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Existence of non-constant stable equilibria in competition-diffusion equations

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1. Introduction

The study of coexistence problem of competing species is one of the main topics in mathematical ecology. In this paper we study the relation between the coexistence of two competing species and the domain shape of their habitat.

The model system which we use here is the following Lotka-Volterra type reaction-diffusion equation:

$$(1.1)\qquad \begin{cases} u_t = d_u \Delta u + u(a_u - b_u u - c_u v), \\ v_t = d_v \Delta v + v(a_v - b_v u - c_v v), & x \in \Omega,\ t > 0, \\ \dfrac{\partial}{\partial \nu} u(t, x) = 0 = \dfrac{\partial}{\partial \nu} v(t, x), & x \in \partial\Omega,\ t > 0, \\ u(0, x) \geq 0,\ v(0, x) \geq 0, & x \in \bar{\Omega}, \end{cases}$$

where Δ is the Laplace operator, Ω a bounded domain in $\mathbf{R}^{n+1}$, u and v the population densities of the two competing species. The constants a_u and a_v are the intrinsic growth rates, b_u, c_v and b_v, c_u the coefficients of intraspecific and interspecific competition, respectively, d_u and d_v the diffusion rates. We assume that all these constants are positive.

By a suitable normalization, we can rewrite (1.1) as

$$(1.2)\qquad \begin{cases} u_t = \varepsilon^2 \Delta u + u(1 - u - cv), \\ v_t = \varepsilon^2 d \Delta v + v(a - bu - v), & x \in \Omega,\ t > 0, \\ \dfrac{\partial}{\partial \nu} u\,(t, x) = 0 = \dfrac{\partial}{\partial \nu} v(t, x), & x \in \partial\Omega,\ t > 0, \\ u(0, x) \geq 0,\ v(0, x) \geq 0, & x \in \bar{\Omega}, \end{cases}$$

where a, b, c, d, ε are positive. The following result is well-known (for example, see de Mottoni [7]):

(I) If $a < \min\{b, 1/c\}$, then $\lim_{t\to\infty}(u(t, x), v(t, x)) = (1, 0)$.

(II) If $b < a < 1/c$, then $\lim_{t\to\infty}(u(t, x), v(t, x)) = \left(\dfrac{1-ac}{1-bc}, \dfrac{a-b}{1-bc}\right)$.

(III) If $1/c < a < b$, then $(1, 0)$ and $(0, a)$ are locally stable.

(IV) If $\max\{b, 1/c\} < a$, then $\lim_{t\to\infty}(u(t, x), v(t, x)) = (0, a)$.

By using Lyapunov's second method, it is also shown that the above results except for Case (III) are independent of the domain shapes. In Case (III), it was shown by Kishimoto and Weinberger [2] that any non-constant stationary solution of (1.2) is unstable if the domain Ω is convex, that is, only one species can survive and the other species becomes extinct due to competition.

Let us consider the case where their habitat is dumbbell-shaped, *i.e.*, the habitat consists of two distinct domains (a domain A and a domain B) and a channel connecting two domains. If the channel is sufficiently narrow, then it is difficult for species on one domain to migrate to another domain through the channel. This implies that the competition is relaxed by the narrow channel and two competing species may show a segregated pattern, that is, one species lives dominantly in domain A and the other dominantly in domain B. In fact, Matano and Mimura [5] proved that there exists a stable non-constant stationary solution if the channel is sufficiently narrow.

We guess that the weaker species can not survive due to competition if the stronger species becomes so active as to migrate to another domain at will through the channel. Conversely, if the stronger species is not so active, two competing species may coexist even in the case where the channel is not so narrow. In other words, the coexistence of two competing species depends on not only the domain shapes but also the magnitude of the competition-diffusion. Our aim is to show how these two effects are concerned with the coexistence.

In this paper, we consider the coexistence problem of two competing species in domains which are non-convex but not necessarily dumbbell-shaped. We shall prove that there exists a stable non-constant stationary solution of (1.2) if the curvature of a boundary of the domain and the magnitude of competition-diffusion are well-balanced in a sense. To show the existence of a stable non-constant stationary solution, we shall make use of the maximum principle (or super-subsolution technique) developed by Matano [4], *i.e.*, we shall construct stable sets of a dynamical system arising from (1.2).

In Section 2 as the first stage, we study travelling wave solutions of a one-dimensional equation. In Section 3, we construct stable sets of (1.2) when Ω is a thin tubular domain or a rotationally symmetric tubular domain. We also study a spatially inhomogeneous one-dimensional equation.

2. Travelling wave solutions of a one-dimensional equation

In this section, we consider the one-dimensional equation

$$\begin{cases} u_t = u_{xx} + f(u, v), \\ dv_t = dv_{xx} + g(u, v), \qquad x \in \mathbf{R},\ t > 0, \end{cases}$$

where $f(u, v) = u(1 - u - cv)$ and $g(u, v) = v(a - bu - v)$. A travelling wave solution $(u, v) = (u(\xi), v(\xi))$, $\xi = x - st$, where s is the propagation speed, satisfies the ordinary differential equation

$$\text{(2.1)} \qquad \begin{cases} 0 = u_{\xi\xi} + su_\xi + f(u, v), \\ 0 = dv_{\xi\xi} + sdv_\xi + g(u, v), \qquad \xi \in \mathbf{R}, \\ (u, v)(-\infty) = (0, a),\ (u, v)(+\infty) = (1, 0). \end{cases}$$

Solutions of this equation will play important roles in the proof of the existence of a stable non-constant stationary solution of (1.2). Since (2.1) does not depend on ξ explicitly, we assume

$$u(0) = 1/2$$

in order to fix the solution. We shall say that (u, v) is (*strictly*) *positively monotone* if u is (strictly) increasing and v is (strictly) decreasing.

From an ecological viewpoint, the propagation speed s represents the difference of the strength of two competing species. If $s > 0$ (resp. $s < 0$), then the travelling wave propagates rightward (resp. leftward). This implies that the species u is weaker (resp. stronger) than the species v. If $s = 0$, then two species are in equilibrium.

Mimura and Fife [6] obtained the following result:

LEMMA 2.1. (Mimura and Fife). *Suppose that $bc > 1$. Then there exists $a \in (1/c, b)$ such that* (2.1) *with $s = 0$ has a strictly positively monotone solution.*

Let b_0, c_0 and d_0 be any constants satisfying

$$\text{(2.2)} \qquad 1 < b_0 c_0 < \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad \frac{b_0 c_0 - 1}{c_0} < d_0 < \frac{b_0}{b_0 c_0 - 1},$$

and let a_0 and (u_0, v_0) be the constant and the strictly positively monotone solution, respectively, of (2.1) with $(b, c, d, s) = (b_0, c_0, d_0, 0)$ given in Lemma 2.1. The eigenvalues of the linearized operator of (2.1) at $(u, v) = (0, a_0)$ (resp. $(1, 0)$) for $(a, b, c, d, s) = (a_0, b_0, c_0, d_0, 0)$ are easily obtained as

$$\pm\sqrt{a_0 c_0 - 1},\ \pm\sqrt{a_0/d_0} \ \text{(resp. } \pm\sqrt{(b_0 - a_0)/d_0},\ \pm 1).$$

From the inequalities (2.2) and $1/c_0 < a_0 < b_0$, we have

$$a_0 c_0 - 1 < a_0/d_0 \quad \text{and} \quad (b_0 - a_0)/d_0 < 1. \tag{2.3}$$

In the following, we fix (a, d) to (a_0, d_0) and take (b, c) as parameters.

If we put $\mathbf{u} = {}^t(u, u_\xi, v, v_\xi)$ and

$$\mathbf{F}(\mathbf{u}; b, c, s) = \begin{pmatrix} u_\xi \\ -u(1 - u - cv) - su_\xi \\ v_\xi \\ -v(a_0 - bu - v)/d_0 - sv_\xi \end{pmatrix},$$

then (2.1) is rewritten as

$$\frac{d}{d\xi}\mathbf{u} = \mathbf{F}(\mathbf{u}; b, c, s). \tag{2.4}$$

Clearly there is a one-to-one correspondence between a solution of (2.1) and a heteroclinic orbit of this 4-dimensional dynamical system which connects $(0, 0, a_0, 0)$ with $(1, 0, 0, 0)$. Hence, by Lemma 2.1, there exists a heteroclinic orbit for (2.4) at $(b, c) = (b_0, c_0)$. We can show that the heteroclinic orbit persists when we take (b, c) as bifurcation parameters in a neighborhood of (b_0, c_0). In fact, we can prove the following theorem by applying Theorem A of Kokubu [3] to our problem.

THEOREM 2.1. *Suppose that* (2.2) *holds. Then there exists a* (b, c)*-family of solutions* $(\hat{u}(\xi; b, c), \hat{v}(\xi; b, c), s(b, c))$ *of* (2.1) *in a neighborhood of* (b_0, c_0) *such that*

$$\begin{aligned} &\lim_{(b,c)\to(b_0,c_0)} \hat{u}(\xi; b, c) = u_0(\xi), \qquad \lim_{(b,c)\to(b_0,c_0)} \hat{v}(\xi; b, c) = v_0(\xi), \\ &\lim_{(b,c)\to(b_0,c_0)} s(b, c) = 0, \\ &\frac{\partial}{\partial b} s(b_0, c_0) < 0, \qquad \frac{\partial}{\partial c} s(b_0, c_0) > 0. \end{aligned} \tag{2.5}$$

Furthermore $(\hat{u}(\xi; b, c), \hat{v}(\xi; b, c))$ *is strictly positively monotone and satisfies*

$$(\hat{u}(\xi; b, c), \hat{v}(\xi; b, c)) = \begin{cases} (0, a_0) + \beta_-(1, e_-(b, c))\exp(\gamma_-(b, c)\xi) \\ \qquad + o(\exp(\gamma_-(b, c)\xi)) \quad \text{as} \quad \xi \longrightarrow -\infty, \\ (1, 0) - \beta_+(1, e_+(b, c))\exp(-\gamma_+(b, c)\xi) \\ \qquad + o(\exp(-\gamma_+(b, c)\xi)) \quad \text{as} \quad \xi \longrightarrow +\infty, \end{cases} \tag{2.6}$$

where β_- *and* β_+ *are certain positive constants and* $\gamma_-(b, c)$, $\gamma_+(b, c)$, $e_-(b, c)$

and $e_+(b, c)$ *are continuous in* (b, c) *and satisfy*

$$
(2.7)\qquad
\begin{aligned}
\gamma_-(b_0, c_0) &= \sqrt{a_0 c_0 - 1},\\
\gamma_+(b_0, c_0) &= \sqrt{(b_0 - a_0)/d_0},\\
e_-(b_0, c_0) &= -\frac{a_0 b_0}{a_0 - d_0(a_0 c_0 - 1)} \ (< 0),\\
e_+(b_0, c_0) &= -\frac{a_0 + d_0 - b_0}{c_0 d_0} \ (< 0).
\end{aligned}
$$

Since the proof of this theorem needs lengthy argument, we state the proof in Appendix.

