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$S\alpha S$ $M(t)$-processes and their canonical representations

Katsuya KOJO

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Introduction

T. Hida, H. Cramér and many other mathematicians have investigated the theory of canonical representations of Gaussian processes. Especially, T. Hida [3] has proved that any purely non-deterministic separable Gaussian process has a unique generalized canonical representation, which is obtained by applying Hellinger-Hahn's theorem to the reproducing kernel Hilbert space made from the covariance function of the process. This representation is called canonical if the multiplicity of the representation is 1 (T. Hida and N. Ikeda [4]). However, it seems that for non-Gaussian processes (especially without 2nd moments), any general theory of canonical representations has not been established yet.

We know that Gaussian random variables are symmetric stable random variables with index $\alpha = 2$. So in this paper, we deal with canonical representations of symmetric-α-stable ($= S\alpha S$) processes ($0 < \alpha \leq 2$).

In Gaussian case, Lévy-McKean's $M(t)$-processes are precious examples to study the theory of canonical representations. The $M(t)$-process is defined as the spherical mean process of the multi-parameter Brownian motion with the spherical harmonic as its weight. N. N. Chentsov [2] found that this Brownian motion can be constructed by integral geometry, and H. P. McKean Jr. [9] used this fact to obtain a causal representation of the $M(t)$-process. We apply this very fact to extend the notions of the multi-parameter Brownian motions and $M(t)$-processes to non-Gaussian $S\alpha S$ case ($0 < \alpha < 2$), and we obtain causal representations of these $M(t)$-processes in the form of

$$X(t) = \int_0^t F(t, u)\, dZ(u).$$

We investigate the canonicalities of these representations by the following methods.

i) Similarly to Gaussian case ($\alpha = 2$), we can consider the closed linear hulls of $\{Z(s); s \leq t\}$ and $\{X(s); s \leq t\}$ respectively for every t. We find whether the hull of $\{X(s); s \leq t\}$ includes the hull of $\{Z(s); s \leq t\}$ for all t or not (the inverse inclusion is trivial). In case that the equality holds (this case we say that the representation is proper), we make the procedure to obtain $\{Z(s);\, s \le t\}$ from $\{X(s);\, s \le t\}$.

ii) In case of $M(t)$-processes, $\{Z(t)\}$ is an SαS process with independent stationary increments (i.e., an SαS motion). For non-Gaussian case ($0 < \alpha < 2$), we apply the Lévy-Itô's theorem on the decomposition of paths to modify $\{Z(t)\}$ into a process whose paths are right continuous and have left limits (this modification is called D-modification in this paper). Using this modification, we obtain a D-modification of $\{X(t)\}$ (to obtain the modification, we apply the integration by parts). And we consider the regularity of paths and we calculate the jumping times and heights of $\{Z(s);\, s \le t\}$ from those of $\{X(s);\, s \le t\}$. This idea is found in P. Lévy [8], and T. Hida and N. Ikeda [4], but cannot be applied to Gaussian case because the paths of Brownian motion are continuous.

Through the argument, we can find whether a causal representation in a certain class is canonical or not. We hope it will be a first step to study the theory of canonical representations of SαS processes.

§ 0. Preliminaries

A real-valued random variable X is called a *symmetric-α-stable* ($=$ *SαS*) random variable if the characteristic function of X is $\exp(-c|z|^{\alpha})$ with some constant $c \ge 0$. The SαS random variable exists if and only if $0 < \alpha \le 2$. When $\alpha = 2$, an SαS random variable is a Gaussian random variable with mean 0.

In this paper, the time domain T is fixed either $[0, \infty)$ or $(-\infty, \infty)$. A stochastic process $\{X(t);\, t \in T\}$ is called an *SαS process* if any finite linear combination $\sum a_j X(t_j)$ ($a_j \in \boldsymbol{R}$, $t_j \in T$) is an SαS random variable. We assume that any SαS process in this paper is separable. Especially, an SαS process with independent stationary increments is unique up to a constant and is called an *SαS motion.*

Let $(S, \mathfrak{B}, \mu)$ be a σ-finite measure space.

Definition 0.1. A random field $\{Y^{\alpha}(B);\, B \in \mathfrak{B}, \mu(B) < \infty\}$ is called an *SαS random measure controlled by* $(S, \mathfrak{B}, \mu)$ if it satisfies the following three conditions:

i) Any finite linear combination $\sum a_j Y^{\alpha}(B_j)$ is an SαS random variable.

ii) The characteristic function of $Y^{\alpha}(B)$ is equal to $\exp(-\mu(B)|z|^{\alpha})$.

iii) If $\{B_j\}_{j=1,2,\ldots}$, $\mu(B_j) < \infty$, is a family of disjoint sets, then $\{Y^{\alpha}(B_j)\}_{j=1,2,\ldots}$ is a family of mutually independent random variables, and if $\mu(\bigcup_j B_j) < \infty$, then $Y^{\alpha}(\bigcup_j B_j) = \sum_j Y^{\alpha}(B_j)$ a.s.

If $\{Y^\alpha(B)\}$ is an SαS random measure controlled by a measure space $(T, \mathfrak{B}, v)$, $X^\alpha(t) \equiv Y^\alpha([0, t])$ if $t \geq 0$, $\equiv Y^\alpha([t, 0])$ if $t < 0$ is called an *SαS process with independent increments controlled by* $(T, \mathfrak{B}, v)$ in this paper.

$L^{(\alpha)}(S, \mathfrak{B}, \mu)$ denotes the family of measurable functions $\left\{f;\ \int_S |f|^\alpha d\mu < \infty\right\}$ equipped with the metric $d^{(\alpha)}(f, g) = \left(\int_S |f - g|^\alpha d\mu\right)^{(1/\alpha)\wedge 1}$ Note that $L^{(\alpha)}(S, \mathfrak{B}, \mu)$ is a Banach space only in case $1 \leq \alpha \leq 2$.

Now we define the Wiener-type stochastic integral $\int_S f dY^\alpha$ of f in $L^{(\alpha)}(S, \mathfrak{B}, \mu)$ with respect to $\{Y^\alpha(B)\}$. If f is a step function $\sum a_j I_{B_j}$, where $\{B_j\}$ is a family of finite disjoint sets and I_B denotes the indicator function of B, then $\int_S f dY^\alpha$ is defined as $\sum a_j Y(B_j)$. For a general f, we take a sequence of step functions $\{f_j\}_{j=1,2,\dots}$ which converges to f in $L^{(\alpha)}$, then $\left\{\int_S f_j dY^\alpha\right\}_{j=1,2\dots}$ converges in p-th order expectation for all $p < \alpha$ (also $p = 2$ when $\alpha = 2$). The convergence does not depend on the selection of $\{f_j\}$, thus we define $\int_S f dY^\alpha$ as this limit. (See M. Schilder [13].)

In this paper, for two processes $\{X(t); t \in T\}$ and $\{\tilde{X}(t); t \in T\}$, $\{X(t)\} \stackrel{d}{=} \{\tilde{X}(t)\}$ means that all finite dimensional distributions are equal to each other.

§1. Representations of SαS processes by causal stochastic integrals

T. Hida [3], and T. Hida and N. Ikeda [4] gave definitions and obtained some propositions on stochastic integral representations of Gaussian processes. We extend them to SαS case.

