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Parabolic index and rough isometries

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Abstract. The parabolic index of a locally finite connected graph of bounded degree is shown to be invariant with respect to rough isomertries. We shall give an elementary proof of the fact that the parabolic index of the graph of the d-dimensional Euclidean lattice $\mathbf{Z}^d$ is equal to d.

1. Introduction

Let $G = \{X, Y, K\}$ be an infinite locally finite connected graph, where X is the countable set of points (= nodes, vertices), Y is the countable set of edges (= arcs) and K is the node-arc incidence function (= matrix). The pair $N = \{G, r\}$ of the graph G and a strictly positive real function r (resistance) on Y is called an infinite network in [13]. Since we always consider the case where $r = 1$ in this paper, we identify the graph G with the network N.

Parabolic and hyperbolic networks (graphs) were studied in the paper [13], to which we refer for all basic notions. We only recall that the parabolic index of the graph G is defined as

$$\operatorname{ind} G = \inf\{p > 1;\ G \text{ is parabolic of order } p\}.$$

A neighbour $z \in X$ of $x \in X$ is a point such that there exists $y \in Y$ which satisfies $K(x, y)K(z, y) = -1$. We say that G is of bounded degree if

$$\sup\{\deg(x);\ x \in X\} \leq M$$

for some constant M, where $\deg(x)$ is the number of neighbours of x.

The geodesic distance between two vertices $x, z \in X$ is the number of edges of the shortest path joining x to z and is denoted by $d_G(x, z)$. In particular, we write $x \sim z$ if x and z are neighbours, i.e., $d_G(x, z) = 1$. Clearly $\{X, d_G\}$ is a metric space.

For a real valued function f on X, the Dirichlet sum of f of order p $(1 < p < \infty)$ is given by

$$D_p(f) = D_p(f;\ G) = \sum_{x \sim t} |f(x) - f(t)|^p.$$

If $D_p(f)$ is finite, we say that f is p-Dirichlet finite. We choose a reference vertex $o \in X$ and define a norm for p-Dirichlet finite functions by

$$\|f\|_{\mathbf{D}^{(p)}(G)} = [|f(o)|^p + D_p(f)]^{1/p}.$$

Let $\mathbf{D}^{(p)}(G)$ denote the Banach space of all real valued, p-Dirichlet finite functions equipped with the norm above. Let $l_0(G)$ denote the linear space of all real valued finitely supported functions on X. We denote by $\mathbf{D}_0^{(p)}(G)$ the closure of $l_0(G)$ in $\mathbf{D}^{(p)}(G)$.

Let $G_1 = \{X_1, Y_1, K_1\}$ and $G_2 = \{X_2, Y_2, K_2\}$ be graphs as above. We say that G_1 is roughly isometric to G_2 if there exists a map (called a rough isometry) ϕ from X_1 to X_2 which satisfies the following two conditions:

(RI-1) There exist constants $a > 0$ and $b \geq 0$ such that

$$a^{-1} d_{G_1}(x, t) - b \leq d_{G_2}(\phi(x), \phi(t)) \leq a d_{G_1}(x, t) + b$$

for all $x, t \in X_1$;

(RI-2) There exists a constant $c \geq 0$ such that, for every $z \in X_2$, there exists $x \in X_1$ which satisfies $d_{G_2}(z, \phi(x)) \leq c$.

It is not difficult to see that if G_1 is roughly isometric to G_2, then G_2 is roughly isometric to G_1; cf. [5, p. 392]. Therefore we say that G_1 and G_2 are roughly isometric. Notice that to be roughly isometric is an equivalence relation.

The notion of a rough isometry was introduced by Kanai [5] and is essentially the same as Gromov's notion of a quasi-isometry [2]. Kanai [6] proved that, for roughly isometric Riemannian manifolds R_1, R_2 of bounded geometry, R_2 is parabolic if so is R_1. The discrete counterpart of Kanai's theorem was proved, among other results, in [10, Theorem 3.2]: for roughly isometric connected graphs G_1 and G_2 of bounded degree, G_2 is recurrent if so is G_1. Remember that a graph G is called recurrent (transient) if the simple random walk on G, which assigns equal probability of passing from a vertex to any of its neighbours, is recurrent (transient). Moreover G is recurrent (transient) if and only if it is parabolic of order 2 (hyperbolic of order 2); see [12], [4] or [11].

