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The conformal factor and a central extension of a formal loop group with values in $PSL(2, \mathbf{R})$

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Introduction

The main objective of this paper is to investigate a relation between a central extension of the Hauser group, which would be a subgroup of "Geroch group", and the conformal factor of the Einstein vacuum filed equations in a 2-dimensional reduction. The conformal factor is considered to be τ-function (for example see [10] [17]) in case of the Einstein vacuum filed equations. As far as the author knows, approaches in this directions were undertaken by P. Breitenlohner and D. Maison [1], K. Okamoto [16] and B. Julia [9].

On the other hand, "solution generating methods" of the stationary axisymmetric Einstein vacuum equations and the Einstein-Maxwell equations have been drastically investigated since Geroch [6] had found that each given stationary axisymmetric solution of the Einstein field equations are accompanied by an infinite family of potentials. Geroch's observation has led to W. Kinnersley's formulation [11] and to the fact that there exists an action of some infinite dimensional group, so called Geroch group, on the space of solutions. Geroch conjecture was proved affirmatively by I. Hauser and F. J. Ernst [8] following Kinnersley's formulation. In [2] H. Doi and K. Okamoto generalized the results of [8] to the case that the field equations take their values in an affine symmetric space, so that a "Kac-Moody" Lie group acts transitively on the space of solutions. However the action of center of the "Kac-Moody" Lie group was trivial. For another formulation and discussions, for example, see Y. S. Wu and M.L. Ge [21].

In the previous paper [7] (cf. [3] [4]), a σ-model with values in an affine symmetric space, that is, $S(U(1) \times U(2)) \backslash SU(1, 2)$ was formulated with a linearization method explored by P. Breitenlohner and D. Maison [1] and a formal loop group method established by K. Takasaki [18]. And a recipe for constructing solutions was given there, which gives the gravitational field interacting with electro-magnetic fields. But no conformal factor was dealt with.

In this paper, a σ-model with values in $K \backslash PSL(2, \mathbf{R})$ is treated with the

$\mathscr{F}\mathscr{G}$ be a formal loop group with values in $PSL(2, \mathbf{R})$. We define a potential space $\mathscr{S}\mathscr{P}$ and the Hauser group $\mathscr{G}^{(\infty)}$ are realized in subspaces of $\mathscr{F}\mathscr{G}$. Then our result is that $\mathscr{G}^{(\infty)}$ acts transitively on $\mathscr{S}\mathscr{P}$, that is, $\mathscr{S}\mathscr{P}$ is an infinite dimensional homogeneous space.

In order to incorporate the conformal factor, we consider a central extension $\mathscr{G}^{(\infty)}$ by the additive group $\mathbf{R}$ and a central extension of $\mathscr{F}\mathscr{G}$ by the additive formal group $F = \mathbf{R}[[z, \rho]]$.

Then for the total space $E(\mathscr{S}\mathscr{P})$ and the centrally extended Hauser group $\mathscr{G}_{ce}^{(\infty)}$ we have the following commutative diagram for $g_{ce} = (g, e^a) \in \mathscr{G}_{ce}^{(\infty)}$:

$$\begin{array}{ccc} E(\mathscr{S}\mathscr{P}) & \xrightarrow{g_{ce}} & E(\mathscr{S}\mathscr{P}) \\ \pi \downarrow & & \pi \downarrow \\ \mathscr{S}\mathscr{P} & \xrightarrow{g} & \mathscr{S}\mathscr{P}. \end{array}$$

And $\mathscr{G}_{ce}^{(\infty)}$ acts transitively on $E(\mathscr{S}\mathscr{P})$.

In our theory, it should be noticed that all the solutions of the Einstein vacuum filed equations don't correspond to elements of the potential space $\mathscr{S}\mathscr{P}$. In fact, there exists no potential of the Weyl solution (see Section 3).

Finally I would like to express my sincere gratitude to Professor K. Okamoto for his suggestion and to T. Hashimoto for discussions with him.

1. Preliminaries

We shall devote this section to a summary of those concepts and results from the theory of the formal loop groups and the central extensions which are needed for our mathematical formulation in this paper. Most of these results are well known and adequately treated in many papers and books (see for example Takasaki's paper [18] for the formal loop groups and [10] [12] [19] for the general theory of Lie groups and Lie algebras).

