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Negative solutions of the generalized Liénard equation

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The aim of this paper is to state the conditions for the existence of negative solutions of the Liénard equation

$$x'' + f(x)x' + g(x) = 0. \tag{1}$$

Throughout this paper we will assume that

$$xf(x) > 0,\ xg(x) > 0 \qquad \text{for all } x \neq 0 \text{ and} \tag{2}$$

f and g are continuous on $R = (-\infty, \infty)$.

Under a negative solution $x(t)$ of (1) we will understand a solution such that $x(t) < 0$ on some interval $[T, \infty)$.

Denote

$$F(x) = \int_0^x f(s)ds, \quad G(x) = \int_0^x g(s)ds, \quad x \in (-\infty, \infty). \tag{3}$$

It follows from the assumptions (2) that $F(x) > 0$, $G(x) > 0$ for all $x \neq 0$, $F(0) = G(0) = 0$, $F(x)$ and $G(x)$ are decreasing for $x < 0$ and increasing for $x > 0$.

In the paper [1] we have made a qualitative analysis of the solutions of (1). In the paper [2] we have considered the behaviour and the existence of positive solutions of (1). In this paper we shall focus our attention on the negative solutions of (1). First, we shall introduce some results from [1] concerning the negative solutions of (1) and we shall use them later.

THEOREM A ([1], Theorem 4.1). Let $x(t)$ be a solution of (1) such that $x(t_0) < 0$, $x'(t_0) \geqslant 0$. Then there exists $\tau > t_0$ such that $x(\tau) = 0$.

COROLLARY 1. Let $x(t) < 0$, $t > t_0$, be a solution of (1). Then $x'(t) < 0$ for $t \geqslant t_0$.

THEOREM B ([1], Theorem 4.3). Let $x(t) < 0$, $t \geqslant t_0$, be a solution of (1). Then $\lim x(t) = -\infty$ as $t \to \infty$.

THEOREM C ([1], Theorem 4.5). Suppose that $F(-\infty) < \infty$ and $\limsup g(x) < 0$ as $t \to -\infty$. Then the equation (1) has no negative solution.

Theorem B says nothing about the boundedness of $x'(t)$. In the following Theorems we will discuss this problem.

THEOREM 1. Let $x(t) < 0$, $t \geq t_0$, be a solution of (1) and let be $F(-\infty) < \infty$. Then $\lim x'(t)$ as $t \to \infty$ exists and is finite.

PROOF. From Theorem B we get that $\lim x(t) = -\infty$ as $t \to \infty$ and from Corollary 1 we get that $x'(t) < 0$ for $t \geq t_0$. Then

$$[F(x(t))]' = f(x(t))x'(t) > 0, \ [x'(t) + F(x(t))]' = -g(x(t)) > 0 \qquad \text{for } t \geq t_0.$$

Thus, $F(x(t))$ and $x'(t) + F(x(t))$ are increasing on $[t_0, \infty)$. Moreover, if $k = F(-\infty)$ we have

$$x'(t_0) + F(x(t_0)) \leq x'(t) + F(x(t)) \leq x'(t) + k < k$$

which implies that $x'(t) + F(x(t))$ is bounded from above and $x'(t)$ is bounded from below. Therefore,

$$\lim x'(t) = \lim [x'(t) + F(x(t))] - \lim F(x(t)) \quad \text{as } t \to \infty$$

exists and is finite.

THEOREM 2. Let $x(t) < 0$, $t \geq t_0$, be a solution of (1) such that $x'(t_0) + F(x(t_0)) \geq 0$. Suppose that $g(x) < F(x)f(x)$ for $x < 0$. Then $\lim x'(t)$ as $t \to \infty$ exists and is finite.