As a generalization of the result [10, Theorem 3.2], we shall show that two roughly isometric graphs of bounded degree have the same parabolic index. It was proved in [9] that the parabolic index of the graph of the d-dimensional Euclidean lattice $\mathbf{Z}^d$ is equal to d. We shall give a more elementary proof of this fact with the aid of the product of flows due to Lyons [8].

In this paper we always assume that a graph has no multiple edges nor self-loops. Thus, for every pair $\{x, z\}$ of nodes, there exists at most one edge y such that $K(x, y) = 1$ and $K(z, y) = -1$. We call x (resp. z) the initial (resp. terminal) point of y and put $y = [x, z]$.

2. Parabolic index and rough isometries

For every graph $G = \{X, Y, K\}$ and every positive integer k, the k-fuzz $G^k = \{X^k, Y^k, K^k\}$ of G is the graph such that $X^k = X$, and $d_{G^k}(x, t) = 1$ if and only if $1 \leq d_G(x, t) \leq k$.

It was proved in [1](see also [10]) that G is parabolic of order 2 if and only if G^k is parabolic of order 2. As a generalization of this, we have the following result.

THEOREM 2.1. *Suppose that G is of bounded degree. Then* $\operatorname{ind} G = \operatorname{ind} G^k$.

PROOF. We first prove that the norms in $\mathbf{D}^{(p)}(G)$ and in $\mathbf{D}^{(p)}(G^k)$ are equivalent. On one hand it is clear that, for every function f on $X = X^k$,

$$D_p(f; G) \leq D_p(f; G^k), \quad \text{so that} \quad \|f\|_{\mathbf{D}^{(p)}(G)} \leq \|f\|_{\mathbf{D}^{(p)}(G^k)}.$$

On the other hand, set for every $y = [z_1, z_2] \in Y^k$

$$U(y, k) = U([z_1, z_2], k) := \{[x_1, x_2] \in Y;\, d_G(x_j, z_i) \leq k, \quad \text{for} \quad i, j = 1, 2\}.$$

Clearly, every edge in G belongs to less than $M^{2(k+1)}$ sets $U(y, k)$. Now, if $d_{G^k}(x, t) = 1$, i.e., $1 \leq d_G(x, t) = n \leq k$, then there is a path in G with vertices $x = x_0 \sim x_1 \sim \cdots \sim x_n = t$. Then we have, with $1/p + 1/q = 1$,

$$\begin{aligned} |f(x) - f(t)|^p &\leq k^{p/q} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|^p \\ &\leq k^{p/q} \sum_{[s,u] \in U([x,t],k)} |f(s) - f(u)|^p. \end{aligned}$$

It follows

$$\begin{aligned} \sum_{[x,t] \in Y^k} |f(x) - f(t)|^p &\leq k^{p/q} \sum_{[x,t] \in Y^k} \sum_{[s,u] \in U([x,t],k)} |f(s) - f(u)|^p \\ &\leq k^{p/q} M^{2(k+1)} \sum_{[s,u] \in Y} |f(s) - f(u)|^p. \end{aligned}$$

Hence

$$\|f\|_{\mathbf{D}^{(p)}(G^k)} \leq k^{1/q} M^{2(k+1)/p} \|f\|_{\mathbf{D}^{(p)}(G)}.$$

Therefore $\mathbf{D}^{(p)}(G)$ contains the same functions as $\mathbf{D}^{(p)}(G^k)$ and $\mathbf{D}_0^{(p)}(G)$ the same functions as $\mathbf{D}_0^{(p)}(G^k)$. Hence by [13, Theorem 3.2], G is parabolic of order p if and only if G^k is parabolic of order p. ■

Let $G_1 = \{X_1, Y_1, K_1\}$ and $G_2 = \{X_2, Y_2, K_2\}$ be graphs as above. We say that a map ψ from X_1 to X_2 is a morphism from G_1 to G_2 if $d_{G_1}(x, t) = 1$ implies $d_{G_2}(\psi(x), \psi(t)) \leq 1$.