Assume that an SαS process $\{X(t); t \in T\}$ $(0 < \alpha \leq 2)$ has the following modification written in the form of stochastic integral

$$X(t) = \int^t F(t, u) dZ(u), \tag{1.1}$$

where

i) $\{Z(t); t \in T\}$ is an SαS process with independent increments controlled by a measure space (T, v),

ii) $F(t, u)$ is a function on $T \times T$ which vanishes on $\{(t, u); u > t\}$ and belongs to $L^{(\alpha)}(T, v)$ as a function of u for every $t \in T$ and $\int^t$ means $\int_{(-\infty, t] \cap T}$

Definition 1.1. The formula (1.1)

$$X(t) = \int^{t} F(t, u) dZ(u),$$

satisfying the above i) and ii), is called a *causal representation of* $\{X(t)\}$.

In non-Gaussian case ($0 < \alpha < 2$), it is unknown whether any SαS process has a causal representation or not. But it is known that any SαS process $\{X(t); t \in T\}$ ($0 < \alpha \le 2$) has a version written in the form of (non-causal) stochastic integral

$$\{X(t)\} \stackrel{d}{=} \left\{ \int_{[0,1]} f(t, u) dZ(u) \right\},$$

where $\{Z(t); t \in [0, 1]\}$ is an SαS motion and $f(t, u)$ belongs to $L^{(\alpha)}[0, 1]$ as a function of u for every $t \in T$ (see J. Kuelbs [7]).

Suppose that $\{X(t); t \in T\}$ is an SαS process with a causal representation (1.1). For every $t \in T$, $\mathfrak{B}_t(X)$ denotes the σ-field generated by SαS random variables $\{X(s); s \le t\}$. It is obvious that

$$\mathfrak{B}_t(X) \subset \mathfrak{B}_t(Z) \qquad \text{for every } t \in T.$$

Definition 1.2. A causal representation (1.1) is called *canonical* (*in the sense of σ-field*) if it satisfies

$$\mathfrak{B}_t(X) = \mathfrak{B}_t(Z) \qquad \text{for every } t \in T.$$

This case we call $\{Z(t)\}$ an *innovation* process of $\{X(t)\}$.

For a given canonical representation of an SαS process, it is a question whether this canonical representation is unique or not. The following proposition would be an answer.

Proposition 1.3. *Suppose that there exist two canonical representations*

$$X(t) = \int^{t} F^{(j)}(t, u) dZ^{(j)}(u) \qquad (j = 1, 2)$$

for an SαS process $\{X(t); t \in T\}$. *Then the formula*

$$\int^{s} F^{(1)}(t, u) dZ^{(1)}(u) = \int^{s} F^{(2)}(t, u) dZ^{(2)}(u) \tag{1.2}$$

is satisfied for every s and t ($s \le t$). (For Gaussian case ($\alpha = 2$), see T. Hida [3].)

PROOF. Fix s and t $(s \leq t)$ arbitrarily. For all $\lambda \in \boldsymbol{R}$, we have

$$E[\exp(i\lambda X(t)) | \mathfrak{B}_s(X)]$$
$$= \exp\left\{ i\lambda \int^s F^{(j)}(t, u) dZ^{(j)}(u) \right\} \exp\left\{ -|\lambda|^\alpha \int_s^t |F^{(j)}(t, u)|^\alpha dv^{(j)}(u) \right\}$$

for each j. Therefore

$$\exp\left\{ i\lambda \left[\int^s F^{(1)}(t, u) dZ^{(1)}(u) - \int^s F^{(2)}(t, u) dZ^{(2)}(u) \right] \right\}$$
$$= \exp\left\{ |\lambda|^\alpha \left[\int_s^t |F^{(1)}(t, u)|^\alpha dv^{(1)}(u) - \int_s^t |F^{(2)}(t, u)|^\alpha dv^{(2)}(u) \right] \right\}$$

for all $\lambda \in \boldsymbol{R}$. We can see the left hand side is complex random variable of absolute value 1 a.s., while the right hand side is real. This means (1.2). □

For every $t \in T$, $\mathfrak{M}_t^\alpha(X)$ denotes the closed linear hull of $\{X(s); s \leq t\}$ in $L^{(\alpha)}$. It is obvious that for the causal representation (1.1),

$$\mathfrak{M}_t^\alpha(X) \subset \mathfrak{M}_t^\alpha(Z) \qquad \text{for every } t \in T.$$

DEFINITION 1.4. A causal representation (1.1) is called *proper* if it satisfies

$$\mathfrak{M}_t^\alpha(X) = \mathfrak{M}_t^\alpha(Z) \qquad \text{for every } t \in T.$$

It is trivial that a proper representation is canonical. For Gaussian case $(\alpha = 2)$, it is well-known that a canonical representation is proper. By contrast, for non-Gaussian case $(0 < \alpha < 2)$, there exist causal representations which are not proper but canonical. We show some examples with such a property in §3.

For Gaussian case, T. Hida [3] gave a criterion to determine whether a given causal representation is proper canonical or not. For $1 < \alpha < 2$, there exists a similar criterion by virtue of the following theory of the projections in Banach space (see I. Singer [14]).

Assume that M_0 is a closed subspace of Banach space $L^{(\alpha)}(T, \mathfrak{B}, v)$ $(1 < \alpha \leq 2)$. For any $f \in L^{(\alpha)}(T, \mathfrak{B}, v)$, f_0 is called a projection of f on M_0 if it minimizes $\int_T |f - f_0|^\alpha dv$ in M_0. For any $f \in L^{(\alpha)}(T, \mathfrak{B}, v)$, the projection f_0 exists uniquely and satisfies

$$\int_T g(f - f_0)^{\langle \alpha - 1 \rangle} dv = 0 \qquad \text{for any } g \in M_0$$

where $x^{\langle \alpha - 1 \rangle} = |x|^{\alpha - 1} \operatorname{sgn}(x)$. (This case it is said that $f - f_0$ is right-orthogonal to M_0.)

We have already known that $\mathfrak{M}_t^\alpha(Z)$ has the norm induced by $L^{(\alpha)}(T, \mathfrak{B}, v)$, so we can apply the theory of projections to the pair $\mathfrak{M}_t^\alpha(Z)$ and its subspace $\mathfrak{M}_t^\alpha(X)$. Therefore, we obtain the following proposition.

PROPOSITION 1.5. *For $1 < \alpha \leq 2$, a causal representation* (1.1) *is proper if and only if, for any $t_0 \in T$, any function $\varphi \in L^{(\alpha)}(T, \mathfrak{B}, v)$ which satisfies*

$$\int^t F(t, \cdot)\varphi^{\langle\alpha-1\rangle} dv = 0 \qquad \textit{for all } \; t \leq t_0$$

is equal to 0 *on* $(-\infty, t_0] \cap T$.

§ 2. SαS $M(t)$-processes

In T. Hida [3], Lévy's $M(t)$-processes provided us precious examples of canonical representations of Gaussian processes. Moreover, H. P. McKean Jr. [9] constructed extended (Gaussian) $M(t)$-processes. He obtained their causal representations and investigated the canonicalities of them. In this section we consider the similar extended $M(t)$-processes in SαS case, which are constructed in the same procedure.

2-1 The constructions of SαS $M(t)$-processes

Lévy's multi-parameter Brownian motion can be constructed by integral geometry (N. N. Chentsov [2]). We construct the similar random field, which we would call the multi-parameter SαS motion, as follows (see S. Takenaka [16]).

Let $\mathscr{H}^n$ be the set of all hyperplanes of codimension 1 in the Euclidean space $\boldsymbol{R}^n (n \geq 1)$. We introduce a parametrization (q, p) in $\mathscr{H}^n$, $q \in S^{n-1}$, $p \geq 0$, as follows:

$$(q, p) \longleftrightarrow h(q, p) = \{x \in \boldsymbol{R}^n; \; -(x \cdot q) + p = 0\}$$

Define a measure μ on $\mathscr{H}^n$ as $d\mu = dq\,dp$ where dq is the normalized uniform measure on S^{n-1} and dp is the Lebesgue measure on $[0, \infty)$. Note that μ is the invariant measure under rotations and parallel transformations in $\mathscr{H}^n$.