PROOF. Let be $x(t) < 0$ for $t \geq t_0$ a solution of (1). Then $x'(t) < 0$ for $t \geq t_0$ (see Corollary 1). Then $[x'(t) + F(x(t))]' = -g(x(t)) > 0$, $[F(x(t))]' = -f(x(t))x'(t) > 0$ for $t \geq t_0$. Therefore, $x'(t) + F(x(t))$ and $F(x(t))$ are increasing for $t \geq t_0$. Thus,

$$x'(t) + F(x(t)) > x'(t_0) + F(x(t_0)) \geq 0, \ x'(t) > -F(x(t))$$

and

$$f(x(t))x'(t) < -F(x(t))f(x(t)) < -g(x(t))$$

which implies that $0 < -f(x(t))x'(t) - g(x(t)) = x''(t)$ for $t > t_0$. Therefore, $x'(t)$ is increasing for $t > t_0$ and being negative, we get that $\lim x'(t) > -\infty$ as $t \to \infty$.

Now, we are going to state the conditions which quarantee the existence of a negative solution $x(t)$ of (1). Assume that $x(t) < 0$, $t \geq t_0$, is a solution of (1). From Corollary 1 we have that $x'(t) < 0$ for $t \geq t_0$. Therefore, $x(t)$ has the inverse function $t = \phi(x)$, $x \leq a = x(t_0) \leq 0$. Let be $y(x) = x'(\phi(x))$, $x \leq a$. Evidently $y(x) < 0$ for $x < a$. It is easy to state that $y(x)$ is a solution of the equation

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y(x)} \qquad \text{on the interval } (-\infty, a). \tag{4}$$

Integration of this equation on $[x, a]$ gives us

$$y(x) = y(a) + F(a) - F(x) + \int_x^a \frac{g(u)}{y(u)}\,du. \tag{5}$$

From this we have that $y(x) - y(a) \geq F(a) - F(x)$,

$$0 > y(x) \geq y(a) + F(a) - F(x).$$

THEOREM 3. Let $k > 1$ and $a < 0$ be constants such that for $x \in (-\infty, a]$

$$F(a) - F(x) \leq \frac{k^2}{k-1}\left(\frac{g(a)}{f(a)} - \frac{g(x)}{f(x)}\right) \tag{6}$$

holds. Then there exists a solution $x(t)$ of (1) such that

$$x(t) < 0, \quad x'(t) < 0 \qquad \text{for all } t \geq t_0 \tag{7}$$

and

$$\lim_{t\to\infty} x(t) = -\infty, \quad x(t_0) = a. \tag{8}$$

PROOF. To prove this theorem is equivalent to finding a solution of the equation (5). Consider the operator

$$Uz(x) = y(a) + F(a) - F(x) + \int_x^a \frac{g(s)}{z(s)}\,ds, \qquad x \leq a, \tag{9}$$

$$-F(a) \leq y(a) \leq -k\frac{g(a)}{f(a)}$$

on the set

$$B = \left\{z(x) \in C((-\infty, a]) \colon -F(x) \leq z(x) \leq -k\frac{g(x)}{f(x)}\right\}, \tag{10}$$

where $C((-\infty, a])$ is the space of all continuous functions with the topology of uniform convergence on compact subintervals of the interval $(-\infty, a]$. It is easy to see that B is convex, closed and bounded in the topology of $C((-\infty, a])$. We will prove that a) $UB \subset B$, b) U is continuous on B, c) UB is relatively compact.

a) $UB \subset B$. Let $z(x) \in B$. Then

$$Uz(x) = y(a) + F(a) - F(x) + \int_x^a \frac{g(s)}{z(s)}\,ds, \qquad x \le a.$$

We have $g(s)/z(s) > 0$ and therefore

$$y(a) + F(a) \ge 0, \quad \int_x^a \frac{g(s)}{z(s)}\,ds > 0.$$

From this it follows that $Uz(x) \ge -F(x)$ for all $x \le a$. Moreover,

$$z(s) \le -k\frac{g(s)}{f(s)}, \qquad s \le a$$

implies that

$$\frac{g(s)}{z(s)} \le -\frac{1}{k}f(s)$$

and

$$\begin{aligned} Uz(x) &\le y(a) + F(a) - F(x) - \frac{1}{k}\int_x^a f(s)ds \\ &= y(a) + F(a) - F(x) - \frac{1}{k}(F(a) - F(x)) \\ &= y(a) + \frac{k-1}{k}(F(a) - F(x)). \end{aligned}$$

