that

$$Y_k \in \mathscr{Y}(G, D, L),\ Y_k = Y_0,\ W_k = [1]_{n,n} \text{ for } 0 \leqq k \leqq u.$$

Since $\boldsymbol{x} = \sum_{k=0}^{u} 2^k z_{k*}$ and $Y = Y_0$, it follows that $Y \in \mathscr{Y}(G, D, L)$ and

$$\begin{aligned} b_i &\geqq (\boldsymbol{a}''_{i*} \circ \boldsymbol{y}''_{i*}) \cdot \boldsymbol{z} = \textstyle\sum_{k=0}^{u} (\Theta_{(nk+1)n1}(\boldsymbol{a}''_{i*} \circ \boldsymbol{y}''_{i*})) \cdot z_{K*} \\ &= \textstyle\sum_{k=0}^{u} 2^k \boldsymbol{a}_{i*} \cdot z_{k*} = \boldsymbol{a}_{i*} \cdot \boldsymbol{x}, \quad i = 1, \ldots, m, \end{aligned}$$

where b_i is the i-th component of $\boldsymbol{b}$, $\boldsymbol{a}''_{i*}$, $\boldsymbol{a}_{i*}$ and $\boldsymbol{y}''_{i*}$ are the i-th row of A'', A and Y'', respectively. In the case of $i > m$, we have

$$\begin{aligned} u_j &\geqq (\boldsymbol{a}''_{m+j,*} \circ \boldsymbol{y}''_{m+j,*}) \cdot \boldsymbol{z} = \textstyle\sum_{k=0}^{u} (\Theta_{(nk+1)n1}(\boldsymbol{a}''_{m+j,*} \circ [1]_n)) \cdot z_{k*} \\ &= \textstyle\sum_{k=0}^{u} 2^k \Theta_{j11}(z_{k*}) = \Theta_{j11}(\boldsymbol{x}), \quad j = 1, \ldots, n. \end{aligned}$$

Thus, the pair of $\boldsymbol{x}$ and Y is a solution of P_b. This shows that assertion (b) is valid. This completes the proof of Lemma 8.

We now introduce a TCP associated with a three-valued IDCP with 0–1 variables.

Definition 11. Let IDCP$(n, m, p, q, A, \boldsymbol{b}, G, D, L)$ be a three-valued problem with 0–1 variables and let $A = [a_{ij}]$, $G = [g_{si}]$, $D = [d_{jr}]$ and $L = [\ell_{sr}]$. The *three-valued coefficient problem associated with the IDCP* is TCP$(n', m', A', \boldsymbol{b}')$, where $n' = 2mn$, $m' = m + pq + mn + 2|\mathfrak{P}_{2mn,2}|$, A' is an $m' \times n'$ matrix whose i-th rows $\boldsymbol{a}'_{i*}$, $i = 1, \ldots, m + pq + mn$, are given by

$$\boldsymbol{a}'_{i*} = \begin{cases} [[0]_{ni-n}, a_{i1}, \ldots, a_{in}, [0]_{2mn-ni}] & \text{for } 1 \leqq i \leqq m, \\ [[0]_{mn}, g_{s1}d_{1r}, \ldots, g_{s1}d_{nr}, \ldots, g_{sm}d_{1r}, \ldots, g_{sm}d_{nr}] & \\ \qquad \text{for } i = m + q(s-1) + r \text{ with } 1 \leqq s \leqq p,\ 1 \leqq r \leqq q, & \\ [[0]_{r-1}, 1, [0]_{mn-1}, -1, [0]_{mn-r}] & \\ \qquad \text{for } i = m + pq + r \text{ with } 1 \leqq r \leqq mn, & \end{cases}$$

and for $i = m + pq + mn + r$ with $1 \leqq r \leqq 2|\mathfrak{P}_{2mn,2}|$, the i-th rows are defined by

$$\Pi_r^*(\boldsymbol{a}_{i*}) = [0]_{2mn-4} \text{ and } \Pi_r(\boldsymbol{a}_{i*}) = \begin{cases} [1, -1, 1, 1] & \text{if } r \leqq |\mathfrak{P}_{2mn,2}|, \\ [-1, 1, 1, 1] & \text{if } r > |\mathfrak{P}_{2mn,2}|, \end{cases}$$

and finally,

$$\boldsymbol{b}' = [\boldsymbol{b}, \ell_{11}, \ldots, \ell_{1q}, \ldots, \ell_{p1}, \ldots, \ell_{pq}, [0]_{mn}, [2]_{2|\mathfrak{P}_{2mn,2}|}],$$

with $\mathfrak{P}_{2mn,2} = \{\Pi = (i, j, mn + i, mn + j) \in \mathfrak{P}_{2mn} | i = j \bmod n\}$ and $\{\Pi_r | r = 1, \ldots, |\mathfrak{P}_{2mn,2}|\} = \mathfrak{P}_{2mn,2}$.