For fixed α $(0 < \alpha \leq 2)$, we have an SαS random measure $\{Y_n^\alpha(B)\}$ with control measure space $(\mathscr{H}^n, \mu)$. For $\boldsymbol{t} \in \boldsymbol{R}^n$, set

$$S_{\boldsymbol{t}} = \{h \in \mathscr{H}^n; h \text{ separates the origin } \boldsymbol{0} \text{ and } \boldsymbol{t}\}$$

and define

$$X_n^\alpha(\boldsymbol{t}) \equiv Y_n^\alpha(S_{\boldsymbol{t}}) = \int_{0 \leq p \leq t(\xi \cdot q)} Y_n^\alpha(dq\,dp) \tag{2.1}$$

where $\boldsymbol{t} = t\xi$; $t \geq 0$, $\xi \in S^{n-1}$.

Then the SαS random field $\{X_n^\alpha(\boldsymbol{t}); \boldsymbol{t} \in \boldsymbol{R}^n\}$ has the following properties:

i) $X_n^\alpha(\boldsymbol{0}) = 0$.

ii) For any $g \in SO(n)$ and $\boldsymbol{a} \in \boldsymbol{R}^n$, we have the formula

$$\{X_n^\alpha(g\boldsymbol{t} + \boldsymbol{a}) - X_n^\alpha(\boldsymbol{a}); \boldsymbol{t} \in \boldsymbol{R}^n\} \stackrel{d}{=} \{X_n^\alpha(\boldsymbol{t}); \boldsymbol{t} \in \boldsymbol{R}^n\}.$$

iii) The characteristic function of $X_n^\alpha(\boldsymbol{t}) - X_n^\alpha(\boldsymbol{s})$ is equal to

$$\exp(-C(n)d(\boldsymbol{t}, \boldsymbol{s})|z|^\alpha),$$

where $C(1) = 1/2$, $C(n) = \Gamma(n/2)\{(n-1)\pi^{1/2}\Gamma((n-1)/2)\}^{-1}$ for $n \geq 2$ and $d(\cdot, \cdot)$ denotes the Euclid distance of $\boldsymbol{R}^n$. This property derives the linear additive property which means that $X_n^\alpha(\boldsymbol{a} + \lambda\boldsymbol{b})$ is an SαS process with independent increments with respect to $\lambda \in \boldsymbol{R}$ for any $\boldsymbol{a}$ and $\boldsymbol{b} \in \boldsymbol{R}^n$.

Especially in Gaussian case ($\alpha = 2$), the Gaussian random field $\{X_n^2(\boldsymbol{t}); \boldsymbol{t} \in \boldsymbol{R}^n\}$ is equal to Lévy's Brownian motion with parameter $\boldsymbol{R}^n$ up to a constant. Furthermore, the uniqueness of the SαS random field with properties i) and iii) is recently proved in T. Mori [10]. So we would call this random field *the SαS motion with parameter* $\boldsymbol{R}^n$.

In Gaussian case ($\alpha = 2$), Lévy-McKean's $M(t)$-process is defined as the spherical mean process of the multi-parameter Brownian motion with the spherical harmonic as its weight. We can extend $M(t)$-processes to SαS case ($0 < \alpha < 2$) by integral geometry as McKean used in [9].

For each $n \geq 1$, let $v_{l,m}^n(\xi)$ be the spherical harmonic on S^{n-1}, where $l(= 0, 1, \cdots)$ is the degree of harmonic and m is the associated multi-suffix. If $n = 1$, l runs only 0 or 1. $v_{l,0}^n$ is called the zonal spherical function which depends only on the colatitude. (For details, see N. J. Vilenkin [18].)

Now we consider that

$$M_{n,l,m}^\alpha(t) \equiv \int_{\xi \in S^{n-1}} X_n^\alpha(t\xi) v_{l,m}^n(\xi)\, d\xi, \qquad t \geq 0, \tag{2.2}$$

where $d\xi$ is the normalized uniform measure on S^{n-1}. The right hand side can be defined as the limit of Riemannian sum in $L^{(\alpha)}$, explained later. We call the SαS process $\{M_{n,l,m}^\alpha(t); t \geq 0\}$ the *SαS* $M(t)$*-process.* Of course, $\{M_{n,0,0}^2(t)\}$ is Lévy's $M(t)$-process and $\{M_{n,l,m}^2(t)\}$ is McKean's $M(t)$-process up to a constant.

Let us calculate the right hand side of (2.2). Using (2.1),

$$M_{n,l,m}^\alpha(t) = \int_{\xi \in S^{n-1}} \left(\int_{0 \leq p \leq t(\xi \cdot q)} Y_n^\alpha(dq\, dp) \right) v_{l,m}^n(\xi)\, d\xi.$$

We can select an appropriate sequence of Riemannian sums

$$\sum_{j=1}^{k} I_{\{(q,p);\, 0 \le p \le t(\xi_{k,j}\cdot q)\}}(q, p)\, v_{l,m}^{n}(\xi_{k,j}) A(B_{k,j})$$

(where $\{B_{k,j}\}_{1 \le j \le k}$ is a partition of S^{n-1}, $\xi_{k,j}$ is an element in $B_{k,j}$ and $A(B_{k,j})$ is the area of $B_{k,j}$), which converges to

$$\int_{(\xi\cdot q)\ge p/t} v_{l,m}^{n}(\xi)\, d\xi$$

uniformly in $(q, p) \in S^{n-1} \times [0, t]$ as the mesh converges to 0 (so that the sequence converges in $L^{(\alpha)}$). Therefore we can exchange the order of the integrations and we have

$$M_{n,l,m}^{\alpha}(t) = \int_{S^{n-1}\times[0,t]} \left(\int_{(\xi\cdot q)\ge p/t} v_{l,m}^{n}(\xi) d\xi \right) Y_n^{\alpha}(dq\,dp).$$

According to McKean [9], for $n \ge 2$,

$$\int_{(\xi\cdot q)\ge p/t} v_{l,m}^{n}(\xi)\, d\xi$$

$$= v_{l,m}^{n}(q) \left(\int_0^{\pi} \sin^{n-2}\theta d\theta \right)^{-1} \int_0^{\cos^{-1}(p/t)} P_l^n(\cos\theta) \sin^{n-2}\theta d\theta,$$

where $P_l^n(x) = C_l^{(n-2)/2}(x)/C_l^{(n-2)/2}(1)$ ($C_b^a(x)$ is the Gegenbauer polynomial). Thus we obtain the following formula which is a causal representation of SαS process $\{M_{n,l,m}^{\alpha}(t);\, t \ge 0\}$:

$$M_{n,l,m}^{\alpha}(t) = \int_0^t F_{n,l}(t, p)\, dZ_{n,l,m}^{\alpha}(p), \qquad (*)$$

where

$$Z_{n,l,m}^{\alpha}(p) \equiv \int_{S^{n-1}} v_{l,m}^{n}(q)\, Y_n^{\alpha}(dq \times [0, p])$$

and

$$F_{1,l}(t, p) \equiv 1/2,$$

$$F_{n,l}(t, p) \equiv \left(\int_0^{\pi} \sin^{n-2}\theta d\theta \right)^{-1} \int_0^{\cos^{-1}(p/t)} P_l^n(\cos\theta) \sin^{n-2}\theta d\theta$$

$$= (-1)^l C(n, l) \left[\int_x^1 \frac{d^l}{dx^l} (1 - x^2)^{l+(n-3)/2} dx \right] \Bigg|_{x=p/t},$$

with a constant $C(n, l) = \Gamma(n/2)\{2^l \pi^{1/2} \Gamma(l + (n-1)/2)\}^{-1}$ for $n \geq 2$. Note that the process $\{Z^\alpha_{n,l,m}(p); p \geq 0\}$ is a 1-parameter SαS process with independent stationary increments, i.e., an SαS motion and that the kernel $F_{n,l}(t, u)$ depends on neither α nor m.