For a morphism ψ from G_1 to G_2, we define the image $G_1' = \{X_1', Y_1', K_1'\}$ *of* G_1 in G_2 with respect to ψ as follows:

$X_1' = \psi(X_1)$, and $y' = [x', t'] \in Y_1'$ if there exists $y = [x, t] \in Y_1$ such that $\psi(x) = x'$ and $\psi(t) = t'$.

It is clear that G_1' is a connected subgraph of G_2 and has no self-loops nor multiple edges.

LEMMA 2.2. *Let G_1 and G_2 be graphs of bounded degree, let ψ be a morphism from G_1 to G_2 and let G_1' be the image of G_1 with respect to ψ. Suppose that there is a constant m such that, for every $x, t \in X$, $\psi(x) = \psi(t)$ implies $d_{G_1}(x, t) \leq m$. Then* $\operatorname{ind} G_1 \leq \operatorname{ind} G_1'$.

PROOF. For every function f' on X_1', we define a function f on X_1 by

$$f(x) = f'(\psi(x)) \qquad \text{for all} \quad x \in X.$$

For every $x', t' \in X_1'$ with $x' \sim t'$, there exist $x, t \in X$ such that $x \sim t$ and $\psi(x) = x'$, $\psi(t) = t'$. We have

$$\sum_{x \sim t} |f(x) - f(t)|^p \leq \sum_{[x', t'] \in Y_1'} \sum_{\substack{\psi(x) = x' \\ \psi(t) = t'}} |f(x) - f(t)|^p$$

$$= \sum_{[x', t'] \in Y_1'} |f'(x') - f'(t')|^p \sum_{\substack{x \in \psi^{-1}(x'), \\ t \in \psi^{-1}(t')}} 1.$$

Let $z' \in X_1'$ and $z \in X_1$ satisfy $\psi(z) = z'$. Then

$$\psi^{-1}(z') \subseteq \{x \in X_1 ;\, d_G(x, z) \leq m\}$$

by our assumption. Therefore Card $\psi^{-1}(z') \leq M^{m+1}$. Here, Card stands for the cardinality. By choosing $o' = \psi(o)$, we get

$$D_p(f; G_1) \leq M^{2(m+1)} D_p(f'; G_1')$$

and hence

$$\| f \|_{\mathbf{D}^{(p)}(G_1)} \leq M^{2(m+1)/p} \| f' \|_{\mathbf{D}^{(p)}(G_1')}.$$

Suppose now that G_1' is parabolic of order p. Then, by [13, Theorem 3.2], there is a sequence $\{f_n'\}$ of real finitely supported functions on X_1' such that

$$\|1 - f_n'\|_{\mathbf{D}^{(p)}(G_1')} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Let f_n and f_n' be related as above. Then f_n is finitely supported on X_1. By the above norm estimation, we see that $\|1 - f_n\|_{\mathbf{D}^{(p)}(G_1)} \to 0$. It follows that $1 \in \mathbf{D}_0^{(p)}(G)$, so that G_1 is parabolic of order p by [13, Theorem 3.2] again. ■

LEMMA 2.3. *Let $G' = \{X', Y', K'\}$ be a subgraph of G. Then* $\operatorname{ind} G' \leq \operatorname{ind} G$.

PROOF. Assume that G is parabolic of order p. Then, there exists a sequence $\{f_n\}$ of real finitely supported functions on X such that $\|1 - f_n\|_{\mathbf{D}^{(p)}(G)} \to 0$ as $n \to \infty$. Denote by g_n the restriction of f_n to X'. Then, g_n is finitely supported and

$$\|1 - g_n\|_{\mathbf{D}^{(p)}(G')} \leq \|1 - f_n\|_{\mathbf{D}^{(p)}(G)} \longrightarrow 0$$

as $n \to \infty$. Hence G' is parabolic of order p by [13, Theorem 3.2]. ■

Now we shall prove our main result.

THEOREM 2.4. *Suppose that G_1 and G_2 are two roughly isometric infinite connected graphs of bounded degree. Then* $\operatorname{ind} G_1 = \operatorname{ind} G_2$.