Although the theory of the formal loop groups is generally formulated with the coefficient field $\mathbf{C}$, the field $\mathbf{R}$ is used in this paper by the reason why we shall discuss about the formal loop groups with values in real Lie groups. But there is no essential change for the theory (see Lemma 1.2).

As to the notation of map, we employ the following rules:

(a) Variables of maps are eliminated except that it is necessary to write the variables.

(b) The spetral parameter of maps is eliminated if the maps have both of its negative and positive powers. Otherwise we shall write it in order to indicate powers which maps have.
However the exceptions exist also.

1.1. Formal loop groups

Following [18], let A be an associative (in general noncommutative) algebra over $\mathbf{R}$ which has a unit element 1 and a filtration $\{A_l\}_{l\in\mathbf{N}\cup\{0\}}$ satisfying the following conditions:

(*i*) $\{A_l\}_{l\in\mathbf{N}\cup\{0\}}$ is decreasing; $A = A_0 \supset A_1 \supset A_2 \supset \cdots$,

(*ii*) $A_l A_{l'} \subset A_{l+l'}$ for $l, l' \in \mathbf{N}\cup\{0\}$,

(*iii*) For any sequence $a_l \in A_l$ ($l\in\mathbf{N}\cup\{0\}$) there exists a unique element $a\in A$ such that $a - \sum_{l=0}^{n} a_l \in A_{n+1}$ for $\forall n \geq 0$.

And for the convenience of our disussions we extend the filtration by defining

$$A_l = A \qquad \text{for } l < 0.$$

The topology induced from the filtration (i.e. $\{A_l\}_{l\in\mathbf{Z}}$ form a system of neighborhoods of the zero element) is complete. Throughout this paper we consider all formal groups to be endowed with the topology induced from the filtration.

Let A be an associative algebra equipped with the filtration $\{A_l\}_{l\in\mathbf{Z}}$. Then A is called the associative filtered algebra.

Let F be an associative filtered algebra over $\mathbf{R}$ with a filtration $\{F_l\}_{l\in\mathbf{Z}}$ and t be a new variable (so-called spectral parameter). The set $\mathscr{F}LG$ satisfying the following conditions is called a formal loop group:

(1.1.1) $\mathscr{F}LG$ consists of elements $\sum_{l\in\mathbf{Z}} g_l t^l (g_l \in \mathrm{gl}(n, F_l)$ for $l\in\mathbf{Z})$,

(1.1.2) $\mathscr{F}LG$ forms a group with respect to the matrix multiplication in the formal power series category.

For the associative filtered algebra F with the filtration $\{F_l\}_{l\in\mathbf{Z}}$ we define

(1.1.3) $\mathscr{F}L\mathscr{G} = \{g = \sum_{l\in\mathbf{Z}} g_l t^l;\ g_l \in \mathrm{gl}(n, F_l),\ g_0$ is invertible in $\mathrm{gl}(n, F)\}$,

(1.1.4) $\mathscr{F}L\mathscr{B} = \{b(t) = \sum_{l\in\mathbf{Z}} b_l t^l \in \mathscr{F}L\mathscr{G};\ b_l = 0$ for $l < 0\}$,

(1.1.5) $\mathscr{F}L\mathscr{N} = \{n(1/t) = \sum_{l\in\mathbf{Z}} n_l t^l \in \mathscr{F}L\mathscr{G};\ n_l = 0$ for $l > 0,\ n_0 = 1\}$.

Then $\mathscr{F}L\mathscr{G}$, $\mathscr{F}L\mathscr{B}$ and $\mathscr{F}L\mathscr{N}$ become the formal loop groups, and $\mathscr{F}L\mathscr{B}$ and $\mathscr{F}L\mathscr{N}$ are subgroups of $\mathscr{F}L\mathscr{G}$. For the above defined formal loop groups we have the Birkhoff decomposition, that is,

LEMMA 1.1. (*see K. Takasaki* [18])
Any element $g \in \mathscr{F}L\mathscr{G}$ *can be uniuely decomposed as*

$$g = n^{-1}(1/t)b(t), \quad n(1/t) \in \mathscr{F}L\mathscr{N}, \quad b(t) \in \mathscr{F}L\mathscr{B}. \tag{1.1.7}$$

As stated above, we discuss the theory of the formal loop groups by taking $\mathbf{R}$ as a coefficient field. It is easy to see that the decomposition of Lemma 1.1 goes on well in case of the formal loop groups with values in real and special linear Lie groups. However we prepare the following lemma directly applicable to our later discussions, and give its proof.