A TCP associated with a three-valued IDCP with 0–1 variables is clearly a TCP, and so it is regarded as an IEP. The following Lemma shows that a solution of a three-valued IDCP with 0–1 variables is obtained from the associated TCP and vice versa.

Lemma 9. *Let IDCP$(n, m, p, q, A, \boldsymbol{b}, G, D, L)$ be a three-valued problem with 0–1 variables and let TCP$(n', m', A', \boldsymbol{b}')$ be the TCP associated with the IDCP. Then we have:*

(a) *If a pair of $\boldsymbol{x} = [x_1, \ldots, x_n]$ and $Y = [y_{ij}]$ is a solution of the IDCP, then $\boldsymbol{z} = [z_{1*}, \ldots, z_{m*}, y_{1*}, \ldots, y_{m*}]$ with $z_{i*} = [x_1 y_{i1}, \ldots, x_n y_{in}]$ and $y_{i*} = [y_{i1}, \ldots, y_{in}]$, $i = 1, \ldots, m$, gives a solution of the TCP.*

(b) *If $\boldsymbol{z} = [z_{1*}, \ldots, z_{m*}, y_{1*}, \ldots, y_{m*}]$ is a solution of the TCP, where $z_{i*} = [z_{i1}, \ldots, z_{in}]$ and $y_{i*} = [y_{i1}, \ldots, y_{in}]$, $i = 1, \ldots, m$, then there exist integers $x_1, \ldots, x_n \in \{0, 1\}$ satisfying*

$$z_{ij} = x_j y_{ij}, \ i = 1, \ldots, m \text{ and } j = 1, \ldots, n,$$

and a pair of $\boldsymbol{x} = [x_1, \dots, x_n]$ *and* $Y = [y_{ij}]$ *is a solution of the IDCP.*

PROOF. To prove the first assertion (a), we write $\boldsymbol{a}_{i*}$ and $\boldsymbol{a}'_{i*}$ for the i-th row of A and A', respectively. Let $G = [g_{si}]$ and $D = [d_{jt}]$. It follows from the definition of A' that

$$\boldsymbol{a}'_{i*} \cdot \boldsymbol{z} = \boldsymbol{a}_{i*} \cdot \boldsymbol{z}_{i*} = \boldsymbol{a}_{i*} \cdot [x_1 y_{i1}, \dots, x_n y_{in}] = (\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{x},$$

for $i = 1, \dots, m$. Hence the i-th component of $A'\boldsymbol{z}^t$ is equal to the i-th component of $(A \circ Y)\boldsymbol{x}^t$ for $i = 1, \dots, m$. We also have the relations

$$\boldsymbol{a}'_{m+q(s-1)+t,*} \cdot \boldsymbol{z} = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} g_{si} y_{ij} d_{jt}, \; s = 1, \dots, p, \; t = 1, \dots, q.$$

This is nothing but the (s, t)-component of the matrix GYD. Finally, we have

$$\boldsymbol{a}'_{m+pq+i,*} \cdot \boldsymbol{z} \leqq 0 \quad \text{for} \quad i = 1, \dots, mn, \quad \text{and}$$

$$\boldsymbol{a}'_{m+pq+mn+i,*} \cdot \boldsymbol{z} \leqq 2 \quad \text{for} \quad i = 1, \dots, 2|\mathfrak{P}_{2mn,2}|.$$

Thus, we obtain $A'\boldsymbol{z}^t \leqq (\boldsymbol{b}')^t$. This completes the proof of assertion (a).