2-2 The canonicalities of the representations (I)

Here we consider the question whether the causal representations (*) are canonical or not. For $n = 1$, it is easy to see that both $(l = 0, 1)$ of the representations (*) are proper canonical. Firstly, we find whether the representations (*) are proper or not for $n \geq 2$.

LEMMA 2.1. *Let $n \geq 2$. For any fixed $t > 0$, we can apply a differential operator*

$$t^{-(n+l-1)} \frac{d}{dt} t^{n+l}$$

to $M^\alpha_{n+2,l,m}(t)$ at t in the sense of $L^{(\alpha)} (0 < \alpha \leq 2)$ and we obtain

$$\left\{t^{-(n+l-1)} \frac{d}{dt} t^{n+l} M^\alpha_{n+2,l,m}(t)\right\} \overset{d}{=} \{K M^\alpha_{n,l,m'}(t)\}$$

with a positive constant $K = K(\alpha, n, l, m, m')$. (Especially, $K = n$ if $\alpha = 2$ or $l = 0$. For Lévy's $M(t)$-process ($\alpha = 2$ and $l = 0$), see T. Hida [3].)

PROOF. Note that the kernel $F_{n,l}(t, u)$ is homogeneous, i.e., it is a function of u/t, therefore

$$\begin{aligned} &t^{n+l} M^\alpha_{n+2,l,m}(t) \\ &= (-1)^l C(n+2, l) \int_0^t \left(\int_u^t \frac{d^l}{dx^l} (t^2 - x^2)^{l+(n-1)/2} dx \right) dZ^\alpha_{n+2,l,m}(u). \end{aligned}$$

Let us consider the right differentiability of $t^{n+l} M^\alpha_{n+2,l,m}(t)$. Fix any $t > 0$ and let $h > 0$.

$$\begin{aligned} &\frac{1}{h}\{(t+h)^{n+l} M^\alpha_{n+2,l,m}(t+h) - t^{n+l} M^\alpha_{n+2,l,m}(t)\} \\ &= (-1)^l C(n+2, l) \times \frac{1}{h}\left\{ \int_t^{t+h} \left(\int_u^{t+h} \frac{d^l}{dx^l} [(t+h)^2 - x^2]^{l+(n-1)/2} dx \right) dZ^\alpha_{n+2,l,m}(u) \right. \\ &\qquad + \int_0^t \left(\int_u^{t+h} \frac{d^l}{dx^l} [(t+h)^2 - x^2]^{l+(n-1)/2} dx \right. \\ &\qquad \left. \left. - \int_u^t \frac{d^l}{dx^l} (t^2 - x^2)^{l+(n-1)/2} dx \right) dZ^\alpha_{n+2,l,m}(u) \right\}. \qquad (2.3) \end{aligned}$$

The first term converges to 0 in $L^{(\alpha)}$ as $h\downarrow 0$ because

$$\frac{d^l}{dx^l}[(t+h)^2-x^2]^{l+(n-1)/2}dx$$
$$=(a\ polynomial\ in\ x,\ h\ and\ t)\times[(t+h)^2-x^2]^{(n-1)/2}$$

and

$$\left\{\frac{1}{h^\alpha}\int_t^{t+h}\left|\int_u^{t+h}\frac{d^l}{dx^l}[(t+h)^2-x^2]^{l+(n-1)/2}dx\right|^\alpha du\right\}^{(1/\alpha)\wedge 1}$$
$$\leq const.\times\left\{h^{-\alpha}\int_t^{t+h}h^{\alpha\cdot[(n-1)/2+1]}du\right\}^{(1/\alpha)\wedge 1}$$
$$\leq const.\times\{h^{\alpha(n-1)/2+1}\}^{(1/\alpha)\wedge 1}.$$

The integrand of the second term of (2.3) converges to

$$\frac{\partial}{\partial t}\left\{(-1)^l C(n+2,\ l)\int_u^t\frac{d^l}{dx^l}(t^2-x^2)^{l+(n-1)/2}dx\right\}$$
$$=(-1)^l C(n+2,\ l)\int_u^t\frac{d^l}{dx^l}\left(\frac{\partial}{\partial t}(t^2-x^2)^{l+(n-1)/2}\right)dx$$
$$=(-1)^l nC(n,\ l)\int_u^t\frac{d^l}{dx^l}(t^2-x^2)^{l+(n-3)/2}dx$$
$$=nt^{n+l-1}F_{n,l}(t,\ u)$$

as $h\downarrow 0$ for every point $u\in[0,\ t]$. The function $F_{n,l}(t,\ u)$ is right continuous in t uniformly on $u\in[0,\ t]$, so we find the second term of (2.3) converges to

$$\int_0^t nt^{n+l-1}F_{n,l}(t,\ u)dZ^\alpha_{n+2,l,m}(u)$$

in $L^{(\alpha)}$. Hence we complete the proof of the right differentiability.

For any $t>0$ and $h>0$, we have the formula

$$-\frac{1}{h}\{(t-h)^{n+l}M^\alpha_{n+2,l,m}(t-h)-t^{n+l}M^\alpha_{n+2,l,m}(t)\}$$
$$=(-1)^l C(n+2,\ l)\times\left(-\frac{1}{h}\right)\left\{\int_{t-h}^t\left(\int_u^t\frac{d^l}{dx^l}(t^2-x^2)^{l+(n-1)/2}dx\right)dZ^\alpha_{n+2,l,m}(u)\right.$$
$$+\int_0^{t-h}\left(\int_u^{t-h}\frac{d^l}{dx^l}[(t-h)^2-x^2]^{l+(n-1)/2}dx\right.$$

$$- \int_u^t \frac{d^l}{dx^l}(t^2 - x^2)^{l+(n-1)/2}\,dx\bigg)dZ^{\alpha}_{n+2,l,m}(u)\bigg\}.$$

Thus to prove the left differentiability, we have only to show similarly that the first and the second term converge to 0 and $\int_0^t nt^{n+l-1}F_{n,l}(t, u)dZ^{\alpha}_{n+2,l,m}(u)$ in $L^{(\alpha)}$ respectively. We complete the proof. □

By this lemma, we can reduce the problem of canonicalities to the case $n = 3$ or $n = 2$ according as n is odd or even respectively.

LEMMA 2.2. *In case* $n = 3$.

i) *If* $l = 0, 1, 2$, *the causal representation* (∗) *is proper for* $0 < \alpha \leq 2$.

ii) *If* $l \geq 3$, *the causal representation* (∗) *is not proper for* $1 < \alpha \leq 2$.

iii) *For any fixed* $t > 0$, $M^{\alpha}_{3,l,m}(t)$ *is differentiable at* t *in* $L^{(\alpha)}$ $(0 < \alpha \leq 2)$.

(Hida [3] and H. P. McKean Jr. [9] for $\alpha = 2$)

CONJECTURE. The causal representation (∗) is not proper for $\alpha = 1$ and (∗) is proper for $0 < \alpha < 1$.