Let $(\hat{b}(\varepsilon), \hat{c}(\varepsilon))$ be a function which depends on a small parameter ε smoothly and satisfies

$$
(2.8)\qquad (\hat{b}(\varepsilon), \hat{c}(\varepsilon)) = (b_0 + b_1\varepsilon + o(\varepsilon), c_0 + c_1\varepsilon + o(\varepsilon)) \quad \text{as} \quad \varepsilon \longrightarrow 0,
$$

where b_1 and c_1 are constants. Then, by virtue of Theorem 2.1, the propagation speed $s(\hat{b}(\varepsilon), \hat{c}(\varepsilon))$ of $(\hat{u}(\xi; \hat{b}(\varepsilon), \hat{c}(\varepsilon)), \hat{v}(\xi; \hat{b}(\varepsilon), \hat{c}(\varepsilon)))$ satisfies

$$
(2.9)\qquad s(\hat{b}(\varepsilon), \hat{c}(\varepsilon)) = (M_b b_1 + M_c c_1)\varepsilon + o(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0,
$$

where we put $M_b = \dfrac{\partial}{\partial b} s(b_0, c_0) < 0$ and $M_c = \dfrac{\partial}{\partial c} s(b_0, c_0) > 0$.

Throughout this paper, we shall use the following notations:

b_0, c_0 and d_0 are constants which satisfy (2.2).

a_0 and (u_0, v_0) are the constant and the strictly positively monotone solution, respectively, of (2.1) with $(b, c, d, s) = (b_0, c_0, d_0, 0)$ given in Lemma 2.1.

$(\hat{u}, \hat{v})$ is the strictly positively monotone solution of (2.1) with $(a, b, c, d, s) = (a_0, b, c, d_0, s(b, c))$ given in Theorem 2.1.

$(\hat{b}(\varepsilon), \hat{c}(\varepsilon))$ is a function of ε which is represented as (2.8).

3. Existence of stable stationary solutions

3.1. Supersolutions and subsolutions

We introduce an order relation into the space $C(\bar{\Omega}) \times C(\bar{\Omega})$ in the following manner:

$$
(u_1, v_1) \geq (u_2, v_2) \Longleftrightarrow u_1(x) \geq u_2(x) \quad \text{and} \quad v_1(x) \leq v_2(x) \ \text{in} \ \bar{\Omega}.
$$

Let $\Phi(t)$ be a local semiflow on $C(\bar{\Omega}) \times C(\bar{\Omega})$ defined by (1.2).

DEFINITION 3.1. $q \in C(\bar{\Omega}) \times C(\bar{\Omega})$ is called a *(time-independent) supersolution* (resp. *subsolution*) of (1.2) if

$$\Phi(t) q \leq q \text{ (resp. } \Phi(t) q \geq q) \qquad \text{for all } t \geq 0.$$

If, in addition, q is not a stationary solution of (1.2), then it is called a *strict supersolution* (resp. *strict subsolution*).

The following result was proved by Matano [4]:

LEMMA 3.1. (Matano). *Let $\bar{q}, \underline{q} \in C(\bar{\Omega}) \times C(\bar{\Omega})$ be a strict supersolution and a strict subsolution of* (1.2) *respectively satisfying $\bar{q} \geq \underline{q}$, $\bar{q} \neq \underline{q}$. Then there exists a stable stationary solution $p = (u, v)$ of* (1.2) *satisfying $\bar{q} \geq p \geq \underline{q}$.*

From the maximum principle in a generalized sense, the following is obtained.

LEMMA 3.2. (Matano and Mimura [5]). *Suppose that (u, v) is a continuous piecewise C^2-class function satisfying*

$$\begin{cases} \varepsilon^2 \Delta u + f(u, v) \leq 0, \\ \varepsilon^2 d \Delta v + g(u, v) \geq 0, \qquad x \in \Omega \end{cases}$$

(in the sense of distribution) and

$$\frac{\partial}{\partial \nu} u(t, x) \geq 0 \geq \frac{\partial}{\partial \nu} v(t, x), \qquad x \in \partial\Omega.$$

Then (u, v) is a supersolution of (1.2). *If (u, v) satisfies the reversed differential inequalities, then it is a subsolution of* (1.2).

3.2. One-dimensional equations

In this subsection, we consider the equation

$$\text{(3.1)} \qquad \begin{cases} u_t = \varepsilon^2 u_{xx} + \varepsilon^2 \gamma(x) u_x + f(u, v), \\ v_t = \varepsilon^2 d_0 v_{xx} + \varepsilon^2 d_0 \gamma(x) v_x + g(u, v), & x \in (0, L),\ t > 0, \\ u_x(t, x) = 0 = v_x(t, x), & x = 0, L,\ t > 0, \end{cases}$$

where $\gamma(x) \in C([0, L])$, $f(u, v) = u(1 - u - cv)$ and $g(u, v) = v(a_0 - bu - v)$. We shall construct a supersolution and a subsolution of this equation by using the technique developed in [9].

Let $\bar{x} \in (0, L)$ and $\ell > 0$ satisfy $[\bar{x} - \ell, \bar{x} + \ell] \subset (0, L)$, and let $\varepsilon > 0$ be a small parameter. Put $(\hat{u}^\varepsilon(x; y), \hat{v}^\varepsilon(x; y)) = (\hat{u}((x - y)/\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon)), \hat{v}((x - y)/$

$\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon)))$ and define $(\bar{U}, \bar{V}) = (\bar{U}(x; \bar{x}, \varepsilon, \ell), \bar{V}(x; \bar{x}, \varepsilon, \ell))$ by

$$\bar{U}(x; \bar{x}, \varepsilon, \ell) = \begin{cases} \hat{u}^\varepsilon(\bar{x} - \ell; \bar{x}) + \bar{P}_u \ell^2/4 & x \in [0, \bar{x} - \ell], \\ \hat{u}^\varepsilon(x; \bar{x}) + \bar{P}_u(x - \bar{x} + \ell/2)^2 & x \in (\bar{x} - \ell, \bar{x} - \ell/2], \\ \hat{u}^\varepsilon(x; \bar{x}) & x \in (\bar{x} - \ell/2, \bar{x} + \ell/2], \\ \hat{u}^\varepsilon(x; \bar{x}) + \bar{Q}_u(x - \bar{x} - \ell/2)^2 & x \in (\bar{x} + \ell/2, \bar{x} + \ell], \\ 1 & x \in (\bar{x} + \ell, L], \end{cases}$$

$$\bar{V}(x; \bar{x}, \varepsilon, \ell) = \begin{cases} \hat{v}^\varepsilon(\bar{x} - \ell; \bar{x}) - \bar{P}_v \ell^2/4 & x \in [0, \bar{x} - \ell], \\ \hat{v}^\varepsilon(x; \bar{x}) - \bar{P}_v(x - \bar{x} + \ell/2)^2 & x \in (\bar{x} - \ell, \bar{x} - \ell/2], \\ \hat{v}^\varepsilon(x; \bar{x}) & x \in (\bar{x} - \ell/2, \bar{x} + \ell/2], \\ \hat{v}^\varepsilon(x; \bar{x}) - \bar{Q}_v(x - \bar{x} - \ell/2)^2 & x \in (\bar{x} + \ell/2, \bar{x} + \ell], \\ 0 & x \in (\bar{x} + \ell, L], \end{cases}$$

where $\bar{P}_u, \bar{Q}_u, \bar{P}_v, \bar{Q}_v > 0$ are constants given by

$$\bar{P}_u = \frac{1}{\varepsilon\ell}\hat{u}_\xi(-\ell/\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon)), \qquad \bar{Q}_u = \frac{4}{\ell^2}\{1 - \hat{u}(\ell/\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon))\},$$

$$\bar{P}_v = -\frac{1}{\varepsilon\ell}\hat{v}_\xi(-\ell/\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon)), \qquad \bar{Q}_v = \frac{4}{\ell^2}\hat{v}(\ell/\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon))$$

(see Figure 1). We note that $(\bar{U}, \bar{V})$ is continuous in ξ and is positively monotone for sufficiently small $\varepsilon > 0$.

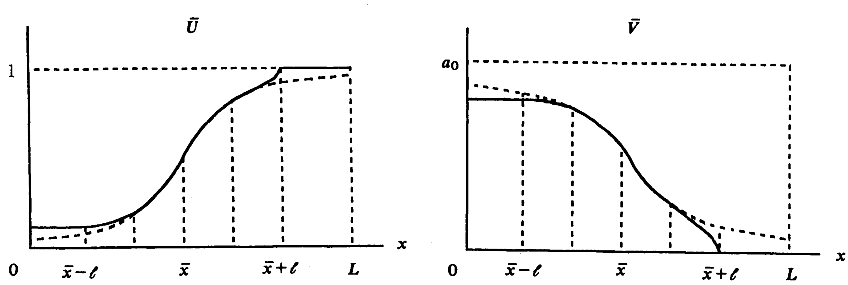

Figure 1 Supersolution $(\bar{U}, \bar{V})$

LEMMA 3.3. *Suppose that* $M_b b_1 + M_c c_1 > \gamma(x)$ *for all* $x \in [\bar{x} - \ell, \bar{x} + \ell] \subset (0, L)$ *for some* $\bar{x} \in (0, L)$ *and* $\ell > 0$. *Then there exists* $\bar{\varepsilon} = \bar{\varepsilon}(\ell) > 0$ *such that,* *if* $0 < \varepsilon \le \bar{\varepsilon}$, $(\bar{U}(x; \bar{x}, \varepsilon, \ell), \bar{V}(x; \bar{x}, \varepsilon, \ell))$ *defined as above is a strict supersolution of* (3.1).

PROOF. By assumption and (2.9), we have $\varepsilon\gamma(x) - s(\hat{b}(\varepsilon), \hat{c}(\varepsilon)) < 0$ for $x \in [\bar{x} - \ell, \bar{x} + \ell]$ if $\varepsilon > 0$ is sufficiently small. According to the asymptotic behavior of $(\hat{u}(\xi; b, c), \hat{v}(\xi; b, c))$ as $\xi \to \pm\infty$ (see (2.6)), we have the following:

(i) $\bar{P}_v/\bar{P}_u = -e_-(b_0, c_0) + o(1)$, $\bar{Q}_v/\bar{Q}_u = -e_+(b_0, c_0) + o(1)$,

(ii) $(\bar{U}, \bar{V}) = (0, a_0) + o(1)$ and $(\bar{V} - a_0)/\bar{U} = e_-(b_0, c_0) + o(1)$ for $x \in (0, \bar{x} - \ell)$,

(iii) $\varepsilon\bar{P}_u/\hat{u}^\varepsilon_\xi = o(1)$ for $x \in (\bar{x} - 3\ell/4, \bar{x} - \ell/2)$,

(iv) $\varepsilon\bar{Q}_u/\hat{u}^\varepsilon_\xi = o(1)$ for $x \in (\bar{x} + \ell/2, \bar{x} + 3\ell/4)$,

as $\varepsilon \downarrow 0$. Moreover, from (2.2) and (2.3), we have the following inequalities:

$$f_u(0, a_0) + f_v(0, a_0)e_-(b_0, c_0) = -(a_0c_0 - 1) < 0,$$

$$g_u(0, a_0) + g_v(0, a_0)e_-(b_0, c_0) = \frac{a_0b_0d_0(a_0c_0 - 1)}{a_0 - d_0(a_0c_0 - 1)} > 0,$$

$$f_u(1, 0) + f_v(1, 0)e_+(b_0, c_0) = -\frac{b_0 - a_0}{d_0} < 0,$$

$$g_u(1, 0) + g_v(1, 0)e_+(b_0, c_0) = \frac{(b_0 - a_0)(a_0 + d_0 - b_0)}{c_0d_0} > 0.$$

Hence the following estimates are obtained as $\varepsilon \downarrow 0$.