PROOF. Since G_1 is roughly isometric to G_2, there exists a rough isometry ϕ from X_1 to X_2. For an integer k such that $k \geq a + b$, it is easily seen by (RI-1) that ϕ is a morphism from G_1 to the k-fuzz G_2^k of G_2. Let G_1' be the image of G_1 with respect to ϕ. For every $x, t \in X_1$ such that $\phi(x) = \phi(t)$, we have by (RI-1)

$$a^{-1} d_{G_1}(x, t) - b \leq d_{G_2}(\phi(x), \phi(t)) = 0,$$

so that $d_{G_1}(x, t) \leq ab$. Thus we may apply Lemma 2.2 with $m = ab$. Hence $\operatorname{ind} G_1 \leq \operatorname{ind} G_1'$. Since G_1' is a subgraph of G_2^k, it follows from Lemma 2.3 that $\operatorname{ind} G_1 \leq \operatorname{ind} G_2^k$. Finally, by Theorem 2.1, $\operatorname{ind} G_1 \leq \operatorname{ind} G_2$. The reverse inequality follows by exchanging the roles of G_1 and G_2, since G_2 is roughly isometric to G_1. ■

3. Parabolic index of the Euclidean lattice domains

Let $G = \{X, Y, K\}$ be a connected graph with no self-loops and no multiple edges. For a real valued function u on $X \times X$ which satisfies condition

(F) $$u(x, z) = -u(z, x) \quad \text{for all} \quad x, z \in X,$$

the boundary ∂u of u is defined as the function on X given by

$$\partial u(x) = \sum_{z \sim x} u(x, z) \qquad \text{for all} \quad x \in X.$$

Notice that for the function W on Y defined by

$$W(y) = u(x, z) \qquad \text{if} \quad y = [x, z],$$

we have

$$\partial u(x) = \sum_{y \in Y} K(x, y) W(y).$$

Choose any reference vertex $o \in X$. A function u on $X \times X$ satisfying condition (F) and such that $u(x, z) = 0$ if x and z are not neighbours is called a flow of value m from o to the ideal boundary of G if

$$\partial u(x) = 0 \qquad \text{if} \quad x \neq o,$$

$$\partial u(o) = m.$$

Now we recall the notion of the product of flows due to Lyons [8]. Let $G = \{X, Y, K\}$ and $G' = \{X', Y', K'\}$ be graphs. Define $G \times G'$ as the graph whose vertex set is the Cartesian product $X \times X'$ and let two vertices (x, x') and (z, z') be connected by an edge if and only if $x = z$ and $x' \sim z'$ or if $x' = z'$ and $x \sim z$.

Denote by $\mathbf{Z} = G^{(1)}$ be the Euclidean 1-dimensional lattice graph, whose vertices are the integers and whose edges are line segments joining consecutive integers. Let u and u' be functions on $(X \times \mathbf{Z}) \times (X \times \mathbf{Z})$ and on $(X' \times \mathbf{Z}) \times (X' \times \mathbf{Z})$ respectively, satisying the condition (F). In particular they might be flows from from $(o, 0)$ to the ideal boundary of $G \times \mathbf{Z}$ and from $(o', 0)$ to the ideal boundary of $G' \times \mathbf{Z}$ respectively. Lyons [8] defined the product $w = u * u'$ of u and u' as a function on $(X \times X' \times \mathbf{Z}) \times (X \times X' \times \mathbf{Z})$ in the following way

$$w((x, x', n), (x, x', n \pm 1)) = \pm 2u((x, n), (x, n \pm 1))u'((x', n), (x', n \pm 1)),$$

$$\begin{aligned} w((x, x', n), (z, x', n)) = u((x, n), (z, n))u'((x', n), (x', n + 1)) - \\ u((x, n), (z, n))u'((x', n), (x', n - 1)), \end{aligned}$$

$$\begin{aligned} w((x, x', n), (x, z', n)) = u'((x', n), (z', n))u((x, n), (x, n + 1)) - \\ u'((x', n), (z', n))u((x, n), (x, n - 1)) \end{aligned}$$

whenever $x \sim z$ or $x' \sim z'$. Set $w = 0$ otherwise. Note that w satisfies (F).

The following lemma is an immediate consequence of the definition.