PROOF. i). We already know that

$$F_{3,0}(t, u) = C(3, 0)\left(1 - \frac{u}{t}\right), \quad F_{3,1}(t, u) = C(3, 1)\left(1 - \frac{u^2}{t^2}\right) \quad \text{and}$$

$$F_{3,2}(t, u) = 4C(3, 2)\left(\frac{u}{t} - \frac{u^3}{t^3}\right).$$

So we can easily show that

$$\frac{d}{dt}tM^{\alpha}_{3,0,0}(t) = C(3, 0)\,Z^{\alpha}_{3,0,0}(t),$$

$$t^{-1}\frac{d}{dt}t^2M^{\alpha}_{3,1,m}(t) = 2C(3, 1)\,Z^{\alpha}_{3,1,m}(t) \quad \text{and}$$

$$t^{-1}\frac{d}{dt}t^3M^{\alpha}_{3,2,m}(t) = 2C(3, 2)\int_0^t u\,dZ^{\alpha}_{3,2,m}(u)$$

for every $t > 0$ in $L^{(\alpha)}$ $(0 < \alpha \leq 2)$. Now it is clear that (∗) is proper if $l = 0, 1$. If $l = 2$, using the equation

$$\int_0^t s^{-2}\left(\int_0^s u\,du\right)ds = \int_0^t\left(1 - \frac{u}{t}\right)du,$$

we have

$$\int_0^t s^{-2}\left(\int_0^s u dZ^{\alpha}_{3,2,m}(u)\right)ds = \int_0^t \left(1-\frac{u}{t}\right)dZ^{\alpha}_{3,2,m}(u)$$

for every $t>0$, where the integration in ds means the limit of Riemannian sum in $L^{(\alpha)}$. The right hand side belongs to $\mathfrak{M}^{\alpha}_t(M^{\alpha}_{3,2,m})$ for every $t>0$ and the kernel is equal to $F_{3,0}(t, u)$ up to a constant. Hence we show that (*) is proper for $l=2$.

ii). For a fixed $t_0>0$, let us compute the inner product between $F_{3,l}(t, u)$ $(0<t\le t_0)$ and u^j $(0\le j\le l-2)$ on $[0, t]$.

$$\int_0^t F_{3,l}(t, u)u^j du = (-1)^{l+1}C(3, l)\int_0^t\left[\frac{d^{l-1}}{dx^{l-1}}(1-x^2)^l\right]\Bigg|_{x=u/t} u^j du \qquad (l\ge 3)$$

$$= const. \times t^{j-1}\left[\frac{d^{l-j-2}}{dx^{l-j-2}}(1-x^2)^l\right]\Bigg|_{x=0}.$$

Using a recurrence property, it can be showed that the value is 0 for all $0<t\le t_0$ if j is even or odd, according as l is odd or even respectively. This implies that $u^{j/(\alpha-1)}$ is right-orthogonal to $F_{3,l}(t, u)$ in $L^{(\alpha)}[0, t]$ $(1<\alpha\le 2)$. We apply Proposition 1.5 and complete the proof of ii).

iii) can be proved similarly to Lemma 2.1. □

LEMMA 2.3. *In case* $n=2$, *then the statements* i) *and* ii) *of Lemma* 2.2 *also hold.* (McKean [9] for $\alpha=2$)

PROOF. i). We already know that

$$F_{2,0}(t, u) = C(2, 0)\cos^{-1}\frac{u}{t},\quad F_{2,1}(t, u) = C(2, 1)\left\{1-\left(\frac{u}{t}\right)^2\right\}^{1/2} \quad \text{and}$$

$$F_{2,2}(t, u) = 3C(2, 2)\frac{u}{t}\left\{1-\left(\frac{u}{t}\right)^2\right\}^{1/2}.$$

And we can show that

$$\int_0^t \frac{s}{t(t^2-s^2)^{1/2}}ds\int_0^s \cos^{-1}\frac{u}{s}du = \frac{\pi}{2}\int_0^t\left(1-\frac{u}{t}\right)du,$$

$$\int_0^t \frac{1}{(t^2-s^2)^{1/2}}ds\int_0^s\left\{1-\left(\frac{u}{s}\right)^2\right\}^{1/2}du = \frac{\pi}{2}\int_0^t\left(1-\frac{u}{t}\right)du \quad \text{and}$$

$$\int_0^t \frac{t}{s(t^2-s^2)^{1/2}}ds\int_0^s \frac{u}{s}\left\{1-\left(\frac{u}{s}\right)^2\right\}^{1/2}du = \frac{\pi}{4}\int_0^t\left\{1-\left(\frac{u}{t}\right)^2\right\}du.$$

Put $dZ^{\alpha}_{2,l,m}(u)$ $(l=0, 1, 2)$ in place of du in these three formulas, where the above integral operators in ds act in $L^{(\alpha)}$. Thus we know that

$$\int_0^t \left(1 - \frac{u}{t}\right) dZ_{2,0,0}^{\alpha}(u) \in \mathfrak{M}_t^{\alpha}(M_{2,0,0}^{\alpha}),$$

$$\int_0^t \left(1 - \frac{u}{t}\right) dZ_{2,1,m}^{\alpha}(u) \in \mathfrak{M}_t^{\alpha}(M_{2,1,m}^{\alpha}) \quad \text{and}$$

$$\int_0^t \left\{1 - \left(\frac{u}{t}\right)^2\right\} dZ_{2,2,m}^{\alpha}(u) \in \mathfrak{M}_t^{\alpha}(M_{2,2,m}^{\alpha})$$

for every $t > 0$. Now we can easily obtain the innovations $\{Z_{2,l,m}^{\alpha}(t)\}$ ($l = 0, 1, 2$), similarly to i) of Lemma 2.2.

ii). It is easily proved that $u^{j/(\alpha-1)}$ $(0 \leq j \leq l-2)$ is right-orthogonal to $F_{2,l}(t, u)$ in $L^{(\alpha)}[0, t]$ $(1 < \alpha \leq 2)$ for any $t > 0$ if j is even or odd, according as l is odd or even respectively, in the same way as the proof of ii) of Lemma 2.2. This implies ii). □

Lemmas 2.1 ~ 2.3 imply the following theorem.

THEOREM 2.4. *Let* $n \geq 2$.

i) *If* $l = 0, 1, 2$, *the causal representation* $(*)$

$$M_{n,l,m}^{\alpha}(t) = \int_0^t F_{n,l}(t, u) dZ_{n,l,m}^{\alpha}(u)$$

is proper for $0 < \alpha \leq 2$.

ii) *If* $l \geq 3$, *the causal representation* $(*)$ *is not proper for* $1 < \alpha \leq 2$.

iii) *If* n *is odd* $(= 2d + 1)$, *then* $M_{n,l,m}^{\alpha}(t)$ *is* d*-times differentiable at* t *in* $L^{(\alpha)}$ $(0 < \alpha \leq 2)$ *for any fixed* $t > 0$. *If* n *is even* $(= 2d)$, *then* $M_{n,l,m}^{\alpha}(t)$ *is* $(d-1)$*-times differentiable at* t *in* $L^{(\alpha)}$ $(0 < \alpha \leq 2)$ *for any fixed* $t > 0$.
(Hida [3] and McKean [9] for $\alpha = 2$)

§ 3. Regularities of paths and canonicalities of representations

In Gaussian case ($\alpha = 2$), to know whether a causal representation is canonical or not, we have only to apply Proposition 1.5 to check whether it is proper or not. On the other hand for non-Gaussian case ($0 < \alpha < 2$), by observing the regularity of paths of the process, we can prove that a causal representation which belongs to a certain class is canonical even if it is not proper (see P. Lévy [8] and T. Hida and N. Ikeda [4]).

3-1 Regularities of paths of certain SαS processes

Firstly, we apply the Lévy-Itô's theorem on the decomposition of paths to an SαS motion.

Let T' be a subinterval in $[0, \infty)$, then $D(T')$ denotes the set of functions which are right continuous and have left limits at all points in T'. If T' is compact, $D(T')$ has a norm of uniformly convergence on T', i.e., $\|f\|_\infty = \sup_{t \in T'} |f(t)|$ for $f \in D(T')$. A stochastic process on T' is called a $D(T')$-process if its almost all paths belong to $D(T')$.