(i) For $x \in (0, \bar{x} - \ell)$,

$$\begin{aligned}
\varepsilon^2\bar{U}_{xx} + \varepsilon^2\gamma(x)\bar{U}_x + f(\bar{U}, \bar{V}) &= f(\bar{U}, \bar{V}) \\
&= f_u(0, a_0)\bar{U} + f_v(0, a_0)(\bar{V} - a_0) + o(|\bar{U}|, |\bar{V} - a_0|) \\
&= \bar{U}\{f_u(0, a_0) + f_v(0, a_0)e_-(b_0, c_0) + o(1)\} \\
&< 0.
\end{aligned}$$

(ii) For $x \in (\bar{x} - \ell, \bar{x} - 3\ell/4)$,

$$\begin{aligned}
&\varepsilon^2\bar{U}_{xx} + \varepsilon^2\gamma(x)\bar{U}_x + f(\bar{U}, \bar{V}) \\
&\quad = \{\varepsilon\gamma(x) - s(\hat{b}(\varepsilon), \hat{c}(\varepsilon))\}\hat{u}^\varepsilon_\xi + f(\bar{U}, \bar{V}) - f(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \\
&\qquad + 2\varepsilon^2\bar{P}_u\{1 + \gamma(x)(x - \bar{x} + \ell/2)\} \\
&\quad < \{f_u(0, a_0)\bar{P}_u - f_v(0, a_0)\bar{P}_v + o(|\bar{P}_u|, |\bar{P}_v|)\}(x - \bar{x} + \ell/2)^2 \\
&\qquad + 2\varepsilon^2\bar{P}_u\{1 + \gamma(x)(x - \bar{x} + \ell/2)\} \\
&\quad < \frac{\ell^2}{16}\bar{P}_u\{f_u(0, a_0) + f_v(0, a_0)e_-(b_0, c_0) + o(1)\}
\end{aligned}$$

$$< 0.$$

(iii) For $x \in (\bar{x} - 3\ell/4, \bar{x} - \ell/2)$,

$$\begin{aligned}
&\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) \\
&\quad = \{\varepsilon\gamma(x) - s(\hat{b}(\varepsilon), \hat{c}(\varepsilon))\} \hat{u}^\varepsilon_\xi + 2\varepsilon^2 \bar{P}_u \{1 + \gamma(x)(x - \bar{x} + \ell/2)\} \\
&\qquad + f(\bar{U}, \bar{V}) - f(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \\
&\quad < \varepsilon\{\gamma(x) - M_b b_1 - M_c c_1 + o(1)\} \hat{u}^\varepsilon_\xi \\
&\qquad + \bar{P}_u \{f_u(0, a_0) + f_v(0, a_0) e_-(b_0, c_0) + o(1)\}(x - \bar{x} + \ell/2)^2 \\
&\quad < 0.
\end{aligned}$$

(iv) For $x \in (\bar{x} - \ell/2, \bar{x} + \ell/2)$,

$$\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) = \{\varepsilon\gamma(x) - s(\hat{b}(\varepsilon), \hat{c}(\varepsilon))\} \hat{u}^\varepsilon_\xi < 0.$$

(v) For $x \in (\bar{x} + \ell/2, \bar{x} + 3\ell/4)$,

$$\begin{aligned}
&\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) \\
&\quad = \{\varepsilon\gamma(x) - s(\hat{b}(\varepsilon), \hat{c}(\varepsilon))\} \hat{u}^\varepsilon_\xi + 2\varepsilon^2 \bar{Q}_u \{1 + \gamma(x)(x - \bar{x} - \ell/2)\} \\
&\qquad + f(\bar{U}, \bar{V}) - f(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \\
&\quad < \varepsilon\{\gamma(x) - M_b b_1 - M_c c_1 + o(1)\} \hat{u}^\varepsilon_\xi \\
&\qquad + \bar{Q}_u \{f_u(1, 0) + f_v(1, 0) e_+(b_0, c_0) + o(1)\}(x - \bar{x} - \ell/2)^2 \\
&\quad < 0.
\end{aligned}$$

(vi) For $x \in (\bar{x} + 3\ell/4, \bar{x} + \ell)$,

$$\begin{aligned}
&\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) \\
&\quad = \{\varepsilon\gamma(x) - s(\hat{b}(\varepsilon), \hat{c}(\varepsilon))\} \hat{u}^\varepsilon_\xi + f(\bar{U}, \bar{V}) - f(\hat{u}^\varepsilon, \hat{v}^\varepsilon) \\
&\qquad + 2\varepsilon^2 \bar{Q}_u \{1 + \gamma(x)(x - \bar{x} - \ell/2)\} \\
&\quad < \{f_u(1, 0)\bar{Q}_u - f_v(1, 0)\bar{Q}_v + o(|\bar{Q}_u|, |\bar{Q}_v|)\}(x - \bar{x} - \ell/2)^2 \\
&\qquad + 2\varepsilon^2 \bar{Q}_u \{1 + \gamma(x)(x - \bar{x} - \ell/2)\} \\
&\quad < \frac{\ell^2}{16} \bar{Q}_u \{f_u(1, 0) + f_v(1, 0) e_+(b_0, c_0) + o(1)\} \\
&\quad < 0.
\end{aligned}$$

(vii) For $x \in (\bar{x} + \ell, L)$,

$$\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) = 0.$$

Therefore the following inequality holds if $\varepsilon > 0$ is sufficiently small:

$$\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) = 0$$

for all $x \in (0, L) \setminus \{\bar{x} \pm \ell, \bar{x} \pm \ell/2\}$. Moreover, we have

$$[\bar{U}_x] = \begin{cases} -\dfrac{1}{\varepsilon} \hat{u}_\xi(\ell/\varepsilon; \hat{b}(\varepsilon), \hat{c}(\varepsilon)) - \bar{Q}_u \ell \ (< 0) & x = \bar{x} + \ell, \\ 0 & x = \bar{x} \pm \ell/2, \ \bar{x} - \ell, \end{cases}$$

where $[u](x) = u(x + 0) - u(x - 0)$. In summary, we have shown that the following inequality holds in the sense of distribution:

$$\varepsilon^2 \bar{U}_{xx} + \varepsilon^2 \gamma(x) \bar{U}_x + f(\bar{U}, \bar{V}) \leq 0 \ (\not\equiv 0) \qquad \text{for all } x \in (0, L).$$

Similarly, we can prove that the following inequality holds in the sense of distribution:

$$\varepsilon^2 d_0 \bar{V}_{xx} + \varepsilon^2 d_0 \gamma(x) \bar{V}_x + g(\bar{U}, \bar{V}) \geq 0 \ (\not\equiv 0) \qquad \text{for all } x \in (0, L).$$

Finally, from the definition of $(\bar{U}, \bar{V})$, we have

$$\bar{U}_x(x; \bar{x}, \varepsilon, \ell) = \bar{V}_x(x; \bar{x}, \varepsilon, \ell) = 0, \qquad x = 0, L.$$

Now, by virtue of Lemma 3.2, if $\varepsilon > 0$ is sufficiently small, $(\bar{U}(x; \bar{x}, \varepsilon, \ell), \bar{V}(x; \bar{x}, \varepsilon, \ell))$ defined as above is a supersolution of (3.1).

Let $(\underline{U}, \underline{V}) = (\underline{U}(x; \underline{x}, \varepsilon, \ell), \underline{V}(x; \underline{x}, \varepsilon, \ell))$ be a function defined by

$$\underline{U}(x; \underline{x}, \varepsilon, \ell) = \begin{cases} 0 & x \in [0, \underline{x} - \ell], \\ \hat{u}^\varepsilon(x; \underline{x}) - \underline{P}_u(x - \underline{x} + \ell/2)^2 & x \in (\underline{x} - \ell, \underline{x} - \ell/2], \\ \hat{u}^\varepsilon(x; \underline{x}) & x \in (\underline{x} - \ell/2, \underline{x} + \ell/2], \\ \hat{u}^\varepsilon(x; \underline{x}) - \underline{Q}_u(x - \underline{x} - \ell/2)^2 & x \in (\underline{x} + \ell/2, \underline{x} + \ell], \\ \hat{u}^\varepsilon(\underline{x} + \ell; \underline{x}) - \underline{Q}_u \ell^2/4 & x \in (\underline{x} + \ell, L], \end{cases}$$

$$\underline{V}(x; \underline{x}, \varepsilon, \ell) = \begin{cases} a_0 & x \in [0, \underline{x} - \ell], \\ \hat{v}^\varepsilon(x; \underline{x}) + \underline{P}_v(x - \underline{x} + \ell/2)^2 & x \in (\underline{x} - \ell, \underline{x} - \ell/2], \\ \hat{v}^\varepsilon(x; \underline{x}) & x \in (\underline{x} - \ell/2, \underline{x} + \ell/2], \\ \hat{v}^\varepsilon(x; \underline{x}) + \underline{Q}_v(x - \underline{x} - \ell/2)^2 & x \in (\underline{x} + \ell/2, \underline{x} + \ell], \\ \hat{v}^\varepsilon(\underline{x} + \ell; \underline{x}) + \underline{Q}_v \ell^2/4 & x \in (\underline{x} + \ell, L], \end{cases}$$

where $\underline{P}_u$, $\underline{Q}_u$, $\underline{P}_v$, $\underline{Q}_v > 0$ are given by

$$\underline{P}_u = \frac{4}{\ell^2}\hat{u}(-\ell/\varepsilon;\, \hat{b}(\varepsilon),\, \hat{c}(\varepsilon)), \qquad \underline{Q}_u = \frac{1}{\varepsilon\ell}\hat{u}_\xi(\ell/\varepsilon;\, \hat{b}(\varepsilon),\, \hat{c}(\varepsilon)),$$

$$\underline{P}_v = \frac{4}{\ell^2}\{a_0 - \hat{v}(-\ell/\varepsilon;\, \hat{b}(\varepsilon),\, \hat{c}(\varepsilon))\}, \qquad \underline{Q}_v = -\frac{1}{\varepsilon\ell}\hat{v}_\xi(\ell/\varepsilon;\, \hat{b}(\varepsilon),\, \hat{c}(\varepsilon))$$

(see Figure 2). Then $(\underline{U}(x;\, \underline{x},\, \varepsilon,\, \ell),\, \underline{V}(x;\, \underline{x},\, \varepsilon,\, \ell))$ is continuous and positively monotone for sufficiently small $\varepsilon > 0$. Similar to Lemma 3.3, the following lemma holds:

LEMMA 3.4. *Suppose that* $M_b b_1 + M_c c_1 < \gamma(x)$ *for* $x \in [\underline{x} - \ell, \underline{x} + \ell] \subset (0, L)$ *for some* $\underline{x} \in (0, L)$ *and* $\ell > 0$. *Then there exists* $\underline{\varepsilon} = \underline{\varepsilon}(\ell) > 0$ *such that, if* $0 < \varepsilon \le \underline{\varepsilon}$, $(\underline{U}(x;\, \underline{x},\, \varepsilon,\, \ell),\, \underline{V}(x;\, \underline{x},\, \varepsilon,\, \ell))$ *defined as above is a strict subsolution of* (3.1).