LEMMA 3.1. *Let u and u' be two functions as above and let $w = u * u'$ be their Lyons product. Then, for every $(x, x', n) \in X \times X' \times \mathbf{Z}$*

$$\partial w(x, x', n) = \partial u(x, n)(u'((x', n), (x', n+1)) - u'((x, n), (x', n-1))) +$$
$$\partial u'(x', n)(u((x, n), (x, n+1)) - u((x, n), (x, n-1))).$$

If u and u' are flows from $(o, 0)$ to the ideal boundary of $G \times \mathbf{Z}$ and from $(o', 0)$ to the ideal boundary of $G' \times \mathbf{Z}$ respectively, then $\partial w(x, x', n) = 0$ for all $n \neq 0$ and all $(x, x') \in X \times X'$. In the cases of interest it turns out that w is actually a flow from $(o, o', 0)$ to the ideal boundary of $G \times G' \times \mathbf{Z}$.

We will apply Lyon's method to prove that the parabolic index of the d-dimensional Euclidean lattice is exactly d.

We will denote by $G^{(d)} = \{X^{(d)}, Y^{(d)}, K^{(d)}\}$ the graph of the d-dimensional Euclidean lattice $\mathbf{Z}^d$. More precisely, let $e^{(1)}, \ldots, e^{(d)}$ be the standard base of the d-dimensional Euclidean space $\mathbf{R}^d$, i.e., the k-th component of $e^{(j)}$ is 1 for $k = j$ and 0 for $k \neq j$. For $a, b \in \mathbf{R}^d$, let $[a, b]$ denote the directed line segment from a to b. If x is a vector in $\mathbf{R}^d$ we will write $x = (x_1, x_2, \ldots, x_d)$, i.e. we will denote by x_j the j-th coordinate of x. For each $j = 1, \ldots, d$, set

$$S^{(d)}_{j,+} = \{[x, x + e^{(j)}];\ x \in X^{(d)},\ x_j \geq 0\},$$
$$S^{(d)}_{j,-} = \{[x, x - e^{(j)}];\ x \in X^{(d)},\ x_j \leq 0\},$$
$$S^{(d)}_{j} = S^{(d)}_{j,+} \cup S^{(d)}_{j,-}.$$

With these notation, we take

$$X^{(d)} = \mathbf{Z}^d, \quad Y^{(d)} = \cup_{j=1}^{d} S^{(d)}_j.$$

THEOREM 3.2. $\operatorname{ind} G^{(d)} = d$.

PROOF. The inequality $\operatorname{ind} G^{(d)} \leq d$ was proved in [9] by using a geometric criterion in [13]. Here we give an elementary proof of the reverse inequality.

Let o be the origin of the axes. In order to construct a flow from o to the ideal boundary of $G^{(d)}$, let us put for $d \geq 2$

$$\Omega^{(d)} = \{(v_1, \ldots, v_{d-1}, n) \in \mathbf{Z}^d;\ |v_i| \leq n \quad (i = 1, \ldots, d-1)\}.$$

Let u_2 be the real valued function on $Y^{(2)}$ which satisfies condition (F) and takes the values for $n \geq 0$

$$u_2((\mu, n), (\mu, n+1)) = 1/(2n+1) \qquad \text{for} \quad |\mu| \leq n;$$

$$u_2((\mu, n), (\mu, n+1)) = 0 \qquad \text{for} \quad |\mu| > n;$$

$$u_2((\mu, n), (\mu+1, n)) = \frac{2\mu+1}{(2n-1)(2n+1)} \qquad \text{for} \quad 0 \le \mu \le n-1;$$

$$u_2((\mu, n), (\mu-1, n)) = \frac{-2\mu+1}{(2n-1)(2n+1)} \qquad \text{for} \quad -n+1 \le \mu \le 0;$$

$$u_2((\mu, n), (\mu \pm 1, n)) = 0 \qquad \text{for} \quad |\mu| > n.$$

For $n < 0$, let

$$u_2((\mu, n), (\mu, n+1)) = u_2((\mu, n), (\mu \pm 1, n)) = 0 \qquad \text{for all} \quad \mu.$$

It is easily seen that $\partial u_2(a) = 0$ for all $a \in X^{(2)}$, $a \neq o$ and $\partial u_2(o) = 1$. Then u_2 is a flow of value 1 from o to the ideal boundary of $G^{(2)}$.

