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Admissibility of difference approximations for scalar conservation laws

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Introduction

This paper is concerned with difference approximations for initial value problems for scalar conservation laws of the form

$$(0.1)\qquad \begin{cases} u_t + f(u)_x = 0, & x \in \boldsymbol{R},\ t > 0, \\ u(x, 0) = u_0(x), & x \in \boldsymbol{R}, \end{cases}$$

where $u = u(x, t)$ is an unknown function, the *flux function* $f: \boldsymbol{R} \to \boldsymbol{R}$ is a function of C^1-class and the initial function u_0 is a bounded measurable function of bounded variation. Various types of difference approximations have been investigated by many authors. We refer the readers to, for instance, [2, 4, 5, 6, 7, 10, 14, 15, 16, 20, 21, 23, 24, 28, 29].

In this paper we study difference approximations *in viscous form*, namely,

$$(0.2)\qquad \begin{aligned} u_i^{n+1} = u_i^n &- \frac{\lambda}{2}\{f(u_{i+1}^n) - f(u_{i-1}^n)\} \\ &+ \frac{\lambda}{2}\{a_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n) - a_{i-\frac{1}{2}}^n(u_i^n - u_{i-1}^n)\}, \qquad n, i \in \boldsymbol{Z},\ n \geq 0, \end{aligned}$$

where initial values u_i^0 are given data and $\lambda = \dfrac{\Delta t}{\Delta x}$ is a fixed constant, Δx the mesh size in space-direction and Δt in time-direction. Each of $\lambda a_{i+\frac{1}{2}}^n$ is called a numerical viscosity coefficient [8, 24, 29]. For the initial values u_i^0, we assume that

$$(0.3)\qquad m \leq u_i^0 \leq M, \qquad i \in \boldsymbol{Z},$$

for some constants m and M both independent of Δx, and

$$(0.4)\qquad \sup_{\Delta x} \sum_{i \in \boldsymbol{Z}} |u_{i+1}^0 - u_i^0| < +\infty.$$

On the mesh ratio λ, we impose so called CFL condition

$$\lambda \max_{m \leq s \leq M} |f'(s)| \leq 1. \tag{0.5}$$

It should be mentioned that the difference approximation (0.2) can be rewritten in a conservative form;

$$u_i^{n+1} = u_i^n - \lambda\{\bar{f}^n_{i+\frac{1}{2}} - \bar{f}^n_{i-\frac{1}{2}}\}, \qquad n, i \in \boldsymbol{Z},\ n \geq 0,$$

where each numerical flux $\bar{f}^n_{i+\frac{1}{2}}$ is defined by

$$\bar{f}^n_{i+\frac{1}{2}} = \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} - \frac{1}{2}a^n_{i+\frac{1}{2}}(u^n_{i+1} - u^n_i).$$

The initial value problem (0.1) can not always have a smooth solution for all time, even if the initial function is smooth. Therefore, weak solutions should be formulated in such a way that they can admit discontinuities. Usually, weak solutions to (0.1) are considered in the sense of distributions. The initial value problem (0.1) may possess infinitely many weak solutions and so, an additional condition, called *entropy condition*, is imposed to select physically relevant solutions. We here employ the entropy condition formulated by Lax [13]. The entropy condition is a condition which requires a weak solution $u = u(x, t)$ to satisfy the inequality

$$U(u)_t + F(u)_x \leq 0 \quad \text{(in distribution sense)} \tag{0.6}$$

for any pair (U, F) of functions such that U is convex and $F' = U'f'$. Such a pair (U, F) is called an *entropy pair.*

The admissibility of difference approximations is a notion for the difference approximations whose difference solutions converge to physically relevant solutions. The convergence is derived from the compactness of the difference solutions and the compactness problem is reduced to the stability problem as shown in Oleĭnik [21]. (See also Crandall and Majda [2], Glimm [6] and Smoller [27].) Therefore, we say that the difference approximation (0.2) is admissible if it is stable and consistent with entropy condition.

Our main purpose is here to characterize admissible difference approximations in terms of numerical viscosity coefficients. For this purpose, we investigate the stability and the consistency with entropy condition. The stability part is already treated by several authors. Here, we state a stability result obtained by LeRoux [16]: *The difference approximation* (0.2) *is L^∞-stable and TV* (*total variation*)*-stable if each of the coefficients $a^n_{i+\frac{1}{2}}$ satisfies*

$$a^{MR}(u_i^n, u_{i+1}^n) \leq a^n_{i+\frac{1}{2}} \leq \frac{1}{\lambda}, \tag{0.7}$$

where $a^{MR}(u_i^n, u_{i+1}^n)$ is defined by

$$a^{MR}(u_i^n, u_{i+1}^n) = \left| \int_0^1 f'(u_i^n + (u_{i+1}^n - u_i^n)\theta)\, d\theta \right| = \left| \frac{f(u_{i+1}^n) - f(u_i^n)}{u_{i+1}^n - u_i^n} \right|$$

and $\lambda a^{MR}(u_i^n, u_{i+1}^n)$ *is the numerical viscosity coefficient of Murmann-Roe scheme* [18, 26]. However, the main and difficult part is to investigate the consistency with entropy condition.

In what follows, we mainly focus our consideration on the consistency with entropy condition. There are a number of results in this direction. Crandall and Majda [2] proved that solutions to monotone difference approximations converge to physically relevant solutions. Well-known schemes such as Lax-Friedrichs scheme, Engquist-Osher scheme and Godunov scheme are monotone difference approximations. Tadmor [29] discussed a relationship between the consistency with entropy condition and numerical viscosity coefficients. He proved that the difference approximation (0.2) is consistent with entropy condition, provided that each of the coefficients $a^n_{i+\frac{1}{2}}$ satisfies the condition

$$a^G(u_i^n, u_{i+1}^n) \le a^n_{i+\frac{1}{2}} \le \frac{1}{2\lambda}, \tag{0.8}$$

where $a^G(u_i^n, u_{i+1}^n)$ is defined by

$$a^G(u_i^n, u_{i+1}^n) = \max_{(s-u_i^n)(s-u_{i+1}^n) \le 0} \frac{f(u_i^n) + f(u_{i+1}^n) - 2f(s)}{u_{i+1}^n - u_i^n}$$

and $\lambda a^G(u_i^n, u_{i+1}^n)$ is the numerical viscosity coefficient of Godunov scheme [7]. His result is a fully discrete version of the previous work of Osher [22] on semidiscrete approximations.

We deal with the consistency with entropy condition in the case of arbitrary flux functions and in the particular case of strictly convex flux functions. The consistency with entropy condition is introduced in the following way. We say that the difference approximation (0.2) is consistent with entropy condition if for an entropy pair (U, F) there exist real numbers $A^n_{i+\frac{1}{2}}$ such that they are bounded by a universal constant depending on U, and such that the *numerical entropy inequality* holds;

$$\begin{aligned} &U(u_i^{n+1}) - U(u_i^n) + \frac{\lambda}{2}\{F(u_{i+1}^n) - F(u_{i-1}^n)\} \\ &- \frac{\lambda}{2}\{A^n_{i+\frac{1}{2}}(u_{i+1}^n - u_i^n) - A^n_{i-\frac{1}{2}}(u_i^n - u_{i-1}^n)\} \le 0, \qquad n, i \in \boldsymbol{Z},\ n \ge 0. \end{aligned} \tag{0.9}$$

If we define a *numerical entropy flux* $\bar{F}^n_{i+\frac{1}{2}}$ by

$$\bar{F}^n_{i+\frac{1}{2}} = \frac{1}{2}\{F(u^n_i) + F(u^n_{i+1})\} - \frac{1}{2}A^n_{i+\frac{1}{2}}(u^n_{i+1} - u^n_i), \tag{0.10}$$

then we can rewrite the numerical entropy inequality (0.9) as follows;

$$U(u^{n+1}_i) - U(u^n_i) + \lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\} \le 0. \qquad n, i \in \boldsymbol{Z},\ n \ge 0. \tag{0.11}$$

It is easily seen that if bounded solutions to difference approximation (0.2) contain a subsequence converging in $L^1_{loc}(\boldsymbol{R} \times [0, \infty))$ and the difference approximation is consistent with entropy condition, then the limit function satisfies the entropy condition, *i.e.*, it becomes a physically relevant solution (see Theorem 4.1). For this reason, we say that the difference approximation (0.2) is *strongly admissible* if the stability condition (0.7) is satisfied and the numerical entropy inequality (0.9) is satisfied for *every* entropy pair.

For general flux functions, we obtain the following result which extends Tadmor's result [29] and is best possible for the admissibility of difference approximations in viscous form.

THEOREM 4.3. *If the condition*

$$a^G(u^n_i, u^n_{i+1}) \le a^n_{i+\frac{1}{2}} \le \frac{1}{\lambda} \tag{0.12}$$

holds for all n and i, then the difference approximation (0.2) *is strongly admissible.*

In order to treat the consistency with entropy condition for small coefficients $a^n_{i+\frac{1}{2}}$, we restrict ourselves to strictly convex flux functions. When the flux function f is strictly convex, the entropy inequality (0.6) for only one entropy pair (U, F) with U strictly convex ensures the physical relevance of solutions (see Theorem 1.1). Taking this fact into account, we say that the difference approximation (0.2) is *admissible* if the stability condition (0.7) is satisfied and the numerical entropy inequality (0.9) is satisfied for some entropy pair (U, F) with U strictly convex. The admissibility of difference approximations guarantees the convergence of difference solutions to physically relevant solutions (see Theorem 5.1).

The next main result shows that difference approximations in viscous form with small coefficients $a^n_{i+\frac{1}{2}}$ can be admissible, provided that the flux function f is strictly convex. This fact is remarkable from the theoretical and computational point of view.

THEOREM 5.2. *Suppose that the flux function f is strictly convex. Let $\varepsilon \in (0, 1)$. If the condition*

$$\max\left\{a^{MR}(u_i^n, u_{i+1}^n), \frac{\varepsilon}{\lambda}\cdot\operatorname{sgn}(u_{i+1}^n - u_i^n)\right\} \le a_{i+\frac{1}{2}}^n \le \frac{1}{\lambda} \tag{0.13}$$

holds for all n and i, then the difference approximation (0.2) *is admissible.*

Note that Murmann-Roe scheme preserves stationary inverse (physically irrelevant) shocks and hence is not admissible. Condition (0.13) asserts that $a^{MR}(u_i^n, u_{i+1}^n)$ should be replaced by $\frac{\varepsilon}{\lambda}$ if $u_i^n < u_{i+1}^n$ and $a^{MR}(u_i^n, u_{i+1}^n) < \frac{\varepsilon}{\lambda}$. Such replacement may be necessary around nearly stationary inverse shocks. Theorem 5.2 implies that Murmann-Roe scheme becomes admissible if its defect of preserving stationary inverse shocks is removed by adding a little amount of numerical viscosity. In particular, it follows that Harten scheme [8, 9] is admissible.

It should be emphasized that in the proof of Theorem 5.2 a particular entropy pair (U, F) is constructed according to the flux function f, the constants m, M, λ and ε. Hence our approach is essentially different from usual approach in which the numerical entropy condition is investigated for an entropy pair fixed in advance (see *e.g.* Majda and Osher [17], Osher and Tadmor [24]).

Finally, we illustrate our method for obtaining the main results mentioned above. In our discussion, a set of modified flux functions plays an important role. The use of modified flux functions enables us to clarify the role of each coefficient $a_{i+\frac{1}{2}}^n$ geometrically and to recover some information which has been lost in the process of discretization. In other words, by using appropriate modified flux functions, we can draw out useful information concerning the consistency with entropy condition. It should be remarked that modified flux functions were used in Osher and Tadmor's work [24].

This paper is organized as follows:

Section 1. Solutions of conservation laws.
Section 2. Difference approximations in viscous form.
Section 3. Relationships between numerical viscosity coefficients and modified flux functions.
Section 4. Strong admissibility of difference approximations.
Section 5. Admissibility of difference approximations with small viscosity coefficients.
Section 6. The construction of a particular entropy function.

1. Solutions of conservation laws

In this section we review some properties of solutions to the initial value problem for scalar conservation laws of the form

$$(1.1) \qquad \begin{cases} u_t + f(u)_x = 0, & x \in \boldsymbol{R},\ t > 0, \\ u(x, 0) = u_0(x), & x \in \boldsymbol{R}, \end{cases}$$

where $u = u(x, t)$ is an unknown function, $f: \boldsymbol{R} \to \boldsymbol{R}$ is a function of C^1-class and the initial function u_0 is a bounded measurable function. The function f is called the *flux function.*

It is well known that the initial value problem (1.1) can not always have smooth solutions for all time, even if the initial function u_0 is smooth. In order to allow discontinuities in solutions, the notion of weak solutions is usually employed.

DEFINITION 1.1. A bounded measurable function $u(x, t)$ ($x \in \boldsymbol{R}$, $t \geq 0$) is called a *weak solution* to the initial value problem (1.1), if it satisfies the equality

$$(1.2) \qquad \int_0^\infty \int_{-\infty}^\infty \{\phi_t(x, t)u(x, t) + \phi_x(x, t)f(x, t)\}\, dx dt + \int_{-\infty}^\infty \phi(x, 0)u_0(x)\, dx = 0$$

for any function $\phi \in C^\infty(\boldsymbol{R} \times [0, \infty))$ vanishing for $|x| + t$ large enough.

The initial value problem (1.1) may possess infinitely many weak solutions [16]. Accordingly, so called entropy condition is imposed on weak solutions to select physically relevant solutions to the initial value problem (1.1) [11, 13, 21, 30].

By $BV(\Omega)$ we mean the space of bounded measurable functions defined on $\Omega \subset \boldsymbol{R}^n$ whose generalized derivatives are finite measures. See Vol'pert [30, 31]. In what follows, we restrict ourselves to initial functions belonging to $BV(\boldsymbol{R})$ and employ the notion of admissible solutions (cf. [3]).

Let (U, F) be a pair of real-valued functions defined on $\boldsymbol{R}$. The pair (U, F) is called an *entropy pair* if

(1) U is a convex function and
(2) $F' = U'f'$.

The functions U and F are called an *entropy function* and an *entropy flux function*, respectively.

DEFINITION 1.2. Let $u_0 \in BV(\boldsymbol{R})$. A weak solution $u = u(x, t)$ is said to be an admissible solution if it satisfies the following conditions:

(i) $u \in BV(\boldsymbol{R} \times [0, T])$ for all $T > 0$.
(ii) For any entropy pair (U, F) and any non-negative function $\phi \in C_0^\infty(\boldsymbol{R} \times (0, \infty))$,

$$\int_0^\infty \int_{-\infty}^\infty \{\phi_t(x, t)U(u(x, t)) + \phi_x(x, t)F(u(x, t))\}\, dxdt \geq 0. \tag{1.3}$$

Condition (ii) is called the entropy condition. Inequality (1.3) is called the *entropy inequality* and is equivalent to the inequality

$$U(u)_t + F(u)_x \leq 0 \tag{1.4}$$

in the sense of Radon measure. The existence and uniqueness of admissible solutions are guaranteed by well-known results [11, 21, 30]. Note that by (1.2) and condition (i), $u(\cdot, t) \to u_0(\cdot)$ in L^1_{loc} as $t \to 0+$.