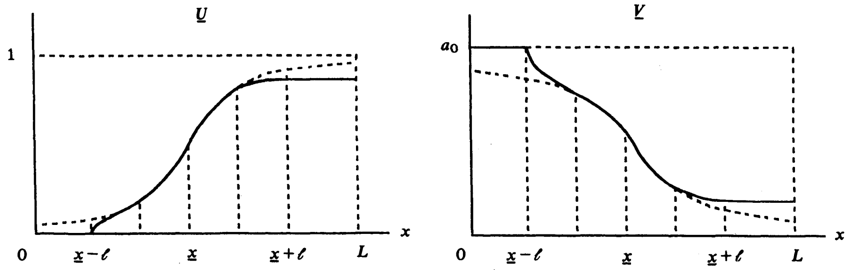

Figure 2 Subsolution $(\underline{U}, \underline{V})$

As a direct consequence of Lemmas 3.1, 3.3 and 3.4, we obtain the following result:

THEOREM 3.1. *Suppose that*

(i) *there exists* $x_0 \in (0, L)$ *such that* $\gamma(x)$ *is strictly increasing in a neighborhood of* $x = x_0$, and

(ii) $M_b b_1 + M_c c_1 = \gamma(x_0)$.

Then there exists $\varepsilon_0 > 0$ *such that, if* $0 < \varepsilon \le \varepsilon_0$, (3.1) *with* $(a, b, c, d) = (a_0, \hat{b}(\varepsilon), \hat{c}(\varepsilon), d_0)$ *has a stable non-constant stationary solution.*

PROOF. By assumption (ii) and (2.9), there exists a solution family $(\hat{u}(\xi;\, \hat{b}(\varepsilon),\, \hat{c}(\varepsilon)),\, \hat{v}(\xi;\, \hat{b}(\varepsilon),\, \hat{c}(\varepsilon)),\, s(\hat{b}(\varepsilon),\, \hat{c}(\varepsilon)))$ of (2.1) such that

$$s(\hat{b}(\varepsilon), \hat{c}(\varepsilon)) = \varepsilon\gamma(x_0) + o(\varepsilon) \qquad \text{as } \varepsilon \longrightarrow 0.$$

By assumption (i), we can choose $\bar{x}$, $\underline{x} \in (0, L)$ and $\ell > 0$ such that

$$(3.2) \qquad \gamma(x) \begin{cases} < \gamma(x_0) & x \in (\bar{x} - \ell, \bar{x} + \ell) \subset (0, x_0), \\ > \gamma(x_0) & x \in (\underline{x} - \ell, \underline{x} + \ell) \subset (x_0, L). \end{cases}$$

From Lemmas 3.3 and 3.4, for sufficiently small $\varepsilon > 0$, there exist a strict supersolution $(\bar{U}(x; \bar{x}, \varepsilon, \ell), \bar{V}(x; \bar{x}, \varepsilon, \ell))$ and a strict subsolution $(\underline{U}(x; \underline{x}, \varepsilon, \ell), \underline{V}(x; \underline{x}, \varepsilon, \ell))$ of (3.1) for $(a, b, c, d) = (a_0, \hat{b}(\varepsilon), \hat{c}(\varepsilon), d_0)$. Moreover, it is easy to see from the definitions of $(\bar{U}, \bar{V})$ and $(\underline{U}, \underline{V})$ that

$$(\bar{U}(x; \bar{x}, \varepsilon, \ell), \bar{V}(x; \bar{x}, \varepsilon, \ell)) > (\underline{U}(x; \underline{x}, \varepsilon, \ell), \underline{V}(x; \underline{x}, \varepsilon, \ell))$$

for all $x \in [0, L]$. Hence, by Lemma 3.1, there exists a stable non-constant stationary solution of (3.1).

REMARK 3.1. By applying the result of Kishimoto and Weinberger [2] to (3.1), we see that any non-constant stationary solution of (3.1) is unstable if $\gamma'(x) \leq 0$ for all $x \in (0, L)$. Theorem 3.1 implies that their result is optimal.

3.3. Thin tubular domains

In this subsection, we consider the existence of a stable non-constant stationary solution of (1.2) in the case where Ω is a thin tubular domain. We characterize $\Omega = \Omega(\mu)$ as follows. Let $p(x)(x \in [0, L])$ be a smooth curve in $\mathbf{R}^{n+1}$ which does not intersect itself, where x denotes the length parameter (*i.e.*, $|p_x(x)| = 1$). Let $\{q_j(x)\}_{j=1}^n$ be an orthonormal basis of an n-dimensional normal plane at $p(x)$, and let $D(x) \subset \mathbf{R}^n$ be a simply connected bounded domain with a smooth boundary. We assume that $p(x)$, $\{q_j(x)\}_{j=1}^n$ and $D(x)$ depend on $x \in [0, L]$ smoothly. We define

$$(3.3) \qquad \Omega = \Omega(\mu) = \{p(x) + \mu \textstyle\sum_{j=1}^n y_j q_j(x) \,|\, y = (y_1, \ldots, y_n) \in D(x),\ x \in (0, L)\},$$

and

$$\alpha(x) = \int_{D(x)} dy, \qquad x \in [0, L].$$

We introduce the equation

$$(3.4) \qquad \begin{cases} u_t = \varepsilon^2 \dfrac{1}{\alpha(x)} \{\alpha(x) u_x\}_x + f(u, v), \\ v_t = \varepsilon^2 d_0 \dfrac{1}{\alpha(x)} \{\alpha(x) v_x\}_x + g(u, v), \qquad x \in (0, L),\ t > 0, \\ u_x(t, x) = 0 = v_x(t, x), \qquad x = 0, L,\ t > 0. \end{cases}$$

It was shown in the proof of Theorem 1.1 of Yanagida [10] that a supersolution (resp. a subsolution) of (1.2) can be constructed by slightly perturbing a strict supersolution (resp. a strict subsolution) of (3.4) in the direction of the normal plane of the curve. (In [10], scalar reaction-diffusion equations are studied, but his technique can be also applied to competition-diffusion equations without much change.)

Hence the following result is obtained:

LEMMA 3.5. *Suppose that* (3.4) *has a strict supersolution* $(\bar{u}, \bar{v})$ *and a strict subsolution* $(\underline{u}, \underline{v})$ *which satisfy* $(\bar{u}, \bar{v}) > (\underline{u}, \underline{v})$. *Then, there exists* $\mu_0 > 0$ *such that, if* $0 < \mu \leq \mu_0$, (1.2) *with* Ω *given by* (3.3) *has a stable stationary solution.*

As a direct consequence of this lemma and Theorem 3.1, we obtain the following result.

THEOREM 3.2. *Suppose that*

(i) Ω *is given by* (3.3),

(ii) *there exists* $x_0 \in (0, L)$ *such that* $\alpha'(x)/\alpha(x)$ *is strictly increasing in a neighborhood of* $x = x_0$, *and*

(iii) $M_b b_1 + M_c c_1 = \alpha'(x_0)/\alpha(x_0)$.

Then there exist $\varepsilon_0 > 0$ *and* $\mu_0 = \mu_0(\varepsilon) > 0$ *such that, if* $0 < \varepsilon \leq \varepsilon_0$ *and* $0 < \mu \leq \mu_0$, (1.2) *with* $(a, b, c, d) = (a_0, \hat{b}(\varepsilon), \hat{c}(\varepsilon), d_0)$ *has a stable non-constant stationary solution.*

By taking $(\hat{b}(\varepsilon), \hat{c}(\varepsilon)) \equiv (b_0, c_0)$, we obtain the following corollary:

COROLLARY 3.1. *Suppose that*

(i) Ω *is given by* (3.3), *and*

(ii) *there exists* $x_0 \in (0, L)$ *such that* $\alpha(x)$ *attains its strict local minimum at* $x = x_0$.

Then there exist $\varepsilon_0 > 0$ *and* $\mu_0 = \mu_0(\varepsilon) > 0$ *such that, if* $0 < \varepsilon \leq \varepsilon_0$ *and* $0 < \mu \leq \mu_0$, (1.2) *with* $(a, b, c, d) = (a_0, b_0, c_0, d_0)$ *has a stable non-constant stationary solution.*

3.4. Rotationally symmetric tubular domains

In this subsection, we consider the existence of a stable non-constant stationary solution of (1.2) in the case where Ω is a rotationally symmetric domain given by

(3.5) $$\Omega = \{(x, y) \in \mathbf{R} \times \mathbf{R}^n \mid 0 < x < L, |y| < \alpha(x)\},$$

where $\alpha(x) > 0$ is a C^1-class function.

Let $\kappa(x)$, $\rho(x)$, $\varphi(x)$ and $\psi(x)$ be functions on $[0, L]$ defined by

$$\kappa(x) = \frac{\alpha'(x)}{\alpha(x)\sqrt{\alpha'(x)^2 + 1}},$$

$$\rho(x) = 1/\kappa(x) \quad \text{if } \alpha'(x) \neq 0,$$

$$\varphi(x) = x - \frac{\alpha(x)}{\alpha'(x)} \quad \text{if } \alpha'(x) \neq 0,$$

$$\psi(x) = x + \frac{\alpha(x)\alpha'(x)}{\sqrt{\alpha(x)^2 + 1} + 1} \quad (= \varphi(x) + \rho(x) \text{ if } \alpha'(x) \neq 0)$$

(see Figure 3). We have the following result:

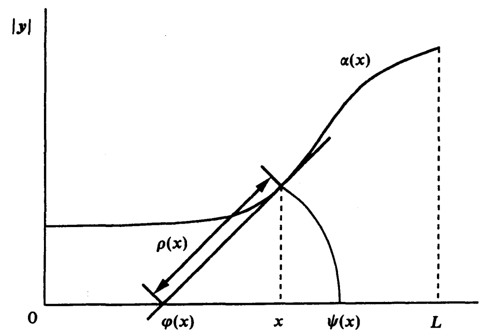

Figure 3 Rotationally symmetric tubular domain Ω

THEOREM 3.3. *Suppose that*

(i) Ω *is given by* (3.5),

(ii) *there exists* $x_0 \in (0, L)$ *such that* $\kappa(x)$ *is strictly increasing in a neighborhood of* $x = x_0$, *and*

(iii) $M_b b_1 + M_c c_1 = n\kappa(x_0)$.

In case $\alpha'(x_0) \neq 0$, *suppose further that*

(iv) $0 < \psi(x_0) < L$, *and*

(v) $(x - \varphi(x_0))^2 + \alpha(x)^2 > \rho(x_0)^2$ *for* $\min\{x_0, \psi(x_0)\} < x < \max\{x_0, \psi(x_0)\}$.

Then there exists $\varepsilon_0 > 0$ *such that, if* $0 < \varepsilon \leq \varepsilon_0$, (1.2) *with* $(a, b, c, d) = (a_0, \hat{b}(\varepsilon),$

$\hat{c}(\varepsilon)$, d_0) *has a stable non-constant stationary solution.*

PROOF. By the symmetric transformation with respect to the plane $x = L/2$ if necessary, we may assume that $\alpha'(x_0) \geq 0$.

By assumption (iii) and (2.9), there exists a solution family $(\hat{u}(\xi;\hat{b}(\varepsilon), \hat{c}(\varepsilon)), \hat{v}(\xi;\hat{b}(\varepsilon), \hat{c}(\varepsilon)), s(\hat{b}(\varepsilon), \hat{c}(\varepsilon)))$ of (2.1) such that

$$s(\hat{b}(\varepsilon), \hat{c}(\varepsilon)) = \varepsilon n\kappa(x_0) + o(\varepsilon) \qquad \text{as} \quad \varepsilon \longrightarrow 0.$$

By assumption (ii), there exist $\delta_0 > 0$ sufficiently small and $\bar{x}, \underline{x} \in (0, L)(\bar{x} < x_0 < \underline{x})$ close to x_0 such that $\kappa(x)$ is strictly increasing in $[\bar{x} - \delta_0, \bar{x}] \cup [\underline{x}, \underline{x} + \delta_0] \subset (0, L)$ and satisfies

$$\kappa(\bar{x}) < \kappa(x_0) < \kappa(\underline{x}). \tag{3.6}$$

It is easy to check that, if $\alpha'(x) \neq 0$ in $[\bar{x} - \delta_0, \bar{x}]$ (resp. $[\underline{x}, \underline{x} + \delta_0]$), then $\alpha'(x)$ is increasing in $[\bar{x} - \delta_0, \bar{x}]$ (resp. $[\underline{x}, \underline{x} + \delta_0]$).