Next, we show the second assertion (b). The hypothesis of the lemma implies $A'\boldsymbol{z}^t \leqq (\boldsymbol{b}')^t$. Therefore, we immediately get

$$\boldsymbol{a}_{i*} \cdot \boldsymbol{z}_{i*} \leqq b_i \quad \text{for} \quad 1 \leqq i \leqq m \tag{3.6}$$

and $Y \in \mathscr{Y}(G, D, L)$, where b_i is the i-th component of $\boldsymbol{b}$. We also have

$$\boldsymbol{z}_{i*} \leqq \boldsymbol{y}_{i*}, \quad i = 1, \dots, m, \quad \text{and} \tag{3.7}$$

$$z_{ij} - z_{kj} + y_{ij} + y_{kj} \leqq 2, \quad i, \, k = 1, \dots, m, \quad j = 1, \dots, n. \tag{3.8}$$

Thus, it is sufficient to show that there exist $x_1, \dots, x_n \in \{0, 1\}$ such that

$$z_{ij} = x_j y_{ij} \quad \text{for} \quad i = 1, \dots, m. \tag{3.9}$$

In fact, once (3.9) is obtained, then we infer from (3.6) that

$$(\boldsymbol{a}_{i*} \circ \boldsymbol{y}_{i*}) \cdot \boldsymbol{x} = \boldsymbol{a}_{i*} \cdot \boldsymbol{z}_{i*} \leqq b_i.$$

This shows that the pair of $\boldsymbol{x}$ and Y is a solution of the IDCP. Define x_j, $j = 1, \dots, n$, by

$$x_j = \begin{cases} 0 & \text{if } z_{ij} = 0, \; i = 1, \dots, m, \\ 1 & \text{otherwise.} \end{cases}$$

If $x_j = 0$ for some j, then (3.9) holds for j and for all i with $1 \leqq i \leqq m$. We now suppose that $x_j = 1$, and that $z_{kj} \neq x_j y_{kj}$ for some k. Since $x_j = 1$, there must exist i such that $z_{ij} = 1$. From the inequalities in (3.7), we have

$$0 \leqq z_{kj} < y_{kj} \leqq 1 \text{ and } 1 \leqq z_{ij} \leqq y_{ij} \leqq 1.$$

From this it follows that $y_{ij} = 1$, $i \neq k$ and $z_{ij} - z_{kj} + y_{ij} + y_{kj} = 3$. This contradicts the inequalities in (3.8). This means that (3.9) holds. This proves the lemma.

THEOREM 3. *Any indeterminate coefficient problem with boundedness conditions is equivalent to some three-valued coefficient problem.*

PROOF. For each P in the class $\mathscr{P}_b$, we assign a problem $\varphi_0(P)$ in the class $\mathscr{P}_{01}$ which is associated with the problem P. Then φ_0 defines a mapping from $\mathscr{P}_b$ into $\mathscr{P}_{01}$. In view of the definition 10, the mapping φ_0 is well-defined. Likewise, we can define two mappings φ_3 from $\mathscr{P}_{ID}$ into $\mathscr{P}_3$ and φ_T from $\mathscr{P}_3 \cap \mathscr{P}_{01}$ into $\mathscr{P}_T$. Let φ be the composite mapping $\varphi_T \cdot \varphi_3 \cdot \varphi_0$. Then φ is a mapping from $\mathscr{P}_b$ into $\mathscr{P}_T$ and it follows from Lemmas 7, 8 and 9 that

$$\mathscr{P}_b \underset{\varphi_3 \cdot \varphi_0}{\longmapsto} \mathscr{P}_{01} \cap \mathscr{P}_3 \quad \text{and} \quad \mathscr{P}_{01} \cap \mathscr{P}_3 \underset{\varphi_T}{\longmapsto} \mathscr{P}_T.$$

Thus, Theorem 3 is established.

The transformations described in assertions (a) of Lemmas 7, 8 and 9 are the inverse transformations given in assertions (b) of Lemmas 7, 8 and 9, respectively. Thus, the above theorem proves the third assertion of Theorem A stated in Section 1.

4. Conditions for the Existence of Solutions of TCP

In this section we investigate various conditions for the existence of solutions of the three-valued coefficient problem for the integers

(4.1) $n > 0$, $m > 0$, $a_{ij} \in \{-1, 0, 1\}$ and b_i for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$.

Without loss of generality, we can assume that $b_i < 0$ for $1 \leqq i \leqq m$. In fact, if $b_h \geqq 0$ for some h with $1 \leqq h \leqq m$, we can formulate an equivalent problem P' for the integers $n' = n + b_h + 1$, $m' = m + b_h + 1$, a'_{ij} and b'_i given by

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leqq i \leqq m \text{ and } 1 \leqq j \leqq n, \\ -1 & \text{if } i = h \text{ and } n + 1 \leqq j \leqq n + b_h + 1, \\ -1 & \text{if } m + 1 \leqq i \leqq m + b_h + 1 \text{ and } j = i - m + n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b'_i = \begin{cases} b_i & \text{if } 1 \leqq i \leqq m \text{ and } i \neq h, \\ -1 & \text{if } i = h \text{ or } m + 1 \leqq i \leqq m + b_h + 1. \end{cases}$$

Clearly, P' is a TCP and equivalent to the original TCP.