REMARK 1.1. Let u be a weak solution such that $u \in BV(\boldsymbol{R} \times [0, T])$ for any $T > 0$. Let $\Gamma(u)$ be the set of jump points of u in the sense of Vol'pert [30, 31], $(x_0, t_0) \in \Gamma(u)$ and let $v = (v_x, v_t)$ be a defining vector at (x_0, t_0) such that $v_x > 0$. Moreover, let u_+ and u_- be the approximate limits of u with respect to half-spaces $\{(x, t); (x - x_0, t - t_0) \cdot v \geq 0\}$ and $\{(x, t); (x - x_0, t - t_0) \cdot v \leq 0\}$, respectively. The entropy inequality (1.3) is equivalent to the inequality

$$\sigma\{U(u_+) - U(u_-)\} - \{F(u_+) - F(u_-)\} \geq 0$$

for almost every $(x_0, t_0) \in \Gamma(u)$ with respect to 1-dimensional Hausdorff measure, where $\sigma = -\dfrac{v_t}{v_x}$ is the shock speed at (x_0, t_0). For details, see [30, 31].

For each $k \in \boldsymbol{R}$, set

$$\begin{cases} U(s; k) = (s - k)^+ \\ F(s; k) = \chi^+(s - k)\{f(s) - f(k)\}, \qquad s \in \boldsymbol{R}. \end{cases} \tag{1.5}$$

Here s^+ and $\chi^+(s)$ are respectively defined by

$$s^+ = \begin{cases} s & \text{for } s \geq 0 \\ 0 & \text{for } s < 0, \end{cases}$$

$$\chi^+(s) = \begin{cases} 1 & \text{for } s \geq 0 \\ 0 & \text{for } s < 0. \end{cases} \tag{1.6}$$

It is easy to see that $(U(s; k), F(s; k))$ is an entropy pair for all $k \in \boldsymbol{R}$.

Since any convex function U is generated by linear functions and functions of the form $(s - k)^+$, we have the following result.

PROPOSITION 1.1 (*cf.* [11]). *Let $u_0 \in BV(\boldsymbol{R})$ and u be a weak solution to the initial value problem* (1.1). *If the entropy inequality* (1.3) *is satisfied for the family $\{(U(s; k), F(s; k)); k \in \boldsymbol{R}\}$, then u is an admissible solution.*

It should be noted that the uniqueness of weak solutions is not in general guaranteed by the entropy inequality (1.3) or (1.4) for a single entropy pair, as the following example shows.

EXAMPLE 1.1. Let $f(s)=(s-1)^2(s+1)^2$ for $s \in \boldsymbol{R}$. Define

$$u_0(x)=\begin{cases} -\dfrac{6}{5} & \text{if } x>0 \\ \dfrac{6}{5} & \text{if } x<0 \end{cases}$$

and

$$u(x, t)=\begin{cases} -\dfrac{6}{5} & \text{if } x>0 \\ \dfrac{6}{5} & \text{if } x<0,\ t \geq 0. \end{cases}$$

Then the function $u=u(x, t)$ gives a weak solution to the initial value problem (1.1) and the entropy inequality (1.4) is satisfied for an entropy pair

$$(U_1(s), F_1(s))=\left(s^4, 16 s^5\left\{\frac{1}{7} s^2-\frac{1}{5}\right\}\right).$$

But the weak solution u is not an admissible solution, because inequality (1.4) is not satisfied for another entropy pair

$$(U_2(s),\ F_2(s))=\left(s^2, 8 s^3\left\{\frac{1}{5} s^2-\frac{1}{3}\right\}\right).$$

In fact, noting that the shock speed $\sigma=0$, we have the following inequalities;

$$\sigma\left\{U_1\left(-\frac{6}{5}\right)-U_1\left(\frac{6}{5}\right)\right\}-\left\{F_1\left(-\frac{6}{5}\right)-F_1\left(\frac{6}{5}\right)\right\}=\frac{32}{175}\left(\frac{6}{5}\right)^5>0,$$

$$\sigma\left\{U_2\left(-\frac{6}{5}\right)-U_2\left(\frac{6}{5}\right)\right\}-\left\{F_2\left(-\frac{6}{5}\right)-F_2\left(\frac{6}{5}\right)\right\}=-\frac{272}{375}\left(\frac{6}{5}\right)^3<0.$$

However, in case that the flux function f is strictly convex, the uniqueness of weak solutions is guaranteed by the entropy condition for a single entropy pair with its entropy function strictly convex. More precisely, we have the following theorem.

THEOREM 1.1. *Suppose that the flux function f is strictly convex. Let $u_0 \in BV(\boldsymbol{R})$ and u be a weak solution to the initial value problem* (1.1) *such that*

$u \in BV(\boldsymbol{R} \times [0, T])$ *for any* $T > 0$. *If the entropy inequality* (1.3) *is satisfied for some entropy pair* (U, F) *with* U *strictly convex, then* u *is an admissible solution.*

PROOF. We use the notations in Remark 1.1. Let $(x_0, t_0) \in \Gamma(u)$, σ be the shock speed at (x_0, t_0), and let u_+ and u_- be the approximate limits of u at (x_0, t_0). Suppose that

$$\sigma\{U(u_+) - U(u_-)\} - \{F(u_+) - F(u_-)\} \geq 0 \tag{1.7}$$

for some entropy pair (U, F) with U strictly convex. In view of the strict convexity of f, it suffices to show that $u_+ < u_-$. To this end, define a linear function g by

$$g(s) = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \cdot (s - u_-) + f(u_-), \qquad s \in \boldsymbol{R}.$$

Note that $g'(s) \equiv \dfrac{f(u_+) - f(u_-)}{u_+ - u_-} = \sigma$ by Rankin-Hugoniot's relation. Since U' is increasing and f, g are locally Lipschitz continuous, we obtain

$$\begin{aligned}
&\sigma\{U(u_+) - U(u_-)\} - \{F(u_+) - F(u_-)\} \\
&= \int_{u_-}^{u_+} \sigma U'(s)\,ds - \int_{u_-}^{u_+} f'(s)U'(s)\,ds \\
&= \int_{u_-}^{u_+} \{g'(s) - f'(s)\}\,U'(s)\,ds \\
&= -\int_{u_-}^{u_+} \{g(s) - f(s)\}\,dU'(s),
\end{aligned}$$

where the last integral is taken in the sense of Stieltjes. This together with (1.7) yields

$$\int_{u_-}^{u_+} \{g(s) - f(s)\}\,dU'(s) \leq 0.$$

Since $g(s) > f(s)$ for all s between u_+ and u_-, we conclude that $u_+ < u_-$. This completes the proof.

2. Difference approximations in viscous form

In this section we formulate difference approximations for the initial value problem (1.1) and give some remarks on the difference approximations.

Let Δx and Δt be mesh sizes in space and time, respectively, and let the mesh ratio $\lambda = \dfrac{\Delta t}{\Delta x}$ be fixed. We are concerned with difference approximations *in viscous form*

$$
(2.1)\qquad \begin{aligned} u_i^{n+1} = u_i^n &- \frac{\lambda}{2}\{f(u_{i+1}^n) - f(u_{i-1}^n)\} \\ &+ \frac{\lambda}{2}\{a_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n) - a_{i-\frac{1}{2}}^n(u_i^n - u_{i-1}^n)\}, \qquad n,\ i \in \boldsymbol{Z},\ n \geq 0, \end{aligned}
$$

where the initial values u_i^0 are given data and each u_i^n is understood to be defined at a grid point $(i\Delta x, n\Delta t)$. $\lambda a_{i+\frac{1}{2}}^n$ is called a numerical viscosity coefficient. Let m and M be constants such that $m < M$. In the arguments developed below, we assume that

$$
(2.2)\qquad m \leq u_i^0 \leq M, \qquad i \in \boldsymbol{Z},
$$

and

$$
(2.3)\qquad \sup_{\Delta x} \sum_{i \in \boldsymbol{Z}} |u_{i+1}^0 - u_i^0| < +\infty.
$$

On the mesh ratio λ, we impose so called CFL condition

$$
(2.4)\qquad \lambda \max_{m \leq s \leq M} |f'(s)| \leq 1.
$$

We can rewrite the difference approximation (2.1) in a conservative form;

$$
(2.5)\qquad u_i^{n+1} = u_i^n - \lambda\{\bar{f}_{i+\frac{1}{2}}^n - \bar{f}_{i-\frac{1}{2}}^n\}, \qquad n,\ i \in \boldsymbol{Z},\ n \geq 0,
$$

where each *numerical flux* $\bar{f}_{i+\frac{1}{2}}^n$ is defined by

$$
(2.6)\qquad \bar{f}_{i+\frac{1}{2}}^n = \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} - \frac{1}{2}a_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n).
$$

If each numerical flux $\bar{f}_{i+\frac{1}{2}}^n$ is determined by a function h_f of $2l$-variables, called a numerical flux function, *i.e.*, if

$$
\bar{f}_{i+\frac{1}{2}}^n = h_f(u_{i-l+1}^n, \ldots, u_{i+l}^n),
$$

then the coefficient $a_{i+\frac{1}{2}}^n$ is also determined by a function a_f of $2l$-variables;

$$
\begin{aligned} a_{i+\frac{1}{2}}^n &= a_f(u_i^n, \ldots, u_{i+1}^n) \\ &= \frac{f(u_i^n) + f(u_{i+1}^n) - 2h_f(u_{i-l+1}^n, \ldots, u_{i+l}^n)}{u_{i+1}^n - u_i^n}. \end{aligned}
$$

We do not assume that the numerical fluxes $\bar{f}^{\,n}_{i+\frac{1}{2}}$ in (2.6) are determined by a numerical flux function.

We recall the following results on L^∞-stability and TV (total variation)-stability. For the proofs, see *e.g.* [8, 9, 16].

LEMMA 2.1. *If each of the coefficients* $a^n_{i+\frac{1}{2}}$ *satisfies the condition*

$$\left|\int_0^1 f'(u^n_i + (u^{n+1}_i - u^n_i)\theta)d\theta\right| \le a^n_{i+\frac{1}{2}} \le \frac{1}{\lambda}, \tag{2.7}$$

then

$$\inf_{i\in \boldsymbol{Z}} u^0_i \le \inf_{i\in \boldsymbol{Z}} u^n_i \le \inf_{i\in \boldsymbol{Z}} u^{n+1}_i \le \sup_{i\in \boldsymbol{Z}} u^{n+1}_i \le \sup_{i\in \boldsymbol{Z}} u^n_i \le \sup_{i\in \boldsymbol{Z}} u^0_i \tag{2.8}$$

for all $n \ge 0$.

LEMMA 2.2. *If each of the coefficients* $a^n_{i+\frac{1}{2}}$ *satisfies condition* (2.7) *in Lemma* 2.1, *then the difference approximation* (2.1) *is TVD* (*total variation diminishing*), *i.e.*, *the inequality*

$$\sum_{i\in \boldsymbol{Z}} |u^{n+1}_{i+1} - u^{n+1}_i| \le \sum_{i\in \boldsymbol{Z}} |u^n_{i+1} - u^n_i| \le \sum_{i\in \boldsymbol{Z}} |u^0_{i+1} - u^0_i| \tag{2.9}$$

holds for all $n \ge 0$.

In the rest of this section, we give some examples of difference approximations in viscous form.

1) *Lax-Friedrichs scheme.*

Lax-Friedrichs scheme is defined by

$$u^{n+1}_i = \frac{1}{2}\{u^n_{i+1} + u^n_{i-1}\} - \frac{\lambda}{2}\{f(u^n_{i+1}) - f(u^n_{i-1})\}.$$

The coefficients $a^n_{i+\frac{1}{2}} = a^{LE}(u^n_i, u^n_{i+1})$ corresponding to Lax-Friedrichs scheme does not depend on u^n_i and u^n_{i+1};

$$a^{LF} \equiv \frac{1}{\lambda}.$$

This difference scheme first appeared in Lax's paper [12]. With the aid of Lax-Friedrichs scheme, Oleĭnik [21] proved the existence and uniqueness of solutions to scalar conservation laws. It should be mentioned that Lax-Friedrichs scheme is essentially defined on a staggered mesh. In other words, there is no relation between u^n_i and u^m_j if $n + i \not\equiv m + j \pmod 2$.

2) *Engquist-Osher scheme.*

Engquist-Osher [5] scheme is the difference approximation (2.5) with the numericl fluxes

$$\bar{f}^{\,n}_{i+\frac{1}{2}} = \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} - \frac{1}{2}\int_{u_i^n}^{u_{i+1}^n} |f'(s)|\, ds. \tag{2.10}$$

Hence each of the coefficients $a^n_{i+\frac{1}{2}} = a^{EO}(u_i^n, u_{i+1}^n)$ is given by

$$a^{EO}(u_i^n, u_{i+1}^n) = \int_0^1 |f'(u_i^n + (u_{i+1}^n - u_i^n)\theta)|\, d\theta. \tag{2.11}$$

3) *Godunov scheme.*

Let $w = w(x, t; u_L, u_R)$ be the exact solution to the Riemann problem:

$$\begin{cases} w_t + f(w)_x = 0, \\ w(x, 0) = \begin{cases} u_L & \text{for } x < 0 \\ u_R & \text{for } x > 0. \end{cases} \end{cases} \tag{2.12}$$

Then, Godunov scheme [7] is defined by

$$\begin{aligned} u_i^{n+1} = \frac{1}{\Delta x}\Bigg[&\int_0^{\frac{1}{2}\Delta x} w(x, \Delta t; u_{i-1}^n, u_i^n)\, dx \\ &+ \int_{-\frac{1}{2}\Delta x}^0 w(x, \Delta t; u_i^n, u_{i+1}^n)\, dx \Bigg]. \end{aligned} \tag{2.13}$$

It is known that Godunov scheme can be rewritten in a conservative form. The numerical flux $\bar{f}^{\,n}_{i+\frac{1}{2}}$ is given by

$$\bar{f}^{\,n}_{i+\frac{1}{2}} = \begin{cases} \min\limits_{u_i^n \le s \le u_{i+1}^n} f(s) & \text{if } u_i^n \le u_{i+1}^n \\ \max\limits_{u_{i+1}^n \le s \le u_i^n} f(s) & \text{if } u_{i+1}^n < u_i^n, \end{cases} \tag{2.14}$$

and hence each of the coefficients $a^n_{i+\frac{1}{2}} = a^G(u_i^n, u_{i+1}^n)$ is written as

$$\begin{aligned} &a^G(u_i^n, u_{i+1}^n) \\ &= \begin{cases} \max\limits_{(s-u_i^n)(s-u_{i+1}^n) \le 0} \dfrac{f(u_i^n) + f(u_{i+1}^n) - 2f(s)}{u_{i+1}^n - u_i^n} & \text{if } u_i^n \ne u_{i+1}^n \\ |f'(u_i^n)| & \text{if } u_i^n = u_{i+1}^n. \end{cases} \end{aligned} \tag{2.15}$$

It should be noted that

$$a^G(u_i^n, u_{i+1}^n) = a^{EO}(u_i^n, u_{i+1}^n),$$

provided that f is convex and $u_i^n < u_{i+1}^n$ (resp. f is concave and $u_{i+1}^n < u_i^n$).