First we consider the case where $\alpha'(x_0) > 0$. Then, by taking δ_0 and $\bar{x}$ suitably, we may assume that $\alpha'(x)$ is increasing in $[\bar{x} - \delta_0, \bar{x}] \cup [\underline{x}, \underline{x} + \delta_0]$. Moreover, by assumptions (iv) and (v), the spherical segment

$$\{(x, y) \mid (x - \varphi(x_0))^2 + |y|^2 = \rho(x_0)^2, x > x_0\}$$

is entirely contained in Ω. By (3.6), we can choose positive constants ℓ, $\bar{\gamma}$ and $\underline{\gamma}$ such that

$$\begin{aligned} &0 < \frac{n}{\rho(\bar{x}) - 2\ell} < \bar{\gamma} < n\kappa(x_0) < \underline{\gamma} < \frac{n}{\rho(\underline{x}) + 2\ell}, \\ &\bar{x} - \delta_0 \leq \max\{x \mid x \leq \bar{x}, x \in S(\bar{x}, -2\ell)\} (\equiv \bar{x}_\ell), \\ &\underline{x} + \delta_0 \geq \max\{x \mid x \geq \underline{x}, x \in S(\underline{x}, 2\ell)\} (\equiv \underline{x}_\ell) \end{aligned} \tag{3.7}$$

where

$$S(x, \ell) = \{y \mid 0 \leq y \leq L, |\rho(x) + \ell| = \sqrt{(y - \varphi(x))^2 + \alpha(y)^2}\}.$$

Let us consider the equation

$$\begin{cases} u_t = \varepsilon^2 u_{rr} + \varepsilon^2 \bar{\gamma} u_r + f(u, v), \\ v_t = \varepsilon^2 d_0 v_{rr} + \varepsilon^2 d_0 \bar{\gamma} v_r + g(u, v), \\ \qquad r \in (\rho(\bar{x}) - 2\ell - \delta, \rho(\bar{x}) + \delta), \quad t > 0, \\ u_r = 0 = v_r, \quad x = \rho(\bar{x}) - 2\ell - \delta, \rho(\bar{x}) + \delta, \quad t > 0, \end{cases} \tag{3.8}$$

where δ is an arbitrary positive constant and

$$r = \sqrt{(x - \varphi(\bar{x}))^2 + |y|^2}.$$

Similar to Lemma 3.3, this equation has a supersolution $(\bar{U}(r), \bar{V}(r)) = (\bar{U}(r; \rho(\bar{x}) - \ell, \varepsilon, \ell), \bar{V}(r; \rho(\bar{x}) - \ell, \varepsilon, \ell))$.

Let $\Omega_1, \Omega_2, \Omega_3$ be subdomains of Ω defined by

$$\Omega_1 = \{(x, y) \in \Omega \mid x > \bar{x}, r > \rho(\bar{x})\},$$

$$\Omega_2 = \{(x, y) \in \Omega \setminus \overline{\Omega_1} \mid x > \bar{x}_\ell, \rho(\bar{x}) - 2\ell < r < \rho(\bar{x})\},$$

$$\Omega_3 = \Omega \setminus \overline{(\Omega_1 \cup \Omega_2)}$$

(see Figure 4). Now we define $(\bar{u}(x, y), \bar{v}(x, y))$ by

$$(\bar{u}(x, y), \bar{v}(x, y)) = \begin{cases} (1, 0) & (x, y) \in \Omega_1, \\ (\bar{U}(r), \bar{V}(r)) & (x, y) \in \Omega_2, \\ (\bar{U}(\rho(\bar{x}) - 2\ell), \bar{V}(\rho(\bar{x}) - 2\ell)) & (x, y) \in \Omega_3. \end{cases}$$

We shall prove that $(\bar{u}(x, y), \bar{v}(x, y))$ defined as above is a supersolution of (1.2).

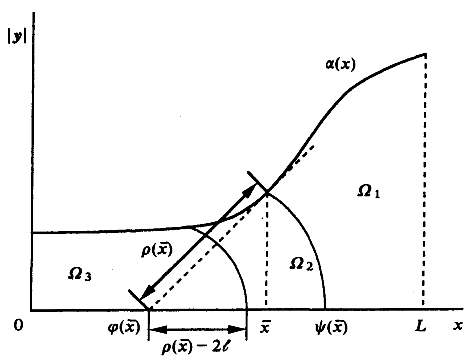

Figure 4 Position of Ω_1, Ω_2 and Ω_3

First we prove that $(\bar{u}(x, y), \bar{v}(x, y))$ satisfies

$$\text{(3.9)} \quad \begin{aligned} &\varepsilon^2 \Delta \bar{u} + f(\bar{u}, \bar{v}) \leq 0, \\ &\varepsilon^2 d_0 \Delta \bar{v} + g(\bar{u}, \bar{v}) \geq 0, \qquad (x, y) \in \Omega, \end{aligned}$$

in the sense of distribution. By substitution, we obtain

$$\varepsilon^2 \Delta \bar{u} + f(\bar{u}, \bar{v})$$
$$= \begin{cases} f(1, 0) = 0 & (x, y) \in \Omega_1, \\ \varepsilon^2 \bar{U}_{rr} + \varepsilon^2 \frac{n}{r} \bar{U}_r + f(\bar{U}, \bar{V}) & (x, y) \in \Omega_2, \\ f(\bar{U}(\rho(\bar{x}) - 2\ell), \bar{V}(\rho(\bar{x}) - 2\ell)) < 0 & (x, y) \in \Omega_3. \end{cases}$$

Here, by (3.7), we have

$$\frac{n}{r} < \bar{\gamma} \qquad \text{for } (x, y) \in \Omega_2.$$

Hence, by using the fact that $(\bar{U}(r), \bar{V}(r))$ is a monotone supersolution of (3.8), we obtain

$$\varepsilon^2 \bar{U}_{rr} + \varepsilon^2 \frac{n}{r} \bar{U}_r + f(\bar{U}, \bar{V})$$
$$< \varepsilon^2 \bar{U}_{rr} + \varepsilon^2 \bar{\gamma} \bar{U}_r + f(\bar{U}, \bar{V}) \leq 0 \qquad \text{for } (x, y) \in \Omega_2.$$

Hence it holds that

$$\varepsilon^2 \Delta \bar{u} + f(\bar{u}, \bar{v}) \leq 0, \qquad (x, y) \in \Omega,$$

in the sense of distribution.

Similarly we can also prove

$$\varepsilon^2 d_0 \Delta \bar{v} + g(\bar{u}, \bar{v}) \geq 0, \qquad (x, y) \in \Omega,$$

in the sense of distribution. Hence (3.9) holds.

Next we prove that

$$\text{(3.10)} \qquad \frac{\partial}{\partial \nu} \bar{u} \geq 0 \quad \text{and} \quad \frac{\partial}{\partial \nu} \bar{v} \leq 0, \qquad (x, y) \in \partial\Omega.$$

The outer unit normal vector ν at $(x, y) \in \{(x, y) | 0 < x < L, |y| = \alpha(x)\}$ is represented as

$$\nu = \frac{1}{\sqrt{\alpha'(x)^2 + 1}} (-\alpha'(x), y/|y|).$$

On the other hand, we have

$$\nabla \bar{u} = \frac{\bar{U}_r}{r} (x - \varphi(\bar{x}), y), \qquad (x, y) \in \Omega_2.$$

Hence, on $\partial\Omega \cap \partial\Omega_2$, we obtain

$$\begin{aligned}\frac{\partial}{\partial \nu}\bar{u} &= \nu \cdot \nabla \bar{u} \\ &= \frac{\bar{U}_r}{r\sqrt{\alpha'(x)^2+1}}\{-\alpha'(x)(x-\varphi(\bar{x}))+\alpha(x)\}.\end{aligned}$$

Since $\{x \mid (x, y) \in \partial\Omega \cap \partial\Omega_2\} \subset [\bar{x}-\delta_0, \bar{x}]$ and since $\alpha'(x)$ is strictly increasing in $[\bar{x}-\delta_0, \bar{x}]$ (*i.e.*, $\alpha(x)$ is convex),

$$\alpha'(x) < \alpha'(\bar{x}) = \frac{\alpha(\bar{x})}{\bar{x}-\varphi(\bar{x})} \le \frac{\alpha(x)}{x-\varphi(\bar{x})}$$

for all $x \in \{x \mid (x, y) \in \partial\Omega_2 \cap \partial\Omega\}$. This means that $\frac{\partial}{\partial \nu}\bar{u} \ge 0$ on $\partial\Omega \cap \partial\Omega_2$. Similarly we obtain $\frac{\partial}{\partial \nu}\bar{v} \le 0$ on $\partial\Omega_2 \cap \partial\Omega$. Moreover, since $(\bar{u}, \bar{v})$ is constant in $\Omega_1 \cup \Omega_3$, we have $\frac{\partial}{\partial \nu}\bar{u} = \frac{\partial}{\partial \nu}\bar{v} = 0$ on $\partial\Omega \cap (\partial\Omega_1 \cup \partial\Omega_3)$. Thus we have shown that (3.10) holds. Hence, by Lemma 3.2, (3.9) and (3.10), $(\bar{u}(x, y), \bar{v}(x, y))$ is a strict supersolution of (1.2).

Next let $(\underline{U}(r), \underline{V}(r)) = (\underline{U}(r; \rho(\underline{x})+\ell, \varepsilon, \ell), \underline{V}(r; \rho(\underline{x})+\ell, \varepsilon, \ell))$ be a subsolution of

$$\begin{cases} u_t = \varepsilon^2 u_{rr} + \varepsilon^2 \underline{\gamma} u_r + f(u, v), \\ v_t = \varepsilon^2 d_0 v_{rr} + \varepsilon^2 \underline{\gamma} v_r + g(u, v), \\ \qquad r \in (\rho(\underline{x})-\delta, \rho(\underline{x})+2\ell+\delta), \qquad t > 0, \\ u_r = 0 = v_r, \quad r = \rho(\underline{x})-\delta, \rho(\underline{x})+2\ell+\delta, \qquad t > 0, \end{cases}$$

where δ is an arbitrary positive constant and

$$r = \sqrt{(x-\varphi(\underline{x}))^2 + |y|^2}.$$

Let $\Omega_4, \Omega_5, \Omega_6$ be subdomains of Ω defined by

$$\Omega_4 = \{(x, y) \in \Omega \mid x > \underline{x}_\ell, r > \rho(\underline{x}) + 2\ell\},$$

$$\Omega_5 = \{(x, y) \in \Omega \setminus \overline{\Omega_4} \mid x > \underline{x}, \rho(\underline{x}) < r < \rho(\underline{x}) + 2\ell\},$$

$$\Omega_6 = \Omega \setminus \overline{(\Omega_4 \cup \Omega_5)},$$

and let $(\underline{u}(x, y), \underline{v}(x, y))$ be defined by

$$(\underline{u}(x, y), \underline{v}(x, y)) = \begin{cases} (\underline{U}(\rho(\underline{x}) + 2\ell), \underline{V}(\rho(\underline{x}) + 2\ell)) & (x, y) \in \Omega_4, \\ (\underline{U}(r), \underline{V}(r)) & (x, y) \in \Omega_5, \\ (0, a_0) & (x, y) \in \Omega_6. \end{cases}$$

Then we can show that $(\underline{u}(x, y), \underline{v}(x, y))$ is a strict subsolution of (1.2).