Thus, we consider TCP$(n, m, A, \boldsymbol{b})$ in which $A = [a_{ij}]$, $\boldsymbol{b} = [b_i]$ for the integers

(4.2) $n > 0$, $m > 0$, $a_{ij} \in \{-1, 0, 1\}$ and $b_i < 0$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$.

We write I and J for the sets $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$, respectively. Given an n-dimensional 0–1 vector $\boldsymbol{x}$, the symbol $J(\boldsymbol{x})$ denotes the subset $\{j \in J \mid x_j = 1\}$ of J.

DEFINITION 12. Let TCP$(n, m, A, \boldsymbol{b})$ be formulated for (4.2). A subset J' of J is called a *weakly removable set* for the TCP, if for each j in J' there exists i in I such that

$$a_{ij} \geqq 0 \text{ and } \textstyle\sum_{k \in J - J'} a_{ik} > b_i - a_{ij}.$$

DEFINITION 13. Let TCP$(n, m, A, \boldsymbol{b})$ be formulated for (4.2). A subset J' of J is called a *strongly removable set* for the TCP, if for each j in J' there exists i in I such that

$$a_{ij} = 1 \text{ and } \textstyle\sum_{k \in J - J'} a_{ik} \geqq b_i.$$

EXAMPLE 5. Consider the problem TCP$(3, 2, A, \boldsymbol{b})$ in which the matrix A and the vector $\boldsymbol{b}$ are given by

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = [-1, -1].$$

Then $\{2\}$ and $\{2, 3\}$ are the strongly removable sets for the TCP, and $\{1\}$, $\{2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$ are weakly removable sets for the TCP.

DEFINITION 14. A weakly (resp. strongly) removable set J' for a TCP is said to be *maximal* if for any k in $J - J'$ the set $J' \cup \{k\}$ is not a weakly (resp. strongly) removable set for the TCP.

In order to investigate removable sets for TCPs, we consider maximal solutions of TCP.

DEFINITION 15. A solution $\boldsymbol{x}$ of TCP$(n, m, A, \boldsymbol{b})$ is said to be *maximal*, if there are no solutions $\boldsymbol{y}$ such that $J(\boldsymbol{y}) = J(\boldsymbol{x}) \cup \{k\}$ for k in $J - J(\boldsymbol{x})$.

In what follows, we denote by $\mathscr{S}^*(\text{TCP}(n, m, A, \boldsymbol{b}))$, or shortly $\mathscr{S}^*(\text{TCP})$, the set of all maximal solutions of the TCP. The following example shows the sets of $\mathscr{S}(\text{TCP})$ and $\mathscr{S}^*(\text{TCP})$ of a TCP.

EXAMPLE 6. Consider the problem TCP$(6, 4, A, \boldsymbol{b})$ in which

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & -1 \end{bmatrix} \text{ and } \boldsymbol{b} = [-1, -1, -1, -1].$$

Then

$$\begin{aligned}\mathscr{S}(\mathrm{TCP}) = \{&[0,0,0,0,1,1],\ [0,0,0,1,1,1],\ [0,0,1,0,1,1],\\ &[1,0,0,1,1,1],\ [1,0,1,1,1,1],\ [1,1,0,0,1,1],\\ &[1,1,1,0,1,1]\}\end{aligned}$$

and

$$\mathscr{S}^*(\mathrm{TCP}) = \{[1,1,1,0,1,1],\ [1,0,1,1,1,1],\ [0,0,1,0,1,1]\}.$$

LEMMA 10. *Let $\boldsymbol{x}$ be a maximal solution of $TCP(n, m, A, \boldsymbol{b})$. Then the subset $J - J(\boldsymbol{x})$ of J is maximal as a strongly removable set for the TCP.*