4) *Murmann-Roe* (*or generalized Courant-Isaacson-Rees*) *scheme.*

The coefficients $a_{i+\frac{1}{2}}^n = a^{MR}(u_i^n, u_{i+1}^n)$ corresponding to Murmann-Roe scheme are given by

$$a^{MR}(u_i^n, u_{i+1}^n) = \left| \int_0^1 f'(u_i^n + (u_{i+1}^n - u_i^n)\theta)\, d\theta \right|. \tag{2.16}$$

Murmann-Roe scheme was proposed by Murmann [18] in the scalar case and was extended by Roe [25, 26] to hyperbolic systems of conservation laws. See also Courant, Isaacson and Rees [1]. It should be noted that the above coefficient $a^{MR}(u_i^n, u_{i+1}^n)$ is the least one satisfying condition (2.7) and Murmann-Roe scheme admits stationary inverse (physically irrelevant) shocks.

5) *Harten scheme.*

The coefficients $a_{i+\frac{1}{2}}^n$ corresponding to Harten scheme are given by

$$a_{i+\frac{1}{2}}^n = \frac{1}{\lambda} Q\left(\lambda \int_0^1 f'(u_i^n + (u_{i+1}^n - u_i^n)\theta)\, d\theta \right), \tag{2.17}$$

where Q is a function defined on $[-1, 1]$ such that $Q(0) \neq 0$ and $|\alpha| \leq Q(\alpha) \leq 1$ for $\alpha \in [0, 1]$. It is clear that Harten scheme satisfies condition (2.7) and hence is TVD (See Harten [8, 9]). For $\varepsilon \in (0, 1)$, Harten gave two examples of such function Q;

$$Q(\alpha) = \max\{|\alpha|, \varepsilon\}, \qquad \alpha \in [-1, 1], \tag{2.18}$$

and

$$Q(\alpha) = \begin{cases} |\alpha| & \text{for } \varepsilon \leq |\alpha| \leq 1 \\ \dfrac{\alpha^2}{2\varepsilon} + \dfrac{\varepsilon}{2} & \text{for } |\alpha| \leq \varepsilon. \end{cases} \tag{2.19}$$

Since $Q(0) \neq 0$, Harten scheme does not admit stationary inverse shocks. However, it has remained to show whether solutions to Harten scheme converge to physically relevant solutions.

6) *Lax-Wendroff scheme.*

The coefficients $a_{i+\frac{1}{2}}^n = a^{LW}(u_i^n, u_{i+1}^n)$ corresponding to Lax-Wendroff scheme [14] are defined by

$$a^{LW}(u_i^n, u_{i+1}^n) = \lambda f'\left(\frac{u_i^n + u_{i+1}^n}{2} \right) \left| \int_0^1 f'(u_i^n + (u_{i+1}^n - u_i^n)\theta)\, d\theta \right|.$$

The coefficients $a^{LW}(u_i^n, u_{i+1}^n)$ do not satisfy condition (2.7) in general. By this fact, Lax-Wendroff scheme may produce overshoots and undershoots near shocks, although it is second order accurate.

REMARK 2.1. As is easily seen, the following inequality holds:

$$a^{LW} \leq a^{MR} \leq a^G \leq a^{EO} \leq a^{LF} \equiv \frac{1}{\lambda}. \tag{2.20}$$

This simple fact is useful for the analysis of difference approximations in viscous form.

3. Relationships between numerical viscosity coefficients and modified flux functions

As mentioned in the previous section, coefficients $a^n_{i+\frac{1}{2}}$ give us important information on the stability of difference approximations. In order to get more information from the coefficients $a^n_{i+\frac{1}{2}}$, we here introduce a general class of modified flux functions and investigate relationships between numerical viscosity coefficients and modified flux functions. Modified flux functions give a geometrical interpretation of numerical viscosity produced by difference approximations. In particular, a certain set of modified flux functions plays an important role in later arguments and provides us with a unified approach to difference approximations.

It is convenient to use the following notations. Let u_L and u_R be real numbers, and let g be a locally Lipschitz continuous function defined on $\boldsymbol{R}$. We define $a^{EO}(u_L, u_R; g)$ by

$$a^{EO}(u_L, u_R; g) = \int_0^1 \left| g'(u_L + (u_R - u_L)\theta) \right| d\theta \tag{3.1}$$

and $a^G(u_L, u_R; g)$ by

$$a^G(u_L, u_R; g) = \begin{cases} \max\limits_{(s-u_L)(s-u_R)\leq 0} \dfrac{g(u_L) + g(u_R) - 2g(s)}{u_R - u_L} & \text{if } u_L \neq u_R \\ |g'(u_L)| & \text{if } u_L = u_R. \end{cases} \tag{3.2}$$

Now, we give the definition of modified flux functions. Throughout the rest of this paper, we denote by $\{u_i^n\}$ the solution to the difference approximation (2.1).

DEFINITION 3.1. A locally Lipschitz continuous function g defined on $\boldsymbol{R}$

is called a *modified flux function associated with* $f(u_i^n)$, $f(u_{i+1}^n)$ *and* $a_{i+\frac{1}{2}}^n$, if it satisfies the following conditions:

(i) $g(u_i^n) = f(u_i^n)$ and $g(u_{i+1}^n) = f(u_{i+1}^n)$.
(ii) $g(s) = f(s)$ if s lies outside the interval between u_i^n and u_{i+1}^n.
(iii) $a^{EO}(u_i^n, u_{i+1}^n; g) = a^G(u_i^n, u_{i+1}^n; g) = a_{i+\frac{1}{2}}^n$.

We sometimes use an abbreviated terminology, a *modified flux function associated with* $a_{i+\frac{1}{2}}^n$, if there is no ambiguity. Modified flux functions associated with $a_{i+\frac{1}{2}}^n$ do not exist in general. In fact, we have the following proposition, which characterizes the existence of modified flux functions (see also [24]). However, it should be noted that a modified flux function associated with $a_{i+\frac{1}{2}}^n$ is not uniquely determined.

PROPOSITION 3.1. *Let* $u_i^n \neq u_{i+1}^n$. *There exists a modified flux function* g *associated with* $f(u_i^n)$, $f(u_{i+1}^n)$ *and* $a_{i+\frac{1}{2}}^n$, *if and only if*

$$a^{MR}(u_i^n, u_{i+1}^n) \leq a_{i+\frac{1}{2}}^n, \tag{3.3}$$

where $a^{MR}(u_i^n, u_{i+1}^n)$ *is the coefficient defined by* (2.16).

PROOF. If there exists a modified flux function g associated with $f(u_i^n)$, $f(u_{i+1}^n)$ and $a_{i+\frac{1}{2}}^n$, then we obtain

$$\begin{aligned}
a_{i+\frac{1}{2}}^n &= a^{EO}(u_i^n, u_{i+1}^n; g) \\
&= \int_0^1 |g'(u_i^n + (u_{i+1}^n - u_i^n)\theta)|\, d\theta \\
&\geq \left| \int_0^1 g'(u_i^n + (u_{i+1}^n - u_i^n)\theta)\, d\theta \right| \\
&= \left| \frac{g(u_{i+1}^n) - g(u_i^n)}{u_{i+1}^n - u_i^n} \right| \\
&= \left| \frac{f(u_{i+1}^n) - f(u_i^n)}{u_{i+1}^n - u_i^n} \right| \\
&= \left| \int_0^1 f'(u_i^n + (u_{i+1}^n - u_i^n)\theta)\, d\theta \right| \\
&= a^{MR}(u_i^n, u_{i+1}^n).
\end{aligned}$$

Conversely, suppose that $a^{MR}(u_i^n, u_{i+1}^n) \leq a_{i+\frac{1}{2}}^n$ and $u_i^n < u_{i+1}^n$. Then we obtain

$$\bar{f}_{i+\frac{1}{2}}^n = \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} - \frac{1}{2} a_{i+\frac{1}{2}}^n (u_{i+1}^n - u_i^n)$$

$$\leq \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} - \frac{1}{2}a^{MR}(u_i^n, u_{i+1}^n)(u_{i+1}^n - u_i^n)$$

$$= \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} - \frac{1}{2}|f(u_{i+1}^n) - f(u_i^n)|$$

$$= \min\{f(u_i^n), f(u_{i+1}^n)\}.$$

Hence we can find a number $s_0 \in (u_i^n, u_{i+1}^n)$ and a continuous function g such that

(1) $g(s_0) = \bar{f}^n_{i+\frac{1}{2}}$,
(2) $g(u_i^n) = f(u_i^n)$ and $g(u_{i+1}^n) = f(u_{i+1}^n)$,
(3) $g(s)$ is nonincreasing on $[u_i^n, s_0]$ and nondecreasing on $[s_0, u_{i+1}^n]$, and
(4) $g(s) = f(s)$ for $s \notin (u_i^n, u_{i+1}^n)$.

Then, the function g is a modified flux function associated with $f(u_i^n)$, $f(u_{i+1}^n)$ and $a^n_{i+\frac{1}{2}}$. In fact, we have

$$\begin{aligned}
a^{EO}(u_i^n, u_{i+1}^n; g) &= \int_0^1 |g'(u_i^n + (u_{i+1}^n - u_i^n)\theta)|\,\theta \\
&= \frac{1}{u_{i+1}^n - u_i^n}\int_{u_i^n}^{u_{i+1}^n} |g'(s)|\,ds \\
&= \frac{1}{u_{i+1}^n - u_i^n}\left[\int_{u_i^n}^{s_0}\{-g'(s)\}\,ds + \int_{s_0}^{u_{i+1}^n} g'(s)\,ds\right] \\
&= \frac{g(u_i^n) + g(u_{i+1}^n) - 2g(s_0)}{u_{i+1}^n - u_i^n} \\
&= \frac{f(u_i^n) + f(u_{i+1}^n) - 2\bar{f}^n_{i+\frac{1}{2}}}{u_{i+1}^n - u_i^n} \\
&= a^n_{i+\frac{1}{2}}.
\end{aligned}$$

Since $g(s_0) = \min_{u_i^n \leq s \leq u_{i+1}^n} g(s)$, this means

$$a^{EO}(u_i^n, u_{i+1}^n) = a^G(u_i^n, u_{i+1}^n) = a^n_{i+\frac{1}{2}}.$$

Similarly, we can prove the existence of modified flux functions associated with $a^n_{i+\frac{1}{2}}$ in the case that $u_i^n > u_{i+1}^n$. Thus the proof is completed.

The next proposition is useful in later arguments.

Proposition 3.2. *Let $u_i^n \neq u_{i+1}^n$. There exists a modified flux function g*

associated with $f(u_i^n), f(u_{i+1}^n)$ *and* $a_{i+\frac{1}{2}}^n$ *satisfying the condition*

$$|g'(s)| \leq \frac{1}{\lambda} \quad \textit{for} \quad (s-u_i^n)(s-u_{i+1}^n) \leq 0, \tag{3.4}$$

if and only if

$$a^{MR}(u_i^n, u_{i+1}^n) \leq a_{i+\frac{1}{2}}^n \leq \frac{1}{\lambda}. \tag{3.5}$$

PROOF. Let u_L and u_R be real numbers such that $u_L \neq u_R$, and set

$$u_M = \frac{1}{2}\{u_L + u_R\} - \frac{\lambda}{2}\{f(u_R) - f(u_L)\}.$$

Note that u_M lies between u_L and u_R. We define two functions $g^{MR}(s; u_L, u_R)$ and $g^{LF}(s; u_L, u_R)$ by

$$g^{MR}(s; u_L, u_R) = \begin{cases} \dfrac{f(u_R) - f(u_L)}{u_R - u_L} \cdot (s - u_L) + f(u_L) & \text{for } (s-u_L)(s-u_R) \leq 0 \\ f(s) & \text{otherwise,} \end{cases} \tag{3.6}$$

$$g^{LF}(s; u_L, u_R) = \begin{cases} -\dfrac{1}{\lambda}(s - u_L) + f(u_L) & \text{for } (s-u_L)(s-u_M) \leq 0 \\ \dfrac{1}{\lambda}(s - u_R) + f(u_R) & \text{for } (s-u_M)(s-u_R) \leq 0 \\ f(s) & \text{otherwise.} \end{cases} \tag{3.7}$$

Now, suppose that (3.5) is satisfied. We easily see that $g^{MR}(s; u_i^n, u_{i+1}^n)$ and $g^{LF}(s; u_i^n, u_{i+1}^n)$ are modified flux functions associated with $a^{MR}(u_i^n, u_{i+1}^n)$ and $a^{LF} \equiv \frac{1}{\lambda}$, respectively.

For each $\alpha \in [0, 1]$, we define a function $g^\alpha(s; u_i^n, u_{i+1}^n)$ by

$$g^\alpha(s; u_i^n, u_{i+1}^n) = \alpha g^{LF}(s; u_i^n, u_{i+1}^n) + (1-\alpha) g^{MR}(s; u_i^n, u_{i+1}^n), \qquad s \in \boldsymbol{R}.$$

Then it is clear that every g^α satisfies

$$|(g^\alpha)'(s; u_i^n, u_{i+1}^n)| \leq \frac{1}{\lambda} \qquad \text{for } (s-u_i^n)(s-u_{i+1}^n) \leq 0$$

and

$$a^G(u_i^n, u_{i+1}^n; g^\alpha) = a^{EO}(u_i^n, u_{i+1}^n; g^\alpha).$$

Also, it is evident that the value $a^G(u_i^n, u_{i+1}^n; g^\alpha) = a^{EO}(u_i^n, u_{i+1}^n; g^\alpha)$ varies continuously from $a^{MR}(u_i^n, u_{i+1}^n)$ to $a^{LF} \equiv \frac{1}{\lambda}$ as α varies from 0 to 1. Therefore, it follows that

$$a^G(u_i^n, u_{i+1}^n; g^\alpha) = a^{EO}(u_i^n, u_{i+1}^n; g^\alpha) = a^n_{i+\frac{1}{2}}$$

for some $\alpha \in [0, 1]$, and hence the 'if' part is proved. To prove the 'only if' part, let g be a modified flux function associated with $a^n_{i+\frac{1}{2}}$ satisfying (3.4). It is clear that

$$a^n_{i+\frac{1}{2}} = a^{EO}(u_i^n, u_{i+1}^n; g) = \int_0^1 |g'(u_i^n + (u_i^n + (u_{i+1}^n - u_i^n)\theta)|\, d\theta \le \frac{1}{\lambda}.$$

Since the first inequality in (3.5) is already proved in Proposition 3.1, this completes the proof.

REMARK 3.1. It is interesting to observe that condition (3.5) is the same as the sufficient condition (2.7) for L^∞-stability and TV-stability of difference approximation (2.1).