It is well-known that any SαS motion $\{Z_0(t); t \in [0, \infty)\}$ $(0 < \alpha < 2)$ has a $D([0, \infty))$-modification $\{Z_0^D(t, \omega); t \in [0, \infty)\}$ represented by

$$Z_0^D(t, \omega) = \lim_{l \to \infty} \int_{[0,t]} \int_{|y| \geq 1/l} y N(dudy, \omega)$$

where $N(dudy, \omega)$ is a Poisson random measure with control measure $n(dudy) \propto |y|^{-(\alpha+1)} dudy$ on $[0, \infty) \times (\boldsymbol{R} \setminus \{0\})$ and $\lim_{l \to \infty}$ means that almost all $D[0, \infty)$-paths converge on any compact interval. Note that the random variable $N((s, s'] \times E, \omega)$ is equal to the number of jumps with height in E on time interval $(s, s']$ of path $Z_0^D(\cdot, \omega)$ for any s and s' $(s \leq s')$ and any Borel set E of $\boldsymbol{R} \setminus \{0\}$. (For details, see K. Itô [5] and K. Sato [12].)

With the help of this theory, let us consider the regularity of paths of SαS process $\{X(t); t \in [0, \infty)\}$ which is represented by

$$X(t) = \int_0^t F(t, u)\, dZ_0(u). \tag{3.1}$$

Now we regard that the kernel $F(t, u)$ is a function restricted on $\boldsymbol{D}_0 = \{(t, u); t \geq u \geq 0\} \setminus \{(0, 0)\}$. We use the following notations which mean conditions on the kernel.

k1) $F(t, u)$ is continuous on $\boldsymbol{D}_0$.

k2) For any fixed $t > 0$, $F(t, u)$ is differentiable in u on $[0, t]$ and $\frac{\partial}{\partial u} F(t, u)$ is continuous on $\boldsymbol{D}_0$.

k3) $F(t, t)$ is bounded in the neighborhood of $t = 0$.

k4) $\sup_{u \in [0,t]} \left| \frac{\partial}{\partial u} F(t, u) \right| \leq const. \times t^{-1}$ in the neighborhood of $t = 0$.

k5) $F(t, u)$ belongs to C^2 on $\boldsymbol{D}_0$.

k6) $\frac{\partial}{\partial u} F(t, u)$ is bounded in the neighborhood of $(t, u) = (0, 0)$.

To the next lemma, we apply the integration by parts. The idea is borrowed from K. Takashima [15].

LEMMA 3.1. *Assume that the kernel* $F(t, u)$ *satisfies* k1) *and* k2). *For almost all* $D[0, \infty)$*-paths* $Z_0^D(\cdot, \omega)$, *we define a process* $\{X^D(t, \omega); t > 0\}$ *as*

$$X^D(t, \omega) \equiv F(t, t)Z_0^D(t, \omega) - \int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t, u)\right)du. \tag{3.2}$$

Then $\{X^D(t, \omega)\}$ *is a* $D(0, \infty)$*-modification of* $\{X(t)\}$ *given by* (3.1). *And there exists a relation of jumping times and heights between paths* $X^D(\cdot, \omega)$ *and* $Z_0^D(\cdot, \omega)$ *expressed as*

$$X^D(t, \omega) - X^D(t-, \omega) = F(t, t)\{Z_0^D(t, \omega) - Z_0^D(t-, \omega)\} \qquad a.s. \tag{3.3}$$

Moreover, if $F(t, u)$ *satisfies* k3) *and* k4), *then* $X^D(\cdot, \omega)$ *is right continuous at* $t = 0$ *and* $X^D(0, \omega) = 0$.

PROOF. By the conditions k1) and k2), we can regard that the right hand side of (3.2) is defined in the sense of $L^{(\alpha)}$ for every $t > 0$ and we find that the right hand side is a modification of $\{X(t)\}$. The condition k1) implies that $F(t, t)Z_0^D(t, \omega)$ is a $D(0, \infty)$-process. By k2), $\int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t, u)\right)du$ is well-defined and has finite value for all $t > 0$ for almost all $D[0, \infty)$-paths $Z_0^D(\cdot, \omega)$. Let us show that this term is continuous on $(0, \infty)$ as paths. Fix ω, consider the right continuity at $t > 0$. Let $h > 0$.

$$\begin{aligned}
&\int_{[0,t+h]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t+h, u)\right)du - \int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t, u)\right)du \\
&= \int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t+h, u) - \frac{\partial}{\partial u}F(t, u)\right)du \\
&+ \int_{(t,t+h]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t+h, u)\right)du
\end{aligned}$$

converges to 0 as $h \downarrow 0$ by k2). This term is left continuous at $t > 0$ because

$$\begin{aligned}
&\int_{[0,t-h]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t-h, u)\right)du - \int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t, u)\right)du \\
&= \int_{[0,t-h]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t-h, u) - \frac{\partial}{\partial u}F(t, u)\right)du \\
&- \int_{(t-h,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t, u)\right)du
\end{aligned}$$

converges to 0 as $h \downarrow 0$ by k2). Hence we prove that $\{X^D(t, \omega)\}$ is a $D(0, \infty)$-modification of $\{X(t)\}$.

Assume k3) and k4). Then $F(t, t)Z_0^D(t, \omega)$ is right continuous at $t=0$ by k3). And

$$\left|\int_{[0,h]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(h, u)\right)du\right| \le h \sup_{u\in[0,h]} |Z_0^D(u, \omega)| \sup_{u\in[0,h]} \left|\frac{\partial}{\partial u}F(h, u)\right|$$

converges to 0 as $h\downarrow 0$ by k4). Thus we prove the right continuity of $\{X^D(t, \omega)\}$ at $t=0$. □

Now we consider these two special cases.

1°) $F(t, t) \equiv 0$ on $t\in(0, \infty)$,

2°) $F(t, t) \neq 0$ on $t\in(0, \infty)$.

The case 1°). We have the following corollary by the relation (3.3).

COROLLARY 3.2. *If* $F(t, u)$ *satisfies* k1), k2) *and* 1°), *almost all paths* $X^D(\cdot, \omega)$ *are continuous on* $(0, \infty)$.

Furthermore, we can consider the differentiability of paths.

LEMMA 3.3. *If* $F(t, u)$ *satisfies* k5) *and* 1°), *then the paths* $X^D(\cdot, \omega)$ *have right and left derivatives at all* $t>0$ *and they satisfy*

$$\frac{d}{dt_+}X^D(t, \omega) = \frac{\partial}{\partial t'}F(t', u)\Big|_{t'=t,u=t} Z_0^D(t, \omega) - \int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}\frac{\partial}{\partial t}F(t, u)\right)du \tag{3.4}$$

$$\frac{d}{dt_-}X^D(t, \omega) = \frac{\partial}{\partial t'}F(t', u)\Big|_{t'=t,u=t} Z_0^D(t-, \omega) - \int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}\frac{\partial}{\partial t}F(t, u)\right)du \tag{3.5}$$

Moreover, if $F(t, u)$ *satisfies* k6), *the paths* $X^D(\cdot, \omega)$ *are right differentiable at* $t=0$ *and* $\frac{d}{dt_+}X^D(t, \omega)\Big|_{t=0} = 0$.