PROOF. Let $A = [a_{ij}]$ and $\boldsymbol{b} = [b_1, \ldots, b_m]$. We first assume that $J - J(\boldsymbol{x})$ is not a strongly removable set for the TCP. In view of Definition 13, there exists an element h in $J - J(\boldsymbol{x})$ such that $\sum_{j \in J(\boldsymbol{x})} a_{ij} < b_i$ for any $i \in I$ with $a_{ih} = 1$. This implies that

$$(4.3) \qquad \sum_{j \in J(x)} a_{ij} + 1 \leqq b_i \qquad \text{for} \quad i \in I \text{ with } a_{ih} = 1.$$

Let $\boldsymbol{y}$ be the n-dimensional 0–1 vector satisfying $J(\boldsymbol{y}) = J(\boldsymbol{x}) \cup \{h\}$. Then we have

$$\sum_{j \in J(\boldsymbol{y})} a_{ij} = \sum_{j \in J(\boldsymbol{x})} a_{ij} + 1 \leqq b_i, \qquad \text{for} \quad i \in I \text{ with } a_{ih} = 1.$$

Since $\boldsymbol{x} \in \mathscr{S}(\mathrm{TCP})$, we also have

$$\sum_{j \in J(\boldsymbol{y})} a_{ij} \leqq \sum_{j \in J(\boldsymbol{x})} a_{ij} \leqq b_i, \qquad \text{for} \quad i \in I \text{ with } a_{ih} \leqq 0.$$

The above inequalities together imply that the vector $\boldsymbol{y}$ is a solution of the TCP. This contradicts the fact that $\boldsymbol{x}$ is a maximal solution. We now assume that $J - J(\boldsymbol{x})$ is not maximal. Then there exists an element h in $J(\boldsymbol{x})$ such that $(J - J(\boldsymbol{x})) \cup \{h\}$ is a strongly removable set for the TCP. It follows from Definition 14 that there exists i in I satisfying

$$a_{ih} = 1 \text{ and } \sum_{j \in J - ((J - J(\boldsymbol{x})) \cup \{h\})} a_{ij} \geqq b_i.$$

We then have

$$\sum_{j \in J(\boldsymbol{x})} a_{ij} = \sum_{j \in J(\boldsymbol{x}) - \{h\}} a_{ij} + a_{ih} = \sum_{j \in J - ((J - J(\boldsymbol{x})) \cup \{h\})} a_{ij} + 1 > b_i.$$

This is a contradiction, since x is a solution of the TCP. Thus, $J - J(x)$ is maximal as a strongly removable set for the TCP. This proves the lemma.

LEMMA 11. *Let x be a maximal solution of $TCP(n, m, A, b)$. Then the subset $J - J(x)$ of J is maximal as a weakly removable set for the TCP.*

PROOF. It follows from Definitions 12 and 13 that a strongly removable set is a weakly removable set. Hence we see from Lemma 10 that $J - J(x)$ is a weakly removable set for the TCP. Accordingly, it is sufficient to show that $J - J(x)$ is maximal as a weakly removable set. Let $A = [a_{ij}]$ and $b = [b_1, \dots, b_m]$, where a_{ij} and b_i are integers given in (4.2). Suppose then that there exists an element h in $J(x)$ such that $(J - J(x)) \cup \{h\}$ is a weakly removable set for the TCP. Then there exists an element i in I satisfying

$$a_{ih} \geqq 0 \text{ and } \sum_{j \in J - ((J - J(x)) \cup \{h\})} a_{ij} > b_i - a_{ih}.$$

Hence, we have

$$\sum_{j \in J(x)} a_{ij} = \sum_{j \in J - (J - J(x)) \cup \{h\}} a_{ij} + a_{ih} > b_i.$$

This contradicts the fact that x is a solution of the TCP. Therefore, $J - J(x)$ is maximal as a weakly removable set for the TCP. This completes the proof of the lemma.

The first assertion of Theorem B stated in Section 1 is obtained by combining Lemmas 10 and 11. In addition, we obtain the following result.