Next we introduce special modified flux functions which play an important role in our arguments. To this end, we assume condition (3.5);

$$a^{MR}(u_i^n, u_{i+1}^n) \le a^n_{i+\frac{1}{2}} \le \frac{1}{\lambda}$$

The special modified flux function $g^n_{i+\frac{1}{2}}$ is defined in the following way (see [24]): If $u_i^n \le u_{i+1}^n$, we define

$$g^n_{i+\frac{1}{2}}(s) = \begin{cases} \max\left\{-\frac{1}{\lambda}(s - u_i^n) + f(u_i^n),\ \bar{f}^n_{i+\frac{1}{2}},\ \frac{1}{\lambda}(s - u_{i+1}^n) + f(u_{i+1}^n)\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u_i^n \le s \le u_{i+1}^n \\ f(s) \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{3.8}$$

If $u_i^n > u_{i+1}^n$, we define

$$g^n_{i+\frac{1}{2}}(s) = \begin{cases} \min\left\{\frac{1}{\lambda}(s - u_{i+1}^n) + f(u_{i+1}^n),\ \bar{f}^n_{i+\frac{1}{2}},\ -\frac{1}{\lambda}(s - u_i^n) + f(u_i^n)\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u_{i+1}^n \le s \le u_i^n \\ f(s) \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{3.9}$$

Here $\bar{f}^n_{i+\frac{1}{2}}$ is the numerical flux defined by (2.6).

It is clear that the function $g^n_{i+\frac{1}{2}}$ is a modified flux function associated with $f(u^n_i)$, $f(u^n_{i+1})$ and $a^n_{i+\frac{1}{2}}$, provided that $u^n_i \neq u^n_{i+1}$. Note that $g^n_{i+\frac{1}{2}} \equiv f$ if $u^n_i = u^n_{i+1}$. As is easily seen, the function $g^n_{i+\frac{1}{2}}$ is minimal in the sense that

$$\text{(3.10)} \qquad \operatorname{sgn}(u^n_{i+1} - u^n_i)\{g^n_{i+\frac{1}{2}}(s) - g(s)\} \leq 0, \qquad s \in \boldsymbol{R},$$

for any modified flux function g satisfying (3.4) in Proposition 3.2.

We have an interesting result on the relation between the minimal modified flux function $g^n_{i+\frac{1}{2}}$ and the original flux function f.

PROPOSITION 3.3. *Let $u^n_i \neq u^n_{i+1}$ and let $g^n_{i+\frac{1}{2}}$ be the minimal modified flux function associated with $f(u^n_i)$, $f(u^n_{i+1})$ and $a^n_{i+\frac{1}{2}}$. The inequality*

$$\text{(3.11)} \qquad sgn(u^n_{i+1} - u^n_i)\{g^n_{i+\frac{1}{2}}(s) - f(s)\} \leq 0$$

holds for all $s \in \boldsymbol{R}$ if and only if

$$\text{(3.12)} \qquad a^G(u^n_i, u^n_{i+1}) \leq a^n_{i+\frac{1}{2}}.$$

PROOF. For similarity, we only consider the case that $u^n_i < u^n_{i+1}$. Noting that $u^n_i, u^n_{i+1} \in [m, M]$, we see from the CFL condition (2.4) that

$$-\frac{1}{\lambda}(s - u^n_i) + f(u^n_i) \leq f(s)$$

and

$$\frac{1}{\lambda}(s - u^n_{i+1}) + f(u^n_{i+1}) \leq f(s)$$

for $u^n_i \leq s \leq u^n_{i+1}$. Therefore, by the definition of $g^n_{i+\frac{1}{2}}$, we easily see that inequality (3.11) holds if and only if

$$\text{(3.13)} \qquad \bar{f}^n_{i+\frac{1}{2}} \leq \min_{u^n_i \leq s \leq u^n_{i+1}} f(s).$$

On the other hand, the relation

$$\bar{f}^n_{i+\frac{1}{2}} = \frac{1}{2}\{f(u^n_i) + f(u^n_{i+1})\} - \frac{1}{2}a^n_{i+\frac{1}{2}}(u^n_{i+1} - u^n_i)$$

shows that inequality (3.13) holds if and only if

$$a^G(u^n_i, u^n_{i+1}) \leq a^n_{i+\frac{1}{2}}.$$

This completes the proof.

4. Strong admissibility of difference approximations

In this section we discuss the admissibility of difference approximations in the case of arbitrary flux functions. The notion of admissibility is introduced so that it guarantees the convergence of difference solutions to admissible solutions to the initial value problem (1.1). The main purpose in this section is to characterize strongly admissible difference approximations in terms of numerical viscosity coefficient (Theorem 4.3).

We begin with the definition of strong admissibility.

DEFINITION 4.1. The difference approximation (2.1) is said to be *strongly admissible* if the following conditions are satisfied:

(A1) $a^{MR}(u_i^n, u_{i+1}^n) \le a_{i+\frac{1}{2}}^n \le \dfrac{1}{\lambda}$ for all n and i.

(A2) For any entropy pair (U, F), there exist real numbers $A_{i+\frac{1}{2}}^n$ such that they are uniformly bounded by a universal constant depending on U, and such that the *numerical entropy inequality*

$$
\begin{aligned}
U(u_i^{n+1}) - U(u_i^n) &+ \frac{\lambda}{2}\{F(u_{i+1}^n) - F(u_{i-1}^n)\} \\
&- \frac{\lambda}{2}\{A_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n) - A_{i-\frac{1}{2}}^n(u_i^n - u_{i-1}^n)\} \le 0
\end{aligned}
\tag{4.1}
$$

holds for all n and i.

Condition (A1) is nothing but the stability condition (2.7). Condition (A2) is called the *numerical entropy condition.* We can rewrite the numerical entropy inequality (4.1) in the following form;

$$
U(u_i^{n+1}) - U(u_i^n) + \lambda\{\bar{F}_{i+\frac{1}{2}}^n - \bar{F}_{i-\frac{1}{2}}^n\} \le 0, \tag{4.2}
$$

where each *numerical entropy flux* $\bar{F}_{i+\frac{1}{2}}^n$ is defined by

$$
\bar{F}_{i+\frac{1}{2}}^n = \frac{1}{2}\{F(u_i^n) + F(u_{i+1}^n)\} - \frac{1}{2}A_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n). \tag{4.3}
$$

Let $\Delta = (\Delta x, \Delta t)$ and define a function $u_\Delta(x, t)$ $(x \in \boldsymbol{R},\ t \ge 0)$ by

$$
u_\Delta(x, t) = u_i^n \qquad \text{for } \begin{cases} i\Delta x \le x < (i+1)\Delta x \\ n\Delta t \le t < (n+1)\Delta t. \end{cases} \tag{4.4}
$$

We have the following convergence theorem which justify the use of terminology 'admissible'.

THEOREM 4.1. *Let $u_0 \in BV(\boldsymbol{R})$ and suppose that*

$$u_\Delta(\cdot, 0) \longrightarrow u_0(\cdot) \text{ in } L^1_{loc}(\boldsymbol{R}) \text{ as } \Delta x \longrightarrow 0.$$

If the difference approximation (2.1) *is strongly admissible, then u_Δ converges in $L^1_{loc}(\boldsymbol{R} \times [0, \infty))$ as $\Delta \to (0, 0)$ to the unique admissible solution to the initial value problem* (1.1).

PROOF. From our assumptions, it follows that approximate solutions u_Δ are both L^∞-bounded and TV-bounded. Hence it is shown by a standard argument (see [2, 6, 19, 21]) that $\{u_\Delta\}$ contains a subsequence converging in $L^1_{loc}(\boldsymbol{R} \times [0, \infty))$. Since the coefficients $a^n_{i+\frac{1}{2}}$ are uniformly bounded, it is seen that the limit function of the subsequence is a weak solution to the initial value problem (1.1) (*cf.* [14]). It is also seen from condition (A2) that the limit function satisfies the entropy condition. Therefore, the limit function is an admissible solution to the initial value problem (1.1). Since the admissible solution is unique, the convergence of $\{u_\Delta\}$ is proved.

REMARK 4.1. In view of Proposition 1.1, it is sufficient to assume the numerical entropy inequality for the family $\{(U(\cdot\,; k), F(\cdot\,; k)); k \in \boldsymbol{R}\}$ of entropy pairs defined by (1.5) for the conclusion in Theorem 4.1. In any case, condition (A2) seems somewhat strong. In fact, it might be expected that condition (A1) is derived from (A2) (see Theorem 4.2 and Remark 4.2 for this point). This is the reason for the terminology, 'strongly admissible'.

Now, we discuss the necessity and sufficiency of a certain condition for the strong admissibility of difference approximations. We first prepare a proposition.

PROPOSITION 4.1. *Let (U, F) be an entropy pair and let $u^n_i(\cdot)$ be a function defined by*

$$u^n_i(\theta) = u^n_i - \theta\lambda\{\bar{f}^n_{i+\frac{1}{2}} - \bar{f}^n_{i-\frac{1}{2}}\}, \qquad \theta \in [0, 1].$$

If the numerical entropy inequality (4.2);

$$U(u^{n+1}_i) - U(u^n_i) + \lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\} \leq 0$$

holds, then the inequality

$$U(u^n_i(\theta)) - U(u^n_i) + \theta\lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\} \leq 0$$

holds for all $\theta \in [0, 1]$.

PROOF. For each $\theta \in [0, 1]$, define

$$G_i(\theta) = U(u^n_i(\theta)) - U(u^n_i) + \theta\lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\}.$$

We easily check that $G_i(\cdot)$ is convex on [0, 1] and hence

$$G_i(\theta) \leq \max\{G_i(0), G_i(1)\}, \qquad \text{for } \theta \in [0, 1].$$

Since $G_i(0) = 0$ and $G_i(1) \leq 0$ (note that $u_i^n(1) = u_{i+1}^n$), this means that

$$G_i(\theta) \leq 0, \qquad \text{for } \theta \in [0, 1].$$

This completes the proof.

By using Proposition 4.1, we obtain a necessity condition for the strong admissibility of difference approximations.

THEOREM 4.2. *Suppose that the initial value* $\{u_i^0\}$ *is given by*

$$u_i^0 = \begin{cases} u_L & \textit{for } i \leq 0 \\ u_R & \textit{for } i \geq 1, \end{cases}$$

where u_L *and* u_R *are arbitrary distinct real numbers. If the difference approximation* (2.1) *is strongly admissible, then the coefficient* $a_{\frac{1}{2}}^n$ *satisfies the inequality*

$$a^G(u_0^0, u_1^0) = a^G(u_L, u_R) \leq a_{\frac{1}{2}}^0 \leq \frac{1}{\lambda}.$$

PROOF. Let (U, F) be an entropy pair. For each $i \in \boldsymbol{Z}$ and $\theta \in [0, 1]$, define

$$\begin{aligned} u_i(\theta) &= u_i^0 - \theta\lambda\{\bar{f}_{i+\frac{1}{2}}^0 - \bar{f}_{i-\frac{1}{2}}^0\}, \\ G_i(\theta) &= U(u_i(\theta)) - U(u_i^0) + \theta\lambda\{\bar{F}_{i+\frac{1}{2}}^0 - \bar{F}_{i-\frac{1}{2}}^0\} \end{aligned}$$

and set

$$G(\theta) = G_0(\theta) + G_1(\theta).$$

Noting that

$$\bar{f}_{i+\frac{1}{2}}^0 = \begin{cases} f(u_L) & \text{if } i \leq -1 \\ f(u_R) & \text{if } i \geq 1 \end{cases}$$

and

$$\bar{F}_{i+\frac{1}{2}}^0 = \begin{cases} F(u_L) & \text{if } i \leq -1 \\ F(u_R) & \text{if } i \geq 1, \end{cases}$$

we see that

$$\begin{aligned} G(\theta) &= G_0(\theta) + G_1(\theta) \\ &= U(u_0(\theta)) + U(u_1(\theta)) - \{U(u_L) + U(u_R)\} + \theta\lambda\{F(u_R) - F(u_L)\} \end{aligned}$$

and hence

$$G'(\theta) = -\lambda U'(u_0(\theta))\{\bar{f}^0_{\frac{1}{2}} - \bar{f}^0_{-\frac{1}{2}}\} - \lambda U'(u_1(\theta))\{\bar{f}^0_{\frac{3}{2}} - \bar{f}^0_{\frac{1}{2}}\} + \lambda\{F(u_R) - F(u_L)\}$$

for $\theta \in [0, 1]$. Therefore, we have

$$\begin{aligned} G'(0) &= -\lambda U'(u_L)\{\bar{f}^0_{\frac{1}{2}} - f(u_L)\} - \lambda U'(u_R)\{f(u_R) - \bar{f}^0_{\frac{1}{2}}\} \\ &\quad + \lambda\{F(u_R) - F(u_L)\} \\ &= \lambda \bar{f}^0_{\frac{1}{2}}\{U'(u_R) - U'(u_L)\} \\ &\quad - \lambda\{f(u_R)U'(u_R) - f(u_L)U'(u_L)\} + \lambda\int_{u_L}^{u_R} f'(s)U'(s)\,ds \\ &= \lambda\int_{u_L}^{u_R}\{\bar{f}^0_{\frac{1}{2}} - f(s)\}\,dU'(s). \end{aligned}$$

On the other hand, we see from Proposition 4.1 that

$$G(\theta) = G_0(\theta) + G_1(\theta) \leq 0$$

for $\theta \in [0, 1]$. Since $G(0) = 0$, this inequality implies that

$$G'(0) = \lambda\int_{u_L}^{u_R}\{\bar{f}^0_{\frac{1}{2}} - f(s)\}\,dU'(s) \leq 0.$$

By arbitrariness of U, this yields

$$\bar{f}^0_{\frac{1}{2}} \leq \min_{u_L \leq s \leq u_R} f(s) \qquad \text{if } u_L < u_R$$

$$\bar{f}^0_{\frac{1}{2}} \geq \max_{u_R \leq s \leq u_L} f(s) \qquad \text{if } u_L > u_R$$

or equivalently

$$a^G(u_L, u_R) \leq a^0_{\frac{1}{2}}.$$

By Definition 4.1, it is clear that $a^0_{\frac{1}{2}} \leq \dfrac{1}{\lambda}$. This completes the proof.

REMARK 4.2. Let $a(\cdot, \cdot)$ be a real-valued function defined on $\boldsymbol{R} \times \boldsymbol{R}$ and consider a difference approximation in viscous form with $a^n_{i+\frac{1}{2}}$ replaced by $a(u^n_i, u^n_{i+1})$. If the difference approximation is strongly admissible for every initial value consisting of two real numbers, then it follows from Theorem 4.2 that

$$a^G(u_i^n, u_{i+1}^n) \le a(u_i^n, u_{i+1}^n) \le \frac{1}{\lambda}$$

for all n and i.

We are now in a position to state the first main result in this paper.