Moreover, it is easy to see from the definitions of $(\bar{u}(x, y), \bar{v}(x, y))$ and $(\underline{u}(x, y), \underline{v}(x, y))$ that $(\bar{u}(x, y), \bar{v}(x, y)) > (\underline{u}(x, y), \underline{v}(x, y))$ for all $(x, y) \in \bar{\Omega}$. Hence, by Lemma 3.1, there exists a stable non-constant stationary solution of (1.2).

Next we consider the case where $\alpha'(x_0) = 0$. By (3.6), we have

$$\kappa(\bar{x}) < 0 < \kappa(\underline{x}),$$

i.e. $\alpha'(x)$ is increasing in $[\bar{x} - \delta_0, \bar{x}] \cup [\underline{x}, \underline{x} + \delta_0]$. A strict subsolution of (1.2) can be constructed precisely in the same manner as in the case $\alpha'(x_0) > 0$. A strict supersolution of (1.2) can be constructed with a slight change if we take account of the fact that $\kappa(\bar{x}) < 0$ and $\rho(\bar{x}) < 0$.

We can choose constants $\ell > 0$ and $\bar{\gamma} < 0$ such that

$$\frac{n}{\rho(\bar{x}) - 2\ell} < \bar{\gamma} < 0 \quad \text{and} \quad \bar{x} - \delta_0 \leq \bar{x}_\ell < \bar{x}.$$

Then, similar to Lemma 3.3, (3.8) has a supersolution $(\bar{U}(r), \bar{V}(r)) = (\bar{U}(r; \rho(\bar{x}) - \ell, \varepsilon, \ell), \bar{V}(r; \rho(\bar{x}) - \ell, \varepsilon, \ell))$.

Let Ω_7, $\Omega_8 \Omega_9$ be subdomains of Ω defined by

$$\Omega_7 = \{(x, y) \in \Omega \,|\, x < \bar{x}_\ell, r > -\rho(\bar{x}) + 2\ell\},$$

$$\Omega_8 = \{(x, y) \in \Omega \setminus \overline{\Omega_7} \,|\, x < \bar{x}, -\rho(\bar{x}) < r < -\rho(\bar{x}) + 2\ell\},$$

$$\Omega_9 = \Omega \setminus \overline{(\Omega_7 \cup \Omega_8)},$$

and let $(\bar{u}(x, y), \bar{v}(x, y))$ be defined by

$$(\bar{u}(x, y), \bar{v}(x, y)) = \begin{cases} (\bar{U}(\rho(\bar{x}) - 2\ell), \bar{V}(\rho(\bar{x}) - 2\ell)) & (x, y) \in \Omega_7, \\ (\bar{U}(-r), \bar{V}(-r)) & (x, y) \in \Omega_8, \\ (1, 0) & (x, y) \in \Omega_9. \end{cases}$$

Since $\alpha(x)$ satisfies

$$\alpha'(x) < \alpha'(\bar{x}) = \frac{\alpha(\bar{x})}{\bar{x} - \varphi(\bar{x})} \leq \frac{\alpha(x)}{x - \varphi(\bar{x})}$$

for all $x \in [\bar{x} - \delta_0, \bar{x}]$, we can prove that $(\bar{u}(x, y), \bar{v}(x, y))$ is a strict super-solution of (1.2). We omit details of the proof.

Since $(\bar{u}(x, y), \bar{v}(x, y)) > (\underline{u}(x, y), \underline{v}(x, y))$, by Lemma 3.1, there exists a stable non-constant stationary solution.

By taking $(\hat{b}(\varepsilon), \hat{c}(\varepsilon)) \equiv (b_0, c_0)$, we obtain the following corollary:

COROLLARY 3.2. *Suppose that there exists $x_0 \in (0, L)$ such that $\alpha(x)$ attains its strict local minimum at $x = x_0$. Then there exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon \leq \varepsilon_0$, (1.2) with $(a, b, c, d) = (a_0, b_0, c_0, d_0)$ has a stable non-constant stationary solution.*

REMARK 3.2. As is easily seen from the proof above, it is sufficient for Theorem 3.3 and Corollary 3.2 that the domain Ω is rotationally symmetric only in a neighborhood of x_0.

4. Concluding Remarks

As we have seen in Section 3, whether or not there exists a stable non-constant stationary solution of (1.2) strongly depends on the domain shape and the magnitude of competition-diffusion—at least, as far as the solutions obtained by our techniques are concerned.

When $\alpha(x)$ is a C^2-class function in a neighborhood of $x = x_0$, the assumption (ii) of Theorem 3.2 for the function $\alpha(x)$ is equivalent to

$$\alpha''(x_0)\alpha(x_0) - \alpha'(x_0)^2 > 0, \tag{4.1}$$

while the assumption (ii) of Theorem 3.3 for the function $\kappa(x)$ is replaced by

$$\alpha''(x_0)\alpha(x_0) - \alpha'(x_0)^2 - \alpha'(x_0)^4 > 0. \tag{4.2}$$

The shape of the thin domain defined by (3.3) is quite different from that of the rotationally symmetric domain defined by (3.5). It seems interesting to note that, if the higher order term of $\alpha'(x_0)$ in (4.2) is neglected, then (4.2) reduces to (4.1).

The stable non-constant stationary solutions of (1.2) which we have obtained in this paper arise from the situation in which the effect of the curvature of the boundary and that of the competition-diffusion are well-balanced. If these effects are unbalanced, we guess that such a stable non-constant stationary solution of (1.2) no more exists unless the domain is dumbbell-shaped with a sufficiently narrow channel.

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Appendix.

In this appendix, we state the proof of Theorem 2.1. Rewrite $\mathbf{F}(\mathbf{u}; b, c, s) = \mathbf{f}(\mathbf{u}) + \mathbf{g}(\mathbf{u}; b, c, s)$, where

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} u_\xi \\ -u(1-u-c_0 v) \\ v_\xi \\ -v(a_0 - b_0 u - v)/d_0 \end{pmatrix}$$

and

$$\mathbf{g}(\mathbf{u}; b, c, s) = \begin{pmatrix} 0 \\ -su_\xi + (c-c_0)uv \\ 0 \\ -sv_\xi + (b-b_0)uv/d_0 \end{pmatrix}.$$

Let $J_- = \dfrac{\partial \mathbf{f}}{\partial \mathbf{u}}(0, 0, a_0, 0)$ be a Jacobian matrix of $\mathbf{f}$ at $\mathbf{u} = {}^t(0, 0, a_0, 0)$, and let λ_i and $\mathbf{p}_i$, $i = 1, 2, 3, 4$, be eigenvalues and corresponding eigenvectors, respectively, of J_-. By an elementary calculation, we get

$$\text{(A.1)} \qquad \begin{aligned} \lambda_1 &= -\sqrt{a_0/d_0}, & \mathbf{p}_1 &= {}^t(0, 0, 1, \lambda_1), \\ \lambda_2 &= -\sqrt{a_0 c_0 - 1}, & \mathbf{p}_2 &= {}^t(1, \lambda_2, e_-, e_-\lambda_2), \\ \lambda_3 &= \sqrt{a_0 c_0 - 1}, & \mathbf{p}_3 &= {}^t(1, \lambda_3, e_-, e_-\lambda_3), \\ \lambda_4 &= \sqrt{a_0/d_0}, & \mathbf{p}_4 &= {}^t(0, 0, 1, \lambda_4), \end{aligned}$$

where

$$e_- = -\frac{a_0 b_0}{a_0 - d_0(a_0 c_0 - 1)} < 0.$$

Note that, by (2.3), we have $\lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4$.

Similarly let σ_i and $\mathbf{q}_i$, $i = 1, 2, 3, 4$, be eigenvalues and corresponding eigenvectors, respectively, of the Jacobian matrix $J_+ = \dfrac{\partial \mathbf{f}}{\partial \mathbf{u}}(1, 0, 0, 0)$ of $\mathbf{f}$ at $\mathbf{u} = {}^t(1, 0, 0, 0)$. Then

$$\text{(A.2)} \qquad \begin{aligned} \sigma_1 &= -1, & \mathbf{q}_1 &= {}^t(1, -1, 0, 0), \\ \sigma_2 &= -\sqrt{(b_0 - a_0)/d_0}, & \mathbf{q}_2 &= {}^t(1, \sigma_2, e_+, e_+\sigma_2), \\ \sigma_3 &= \sqrt{(b_0 - a_0)/d_0}, & \mathbf{q}_3 &= {}^t(1, \sigma_3, e_+, e_+\sigma_3), \\ \sigma_4 &= 1, & \mathbf{q}_4 &= {}^t(1, 1, 0, 0), \end{aligned}$$

where $\sigma_1 < \sigma_2 < 0 < \sigma_3 < \sigma_4$ and

$$e_+ = -\frac{a_0 + d_0 - b_0}{c_0 d_0} < 0.$$

Let W_-^s and W_-^u be a stable manifold and an unstable manifold of (2.4) with $(b, c, s) = (b_0, c_0, 0)$ with respect to the equilibrium point $(0, 0, a_0, 0)$. Then, by (A.1), W_-^s and W_-^u are two-dimensional and are tangent to $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, $\mathbf{p}_4$, respectively. Further let W_-^{ss} and W_-^{uu} be a λ_1-stable submanifold of W_-^s and a λ_4-unstable submanifold of W_-^u, respectively. Then W_-^{ss} and W_-^{uu} are one-dimensional and are tangent to $\mathbf{p}_1$ and $\mathbf{p}_4$, respectively.

Similarly, let W_+^s and W_+^u be a stable manifold and an unstable manifold of (2.4) with $(b, c, s) = (b_0, c_0, 0)$ with respect to the equilibrium point $(1, 0, 0, 0)$. Then, by (A.2), W_+^s and W_+^u are two-dimensional and are tangent to $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, $\mathbf{q}_4$, respectively. Further let W_+^{ss} and W_+^{uu} be a σ_1-stable submanifold of W_+^s and a σ_4-unstable submanifold of W_+^u, respectively. Then W_+^{ss} and W_+^{uu} are one-dimensional and are tangent to $\mathbf{q}_1$ and $\mathbf{q}_4$, respectively.

Let $\mathscr{L}$ be a linearized operator of (2.1) with $s = 0$ around $(u, v) = (u_0, v_0)$, and let $\mathscr{L}^*$ be its adjoint operator defined by

$$\mathscr{L}(u, v) = \begin{pmatrix} u_{\xi\xi} + f_u^0 u + f_v^0 v \\ d_0 v_{\xi\xi} + g_u^0 u + g_v^0 v \end{pmatrix},$$

$$\mathscr{L}^*(u, v) = \begin{pmatrix} u_{\xi\xi} + f_u^0 u + g_u^0 v \\ d_0 v_{\xi\xi} + f_v^0 u + g_v^0 v \end{pmatrix},$$

respectively, where $f_u^0(\xi) = \dfrac{\partial f}{\partial u}(u_0(\xi), v_0(\xi))$, and so on. The adjoint equation can be written as

$$\frac{d}{d\xi}\mathbf{v} = -\mathbf{v}\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0), \tag{A.3}$$

where $\mathbf{u}_0 = {}^t(u_0, u_{0\xi}, v_0, v_{0\xi})$ and $\mathbf{v} = (-u_\xi, u, -d_0 v_\xi, d_0 v)$. Differentiating (2.1) with respect to ξ, we see that $(u, v) = (u_{0\xi}(\xi), v_{0\xi}(\xi))$ is a non-trivial bounded solution of $\mathscr{L}(u, v) = 0$. Hence its adjoint equation $\mathscr{L}^*(u, v) = 0$ must also have a non-trivial bounded solution, say $(u^*(\xi), v^*(\xi))$. Then $\mathbf{v}^* = (-u_\xi^*, u^*, -d_0 v_\xi^*, d_0 v^*)$ satisfies (A.3). It can be proved that the non-trivial bounded solution of $\mathscr{L}^*(u, v) = 0$ is unique up to multiplication by constants.