THEOREM 4. *If there exists a solution of $TCP(n, m, A, b)$, then there must exist a subset J' of J such that J' is maximal as a weakly removable set and, at the same time, maximal as a strongly removable set for the TCP.*

EXAMPLE 7. Take the same problem $TCP(6, 4, A, b)$ as in Example 6. Then the strongly removable sets for the TCP are $\{2\}$, $\{3\}$, $\{4\}$ and $\{1, 2, 4\}$, and the weakly removable sets are $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 6\}$, $\{2, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{2, 3, 4, 5\}$, $\{3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$. Therefore, $\{2\}$, $\{3\}$, $\{4\}$ and $\{1, 2, 4\}$ are maximal as a strongly removable set and $\{2\}$, $\{4\}$, $\{1, 2, 4\}$, $\{1, 3, 4, 6\}$, $\{2, 3, 4, 5\}$, $\{3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$ are maximal as a weakly removable set. Hence, $\{2\}$, $\{4\}$ and $\{1, 2, 4\}$ are the maximal sets which are both strongly and weakly removable sets. On the other hand, as seen in Exampe 6, we have

$$\{J - J(x) | x \in \mathcal{S}^*(\text{TCP})\} = \{\{2\}, \{4\}, \{1, 2, 4\}\}.$$

The second assertion of Theorem B gives a condition for checking whether or not a given 0–1 vector is a solution of a TCP.

THEOREM 5. *Let $TCP(n, m, A, \boldsymbol{b})$ be formulated for the integers given by* (4.2) *and let $J_- = \{j \in J \mid \sum_{i \in I} a_{ij} < 0\}$. If the inequality*

$$\sum_{j \in J_- - J'} \sum_{i \in I} a_{ij} > \sum_{i \in I} b_i$$

holds for a subset J' of J, then $J(\boldsymbol{x}) \cap J' \neq \emptyset$ for any solution $\boldsymbol{x}$ of the TCP.

PROOF. Suppose that $J(\boldsymbol{x}) \cap J' = \emptyset$ for some solution $\boldsymbol{x}$ of the TCP. Since $\boldsymbol{x} \in \mathcal{S}(\text{TCP})$, $\sum_{j \in J(\boldsymbol{x})} a_{ij} \leqq b_i$ for each $i \in I$. Hence, we have

$$\sum_{j \in J(\boldsymbol{x})} \sum_{i \in I} a_{ij} \leqq \sum_{i \in I} b_i. \tag{4.4}$$

Since $J(\boldsymbol{x}) \cap J' = \emptyset$, it follows that

$$J(\boldsymbol{x}) \cap (J_- - J') = J(\boldsymbol{x}) \cap J_- - J(\boldsymbol{x}) \cap J' = J(\boldsymbol{x}) \cap J_-.$$

Therefore, we obtain

$$\begin{aligned} \sum_{j \in J(\boldsymbol{x})} \sum_{i \in I} a_{ij} &\geqq \sum_{j \in J(\boldsymbol{x}) \cap J_-} \sum_{i \in I} a_{ij} = \sum_{j \in J(\boldsymbol{x}) \cap (J_- - J')} \sum_{i \in I} a_{ij} \\ &\geqq \sum_{j \in J_- - J'} \sum_{i \in I} a_{ij} > \sum_{i \in I} b_i. \end{aligned}$$

This contradicts the inequality (4.4). Thus, $J(\boldsymbol{x}) \cap J' \neq \emptyset$ is valid for any solution $\boldsymbol{x}$ of the TCP. This completes the proof of Theorrem 5.

The next assertion is an immediate consequence of Theorem 5 in the case of $J' = \emptyset$.

COROLLARY 1. *Let TCP be formulated via* (4.2). *If the inequality*

$$\sum_{j \in J_-} \sum_{i \in I} a_{ij} > \sum_{i \in I} b_i,$$

holds, then there are no solutions of the TCP.

Let J' be a subset of J and p a positive integer. Put

$$D(A, J', p) = \{i \in I \mid e_i \geqq 2g_i^+(A, J') + g_i^0(A, J')\},$$

where

$$e_i = \begin{cases} b_i + p & \text{if } \; p \leqq |J'|, \\ b_i + |J'| & \text{if } \; p > |J'|, \end{cases}$$

$g_i^+(A, J') = |\{j \in J' \mid a_{ij} = 1\}|$ and $g_i^0(A, J') = |\{j \in J' \mid a_{ij} = 0\}|$.

Then the following two propositions can be easily checked.

PROPOSITION 5. *$D(A, J', p_1) \supset D(A, J', p_2)$ for $p_1 \geqq p_2$.*

PROPOSITION 6. *For any $J_p \subset J'$ such that $|J_p| = p$,*

$$\sum_{j\in J_p} a_{ij} \leqq b_i \quad \textit{for any } i\in D(A, J', p).$$

REMARK 2. Since $b_i < 0$ for $i\in I$, we have $|J(\boldsymbol{x})| \geqq \max\{|b_i|\, |i\in I\}$ for any solution $\boldsymbol{x}$ of TCP$(n, m, A, \boldsymbol{b})$. Hence, Propositions 5 and 6 imply that if $i\in D(A, J', \max\{|b_i|\,|i\in I\})$ then the i-th inequality of $A\boldsymbol{x}^t \leqq \boldsymbol{b}^t$ is trivial and we can neglect these rows under the conditions $x_j = 0$, $j\in J - J'$.