THEOREM 4.3. *If the condition*

$$a^G(u_i^n, u_{i+1}^n) \le a^n_{i+\frac{1}{2}} \le \frac{1}{\lambda} \tag{4.5}$$

holds for all n and i, then the difference approximation (2.1) *is strongly admissible.*

Before proceeding the proof of this theorem, we give a remark.

REMARK 4.3. Theorem 4.2 and Theorem 4.3 are summarized as follows: *Condition* (4.5) *is almost necessary and sufficient for the strong admissibility of difference approximation* (2.1). We mention some related results. Osher [22] proved that semidiscrete approximations having more numerical viscosity than that of Godunov's is consistent with entropy condition. The result of Osher was extended by Tadmor [29] to the case of difference approximations (see also Osher and Tadmor [24]). Our results give extensions of their results and seem to be best possible for the strong admissibility of difference approximations in viscous form.

The remaining part of this section is devoted to the proof of Theorem 4.3. What is key to the proof is to find numerical entropy fluxes $\bar{F}^n_{i+\frac{1}{2}}$ with which the numerical entropy inequality (4.2) is satisfied. In the following, we assume condition (4.5). It should be noted that $u_i^n \in [m, M]$ for all n and i. Now, let $g^n_{i+\frac{1}{2}}$ be the minimal modified flux function defined by (3.8) or (3.9). Let $k \in \boldsymbol{R}$ and $(U(\cdot\,; k), F(\cdot\,; k))$ be the entropy pair defined by (1.5). Moreover, set

$$\begin{aligned} \bar{F}^n_{i+\frac{1}{2}}(k) &= \frac{1}{2}\{F(u_i^n; k) + F(u_{i+1}^n; k)\} \\ &\quad - \frac{1}{2}\int_{u_i^n}^{u_{i+1}^n} \chi^+(s-k)|(g^n_{i+\frac{1}{2}})'(s)|\,ds \end{aligned} \tag{4.6}$$

and

$$A^n_{i+\frac{1}{2}}(k) = \int_0^1 \chi^+(u_i^n + (u_{i+1}^n - u_i^n)\theta - k)|(g^n_{i+\frac{1}{2}})'(u_i^n + (u_{i+1}^n - u_i^n)\theta)|\,d\theta. \tag{4.7}$$

It is evident that

and

$$\bar{F}^n_{i+\frac{1}{2}}(k) = \frac{1}{2}\{F(u^n_i; k) + F(u^n_{i+1}; k)\} - \frac{1}{2}A^n_{i+\frac{1}{2}}(k)(u^n_{i+1} - u^n_i)$$

$$0 \le A^n_{i+\frac{1}{2}}(k) \le \frac{1}{\lambda}, \qquad k \in \boldsymbol{R}.$$

Note that $\bar{F}^n_{i+\frac{1}{2}}(k)$ and $A^n_{i+\frac{1}{2}}(k)$ are locally Lipschitz continuous with respect to $k \in \boldsymbol{R}$.

By preparing two lemmas, we first prove the numerical entropy inequality for the entropy pair $(U(\cdot\,; k), F(\cdot\,; k))$;

$$U(u^{n+1}_i; k) - U(u^n_i; k) + \lambda\{\bar{F}^n_{i+\frac{1}{2}}(k) - \bar{F}^n_{i-\frac{1}{2}}(k)\} \le 0. \tag{4.8}$$

In order to deal with the left-hand side of (4.8), we use auxiliary functions $R_+(k; z, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}})$ and $R_-(k; z, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}})$ which are respectively defined by

$$\begin{aligned} &R_+(k; z, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}}) \\ &= \{\chi^+(z-k) - \chi^+(u^n_i - k)\} \int_k^{u^n_i} \{|(g^n_{i+\frac{1}{2}})'(s)| + (g^n_{i+\frac{1}{2}})'(s)\}\,ds \\ &\quad + \{\chi^+(z-k) - \chi^+(u^n_{i+1} - k)\} \int_k^{u^n_{i+1}} \left\{\frac{1}{\lambda} - |(g^n_{i+\frac{1}{2}})'(s)|\right\} ds \\ &\quad + \{\chi^+(z-k) - \chi^+(u^n_i - k)\}\{g^n_{i+\frac{1}{2}}(k) - f(k)\} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &R_-(k; z, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}}) \\ &= \{\chi^+(z-k) - \chi^+(u^n_{i+1} - k)\} \int_k^{u^n_{i+1}} \{|(g^n_{i+\frac{1}{2}})'(s)| - (g^n_{i+\frac{1}{2}})'(s)\}\,ds \\ &\quad + \{\chi^+(z-k) - \chi^+(u^n_i - k)\} \int_k^{u^n_i} \left\{\frac{1}{\lambda} - |(g^n_{i+\frac{1}{2}})'(s)|\right\} ds \\ &\quad - \{\chi^+(z-k) - \chi^+(u^n_{i+1} - k)\}\{g^n_{i+\frac{1}{2}}(k) - f(k)\}, \end{aligned} \tag{4.10}$$

for $k \in \boldsymbol{R}$, where $z \in \boldsymbol{R}$ is a parameter. The following result is elementary but useful in our arguments.

LEMMA 4.1. *We have the equality*

$$\begin{aligned} &U(u^{n+1}_i; k) - U(u^n_i; k) + \lambda\{\bar{F}^n_{i+\frac{1}{2}}(k) - \bar{F}^n_{i-\frac{1}{2}}(k)\} \\ &= \frac{\lambda}{2}\{R_+(k; u^{n+1}_i, u^n_{i-1}, u^n_i, g^n_{i-\frac{1}{2}}) + R_-(k; u^{n+1}_i, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}})\}. \end{aligned} \tag{4.11}$$

PROOF. By the definition of $g^n_{i+\frac{1}{2}}$, it is clear that

$$a^n_{i+\frac{1}{2}}(u^n_{i+1} - u^n_i) = (u^n_{i+1} - u^n_i)\int_0^1 |(g^n_{i+\frac{1}{2}})'(u^n_i + (u^n_{i+1} - u^n_i)\theta)|\,d\theta$$
$$= \int_{u^n_i}^{u^n_{i+1}} |(g^n_{i+\frac{1}{2}})'(s)|\,ds.$$

Therefore, we have

$$\begin{aligned}
U(u^{n+1}_i; k) &= (u^{n+1}_i - k)^+ \\
&= \chi^+(u^{n+1}_i - k)(u^{n+1}_i - k) \\
&= \chi^+(u^n_{i+1} - k)\Bigg[u^n_i - k - \frac{\lambda}{2}\{f(u^n_{i+1}) - f(u^n_{i-1})\} \\
&\quad + \frac{\lambda}{2}\int_{u^n_i}^{u^n_{i+1}} |(g^n_{i+\frac{1}{2}})'(s)|\,ds - \frac{\lambda}{2}\int_{u^n_{i-1}}^{u^n_i} |(g^n_{i-\frac{1}{2}})'(s)|\,ds\Bigg] \\
&= \chi^+(u^{n+1}_i - k)\int_k^{u^n_{i-1}} \frac{\lambda}{2}\{|(g^n_{i-\frac{1}{2}})'(s)| + (g^n_{i-\frac{1}{2}})'(s)\}\,ds \\
&\quad + \chi^+(u^{n+1}_i - k)\int_k^{u^n_i}\left\{1 - \frac{\lambda}{2}|(g^n_{i-\frac{1}{2}})'(s)| - \frac{\lambda}{2}|(g^n_{i+\frac{1}{2}})'(s)|\right\}ds \\
&\quad + \chi^+(u^{n+1}_i - k)\int_k^{u^n_{i+1}} \frac{\lambda}{2}\{|(g^n_{i+\frac{1}{2}})'(s)| - (g^n_{i+\frac{1}{2}})'(s)\}\,ds \\
&\quad + \frac{\lambda}{2}\chi^+(u^{n+1}_i - k)\{g^n_{i-\frac{1}{2}}(k) - f(k)\} \\
&\quad - \frac{\lambda}{2}\chi^+(u^{n+1}_i - k)\{g^n_{i+\frac{1}{2}}(k) - f(k)\}.
\end{aligned}$$

Here we have used the fact that $g^n_{i-\frac{1}{2}}(u^n_{i-1}) = f(u^n_{i-1})$ and $g^n_{i+\frac{1}{2}}(u^n_{i+1}) = f(u^n_{i+1})$.

On the other hand, we have

$$\begin{aligned}
&\bar{F}^n_{i+\frac{1}{2}}(k) \\
&= \frac{1}{2}\{F(u^n_i; k) + F(u^n_{i+1}; k)\} \\
&\quad - \frac{1}{2}\left\{\int_k^{u^n_{i+1}} \chi^+(u^n_{i+1} - k)|(g^n_{i+\frac{1}{2}})'(s)|\,ds - \int_k^{u^n_i} \chi^+(u^n_i - k)|(g^n_{i+\frac{1}{2}})'(s)|\,ds\right\} \\
&= \frac{1}{2}\chi^+(u^n_i - k)\int_k^{u^n_i}\{|(g^n_{i+\frac{1}{2}})'(s)| + f'(s)\}\,ds
\end{aligned}$$

$$-\frac{1}{2}\chi^+(u^n_{i+1}-k)\int_k^{u^n_{i+1}}\{|(g^n_{i+\frac{1}{2}})'(s)|-f'(s)\}\,ds.$$

Hence we obtain

$$\begin{aligned}
&U(u^n_i;k)-\lambda\{\bar F^n_{i+\frac{1}{2}}(k)-\bar F^n_{i-\frac{1}{2}}(k)\}\\
&\quad=\chi^+(u^n_{i-1}-k)\int_k^{u^n_{i-1}}\frac{\lambda}{2}\{|(g^n_{i-\frac{1}{2}})'(s)|+f'(s)\}\,ds\\
&\qquad+\chi^+(u^n_i-k)\int_k^{u^n_i}\left\{1-\frac{\lambda}{2}|(g^n_{i-\frac{1}{2}})'(s)|-\frac{\lambda}{2}|(g^n_{i+\frac{1}{2}})'(s)|\right\}ds\\
&\qquad+\chi^+(u^n_{i+1}-k)\int_k^{u^n_{i+1}}\frac{\lambda}{2}\{|(g^n_{i+\frac{1}{2}})'(s)|-f'(s)\}\,ds\\
&\quad=\chi^+(u^n_{i-1}-k)\int_k^{u^n_{i-1}}\frac{\lambda}{2}\{|(g^n_{i-\frac{1}{2}})'(s)|+(g^n_{i-\frac{1}{2}})'(s)\}\,ds\\
&\qquad+\chi^+(u^n_i-k)\int_k^{u^n_i}\left\{1-\frac{\lambda}{2}|(g^n_{i-\frac{1}{2}})'(s)|-\frac{\lambda}{2}|(g^n_{i+\frac{1}{2}})'(s)|\right\}ds\\
&\qquad+\chi^+(u^n_{i+1}-k)\int_k^{u^n_{i+1}}\frac{\lambda}{2}\{|(g^n_{i+\frac{1}{2}})'(s)|-(g^n_{i+\frac{1}{2}})'(s)\}\,ds\\
&\qquad+\frac{\lambda}{2}\chi^+(u^n_{i-1}-k)\{g^n_{i-\frac{1}{2}}(k)-f(k)\}\\
&\qquad-\frac{\lambda}{2}\chi^+(u^n_{i+1}-k)\{g^n_{i+\frac{1}{2}}(k)-f(k)\}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&U(u^{n+1}_i;k)-U(u^n_i:k)+\lambda\{\bar F^n_{i+\frac{1}{2}}(k)-\bar F^n_{i-\frac{1}{2}}(k)\}\\
&\quad=\{\chi^+(u^{n+1}_i-k)-\chi^+(u^n_{i-1}-k)\}\int_k^{u^n_{i-1}}\frac{\lambda}{2}\{|(g^n_{i-\frac{1}{2}})'(s)|+(g^n_{i-\frac{1}{2}})'(s)\}\,ds\\
&\qquad+\{\chi^+(u^{n+1}_i-k)-\chi^+(u^n_i-k)\}\int_k^{u^n_i}\left\{1-\frac{\lambda}{2}|(g^n_{i-\frac{1}{2}})'(s)|-\frac{\lambda}{2}|(g^n_{i+\frac{1}{2}})'(s)|\right\}ds\\
&\qquad+\{\chi^+(u^{n+1}_i-k)-\chi^+(u^n_{i+1}-k)\}\int_k^{u^n_{i+1}}\frac{\lambda}{2}\{|(g^n_{i+\frac{1}{2}})'(s)|-(g^n_{i+\frac{1}{2}})'(s)\}\,ds\\
&\qquad+\frac{\lambda}{2}\chi^+(u^{n+1}_i-k)-\chi^+(u^n_{i-1}-k)\}\{g^n_{i-\frac{1}{2}}(k)-f(k)\}\\
&\qquad-\frac{\lambda}{2}\{\chi^+(u^{n+1}_i-k)-\chi^+(u^n_{i+1}-k)\}\{g^n_{i+\frac{1}{2}}(k)-f(k)\}
\end{aligned}$$

$$= \frac{\lambda}{2}\{R_+(k; u_i^{n+1}, u_{i-1}^n, u_i^n, g_{i-\frac{1}{2}}^n) + R_-(k; u_i^{n+1}, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n)\}.$$

This completes the proof.

LEMMA 4.2. *Let* $z \in [m, M]$. *If condition* (4.5) *is satisfied, then*

$$R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) \leq 0 \tag{4.12}$$

for all $k \in \boldsymbol{R}$.

PROOF. We only treat $R_+(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n)$, because $R_-(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n)$ can be treated in a similar manner. If $k \notin [m, M]$, then

$$\chi^+(z - k) - \chi^+(u_i^n - k) = \chi^+(z - k) - \chi^+(u_{i+1}^n - k) = 0$$

and hence

$$R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) = 0.$$

Next, let $k \in [m, M]$. By the CFL condition (2.4) and the definition (3.8) (or (3.9)) of $g_{i+\frac{1}{2}}^n$, it is clear that

$$\frac{1}{\lambda} - |(g_{i+\frac{1}{2}}^n)'(s)| \geq 0 \qquad \text{for } s \in [m, M].$$

Therefore, noting that $u_{i+1}^n \in [m, M]$, we obtain

$$\{\chi^+(z - k) - \chi^+(u_{i+1}^n - k)\} \int_k^{u_{i+1}^n} \left\{\frac{1}{\lambda} - |(g_{i+\frac{1}{2}}^n)'(s)|\right\} ds \leq 0.$$

We also see that

$$\{\chi^+(z - k) - \chi^+(u_i^n - k)\} \int_k^{u_i^n} \{|(g_{i+\frac{1}{2}}^n)'(s)| + (g_{i+\frac{1}{2}}^n)'(s)\} ds \leq 0.$$

Consequently, it follows from (4.9) that

$$\begin{aligned} &R_+(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) \\ &\quad \leq \{\chi^+(z - k) - \chi^+(u_i^n - k)\}\{g_{i+\frac{1}{2}}^n(k) - f(k)\}. \end{aligned}$$

It remains to show that

$$\{\chi^+(z - k) - \chi^+(u_i^n - k)\}\{g_{i+\frac{1}{2}}^n(k) - f(k)\} \leq 0.$$

If $(k - u_i^n)(k - u_{i+1}^n) \geq 0$, then

$$g_{i+\frac{1}{2}}^n(k) = f(k),$$

which yields

$$\{\chi^+(z-k)-\chi^+(u_i^n-k)\}\{g^n_{i+\frac{1}{2}}(k)-f(k)\}=0.$$

If $(k-u_i^n)(k-u_{i+1}^n)<0$, We easily see that

$$\chi^+(z-k)-\chi^+(u_i^n-k)=0 \quad \text{or} \quad \operatorname{sgn}(u_{i+1}^n-n_i^n).$$

Since $\operatorname{sgn}(u_{i+1}^n-u_i^n)\{g^n_{i+\frac{1}{2}}(k)-f(k)\}\leq 0$ by Proposition 3.3, this yields

$$\{\chi^+(z-k)-\chi^+(u_i^n-k)\}\{g^n_{i+\frac{1}{2}}(k)-f(k)\}\leq 0.$$

This completes the proof.