It was proved in Theorem A of Kokubu [3] that, when (b, c) is in a neighborhood of (b_0, c_0), the heteroclinic orbit of (2.4) persists if the following conditions are satisfied:

(H1) The heteroclinic orbit $\mathbf{u}_0(\xi)$, $\xi \in \mathbf{R}$, is on neither W_-^{uu} nor W_+^{ss}.

(H2) W_-^u and W_+^s have one-dimensional intersection.

(H3) W_-^u is transversal to W_+^{uu}, and W_+^s is transversal to W_-^{ss}.

(H4) $\mathbf{g}$ satisfies

$$\int_{-\infty}^{+\infty} \mathbf{v}^*(\xi) \frac{\partial}{\partial(s, b, c)} \mathbf{g}(\mathbf{u}_0(\xi); b_0, c_0, 0)\, d\xi \neq 0.$$

Moreover it was also shown in [3] that $s(b, c)$ satisfies

$$\text{(A.4)} \qquad \frac{\partial}{\partial b} s(b_0, c_0) = -\frac{\displaystyle\int_{-\infty}^{+\infty} \mathbf{v}^*(\xi) \frac{\partial}{\partial b} \mathbf{g}(\mathbf{u}_0(\xi); b_0, c_0, 0)\, d\xi}{\displaystyle\int_{-\infty}^{+\infty} \mathbf{v}^*(\xi) \frac{\partial}{\partial s} \mathbf{g}(\mathbf{u}_0(\xi); b_0, c_0, 0)\, d\xi},$$

$$\frac{\partial}{\partial c} s(b_0, c_0) = -\frac{\displaystyle\int_{-\infty}^{+\infty} \mathbf{v}^*(\xi) \frac{\partial}{\partial c} \mathbf{g}(\mathbf{u}_0(\xi); b_0, c_0, 0)\, d\xi}{\displaystyle\int_{-\infty}^{+\infty} \mathbf{v}^*(\xi) \frac{\partial}{\partial s} \mathbf{g}(\mathbf{u}_0(\xi); b_0, c_0, 0)\, d\xi}.$$

We shall prove in the following that the conditions (H1)–(H4) do hold for (2.4).

First we prove that the condition (H1) holds.

LEMMA A.1. *The heteroclinic orbit* $\mathbf{u}_0(\xi)$, $\xi \in \mathbf{R}$, *is on neither* W_-^{uu} *nor* W_+^{ss}.

PROOF. Let us consider the equation

$$\text{(A.5)} \qquad d_0 V_{\xi\xi} + V(a_0 - V) = 0, \qquad \xi \in \mathbf{R}.$$

Then $\mathbf{V} = {}^t(0, 0, V, V_\xi)$ is a solution of (2.4) with $(b, c, s) = (b_0, c_0, 0)$. It is easy to verify that, for an arbitrary constant C, there exists a solution of (A.5) satisfying

$$V(\xi) = a_0 + C \exp(\lambda_4 \xi) + o(\exp(\lambda_4 \xi)) \qquad \text{as } \xi \longrightarrow -\infty.$$

or equivalently,

$$\mathbf{V}(\xi) = {}^t(0, 0, a_0, 0) + C\mathbf{p}_4 \exp(\lambda_4 \xi) + o(\exp(\lambda_4 \xi)) \qquad \text{as } \xi \longrightarrow -\infty.$$

This means that $\mathbf{V}(\xi)$ is on W_-^{uu}. Since W_-^{uu} is one-dimensional, if $\mathbf{u}_0(\xi)$ is on W_-^{uu}, then $\mathbf{u}_0(\xi) \equiv \mathbf{V}(\xi)$ for some C. However this contradicts that (u_0, v_0) is a solution of (2.1). Hence $\mathbf{u}_0(\xi)$ is not on W_-^{uu}.

Similarly, if $\mathbf{u}_0(\xi)$ is on W_+^{ss}, then we can prove that $(u_0, v_0) \equiv (U, 0)$, where U is a solution of

$$U_{\xi\xi} + U(1 - U) = 0, \qquad \xi \in \mathbf{R},$$
$$U(\xi) \longrightarrow 1 \quad \text{as} \quad \xi \longrightarrow +\infty.$$

This contradiction implies that $\mathbf{u}_0(\xi)$ is not on W_+^{ss}.

In order to verify that the condition (H2) holds, we need some lemmas below.

LEMMA A.2. *Any non-trivial bounded solution* (u^*, v^*) *of* $\mathscr{L}^*(u, v) = 0$ *satisfies* $u^*(\xi)v^*(\xi) < 0$ *for all* $\xi \in \mathbf{R}$.

PROOF. First we prove that (u^*, v^*) satisfies $u^*(\xi)v^*(\xi) < 0$ in a neighborhood of $-\infty$. Since the eigenvalues of J_- is all real, neither $u^*(\xi)$ nor $v^*(\xi)$ oscillates in a neighborhood of $-\infty$. Moreover, since $f_v^0 < 0$ and $g_u^0 < 0$, neither $u^*(\xi)$ nor $v^*(\xi)$ is identically zero in a neighborhood of $-\infty$. Hence we can define ξ_u and ξ_v by

$$\xi_u = \inf\{\xi \in \mathbf{R} \mid u^*(\xi) = 0\} \in (-\infty, +\infty],$$
$$\xi_v = \inf\{\xi \in \mathbf{R} \mid v^*(\xi) = 0\} \in (-\infty, +\infty].$$

Without loss of generality, we assume that $\xi_u \leq \xi_v$ and that $u^*(\xi) > 0$ in a neighborhood of $-\infty$.

If $v^*(\xi) > 0$ for all $\xi \in (-\infty, \xi_v)$, then

$$\begin{aligned} 0 &= \int_{-\infty}^{\xi_u} u^*\{(u_{0\xi})_{\xi\xi} + f_u^0 u_{0\xi} + f_v^0 v_{0\xi}\}\, d\xi \\ &= -u_\xi^*(\xi_u)u_{0\xi}(\xi_u) + \int_{-\infty}^{\xi_u} \{f_v^0 v_{0\xi} u^* - g_u^0 u_{0\xi} v^*\}\, d\xi \\ &> 0. \end{aligned}$$

This contradiction implies that $v^*(\xi) < 0$ for all $\xi \in (-\infty, \xi_v)$.

Let us derive a contradiction by assuming that $\xi_u < +\infty$. First we consider the case where $\xi_u = \xi_v$. In this case, integration by parts yields

$$\begin{aligned} 0 &= \int_{-\infty}^{\xi_u} ((u_{0\xi}, v_{0\xi}), \mathscr{L}^*(u^*, v^*))\, d\xi \\ &= u_{0\xi}(\xi_u)u_\xi^*(\xi_u) + v_{0\xi}(\xi_u)v_\xi^*(\xi_u). \end{aligned}$$

Since $u_\xi^*(\xi_u) \leq 0$ and $v_\xi^*(\xi_u) \geq 0$, the above equality holds only if $u_\xi^*(\xi_u) = 0$ and $v_\xi^*(\xi_u) = 0$. By the uniqueness of the solution, this implies that $u^*(\xi) \equiv 0$ and $v^*(\xi) \equiv 0$. This contradicts the fact that (u^*, v^*) is a non-trivial solution.

Next we consider the case where $\xi_u < \xi_v$ and u^* has at least one zero in (ξ_u, ξ_v). Then we have the inequalities $u_\xi^*(\xi_u) \leq 0$ and $u_{\xi\xi}^*(\xi_u) = -g_u^0 v^* < 0$.

Since $u^*(\xi) > 0$ for $\xi < \xi_u$, these inequalities imply that $u_\xi^*(\xi_u) < 0$. If we put

$$\xi_0 = \inf\{\xi \in (\xi_u, \xi_v) | u^*(\xi) = 0\},$$

then $u^*(\xi) < 0$ for $\xi \in (\xi_u, \xi_0)$ so that

$$\begin{aligned} 0 &= \int_{\xi_u}^{\xi_0} u^*\{(u_{0\xi})_{\xi\xi} + f_u^0 u_{0\xi} + f_v^0 v_{0\xi}\}\, d\xi \\ &= [-u_{0\xi}u_\xi^*]_{\xi_u}^{\xi_0} + \int_{\xi_u}^{\xi_0} \{-g_u^0 u_{0\xi} v^* + f_v^0 v_{0\xi} u^*\}\, d\xi \\ &< 0. \end{aligned}$$

This is a contradiction. Hence u^* has no zero in (ξ_u, ξ_v).

Finally we consider the case where $\xi_u < \xi_v$ and u^* has no zero in (ξ_u, ξ_v). Putting $\tilde{u}^*(\xi) = \max\{u^*(\xi), 0\}$ and substituting $(\tilde{u}^*(\xi), v^*(\xi))$ into $\mathscr{L}^*(u, v)$, we obtain

$$\tilde{u}_{\xi\xi}^* + f_u^0 \tilde{u}^* + g_u^0 v^* = \begin{cases} 0 & \xi \in (-\infty, \xi_u), \\ -u_\xi^*(\xi_u)\delta(\xi - \xi_u) & \xi = \xi_u, \\ g_u^0 v^*(> 0) & \xi \in (\xi_u, \xi_v], \end{cases}$$

$$d_0 v_{\xi\xi}^* + f_v^0 \tilde{u}^* + g_v^0 v^* = \begin{cases} 0 & \xi \in (-\infty, \xi_u], \\ -f_v^0 u^*(< 0) & \xi \in (\xi_u, \xi_v], \end{cases}$$

where $\delta(\xi)$ denotes a Dirac's δ-function. Hence we obtain

$$0 < \int_{-\infty}^{\xi_v} ((u_{0\xi}, v_{0\xi}), \mathscr{L}^*(\tilde{u}^*, v^*))\, d\xi = v_{0\xi}(\xi_v) v_\xi^*(\xi_v) \le 0.$$

This contradiction implies that $\xi_u = +\infty$.

Thus the proof is completed.

LEMMA A.3. *Any non-trivial bounded solution* (u, v) *of* $\mathscr{L}(u, v) = 0$ *satisfies* $u(\xi)v(\xi) < 0$ *for all* $\xi \in \mathbf{R}$.

This lemma can be proved in the same manner as Lemma A.2 by replacing the roles of $(u_{0\xi}, v_{0\xi})$ and (u^*, v^*). So we omit the proof.

The following lemma shows that the condition (H2) holds.