Now we define recursively a flow on $G^{(d)}$ for all $d \ge 3$ by setting

$$u_d = u_{d-1} * u_2, \qquad d = 3, 4, \dots$$

We have

$$u_d(x, x+e^{(d)}) = 2^{d-2}(2n+1)^{-d+1} \quad \text{if} \quad x \in \Omega^{(d)} \quad \text{and} \quad x_d = n.$$

If $x_d = z_d = n$, $x, z \in \Omega^{(d)}$ and $x \sim z$,

$$|u_d(x, z)| \le B_d(2n+1)^{-d+1}$$

for some constant B_d independent of n. Moreover, if $x \sim z$,

$$u_d(x, z) = 0 \quad \text{unless} \quad x, z \in \Omega^{(d)}.$$

By Lemma 3.1 and induction one checks easily that

$$\partial u_d(x) = 0 \quad \text{if} \quad x \neq o,$$

$$\partial u_d(o) = 2^{d-2}.$$

Therefore u_d is a flow of value 2^{d-2} from the origin to the ideal boundary of $G^{(d)}$. Set, for every $n = 0, 1, \dots$

$$E_1(n) = \{[x, z] \in Y^{(d)};\ x, z \in \Omega^{(d)},\ x_d = z_d - 1 = n\},$$

$$E_2(n) = \{[x, z] \in Y^{(d)};\ x, z \in \Omega^{(d)},\ x_d = z_d = n\}.$$

Then $\operatorname{Card} E_1(n) = (2n+1)^{d-1}$ and $\operatorname{Card} E_2(n) = 2n(2n+1)^{d-2}$. Let $1 < p < d$ and $1/p + 1/q = 1$. Then, remembering that for every $[x, z] \in Y^{(d)}$, $u_d(x, z) = 0$ unless $\{x, z\} \subset \Omega^{(d)}$, we have

$$
\begin{aligned}
H_q(u_d) &:= \frac{1}{2} \sum_{x \sim t} |u_d(x, t)|^q \\
&= \sum_{n=0}^{\infty} \Big\{ \sum_{[x,t] \in E_1(n)} |u_d(x, t)|^q + \sum_{[x,t] \in E_2(n)} |u_d(x, t)|^q \Big\} \\
&\leq \{2^{(d-2)q} + (B_d)^q\} \sum_{n=0}^{\infty} (2n+1)^{(d-1)(1-q)} < \infty,
\end{aligned}
$$

since $p < d$ implies $(d-1)(1-q) < -1$. Therefore $G^{(d)}$ is hyperboic of order p, $(1 < p < d)$ by [13, Theorem 4.3]. Thus $\operatorname{ind} G^{(d)} \geq d$. ■

4. An application

A tiling $\mathbf{T}$ of the Euclidean plane $\mathbf{R}^2$ is a collection of closed topological disks T, having pairwise disjoint interiors, such that $\cup_{T \in \mathbf{T}} T = \mathbf{R}^2$. We assume that the tiling is normal in the sense [3]. The edges and the vertices of the tiling are the edges and vertices of a plane graph $G = \{X, Y, K\}$ which is locally finite connected and of bounded degree: this graph is called the edge graph of the tiling. See the book [3] for hundreds of figures and examples.

It was proved in [11, Theorem 4] that for every normal tiling $\mathbf{T}$ there is a combinatorially equivalent tiling $\mathbf{T}'$ whose edge graph $G' = \{X', Y', K'\}$ is uniformly imbedded in $\mathbf{R}^2$, i.e. there is a constant k such that for every $p, q \in X'$

$$k^{-1} d_{G'}(p, q) \leq |p - q| \leq k d_{G'}(p, q)$$

where $|p - q|$ denotes the Euclidean distance between p and q.

Therefore the inclusion map is a rough isometry of G' into $\mathbf{R}^2$. Namely G' is roughly isometric to $G^{(2)}$. On the other hand G and G' are isometric graphs. Since rough isometry is an equivalence relation, G is roughly isometric to $G^{(2)}$. Then our Theorems 2.4 and 3.2 imply that $\operatorname{ind} G = 2$.

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