The following proposition is a direct consequence of Lemma 4.1 and Lemma 4.2.

PROPOSITION 4.2. *Let* $k\in R$. *If condition* (4.5) *is satisfied, then the numerical entropy inequality*

$$U(u_i^{n+1};k)-U(u_i^n;k)+\lambda\{\bar{F}^n_{i+\frac{1}{2}}(k)-\bar{F}^n_{i-\frac{1}{2}}(k)\}\leq 0$$

holds for all n and i.

PROOF of THEOREM 4.3. In order to obtain the conclusion in Theorem 4.3, it suffices to prove that condition (A2) is satisfied. To this end, let (U, F) be an arbitrary entropy pair. Without loss of generality, we assume that $U(m)=F(m)=0$ and $U'(s)$ is continuous at $s=m$. We first note that

$$\begin{aligned} U(s) &= U(m)+\int_m^s U'(k)\,dk \\ &= [(k-s)U'(k)]_{k=m}^{k=s}+\int_m^s (s-k)\,dU'(k) \\ &= (s-m)U'(m)+\int_m^M (s-k)^+\,dU'(k) \\ &= (s-m)U'(m)+\int_m^M U(s;k)\,dU'(k) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} F(s) &= F(m)+\int_m^s F'(k)\,dk \\ &= \int_m^s f'(k)U'(k)\,dk \end{aligned}$$

$$(4.14)\qquad = [\{f(k)-f(s)\}U'(k)]_{k=m}^{k=s} + \int_m^s \{f(s)-f(k)\}\,dU'(k)$$

$$= \{f(s)-f(m)\}U'(m) + \int_m^M \chi^+(s-k)\{f(s)-f(k)\}\,dU'(k)$$

$$= \{f(s)-f(m)\}U'(m) + \int_m^M F(s;k)\,dU'(k)$$

for $s\in[m, M]$, where the integral is taken in the sense of Stieltjes.

Now, set

$$(4.15)\qquad \bar{F}^n_{i+\frac{1}{2}} = \frac{1}{2}\{F(u_i^n)+F(u_{i+1}^n)\} - \frac{1}{2}\int_{u_i^n}^{u_{i+1}^n} U'(s)|(g^n_{i+\frac{1}{2}})'(s)|\,ds$$

and

$$(4.16)\qquad A^n_{i+\frac{1}{2}} = \int_0^1 U'(u_i^n+(u_{i+1}^n-u_i^n)\theta)|(g^n_{i+\frac{1}{2}})'(u_i^n+(u_{i+1}^n-u_i^n)\theta)|\,d\theta.$$

Noting that $u_i^n,\ u_{i+1}^n\in[m, M]$, we can easily check that

$$A^n_{i+\frac{1}{2}} = \int_m^M A^n_{i+\frac{1}{2}}(k)\,dU'(k) + a^n_{i+\frac{1}{2}}U'(m),$$

where each of $A^n_{i+\frac{1}{2}}(k)$ is the number defined by (4.7). It is clear that

$$(4.17)\qquad \bar{F}^n_{i+\frac{1}{2}} = \frac{1}{2}\{F(u_i^n)+F(u_{i+1}^n)\} - \frac{1}{2}A^n_{i+\frac{1}{2}}(u_{i+1}^n-u_i^n)$$

and

$$(4.18)\qquad |A^n_{i+\frac{1}{2}}| \le \frac{2}{\lambda}\sup_{m\le s\le M}|U'(s)|.$$

Therefore, we have

$$\bar{F}^n_{i+\frac{1}{2}} = \int_m^M\left[\frac{1}{2}\{F(u_i^n;k)+F(u_{i+1}^n;k)\} - \frac{1}{2}A^n_{i+\frac{1}{2}}(k)(u_{i+1}^n-u_i^n)\right]dU'(k)$$

$$+\frac{1}{2}\left[f(u_i^n)+f(u_{i+1}^n) - a^n_{i+\frac{1}{2}}(u_{i+1}^n-u_i^n) - f(m)\right]U'(m)$$

$$= \int_m^M \bar{F}^n_{i+\frac{1}{2}}(k)\,dU'(k) + \{\bar{f}^n_{i+\frac{1}{2}} - f(m)\}U'(m).$$

Combining above relations, we obtain

$$\begin{aligned}
&U(u_i^{n+1}) - U(u_i^n) + \lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\} \\
&= \int_m^M [U(u_i^{n+1}; k) - U(u_i^n; k) + \lambda\{\bar{F}^n_{i+\frac{1}{2}}(k) - \bar{F}^n_{i-\frac{1}{2}}(k)\}]\, dU'(k) \\
&\quad + [u_i^{n+1} - u_i^n + \lambda\{\bar{f}^n_{i+\frac{1}{2}} - \bar{f}^n_{i-\frac{1}{2}}\}]\, U'(m) \\
&= \int_m^M [U(u_i^{n+1}; k) - U(u_i^n; k) + \lambda\{\bar{F}^n_{i+\frac{1}{2}}(k) - \bar{F}^n_{i-\frac{1}{2}}(k)\}]\, dU'(k).
\end{aligned}$$

Since the integrand is nonpositive by Proposition 4.2, this shows that

$$U(u_i^{n+1}) - U(u_i^n) + \lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\} \leq 0$$

for all n and i. This completes the proof of Theorem 4.3.

5. Admissibility of difference approximations with small viscosity coefficients

In this section and the next section, we discuss the admissibility of difference approximations in case of strictly convex flux functions.

From the theoretical and computational point of view, it is important to study the admissibility of difference approximations with small numerical viscosity coefficients. As was shown in the previous section, if no additional restriction is imposed on the flux function f and some of coefficients $a^n_{i+\frac{1}{2}}$ are less than those of Godunov scheme, then the strong admissibility of the difference approximation may be violated. Accordingly, we restrict ourselves to strictly convex flux functions in order to allow small $a^n_{i+\frac{1}{2}}$. When the flux function f is strictly convex, Theorem 1.1 enables us to discuss the admissibility of difference approximations by a weaker condition than (A2) in Definition 4.1.

We begin with the definition of the admissibility of difference approximations.

DEFINITION 5.1. Let f be strictly convex. The difference approximation (2.1) is said to be *admissible* if the following conditions are satisfied:

(A1) $a^{MR}(u_i^n, u_{i+1}^n) \leq a^n_{i+\frac{1}{2}} \leq \dfrac{1}{\lambda}$ for all n and i.

(A2)′ For some entropy pair (U, F) with U strictly convex, there exist real numbers $A^n_{i+\frac{1}{2}}$ such that they are uniformly bounded by a universal constant depending on U, and such that the *numerical entropy inequality*

$$
\begin{aligned}
&U(u_i^{n+1}) - U(u_i^n) + \frac{\lambda}{2}\{F(u_{i+1}^n) - F(u_{i-1}^n)\} \\
(5.1) \qquad & \\
&- \frac{\lambda}{2}\{A_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n) - A_{i-\frac{1}{2}}^n(u_i^n - u_{i-1}^n)\} \le 0
\end{aligned}
$$

holds for all n and i.

We immediately obtain the following convergence theorem. In the statement we denote by $u_\Delta(x, t)$ the approximate solution defined by (4.4).

THEOREM 5.1. *Let f be strictly convex. Let $u_0 \in BV(\boldsymbol{R})$ and suppose that*

$$u_\Delta(\cdot, 0) \longrightarrow u_0(\cdot) \quad in \quad L^1_{loc}(\boldsymbol{R}) \quad as \quad \Delta x \longrightarrow 0.$$

If the difference approximation (2.1) *is admissible, then u_Δ converges in $L^1_{loc}(\boldsymbol{R} \times [0, \infty))$ as $\Delta \to (0, 0)$ to the unique admissible solution to the initial value problem* (1.1).

The proof is similar to that of Theorem 4.1. What is different is that the uniqueness of a limit function is derived from Theorem 1.1.

The next main result gives a sufficiency condition for the admissibility of difference approximations with small numerical viscosity coefficients.

THEOREM 5.2. *Suppose that the flux function f is strictly convex. Let $\varepsilon \in (0, 1)$. If the condition*

$$\max\left\{a^{MR}(u_i^n, u_{i+1}^n), \frac{\varepsilon}{\lambda}\operatorname{sgn}(u_{i+1}^n - u_i^n)\right\} \le \frac{1}{\lambda} \tag{5.2}$$

holds for all n and i, then the difference approximation (2.1) *is admissible.*

Note that Murmann-Roe scheme preserves stationary inverse (physically irrelevant) shocks and hence is not admissible. Condition (5.2) asserts that $a^{MR}(u_i^n, u_{i+1}^n)$ should be replaced by $\frac{\varepsilon}{\lambda}$ if $u_i^n \le u_{i+1}^n$ and $a^{MR}(u_i^n, u_{i+1}^n) < \frac{\varepsilon}{\lambda}$. Such replacement may be necessary around nearly stationary inverse shocks. From Theorem 5.2, we see that Murmann-Roe scheme becomes admissible if its efect of preserving stationary inverse shocks is removed by adding a small amount of numerical viscosity. In particular, we see that Harten scheme [8, 9] is admissible.

COROLLARY 5.1. *Suppose that the flux function f is strictly convex. Then Harten scheme with coefficients $a_{i+\frac{1}{2}}^n$ defined by* (2.17) *and* (2.18) (*or* (2.19)) *is admissible.*

REMARK 5.1. As is easily seen, it follows from the strict convexity of f that $a^{MR}(u_i^n, u_{i+1}^n) = a^G(u_i^n, u_{i+1}^n)$ in either case of the following;

(a) $u_i^n \geq u_{i+1}^n$,

(b) $s_0 \leq u_i^n < u_{i+1}^n$,

(c) $u_i^n < u_{i+1}^n \leq s_0$,

where s_0 is a real number such that $f'(s_0) = 0$.

In order to prove Theorem 5.2, it is necessary to construct a particular entropy pair (U, F) and numerical entropy fluxes $\bar{F}^n_{i+\frac{1}{2}}$ with which the numerical entropy inequality (5.1) is satisfied. The approach is essentially different from a usual approach, in which the numerical entropy condition is investigated for an entropy pair fixed in advance (see e.g. [17, 24]). Since the argument is complicated, we construct a particular entropy pair in the next section.

In the remaining part of this section, we give key estimates of $R_\pm(k; z, u_i^n, u_{i+1}^n, g^n_{i+\frac{1}{2}})$ in the case that $u_i^n < s_0 < u_{i+1}^n$, where $R_\pm(k; z, u_i^n, u_{i+1}^n, g^n_{i+\frac{1}{2}})$ are the auxiliary functions defined by (4.9) and (4.10), and s_0 is a real number such that $f'(s_0) = 0$. As is seen from Remark 5.1, the other case is reduced to the case treated in Section 4.

Now, let f be strictly convex and s_0 be a real number such that $f'(s_0) = 0$. In the following, we assume that

$$m \leq u_i^n < s_0 < u_{i+1}^n \leq M \tag{5.3}$$

and

$$a^{MR}(u_i^n, u_{i+1}^n) \leq a^n_{i+\frac{1}{2}} < a^G(u_i^n, u_{i+1}^n). \tag{5.4}$$

It is convenient to employ the following notation. We write

$$\begin{cases} \bar{v}_i^n = u_i^n + \lambda\{f(u_i^n) - \bar{f}^n_{i+\frac{1}{2}}\} \\ \hat{v}_{i+1}^n = u_{i+1}^n - \lambda\{f(u_{i+1}^n) - \bar{f}^n_{i+\frac{1}{2}}\}. \end{cases} \tag{5.5}$$

Note that $u_i^n \leq \bar{v}_i^n < \hat{v}_{i+1}^n \leq u_{i+1}^n$ by (5.4) and the minimal modified flux function $g^n_{i+\frac{1}{2}}$ is written as

$$g^n_{i+\frac{1}{2}}(s) = \begin{cases} -\dfrac{1}{\lambda}(s - u_i^n) + f(u_i^n) & \text{for } s \in [u_i^n, \bar{v}_i^n] \\ \bar{f}^n_{i+\frac{1}{2}} & \text{for } s \in [\bar{v}_i^n, \hat{v}_{i+1}^n] \\ \dfrac{1}{\lambda}(s - u_{i+1}^n) + f(u_{i+1}^n) & \text{for } s \in [\hat{v}_{i+1}^n, u_{i+1}^n]. \end{cases} \tag{5.6}$$

Next, let $\bar{w}_i^n$ and $\hat{w}_{i+1}^n$ be real numbers such that

$$
\left\{
\begin{array}{l}
f(\bar{w}_i^n) = f(\hat{w}_{i+1}^n) = \bar{f}_{i+\frac{1}{2}}^n \\
u_i^n \le \bar{w}_i^n < \hat{w}_{i+1}^n \le u_{i+1}^n .
\end{array}
\right.
\tag{5.7}
$$

Note that the existence of $\bar{w}_i^n$ and $\hat{w}_{i+1}^n$ is ensured by (5.3) and (5.4). By the strict convexity, there is no real number satisfying (5.7), except for $\bar{w}_i^n$ and $\hat{w}_{i+1}^n$. It is clear that

$$
u_i^n \le \bar{v}_i^n \le \bar{w}_i^n < \hat{w}_{i+1}^n \le \hat{v}_{i+1}^n \le u_{i+1}^n . \tag{5.8}
$$

LEMMA 5.1. *We have the inequality*

$$
\begin{aligned}
&\max\left\{\frac{1}{\lambda}(\bar{v}_i^n - k),\ \frac{1}{\lambda}(k - \hat{v}_{i+1}^n)\right\} \le \bar{f}_{i+\frac{1}{2}}^n - f(k) \\
&\qquad\qquad \text{for } \frac{\bar{v}_i^n + \bar{w}_i^n}{2} \le k \le \frac{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n}{2} .
\end{aligned}
\tag{5.9}
$$

PROOF. For notational simplicity, we write $I = \left(\dfrac{\bar{v}_i^n + \bar{w}_i^n}{2}, \dfrac{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n}{2}\right)$. Set

$$
p(k) = \frac{1}{\lambda}(\bar{v}_i^n - k) - \{\bar{f}_{i+\frac{1}{2}}^n - f(k)\}
$$

and

$$
q(k) = \frac{1}{\lambda}(k - \hat{v}_{i+1}^n) - \{\bar{f}_{i+\frac{1}{2}}^n - f(k)\}
$$

for $k \in \bar{I}$. Here $\bar{I}$ denotes the closure of I. By the CFL condition (2.4), we easily see that $p(\cdot)$ is nonincreasing on $\bar{I}$. Since $f(\bar{w}_i^n) = \bar{f}_{i+\frac{1}{2}}^n$, we also see that

$$
p(k) \le \frac{1}{\lambda}(\bar{v}_i^n + \bar{w}_i^n - 2k)
$$

for $\dfrac{\bar{v}_i^n + \bar{w}_i^n}{2} \le k \le \bar{w}_i^n$, and hence

$$
p\left(\frac{\bar{v}_i^n + \bar{w}_i^n}{2}\right) \le 0.
$$

Therefore, it is shown that $p(k) \le 0$ for $k \in \bar{I}$. Similarly, we see that $q(\cdot)$ is nondecreasing on $\bar{I}$ and

$$
q\left(\frac{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n}{2}\right) \le 0.
$$

This shows that $q(k) \leq 0$ for $k \in \bar{I}$. Thus the proof is completed.