By using Remark 2 and Theorem 5, we can obtain a simple condition to test the existence of a solution $\boldsymbol{x} = [x_j]$ such that $x_j = 0$ for $j\in J'$ and some subset J' of J.

COROLLARY 2. *Let TCP be formulated through* (4.2), *J' a subset of J and let $I' = D(A, J', \max\{|b_i|\,|i\in I\})$. If the inequality*

$$\sum_{j\in J_- - J'} \sum_{i\in I} a_{ij} - \sum_{j\in J_- - J'} \sum_{i\in I'} a_{ij} > \sum_{i\in I - I'} b_i$$

holds, then $J(\boldsymbol{x})\cap J' \neq \emptyset$ for any solution $\boldsymbol{x}$ of the TCP.

The following result is easily obtained.

THEOREM 6. *For any solution $\boldsymbol{x}$ of TCP$(n, m, A, \boldsymbol{b})$,*

$$\sum_{j\in J(\boldsymbol{x})} \sum_{i\in I} a_{ij} - \sum_{j\in J(\boldsymbol{x})} \sum_{i\in I'} a_{ij} \leqq \sum_{i\in I - I'} b_i$$

holds, where $I' = D(A, J(\boldsymbol{x}), \max\{|b_i|\,|i\in I\})$.

It is clear that $|J(\boldsymbol{x})| \geqq \max\{|b_i|\,|i\in I\}$ for any solution $\boldsymbol{x}$ of TCP$(n, m, A, \boldsymbol{b})$. Conversely, the following Proposition is valid for the case in which $|J'| < \max\{|b_i|\,|i\in I\}$.

PROPOSITION 7. *For any subset J' of J with $|J'| < \max\{|b_i|\,|i\in I\}$, the inequality*

$$\sum_{i\in J'} \sum_{i\in I} a_{ij} - \sum_{j\in J'} \sum_{i\in I'} a_{ij} > \sum_{i\in I - I'} b_i$$

holds, where $I' = D(A, J', \max\{|b_i|\,|i\in I\})$.

PROOF. Since $|J'| < \max\{|b_i|\,|i\in I\}$ and $b_i < 0$ for i in I,

$$|J'| + b_i < 2g_i^+(A, J') + g_i^0(A, J')$$

holds for any i in $I - I'$. Set $g_i^-(A, J') = |\{j\in J'|a_{ij} = -1\}|$. Then we have $|J'| = g_i^-(A, J') + g_i^0(A, J') + g_i^+(A, J')$. Hence,

$$\begin{aligned}\sum_{j\in J'} a_{ij} &= -g_i^-(A, J') + g_i^+(A, J') \\ &= -|J'| + 2g_i^+(A, J') + g_i^0(A, J') > b_i\end{aligned}$$

holds for each i in $I - I'$. Thus, we have

$$\sum_{j\in J'} \sum_{i\in I} a_{ij} - \sum_{j\in J'} \sum_{i\in I'} a_{ij} = \sum_{j\in J'} \sum_{i\in I-I'} a_{ij} > \sum_{i\in I-I'} b_i,$$

and the proof is now complete.

REMARK 3. If there exists a maximal solution $\boldsymbol{x}$ of TCP$(n, m, A, \boldsymbol{b})$, then the following two facts are derived from Theorems 4 and 6:

(a) $J - J(\boldsymbol{x})$ is maximal as a weakly and strongly removable set,

(b) $\sum_{j\in J(\boldsymbol{x})} \sum_{i\in I} a_{ij} - \sum_{j\in J(\boldsymbol{x})} \sum_{i\in I'} a_{ij} \leqq \sum_{i\in I-I'} b_i,$

where $I' = D(A, J', \max\{|b_i|\,|i\in I\})$. It is not known yet whether the converse is true. However it is easily seen that $\boldsymbol{x}$ is a solution of the TCP if (b) holds for an 0–1 vector $\boldsymbol{x}$ with $|J(\boldsymbol{x})| = \max\{|b_i|\,|i\in I\}$.

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