Respecting the fact that $y(a) \le -k\dfrac{g(a)}{f(a)}$ and the condition (6) we get that $Uz(x) \le -k\dfrac{g(x)}{f(x)}$. Thus, $Uz(x) \in B$ and $UB \subset B$.

b) U is continuous on B. Let $\{z_n(x)\}$ be a sequence of the elements from B which converges to $z(x) \in B$ uniformly on each compact subinterval of $(-\infty, a]$. Then

$$|Uz_n(x) - Uz(x)| \le \int_x^a \left|\frac{g(s)}{z_n(s)} - \frac{g(s)}{z(s)}\right| ds.$$

The sequence $\left\{\left|\dfrac{g(s)}{z_n(s)} - \dfrac{g(s)}{z(s)}\right|\right\}$ converges pointwise to zero on $(-\infty, a]$, and

$$\left|\frac{g(s)}{z_n(s)} - \frac{g(s)}{z(s)}\right| \le \left|\frac{g(s)}{z_n(s)}\right| + \left|\frac{g(s)}{z(s)}\right| \le \frac{2}{k}|f(s)|,$$

where $2|f(s)|/k$ is integrable on compact subintervals of $(-\infty, a]$. By Lebesgue's dominated convergence theorem we get that $\{Uz_n(x)\}$ converges to $Uz(x)$ uniformly on each compact subinterval of $(-\infty, a]$.

c) UB is a relatively compact set. This follows from the uniform boundedness of $Uz(x)$ and $(Uz(x))'$ on compact subintervals of $(-\infty, a]$. Indeed, $|Uz(x)| \leq F(x)$ and

$$|(Uz(x))'| \leq f(x) + \left|\frac{g(x)}{z(x)}\right| \leq |f(x)| + \frac{1}{k}|f(x)|$$

$$= \frac{k+1}{k}|f(x)|.$$

Thus, all conditions for use of Schauder's fixed point theorem are satisfied. Therefore, the operator U has a fixed point y in B which is a solution of (5) on the interval $(-\infty, a]$.

REMARK 1. It is possible to take $k = 2$ in (6) because $k^2/(k-1) \geq 4$ for all $k \geq 2$.

REMARK 2. In the case that $x(t_0) = a = 0$ the condition (6) has to be

$$F(x)f(x) \leq \frac{k^2}{k-1}g(x), \qquad x \leq 0, \tag{11}$$

the operator U has to have the form

$$Uz(x) = y(0) - F(x) + \int_x^0 \frac{g(s)}{z(s)}\,ds, \qquad x \leq 0, \quad y(0) < 0 \tag{12}$$

and

$$B = \left\{z(x) \in C((-\infty, 0]) \colon y(0) - F(x) \leq z(x) \leq -k\frac{g(x)}{f(x)}\right\}.$$

Then the conclusion of the Theorem 3 still holds.

EXAMPLE. Let $f(x) = g(x)$, $x \in (-\infty, 0]$. Then the conditions (6) and the condition (11) are satisfied. In this case we have

$$x'' + f(x)x' + f(x) = 0$$

which has the negative solutions

$$x(t) = -t + c.$$

REMARK 3. The operator U in Theorem 3 as well as the operator U in

Remark 2 are monotone on the set B. Therefore, it is possible to prove Theorem 3 by use of the theory of monotone operators.

References

[1] D. Hricišáková, Continuability and (non-) oscillatory properties of solutions of generalized Liénard equation, Hiroshima Math. J., **20** (1990), 11–22.

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