The following result plays an essential role in the proof of Theorem 5.2.

PROPOSITION 5.1. *Let $z \in [m, M]$. Suppose that*

$$m \leq u_i^n < s_0 < u_{i+1}^n \leq M$$

and

$$a^{MR}(u_i^n, u_{i+1}^n) \leq a_{i+\frac{1}{2}}^n < a^G(u_i^n, u_{i+1}^n).$$

Then we have

$$R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) \leq \begin{cases} 0 & \text{for } k \leq \dfrac{\bar{v}_i^n + \bar{w}_i^n}{2} \\ \bar{f}_{i+\frac{1}{2}}^n & \text{for } \dfrac{\bar{v}_i^n + \bar{w}_i^n}{2} \leq k \leq \dfrac{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n}{2} \\ 0 & \text{for } k \geq \dfrac{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n}{2}. \end{cases} \tag{5.10}$$

PROOF. We write $I = \left(\dfrac{\bar{v}_i^n + \bar{w}_i^n}{2}, \dfrac{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n}{2}\right)$. As in the proof of Lemma 4.2, we obtain

$$R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) \leq 0$$

for $k \notin I$. To prove the remaining inequalities, let $k \in I$. Since $\chi^+(u_i^n - k) = 0$, $\chi^+(u_{i+1}^n - k) = 1$ and $g_{i+\frac{1}{2}}^n(k) = \bar{f}_{i+\frac{1}{2}}^n$, it follows from (4.9) that

$$\begin{aligned} R_+&(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) \\ &\leq \{\chi^+(z-k) - 1\} \int_k^{u_{i+1}^n} \left\{\frac{1}{\lambda} - |(g_{i+\frac{1}{2}}^n)'(s)|\right\} ds + \chi^+(z-k)\{\bar{f}_{i+\frac{1}{2}}^n - f(k)\} \\ &\leq \max\left\{\bar{f}_{i+\frac{1}{2}}^n - f(k),\ -\int_k^{u_{i+1}^n} \left\{\frac{1}{\lambda} - |(g_{i+\frac{1}{2}}^n)'(s)|\right\}\right\}. \end{aligned}$$

On the other hand, it follows from (5.6) and Lemma 5.1 that

$$-\int_k^{u_{i+1}^n} \left\{\frac{1}{\lambda} - |(g_{i+\frac{1}{2}}^n)'(s)|\right\} ds = \frac{1}{\lambda}\int_k^{\hat{v}_{i+1}^n} ds = \frac{1}{\lambda}(k - \hat{v}_{i+1}^n) \leq \bar{f}_{i+\frac{1}{2}}^n - f(k).$$

Therefore, we obtain

$$R_+(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) \leq \bar{f}_{i+\frac{1}{2}}^n - f(k).$$

Similarly, we see that

$$\begin{aligned} R_-(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n) &\le \max\left\{\bar{f}_{i+\frac{1}{2}}^n - f(k),\ \int_k^{u_i^n}\left\{\frac{1}{\lambda} - |(g_{i+\frac{1}{2}}^n)'(s)|\right\}ds\right\} \\ &= \max\left\{\bar{f}_{i+\frac{1}{2}}^n,\ \frac{1}{\lambda}(\bar{v}_i^n - k)\right\} \\ &\le \bar{f}_{i+\frac{1}{2}}^n - f(k). \end{aligned}$$

This completes the proof.

6. The construction of a particular entropy function

In this section we give the proof of Theorem 5.2. Under the assumption in Theorem 5.2, we first construct a nonnegative function P satisfying

$$\int_m^M P(k) R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n)\, dk \le 0$$

and then prove the entropy inequality (5.1) for an entropy pair (U, F) such that $U'' = P$. It should be emphasized that the entropy pair (U, F) is constructed according to the strictly convex flux function f, the constants m, M, λ and $\varepsilon \in (0, 1)$.

Without loss of generality, we assume that

$$f(0) = f'(0) = 0.$$

By the strict convexity of f, this implies that

$$f(s) \ge 0$$

for all $s \in \boldsymbol{R}$.

We introduce some notations which are used in this section. Let $h_+(\cdot)$ and $h_-(\cdot)$ be respectively the inverse functions of f restricted to $[0, \infty)$ and restricted to $(-\infty, 0]$. By the definition, it is clear that

$$f(h_+(\tau)) = f(h_-(\tau)) = \tau, \quad \tau \ge 0$$

and

$$\begin{cases} h_+(f(s)) = s, & s \ge 0 \\ h_-(f(s)) = s, & s \le 0. \end{cases}$$

Set

$$h(\tau) = h_+(\tau) - h_-(\tau) \tag{6.1}$$

for $\tau \geq 0$. Note that the following hold:

(1) $h_+(\cdot)$ is monotone increasing and strictly concave on $[0, \infty)$.
(2) $h_-(\cdot)$ is monotone decreasing and strictly convex on $[0, \infty)$.
(3) $h(\cdot)$ is monotone increasing and strictly conave on $[0, \infty)$.

Next let $\varepsilon \in (0, 1)$. We define positive numbers μ_+ and μ_- by

$$\mu_+ = \sup\left\{\tau \in [0, \infty); \left|f'\left(h_+\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\right)\right)\right| \leq \frac{1}{\lambda}\right\} \tag{6.2}$$

and

$$\mu_- = \sup\left\{\tau \in [0, \infty); \left|f'\left(h_-\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\right)\right)\right| \leq \frac{1}{\lambda}\right\}. \tag{6.3}$$

We also write

$$K_+ = \frac{h_+(\mu_+ + \frac{\varepsilon}{2\lambda}h(\mu_+)) - h_+(\mu_+)}{h(\mu_+)} - \frac{\varepsilon}{2}$$

and

$$K_- = -\frac{h_-(\mu_- + \frac{\varepsilon}{2\lambda}h(\mu_-)) - h_-(\mu_-)}{h(\mu_-)} - \frac{\varepsilon}{2}.$$

We have a lemma.

LEMMA 6.1. *Set*

$$K = \min\{K_+, K_-\}. \tag{6.4}$$

Then $K > 0$ *and the following inequalities hold*:

$$\begin{aligned} &h_+\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\right) - \frac{\varepsilon}{2}h(\tau) - h_+(\tau) \geq Kh(\tau) && \textit{for } 0 < \tau \leq \mu_+, \\ &-\left\{h_-\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\right) + \frac{\varepsilon}{2}h(\tau) - h_-(\tau)\right\} \geq Kh(\tau) && \textit{for } 0 < \tau \leq \mu_-. \end{aligned} \tag{6.5}$$

PROOF. Since $h'_+\left(\mu_+ + \dfrac{\varepsilon}{2\lambda}h(\mu_+)\right) = \lambda$ by (6.2), it follows from the concavity of h_+ that

$$\frac{h_+\left(\mu_+ + \frac{\varepsilon}{2\lambda}h(\mu_+)\right) - h_+(\mu_+)}{h(\mu_+)} > \frac{\varepsilon}{2\lambda}h'_+\left(\mu_+ + \frac{\varepsilon}{2\lambda}h(\mu_+)\right) = \frac{\varepsilon}{2}.$$

This means that $K_+ > 0$. Similarly, it is shown that $K_- > 0$. Thus it is proved that $K > 0$.

Next, we prove (6.5). By the strict concavity of h_+, we obtain

$$\begin{aligned}
\frac{h_+\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\right) - h_+(\tau)}{h(\tau)} &= \frac{\varepsilon}{2\lambda}\int_0^1 h'_+\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\theta\right)d\theta \\
&> \frac{\varepsilon}{2\lambda}\int_0^1 h'_+(\mu_+ + \frac{\varepsilon}{2\lambda}h(\mu_+)\theta)\,d\theta \\
&= \frac{h_+\left(\mu_+ + \frac{\varepsilon}{2\lambda}h(\mu_+)\right) - h_+(\mu_+)}{h(\mu_+)} \\
&= K_+ + \frac{\varepsilon}{2}
\end{aligned}$$

for $0 < \tau \le \mu_+$. By the strict convexity of h_-, we similarly obtain

$$\begin{aligned}
-\frac{h_-\left(\tau + \frac{\varepsilon}{2\lambda}h(\tau)\right) - h_-(\tau)}{h(\tau)} &> -\frac{h_-\left(\mu_- + \frac{\varepsilon}{2\lambda}h(\mu_-)\right) - h_-(\mu_-)}{h(\mu_-)} \\
&= K_- + \frac{\varepsilon}{2}
\end{aligned}$$

for $0 < \tau \le \mu_-$. Thus we obtain (6.5). This completes the proof.

Now, we are in a position to define the nonnegative function P. For $s \in \boldsymbol{R}$, set

$$P(s) = \{h(f(s))\}^{\frac{1}{\gamma}}, \tag{6.6}$$

where $\gamma = \frac{1}{8}K^2$ and K is the constant in Lemma 6.1.

Before proceeding the proof of Theorem 5.2, we prepare several lemmas. For notational simplicity, we write

$$\left\{\begin{aligned}
J_+(\tau) &= \int_{I_+(\tau)} P(k)\{\tau - f(k)\}\,dk \qquad \text{for } 0 \le \tau \le \mu_+ \\
J_0(\tau) &= \int_{I_0(\tau)} P(k)\{\tau - f(k)\}\,dk \qquad \text{for } \tau \ge 0 \\
J_-(\tau) &= \int_{I_-(\tau)} P(k)\{\tau - f(k)\}\,dk \qquad \text{for } 0 \le \tau \le \mu_-,
\end{aligned}\right. \tag{6.7}$$

where the intervals of integration are defined by

$$
(6.8)\qquad \begin{cases} I_+(\tau) = \left[h_+(\tau),\ \dfrac{1}{2}\left\{h_+(\tau) + h_+\left(\tau + \dfrac{\varepsilon}{2\lambda}h(\tau)\right) - \dfrac{\varepsilon}{2}h(\tau)\right\}\right] \\ I_0(\tau) \ = [h_-(\tau),\ h_+(\tau)] \\ I_-(\tau) = \left[\dfrac{1}{2}\left\{h_-(\tau) + h_-\left(\tau + \dfrac{\varepsilon}{2\lambda}h(\tau)\right) + \dfrac{\varepsilon}{2}h(\tau)\right\},\ h_-(\tau)\right]. \end{cases}
$$

LEMMA 6.2. *The inequality*

$$
\tau\int_{h_-(\tau)}^{h_+(\tau)} P(k)\,dk - \gamma\tau\{h(\tau)\}^{1+\frac{1}{\gamma}} \leq 0
$$

holds for $\tau \geq 0$.

PROOF. For each $\tau \geq 0$, set

$$
H(\tau) = \int_{h_-(\tau)}^{h_+(\tau)} P(k)\,dk - \gamma\{h(\tau)\}^{1+\frac{1}{\gamma}}.
$$

Since $H(0) = 0$, it suffices to show that $H(\cdot)$ is nonincreasing on $[0, \infty)$. In fact it follows that

$$
\begin{aligned}
H'(\tau) &= h'_+(\tau)P(h_+(\tau)) - h'_-(\tau)P(h_-(\tau)) - (1+\gamma)\{h(\tau)\}^{\frac{1}{\tau}}h'(\tau) \\
&= \{h_+(\tau) - h_-(\tau)\}'(h(\tau)\}^{\frac{1}{\gamma}} - (1+\gamma)h'(\tau)\{h(\tau)\}^{\frac{1}{\gamma}} \\
&= h'(\tau)\{h(\tau)\}^{\frac{1}{\gamma}} - (1+\gamma)h'(\tau)\{h(\tau)\}^{\frac{1}{\gamma}} \\
&= -\gamma h'(\tau)\{h(\tau)\}^{\frac{1}{\gamma}} \\
&\leq 0
\end{aligned}
$$

for $\tau \geq 0$. This completes the proof.

LEMMA 6.3. *The following inequalities hold*:

$$
(6.9)\qquad J_+(\tau) + J_0(\tau) \leq 0 \qquad \textit{for}\ \ 0 \leq \tau \leq \mu_+.
$$

$$
(6.10)\qquad J_-(\tau) + J_0(\tau) \leq 0 \qquad \textit{for}\ \ 0 \leq \tau \leq \mu_-.
$$

PROOF. We prove inequality (6.9). Inequality (6.10) can be proved similarly. Let $0 < \tau \leq \mu_+$. In view of Lemma 6.2, it suffices to show that

$$
J_+(\tau) \leq -\gamma\tau\{h(\tau)\}^{1+\frac{1}{\gamma}}
$$

and

$$J_0(\tau) \leq \tau \int_{h_-(\tau)}^{h_+(\tau)} P(k)\, dk.$$

Regarding

$$\frac{\tau}{h(\tau)} k = \frac{\tau}{h_+(\tau)} \{k - h_+(\tau)\} + \tau$$

as a linear function of $k \in \boldsymbol{R}$, we see that its graph intersects the graph of $f(k)$ at $k = 0$ and $k = h_+(\tau)$. By the convexity of f, we have

$$f(k) \geq \frac{\tau}{h_+(\tau)} \{k - h_+(\tau)\} + \tau, \qquad k \in I_+(\tau),$$

and hence

$$\tau - f(k) \leq -\frac{\tau}{h_+(\tau)} \{k - h_+(\tau)\} \leq -\frac{\tau}{h(\tau)} \{k - h_+(\tau)\} < 0, \qquad k \in I_+(\tau).$$

It is clear that

$$P(k) \geq P(h_+(\tau)) = \{h(\tau)\}^{\frac{1}{\gamma}}, \qquad k \in I_+(\tau).$$

Therefore, taking Lemma 6.1 into account, we obtain

$$\begin{aligned}
J_+(\tau) &\leq P(h_+(\tau)) \int_{I_+(\tau)} \{\tau - f(k)\}\, dk \\
&\leq -\{h(\tau)\}^{\frac{1}{\gamma}} \frac{\tau}{h(\tau)} \int_{I_+(\tau)} \{k - h_+(\tau)\}\, dk \\
&= -\frac{1}{8} \tau \{h(\tau)\}^{\frac{1}{\gamma} - 1} \left\{ h_+\left(\tau + \frac{\varepsilon}{2\lambda} h(\tau)\right) - \frac{\varepsilon}{2} h(\tau) - h_+(\tau) \right\}^2 \\
&\leq -\frac{1}{8} \tau \{h(\tau)\}^{\frac{1}{\gamma} - 1} \{K h(\tau)\}^2 \\
&= -\gamma\tau \{h(\tau)\}^{1 + \frac{1}{\gamma}}.
\end{aligned}$$

Since $f(k) \geq 0$ for $k \in \boldsymbol{R}$, it is clear that

$$J_0(\tau) \leq \tau \int_{h_-(\tau)}^{h_+(\tau)} P(k)\, dk.$$

This completes the proof.