PROOF. *The right differentiability at* $t>0$; *Let* $h>0$, *then by* k5),

$$\frac{1}{h}\int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u}F(t+h, u) - \frac{\partial}{\partial u}F(t, u)\right)du$$

$$= \int_{[0,t]} Z_0^D(u, \omega)\frac{\partial}{\partial u}\frac{\partial}{\partial t}F(t+\theta h, u)\,du \qquad (\text{where } 0<\theta=\theta(h, t, u)<1)$$

$$\longrightarrow \int_{[0,t]} Z_0^D(u, \omega)\frac{\partial}{\partial u}\frac{\partial}{\partial t}F(t, u)\,du \qquad (h\downarrow 0).$$

On the other hand

$$\frac{1}{h}\int_{(t,t+h]} Z_0^D(u,\omega)\frac{\partial}{\partial u}F(t+h,u)\,du$$

$$\longrightarrow Z_0^D(t,\omega)\frac{\partial}{\partial u'}F(t',u')\Big|_{t'\downarrow t,u'\downarrow t} \qquad (h\downarrow 0).$$

By 1°) and k5), we have

$$0=\frac{d}{dt}F(t,t)=\frac{\partial}{\partial t'}F(t',u')\Big|_{t'=t,u'=t}+\frac{\partial}{\partial u'}F(t',u')\Big|_{t'=t,u'=t},$$

so we obtain (3.4).

The left differentiability at $t>0$; By k5),

$$\frac{1}{-h}\int_{[0,t-h]} Z_0^D(u,\omega)\left(\frac{\partial}{\partial u}F(t-h,u)-\frac{\partial}{\partial u}F(t,u)\right)du$$

$$=\int_{[0,t-h]} Z_0^D(u,\omega)\frac{\partial}{\partial u}\frac{\partial}{\partial t}F(t-\theta h,u)\,du$$

$$\longrightarrow \int_{[0,t)} Z_0^D(u,\omega)\frac{\partial}{\partial u}\frac{\partial}{\partial t}F(t,u)\,du \qquad (h\downarrow 0).$$

Now the interval $[0,t)$ can be replaced by $[0,t]$. And

$$-\frac{1}{-h}\int_{(t-h,t]} Z_0^D(u,\omega)\frac{\partial}{\partial u}F(t,u)\,du$$

$$\longrightarrow Z_0^D(t-,\omega)\frac{\partial}{\partial u'}F(t,u')\Big|_{u'\uparrow t} \qquad (h\downarrow 0).$$

So we obtain (3.5). □

Especially, the paths belong to $C^1(0,\infty)$ if

$$\frac{\partial}{\partial t}F(t,u)\Big|_{u=t}\equiv 0 \qquad \text{on } (0,\infty).$$

The case 2°). For simplicity, we assume $F(t,t)\equiv 1$. Then by (3.3), for any fixed $t>0$, $N((s,s']\times E,\omega)$ can be obtained from $\{X^D(r,\omega);\, r\in\boldsymbol{Q}\cap[0,t]\}$ for any $s, s'(\in\boldsymbol{Q}, 0<s<s'\le t)$ and any Borel set E of $\boldsymbol{R}\setminus\{0\}$. For example, if $E=(y_0,\infty)$ $(y_0>0)$,

$$\{\omega;\, N((s,s']\times(y_0,\infty),\omega)\ge 1\}$$

$$= \bigcup_m \bigcap_n \bigcup_{\substack{r,r' \in Q;\\ s<r<r'\le s',\\ r'-r<1/n}} \{\omega;\, X^D(r',\omega) - X^D(r,\omega) > y_0 + 1/m\}.$$

So, for every $t' \in \boldsymbol{Q}$ $(0 < t' \le t)$ and $l \in \boldsymbol{N}$, we calculate

$$\sum_{k=1}^{n} \{X^D(s_{n,k},\omega) - X^D(s_{n,k-1},\omega)\} N\left((s_{n,k-1}, s_{n,k}] \times \left(-\frac{1}{l}, \frac{1}{l}\right)^c, \omega\right),$$

where $\{s_{n,k} \in \boldsymbol{Q};\, 0 \le k \le n\}$ is a partition of $[0, t']$ and the mesh tends to 0 as $n \to \infty$. As the above random variable converges a.s. as $n \to \infty$ for every t', we regard the limit of variables as a stochastic process whose paths belong to $D([0, t])$ a.s. Taking the limit as $l \to \infty$, we obtain the D-modification $\{Z_0^D(s,\omega);\, s \in [0, t]\}$ of $\{Z_0(s);\, s \in [0, t]\}$ (see K. Itô [5] for reference). Thus we have

PROPOSITION 3.4. *If the kernel satisfies* k1), k2) *and* 2°), *then the causal representation* (3.1) *is canonical* (see P. Lévy [8] and T. Hida and N. Ikeda [4]).

3-2 The canonicalities of the representations (II)

For Gaussian case ($\alpha = 2$), as we saw in Theorem 2.4 of subsection 2–2, the representation $(*)$ is not canonical if $n \ge 2$ and $l \ge 3$ (H. P. McKean Jr. [9]). McKean obtained the proper canonical representations of $\{M^2_{n,l,m}(t)\}$ in these cases. For non-Gaussian case $(0 < \alpha < 2)$, we apply the argument of the previous subsection to SαS $M(t)$-processes and their representations $(*)$.

LEMMA 3.5. *In case* $n = 3$.

i) *For all* l, $\{M^\alpha_{3,l,m}(t)\}$ $(0 < \alpha < 2)$ *has a modification whose paths are continuous on* $[0, \infty)$ *and differentiable in both sides at all* $t > 0$. (*The derivatives are not equal to each other. And* $\{tM^\alpha_{3,l,m}(t)\}$ *has a modification whose paths are right differentiable at* $t = 0$.)

ii) *For all* l, *the causal representation* $(*)$ *of* $\{M^\alpha_{3,l,m}(t)\}$ $(0 < \alpha < 2)$ *is canonical.*

PROOF. i) is proved bacause

$$F_{3,l}(t,u) = (-1)^l C(3,l) \int_x^1 \frac{d^l}{dx^l}(1-x^2)^l\, dx \Big|_{x=u/t}$$

satisfies the conditions k1) $\sim$ k5) and 1°). And

$$\frac{\partial}{\partial t} F_{3,l}(t,u)\Big|_{u=t} \ne 0 \qquad \text{on } (0,\infty),$$

so the right and left derivatives are not equal.

ii). Let us consider the right derivative of the C-modification of $\{M^{\alpha}_{3,l,m}(t)\}$ as a process, then the process satisfies 2°). So we apply Proposition 3.4 to obtain $\{Z^{\alpha}_{3,l,m}(t)\}$. □

LEMMA 3.6. *In case* $n = 2$.

i) *For all* l, $\{M^{\alpha}_{2,l,m}(t)\}$ *has a modification whose paths are continuous on* $[0, \infty)$.

ii) *For all* l, *the causal representation* $(*)$ *of* $\{M^{\alpha}_{2,l,m}(t)\}$ *is canonical.*

PROOF. i). Let us prove $\{M^{\alpha}_{2,l,m}(t)\}$ has a modification in the form of (3.2). The kernel

$$F_{2,l}(t, u) = (-1)^l C(2, l) \int_x^1 \frac{d^l}{dx^l}(1 - x^2)^{l-1/2}\, dx \Big|_{x=u/t}$$

satisfies k1) and 1°) (thus the first term of (3.2) vanishes), and is differentiable in u on $[0, t)$ for every $t > 0$. Note that

$$\frac{\partial}{\partial u} F_{2,l}(t, u) = \left(a \text{ polynomial in } \frac{u}{t}\right) \times \left\{1 - \left(\frac{u}{t}\right)^2\right\}^{-1/2} \frac{1}{t}.$$