LEMMA A.4.

$$\begin{aligned} &\textstyle\bigcup_{n=1}^\infty \{(u, v) | (u, v) \text{ is a bounded solution of } \mathscr{L}^n(u, v) = 0\} \\ &\quad = \{\beta(u_{0\xi}, v_{0\xi}) | \beta \in \mathbf{R}\}. \end{aligned}$$

PROOF. Suppose that there exists a non-trivial bounded solution (u, v) of

$\mathscr{L}(u, v) = 0$ which is linearly independent of $(u_{0\xi}, v_{0\xi})$. It is clear that $(U, V) = (u, v) + C(u_{0\xi}, v_{0\xi})$ also is a non-trivial bounded solution of $\mathscr{L}(u, v) = 0$ for an arbitrary constant C. If we take $C = -u(0)/u_{0\xi}(0)$, then $U(0) = 0$. This contradicts Lemma A.3.

Next let (u, v) be a non-trivial bounded solution of $\mathscr{L}^n(u, v) = 0$ for $n \geq 2$ and let (u^*, v^*) be a non-trivial bounded solution of $\mathscr{L}^*(u, v) = 0$. From the above argument, we see that there exists $\beta \in \mathbf{R}$ such that $\mathscr{L}^{n-1}(u, v) = \beta(u_{0\xi}, v_{0\xi})$. Hence

$$\beta \int_{-\infty}^{+\infty} ((u_{0\xi}, v_{0\xi}), (u^*, v^*))\, d\xi = \int_{-\infty}^{+\infty} (\mathscr{L}^{n-2}(u, v), \mathscr{L}^*(u^*, v^*))\, d\xi = 0.$$

Here, by Lemmas A.2 and A.3, we have

$$\int_{-\infty}^{+\infty} ((u_{0\xi}, v_{0\xi}), (u^*, v^*))\, d\xi \neq 0.$$

Hence we obtain $\beta = 0$. Inductively, we obtain $(u, v) \in \{\beta(u_{0\xi}, v_{0\xi}) | \beta \in \mathbf{R}\}$.

The following lemma shows that the former part of the condition (H3) holds.

LEMMA A.5. *Let (u, v) be any solution of $\mathscr{L}(u, v) = 0$ satisfying $(u, v) \to (0, 0)$ as $\xi \to -\infty$. If $|(u, v)| \to \infty$ as $\xi \to +\infty$, then there exists a constant $C_4 \neq 0$ such that*

$$(u(\xi), v(\xi)) = C_4(1, 0) \exp(\sigma_4 \xi) + o(\exp(\sigma_4 \xi)) \qquad \text{as } \xi \longrightarrow +\infty.$$

PROOF. By (A.2), the asymptotic behavior of $\mathbf{u} = {}^t(u, u_\xi, v, v_\xi)$ is represented as

$$\mathbf{u}(\xi) = C_3 \mathbf{q}_3 \exp(\sigma_3 \xi) + C_4 \mathbf{q}_4 \exp(\sigma_4 \xi) + o(\exp(\sigma_3 \xi)) \qquad \text{as } \xi \longrightarrow +\infty,$$

where C_3 and C_4 are certain constants. It is sufficient for the proof of this lemma to show that $C_4 \neq 0$.

Contrary to the conclusion, suppose that $C_4 = 0$. Then, by (A.2), $(u(\xi), v(\xi))$ satisfies $u(\xi) > 0$, $u_\xi(\xi) > 0$, $v(\xi) < 0$ and $v_\xi(\xi) < 0$ in a neighborhood of $+\infty$. On the other hand, by Lemma A.2, we have $u^*(\xi) > 0$, $u^*_\xi(\xi) < 0$, $v^*(\xi) < 0$ and $v^*_\xi(\xi) > 0$ in a neighborhood of $+\infty$. Hence, integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{\xi} ((u, v), \mathscr{L}^*(u^*, v^*))\, d\xi \\ &= u(\xi)u^*_\xi(\xi) + d_0 v(\xi) v^*_\xi(\xi) - u_\xi(\xi) u^*(\xi) - d_0 v_\xi(\xi) v^*(\xi) \\ &< 0 \end{aligned}$$

if $\xi > 0$ is sufficiently large. This contradiction implies that $C_4 \neq 0$.

The following lemma shows that the latter part of the conditio (H3) holds.

LEMMA A.6. *Let* (u, v) *be any solution of* $\mathscr{L}(u, v) = 0$ *satisfying* $(u, v) \to (0, 0)$ *as* $\xi \to +\infty$. *If* $|(u, v)| \to \infty$ *as* $\xi \to -\infty$, *then there exists a constant* $C_1 \neq 0$ *such that*

$$(u(\xi), v(\xi)) = C_1(0, 1)\exp(\lambda_1 \xi) + o(\exp(\lambda_1 \xi)) \qquad \text{as } \xi \longrightarrow -\infty.$$

This lemma can be proved in the same manner as Lemma A.5. So we omit the proof.

Finally, from Lemma A.2 and

$$\frac{\partial}{\partial(s, b, c)}\mathbf{g}(\mathbf{u}_0(\xi); b_0, c_0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ -u_{0\xi} & 0 & u_0 v_0 \\ 0 & 0 & 0 \\ -v_{0\xi} & u_0 v_0/d_0 & 0 \end{pmatrix},$$

it follows that the condition (H4) holds.

Thus we have shown that all the conditions (H1)–(H4) do hold. Moreover, by Lemma A.2 and (A.4), we obtain

$$\frac{\partial}{\partial b}s(b_0, c_0) < 0 \quad \text{and} \quad \frac{\partial}{\partial c}s(b_0, c_0) > 0.$$

The proof of (2.5) is now completed.

Next let us prove that $(\hat{u}, \hat{v})$ is positively monotone in a neighborhood of $(b, c) = (b_0, c_0)$ and satisfies (2.6) and (2.7). Let $\hat{J}_- = \dfrac{\partial \mathbf{F}}{\partial \mathbf{u}}(0, 0, a_0, 0)$ be a Jacobian matrix of $\mathbf{F}$ at $\mathbf{u} = {}^t(0, 0, a_0, 0)$, and let $\hat{\lambda}_i$, and $\hat{\mathbf{p}}_i$, $i = 1, 2, 3, 4$, denote eigenvalues and corresponding eigenvectors of $\hat{J}_-$. If (b, c) is close to (b_0, c_0), by continuity, we may assume that $\hat{\lambda}_1 < \hat{\lambda}_2 < 0 < \hat{\lambda}_3 < \hat{\lambda}_4$.

Similarly let $\hat{J}_+ = \dfrac{\partial \mathbf{F}}{\partial \mathbf{u}}(1, 0, 0, 0)$ be a Jacobian matrix of $\mathbf{F}$ at $\mathbf{u} = {}^t(1, 0, 0, 0)$, and let $\hat{\sigma}_i$, and $\hat{\mathbf{q}}_i$, $i = 1, 2, 3, 4$, denote eigenvalues and corresponding eigenvectors of $\hat{J}_+$. If (b, c) is close to (b_0, c_0), by continuity, we may assume that $\hat{\sigma}_1 < \hat{\sigma}_2 < 0 < \hat{\sigma}_3 < \hat{\sigma}_4$.

By virtue of Lemma A.1, there exist constants $C_- > 0$ and $C_+ > 0$ such that

$$\mathbf{u}_0(\xi) = \begin{cases} {}^t(0, 0, a_0, 0) + C_-\mathbf{p}_3 \exp(\lambda_3 \xi) + o(\exp(\lambda_3 \xi)) \\ \qquad \text{as } \xi \longrightarrow -\infty, \\ {}^t(1, 0, 0, 0) - C_+\mathbf{q}_2 \exp(\sigma_2 \xi) + o(\exp(\sigma_2 \xi)) \\ \qquad \text{as } \xi \longrightarrow +\infty. \end{cases}$$

In view of the proof of Theorem A of Kokubu [3], there exist constants $\beta_- > 0$ and $\beta_+ > 0$ such that $\beta_- \to C_-$ and $\beta_+ \to C_+$ as $(b, c) \to (b_0, c_0)$, and that $\hat{\mathbf{u}} = {}^t(\hat{u}, \hat{u}_\xi, \hat{v}, \hat{v}_\xi)$ satisfies

$$\hat{\mathbf{u}}(\xi) = \begin{cases} {}^t(0, 0, a_0, 0) + \beta_- \hat{\mathbf{p}}_3 \exp(\gamma_- \xi) + o(\exp(\gamma_- \xi)) \\ \qquad\qquad \text{as } \xi \longrightarrow -\infty, \\ {}^t(1, 0, 0, 0) - \beta_+ \hat{\mathbf{q}}_2 \exp(\gamma_+ \xi) + o(\exp(\gamma_+ \xi)) \\ \qquad\qquad \text{as } \xi \longrightarrow +\infty, \end{cases}$$

where

$$\gamma_-(b, c) = \hat{\lambda}_3 \quad \text{and} \quad \gamma_+(b, c) = \hat{\sigma}_2.$$

Moreover the orbit of $\hat{\mathbf{u}}$ converges to that of $\mathbf{u}_0$ as $(b, c) \to (b_0, c_0)$. These imply that there exists ξ_1 independent of (b, c) such that, if (b, c) is close to (b_0, c_0), $\hat{u}(\xi)$ is strictly increasing and $\hat{v}(\xi)$ is strictly decreasing in $(-\infty, \xi_1)$. Similarly we can prove that there exists ξ_2 independent of (b, c) such that, if (b, c) is close to (b_0, c_0), $\hat{u}(\xi)$ is strictly increasing and $\hat{v}(\xi)$ is strictly decreasing in $(\xi_2, +\infty)$.

On the other hand, since (u_0, v_0) is strictly positively monotone and since $\hat{\mathbf{u}}$ depends continuously on (b, c), $\hat{u}(\xi)$ is strictly increasing and $\hat{v}(\xi)$ is strictly decreasing in (ξ_1, ξ_2) if (b, c) is close to (b_0, c_0).

Thus we have shown that $(\hat{u}(\xi), \hat{v}(\xi))$ is strictly positively monotone. Moreover, since $\gamma_- \to \lambda_3$, $\hat{\mathbf{p}}_3 \to \mathbf{p}_3$ and $\gamma_+ \to \sigma_2$, $\hat{\mathbf{q}} \to \mathbf{q}_2$ as $(b, c) \to (b_0, c_0)$, we obtain (2.6) and (2.7).

The proof of Theorem 2.1 is now completed.

References

[1] E. A. Coddington and N. Levinson, Theory of ordinary differential equations, McGraw-Hill, 1955.

[2] K. Kishimoto and H. Weinberger, The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains, J. Differential Equations, **58** (1985), 15–21.

[3] H. Kokubu, Homoclinic and heteroclinic bifurcations of vector fields, Japan J. Appl. Math., **5** (1988), 455–501.

[4] H. Matano, Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems, J. Fac. Sci. Univ. Tokyo, **30** (1984), 645–673.

[5] H. Matano and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, Publ. RIMS Kyoto Univ., **19** (1983), 1049–1079.

[6] M. Mimura and P. C. Fife, A 3-component system of competition and diffusion, Hiroshima Math. J., **16** (1986), 189–207.

[7] P. de Mottoni, Qualitative analysis for some quasilinear parabolic systems, Institute of Math., Polish Academy Sci., zam. 11/79, **190** (1979).

[8] M. H. Protter and H. F. Weinberger, Maximum principles in differential equations,

Prentice-Hall, 1967.

[9] E. Yanagida, Stability of stationary distributions in a space-dependent population growth process, J. Math. Biology, **15** (1982), 37–50.

[10] E. Yanagida, Existence of stable stationary solutions of scalar reaction-diffusion equations in thin tubular domains, Applicable Analysis, **36** (1990), 171–188.

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