LEMMA 6.4. *Suppose that* $m \leq u_i^n < 0 < u_{i+1}^n \leq M$ *and*

$$\max\left\{a^{MR}(u_i^n, u_{i+1}^n), \frac{\varepsilon}{\lambda}\operatorname{sgn}(u_{i+1}^n - u_i^n)\right\} \leq a_{i+\frac{1}{2}}^n < a^G(u_{i+1}^n, u_i^n). \tag{6.11}$$

We have the following:

(i) *If* $f(u_i^n) \leq f(u_{i+1}^n)$, *then*

$$0 < \bar{f}_{i+\frac{1}{2}}^n \leq \mu_+ \quad \text{and} \quad I_+(\bar{f}_{i+\frac{1}{2}}^n) \subset \left[\hat{w}_{i+1}^n, \frac{1}{2}\{\hat{v}_{i+1}^n + \hat{w}_{i+1}^n\}\right].$$

(ii) *If* $f(u_{i+1}^n) \leq f(u_i^n)$, *then*

$$0 < \bar{f}_{i+\frac{1}{2}}^n \leq \mu_- \quad \text{and} \quad I_-(\bar{f}_{i+\frac{1}{2}}^n) \subset \left[\frac{1}{2}\{\bar{v}_i^n + \bar{w}_i^n\}, \bar{w}_i^n\right].$$

PROOF. For similarity, we only prove (i). Suppose that $f(u_i^n) \leq f(u_{i+1}^n)$. Then it follows from (5.7), (6.1) and (6.11) that

$$\begin{aligned}
f(u_{i+1}^n) &\geq \frac{1}{2}\{f(u_i^n) + f(u_{i+1}^n)\} \\
&= \bar{f}_{i+\frac{1}{2}}^n + \frac{1}{2}a_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n) \\
&\geq \bar{f}_{i+\frac{1}{2}}^n + \frac{\varepsilon}{2\lambda}(u_{i+1}^n - u_i^n) \\
&\geq \bar{f}_{i+\frac{1}{2}}^n + \frac{\varepsilon}{2\lambda}(\hat{w}_{i+1}^n - \bar{w}_i^n) \\
&= \bar{f}_{i+\frac{1}{2}}^n + \frac{\varepsilon}{2\lambda}\{h_+(\bar{f}_{i+\frac{1}{2}}^n) - h_-(\bar{f}_{i+\frac{1}{2}}^n)\} \\
&= \bar{f}_{i+\frac{1}{2}}^n + \frac{\varepsilon}{2\lambda}h(\bar{f}_{i+\frac{1}{2}}^n).
\end{aligned}$$

Here we used the fact that $\bar{f}_{i+\frac{1}{2}}^n > \min_{u_i^n \leq s \leq u_{i+1}^n} f(s) = 0$ by the second inequality in (6.11). Therefore, we see that

$$u_{i+1}^n = h_+(f(u_{i+1}^n)) \geq h_+\left(\bar{f}_{i+\frac{1}{2}}^n + \frac{\varepsilon}{2\lambda}h(\bar{f}_{i+\frac{1}{2}}^n)\right) > 0$$

and hence

$$0 < f'\left(h_+\left(\bar{f}^{\,n}_{i+\frac{1}{2}} + \frac{\varepsilon}{2\lambda}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right)\right) \le f'(u^n_{i+1}) \le \frac{1}{\lambda}.$$

Thus it is proved that

$$0 < \bar{f}^{\,n}_{i+\frac{1}{2}} \le \mu_+.$$

We next show that

$$I_+(\bar{f}^{\,n}_{i+\frac{1}{2}}) \subset \left[\hat{w}^n_{i+1}, \frac{1}{2}(\hat{v}^n_{i+1} + \hat{w}^n_{i+1})\right].$$

Since $h_+(f(u^n_{i+1})) \ge \lambda$, it follows from the strict convexity of h_+ that

$$\begin{aligned} u^n_{i+1} - h_+\left(\bar{f}^{\,n}_{i+\frac{1}{2}} + \frac{\varepsilon}{2\lambda}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right) &= h_+(f(u^n_{i+1})) - h_+\left(\bar{f}^{\,n}_{i+\frac{1}{2}} + \frac{\varepsilon}{2\lambda}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right) \\ &> \lambda\left[f(u^n_{i+1}) - \left\{\bar{f}^{\,n}_{i+\frac{1}{2}} + \frac{\varepsilon}{2\lambda}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right\}\right], \end{aligned}$$

which yields

$$h_+\left(\bar{f}^{\,n}_{i+\frac{1}{2}} + \frac{\varepsilon}{2\lambda}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right) - \frac{\varepsilon}{2}h(\bar{f}^{\,n}_{i+\frac{1}{2}}) < u^n_{i+1} - \lambda\{f(u^n_{i+1}) - \bar{f}^{\,n}_{i+\frac{1}{2}}\} = \hat{v}^n_{i+1}.$$

Since $\hat{w}^n_{i+1} = h_+(\bar{f}^{\,n}_{i+\frac{1}{2}})$, it follows that

$$\frac{1}{2}\left\{h_+(\bar{f}^{\,n}_{i+\frac{1}{2}}) + h_+\left(\bar{f}^{\,n}_{i+\frac{1}{2}} + \frac{\varepsilon}{2\lambda}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right) - \frac{\varepsilon}{2}h(\bar{f}^{\,n}_{i+\frac{1}{2}})\right\} \le \frac{1}{2}\{\hat{v}^n_{i+1} + \hat{w}^n_{i+1}\}$$

and hence it is proved that

$$I_+(\bar{f}^{\,n}_{i+\frac{1}{2}}) \subset \left[\hat{w}^n_{i+1}, \frac{1}{2}\{\hat{v}^n_{i+1} + \hat{w}^n_{i+1}\}\right].$$

This completes the proof.

LEMMA 6.5. *Let* $z \in [m, M]$. *If condition* (5.2);

$$\max\left\{a^{MR}(u^n_i, u^n_{i+1}), \frac{\varepsilon}{\lambda}\operatorname{sgn}(u^n_{i+1} - u^n_i)\right\} \le a^n_{i+\frac{1}{2}} \le \frac{1}{\lambda},$$

holds for all n and i, then

$$\int_m^M P(k)R_\pm(k; z, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}})\,dk \le 0 \tag{6.12}$$

for all n and i.

PROOF. We assume that $u_i^n < 0 < u_{i+1}^n$ and

$$\max\left\{a^{MR}(u_i^n, u_{i+1}^n), \frac{\varepsilon}{\lambda}\operatorname{sgn}(u_{i+1}^n - u_i^n)\right\} \le a_{i+\frac{1}{2}}^n < a^G(u_i^n, u_{i+1}^n).$$

Notice that by Remark 5.1, the other case is reduced to Lemma 4.2. We remark that $u_i^n \in [m, M]$ for all n and i by Lemma 2.1. Now suppose that $f(u_i^n) \le f(u_{i+1}^n)$. Then it is clear that

$$m \le \frac{1}{2}(\bar{v}_i^n + \bar{w}_i^n) \le \bar{w}_i^n < \hat{w}_{i+1}^n < \frac{1}{2}(\hat{v}_{i+1}^n + \hat{w}_{i+1}^n) < M$$

(see (5.5) and (5.7)), and it follows from Lemma 6.3 that $0 < \bar{f}_{i+\frac{1}{2}}^n \le \mu_+$ and $I_+(\bar{f}_{i+\frac{1}{2}}^n) \subset \left[\hat{w}_{i+1}^n, \frac{1}{2}(\hat{v}_{i+1}^n + \hat{w}_{i+1}^n)\right]$. Accordingly, we see from Proposition 5.1 and Lemma 6.3 that

$$\int_m^M P(k)R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n)\,dk \le J_0(\bar{f}_{i+\frac{1}{2}}^n) + J_+(\bar{f}_{i+\frac{1}{2}}^n) \le 0.$$

Here we have used the fact $\bar{w}_i^n = h_-(\bar{f}_{i+\frac{1}{2}}^n)$, $\hat{w}_{i+1}^n = h_+(\bar{f}_{i+\frac{1}{2}}^n)$ and $\bar{f}_{i+\frac{1}{2}}^n \le f(k)$ for $k \notin [\bar{w}_i^n, \hat{w}_{i+1}^n]$. Similarly, if $f(u_i^n) \ge f(u_{i+1}^n)$, we see that

$$\int_m^M P(k)R_\pm(k; z, u_i^n, u_{i+1}^n, g_{i+\frac{1}{2}}^n)\,dk \le J_-(\bar{f}_{i+\frac{1}{2}}^n) + J_0(\bar{f}_{i+\frac{1}{2}}^n) \le 0.$$

This completes the proof.

We can now prove Theorem 5.2.

PROOF of THEOREM 5.2. It is trivial that condition (A1) is satisfied. This ensures that $m \le u_i^n \le M$ for all n and i. To prove (A2)′, let (U, F) be an entropy pair such that

$$\begin{cases} U''(s) = P(s) \\ F'(s) = U'(s)f'(s), \qquad s \in \boldsymbol{R}, \end{cases} \tag{6.13}$$

and

$$U(m) = U'(m) = F(m) = 0. \tag{6.14}$$

It is obvious that U is strictly convex. Noting $U(m) = U'(m) = F(m) = 0$, we easily see that

$$
\begin{aligned}
U(s) &= \int_m^s U'(k)\,dk \\
&= \int_m^M P(k)(s-k)^+\,dk \\
&= \int_m^M P(k)\,U(s;k)\,dk
\end{aligned}
$$

and

$$
\begin{aligned}
F(s) &= \int_m^s U'(k)f'(k)\,dk \\
&= \int_m^M P(k)\chi^+(s-k)\{f(s)-f(k)\}\,dk \\
&= \int_m^M P(k)F(s;k)\,dk
\end{aligned}
$$

for $s \in [m, M]$ (see (4.13) and (4.14)).

Now set

$$
\bar{F}^n_{i+\frac{1}{2}} = \frac{1}{2}\{F(u^n_i) + F(u^n_{i+1})\} - \frac{1}{2}\int_{u^n_i}^{u^n_{i+1}} U'(s)|(g^n_{i+\frac{1}{2}})'(s)|\,ds.
$$

and

$$
A^n_{i+\frac{1}{2}} = \int_0^1 U'(u^n_i + (u^n_{i+1} - u^n_i)\theta)|(g^n_{i+\frac{1}{2}})'(u^n_i + (u^n_{i+1} - u^n_i)\theta)|\,d\theta.
$$

Then it is seen that

$$
\bar{F}^n_{i+\frac{1}{2}} = \frac{1}{2}\{F(u^n_i) + F(u^n_{i+1})\} - \frac{1}{2}A^n_{i+\frac{1}{2}}(u^n_{i+1} - u^n_i)
$$

and

$$
|A^n_{i+\frac{1}{2}}| \le \frac{1}{\lambda}\max_{m \le s \le M}|U'(s)|.
$$

It remains to show the numerical entropy inequality (5.1). By an elementary calculation, we have

$$
\begin{aligned}
A^n_{i+\frac{1}{2}}(u^n_{i+1} - u^n_i) &= \frac{1}{2}\int_{u^n_i}^{u^n_{i+1}} U'(s)|(g^n_{i+\frac{1}{2}})'(s)|\,ds \\
&= \frac{1}{2}\int_{u^n_i}^{u^n_{i+1}} \left\{\int_m^s U''(k)\,dk\right\}|(g^n_{i+\frac{1}{2}})'(s)|\,ds \\
&= \frac{1}{2}\int_{u^n_i}^{u^n_{i+1}} \left\{\int_m^M \chi^+(s-k)P(k)\,dk\right\}|(g^n_{i+\frac{1}{2}})'(s)|\,ds \\
&= \frac{1}{2}\int_m^M P(k)\left\{\int_{u^n_i}^{u^n_{i+1}} \chi^+(s-k)|(g^n_{i+\frac{1}{2}})'(s)|\,ds\right\}dk,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\bar{F}^n_{i+\frac{1}{2}} \\
&\quad = \frac{1}{2}\int_m^M P(k)F(u^n_i;k)\,dk + \frac{1}{2}\int_m^M P(k)F(u^n_{i+1};k)\,dk \\
&\quad\quad - \frac{1}{2}\int_m^M P(k)\int_{u^n_i}^{u^n_{i+1}} \chi^+(s-k)|(g^n_{i+\frac{1}{2}})'(s)|\,ds\,dk \\
&\quad = \frac{1}{2}\int_m^M P(k)\left\{F(u^n_i;k) + F(u^n_{i+1};k) - \int_{u^n_i}^{u^n_{i+1}} \chi^+(s-k)|(g^n_{i+\frac{1}{2}})'(s)|\,ds\right\}dk \\
&\quad = \int_m^M P(k)\bar{F}^n_{i+\frac{1}{2}}(k)\,dk.
\end{aligned}
$$

Therefore, we see from Lemma 4.1 that

$$
\begin{aligned}
&U(u^{n+1}_i) - U(u^n_i) + \lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i+\frac{1}{2}}\} \\
&\quad = \int_m^M P(k)[U(u^{n+1}_i;k) - U(u^n_i;k) + \lambda\{\bar{F}^n_{i+\frac{1}{2}}(k) - \bar{F}^n_{i-\frac{1}{2}}(k)\}]\,dk \\
&\quad = \frac{1}{2}\int_m^M P(k)\{R_+(k;u^{n+1}_i, u^n_{i-1}, u^n_i, g^n_{i-\frac{1}{2}}) \\
&\quad\quad + R_-(k;u^{n+1}_i, u^n_i, u^n_{i+1}, g^n_{i+\frac{1}{2}})\}\,dk.
\end{aligned}
$$

By Lemma 6.5, this means that

$$
U(u^{n+1}_i) - U(u^n_i) + \lambda\{\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\} \le 0
$$

for all n and i. This completes the proof of Theorem 5.2.

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