So, according as $\frac{\partial}{\partial u} F_{2,l}(t, u) \to \infty$ or $-\infty$ as $u \uparrow t$ (whether the limit is ∞ or $-\infty$ depends only on l.), we have some $\varepsilon = \varepsilon(t, l) > 0$ such that $\frac{\partial}{\partial u} F_{2,l}(t', u)$ increases or decreases monotonously in u and decreases or increases monotonously in t' on $\{(t', u); t - \varepsilon \leq u < t' \leq t + \varepsilon\}$ respectively.
Hence the second term of (3.2)

$$\int_{[0,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u} F_{2,l}(t, u)\right) du, \tag{3.6}$$

where $\{Z_0^D(t, \omega)\}$ is a D-modification of $\{Z^{\alpha}_{2,l,m}(t)\}$, is well-defined for all $t > 0$ because

$$\left| \int_{[t-\varepsilon,t]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u} F_{2,l}(t, u)\right) du \right|$$
$$\leq \sup_{u \in [t-\varepsilon, t]} |Z_0^D(u, \omega)| \left| \int_{t-\varepsilon}^t \frac{\partial}{\partial u} F_{2,l}(t, u)\, du \right| < \infty.$$

Let us prove the right continuity of (3.6) at $t > 0$. Let h be $0 < h < \varepsilon$ then

$$\int_{[0,t-\varepsilon]} Z_0^D(u, \omega)\left(\frac{\partial}{\partial u} F_{2,l}(t + h, u) - \frac{\partial}{\partial u} F_{2,l}(t, u)\right) du \longrightarrow 0 \qquad (h \downarrow 0).$$

By k 1),

$$\left|\int_{(t-\varepsilon,t]} Z_0^D(u,\omega)\left(\frac{\partial}{\partial u}F_{2,l}(t+h,u)-\frac{\partial}{\partial u}F_{2,l}(t,u)\right)du\right|$$
$$\leq \sup_{u\in(t-\varepsilon,t]}|Z_0^D(u,\omega)|\left|\int_{t-\varepsilon}^{t}\left(\frac{\partial}{\partial u}F_{2,l}(t+h,u)-\frac{\partial}{\partial u}F_{2,l}(t,u)\right)du\right| \longrightarrow 0 \qquad (h\downarrow 0),$$

and

$$\left|\int_{(t,t+h]} Z_0^D(u,\omega)\left(\frac{\partial}{\partial u}F_{2,l}(t+h,u)\right)du\right|$$
$$\leq \sup_{u\in[t,t+h]}|Z_0^D(u,\omega)|\left|\int_{t}^{t+h}\frac{\partial}{\partial u}F_{2,l}(t+h,u)\,du\right| \longrightarrow 0 \qquad (h\downarrow 0).$$

To prove the left continuity of (3.6) at $t>0$, we have only to let h be $0<h<\varepsilon$ and prove similarly that

$$\int_{[0,t-\varepsilon]} Z_0^D(u,\omega)\left(\frac{\partial}{\partial u}F_{2,l}(t-h,u)-\frac{\partial}{\partial u}F_{2,l}(t,u)\right)du,$$
$$\int_{(t-\varepsilon,t-h]} Z_0^D(u,\omega)\left(\frac{\partial}{\partial u}F_{2,l}(t-h,u)-\frac{\partial}{\partial u}F_{2,l}(t,u)\right)du \qquad \text{and}$$
$$\int_{(t-h,t]} Z_0^D(u,\omega)\left(\frac{\partial}{\partial u}F_{2,l}(t,u)\right)du$$

converge to 0 as $h\downarrow 0$.

Using the fact that $\int_0^t\left|\frac{\partial}{\partial u}F_{2,l}(t,u)\right|du$ is bounded (constant in fact) in the neighborhood of $t=0$, we show the right continuity at $t=0$. Hence i) is proved.

ii). The proof is similar to i) of Lemma 2.3. We apply an integral operator $t^{-(l-1)}\int_0^t \frac{s^{l-1}}{(t^2-s^2)^{1/2}}ds$ to $\{M_{2,l,m}^{\alpha}(s);\,0<s\leq t\}$ $(l\geq 1)$ and we obtain a new process with a causal representation whose kernel is a polynomial in u/t (like the odd dimensional cases). The kernel of the new process satisfies either 1°) or 2°). In the case 2°), we apply Theorem 3.4 to finish the proof. In the case 1°), we have only to differentiate the process a certain times until 2°) is satisfied. □

If $n\geq 4$, the kernel $F_{n,l}(t,u)$ satisfies k 5) and the reduction formula below (see the proof of Lemma 2.1).

$$\frac{\partial}{\partial t} F_{n+2,l}(t, u) = 2n F_{n,l+1}(t, u) \frac{u}{t^2} \qquad \text{for } n \geq 2.$$

Finally, we have the following theorem.

THEOREM 3.7. *For* $0 < \alpha < 2$.

i) *For all n and l, the causal representation* (∗)

$$M^{\alpha}_{n,l,m}(t) = \int_0^t F_{n,l}(t, u)\, dZ^{\alpha}_{n,l,m}(u)$$

is canonical.

ii) *If n is odd* ($= 2d + 1$, $d \geq 1$), *then* $\{M^{\alpha}_{n,l,m}(t)\}$ *has a modification whose paths belong to* $C^{d-1}(0, \infty)$ *and d-times differentiable in both sides at all* $t > 0$. ($\{t^d M^{\alpha}_{n,l,m}(t)\}$ *has a modification whose paths belong to* $C^{d-1}[0, \infty)$ *and d-times differentiable in both sides at all* $t \geq 0$.) *If n is even* ($= 2d$), *then* $\{M^{\alpha}_{n,l,m}(t)\}$ *has a modification whose paths belong to* $C^{d-1}(0, \infty)$. ($\{t^{d-1} M^{\alpha}_{n,l,m}(t)\}$ *has a modification whose paths belong to* $C^{d-1}[0, \infty)$.)

Let us sum up the results of the path properties of $\{M^{\alpha}_{n,l,m}(t);\, t \geq 0\}$ and the canonicalities of their causal representations (∗) as the following list.

<table>
<tr><td rowspan="2">n</td><td>l</td><td colspan="2">$l = 0, 1, 2$</td><td colspan="3">$l \geq 3$</td></tr>
<tr><td>α</td><td>$0 < \alpha < 2$</td><td>$\alpha = 2$</td><td>$0 < \alpha \leq 1$</td><td>$1 < \alpha < 2$</td><td>$\alpha = 2$</td></tr>
<tr><td rowspan="3">$n = 1$
$(l = 0, 1)$</td><td>paths</td><td>D</td><td>C</td><td colspan="3" rowspan="3">———</td></tr>
<tr><td rowspan="2">(∗)</td><td colspan="2">proper</td></tr>
<tr><td colspan="2">canonical</td></tr>
<tr><td rowspan="3">n: even
$(= 2d)$</td><td>paths</td><td>C^{d-1}</td><td>C^{d-1}</td><td colspan="2">C^{d-1}</td><td>C^{d-1}</td></tr>
<tr><td rowspan="2">(∗)</td><td colspan="2">proper</td><td>unknown</td><td colspan="2">not proper</td></tr>
<tr><td colspan="2">canonical</td><td colspan="2">canonical</td><td>not canonical</td></tr>
<tr><td rowspan="3">n: odd
$(= 2d + 1)$</td><td>paths</td><td>C^{d-1}</td><td>C^{d}</td><td colspan="2">C^{d-1}</td><td>C^{d}</td></tr>
<tr><td rowspan="2">(∗)</td><td colspan="2">proper</td><td>unknown</td><td colspan="2">not proper</td></tr>
<tr><td colspan="2">canonical</td><td colspan="2">canonical</td><td>not canonical</td></tr>
</table>

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Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 724, *Japan*