LEMMA 1.2. *Let* $\mathscr{F}\mathscr{G}$ *be a formal loop group with values in* $SL(n, \mathbf{R})$ *defined by an associative filtered algebra* F *with a filtration* $\{F_l\}_{l \in \mathbf{Z}}$, *and let* $\mathscr{F}\mathscr{B}$ *and* $\mathscr{F}\mathscr{N}$ *be its subgroups defined by the definitions* (1.1.4) *and* (1.1.5), *respectively. Then, any element* g *of* $\mathscr{F}\mathscr{G}$ *can be uniquely decomposed as*

$$g = n^{-1}(1/t)b(t), \quad n(1/t) \in \mathscr{F}\mathscr{N}, \quad b(t) \in \mathscr{F}\mathscr{B}. \tag{1.1.6}$$

PROOF. It is noticed that if the decomposition is possible, its uniqueness holds. Let $\mathscr{F}L\mathscr{G}$, $\mathscr{F}L\mathscr{B}$ and $\mathscr{F}L\mathscr{N}$ be the same formal loop groups in Lemma 1.1. First we prove the decomposability in case of the coefficient field $\mathbf{R}$. Since $\mathbf{R}$ is considered to be included in $\mathbf{C}$, $g \in \mathscr{F}\mathscr{G}$ is an element of $\mathscr{F}L\mathscr{G}$. Therefore, by Lemma 1.1, g is uniquely decomposed as

$$g = n^{-1}(1/t)b(t), \quad n(1/t) \in \mathscr{F}L\mathscr{N}, \quad b(t) \in \mathscr{F}L\mathscr{B}.$$

On the other hand, there is also the unique decomposition $\bar{g} = \bar{n}^{-1}(1/t)\bar{b}(t)$ for $\bar{g} \in \mathscr{F}L\mathscr{G}$. Since $g = \bar{g}$ and the uniqueness of decomposition, we have $n(1/t) = \bar{n}(1/t)$ and $b(t) = \bar{b}(t)$.

Next we prove the decomposition in $\mathscr{F}\mathscr{G}$. Since any element $g \in \mathscr{F}\mathscr{G}$ belongs to $\mathscr{F}L\mathscr{G}$, we have the unique decomposition (1.1.7). Let $\mathscr{F}F$, $\mathscr{F}B$ and $\mathscr{F}N$ be the formal loop groups with respect to the Laurent power series. Then it is clear that $\det : \mathscr{F}L\mathscr{G} \to \mathscr{F}F$ is a well-defined homomorphism. So taking the determinant of the both sides of (1.1.7), we have

$$1 = (\det n(1/t))^{-1} \det b(t)$$

which is the decomposition in $\mathscr{F}F$. Following the trivial decomposition of 1 and its uniqueness in $\mathscr{F}F$, we conclude that $\det n(1/t) = \det b(t) = 1$. □

1.2. Formal loop Lie algebras

Let F and t be the same as in Subsection 1.1. We call the set $\mathscr{F}L\mathfrak{g}$ satisfying the following conditions a formal loop Lie algebra:

(1.2.1) $\mathscr{F}L\mathfrak{g}$ consists of elements $\sum_{l\in\mathbf{Z}} X_l t^l (X_l \in \mathfrak{gl}(n, F_l)$ for $l \in \mathbf{Z})$,

(1.2.2) $\mathscr{F}L\mathfrak{g}$ forms a Lie algebra with the Lie bracket $[X, Y] = XY - XY$ in the formal power series category.

Let F be the associative filtered algebra with the filtration $\{F_l\}_{l\in\mathbf{Z}}$ and let

$$\mathscr{F}L\mathfrak{g} = \{\sum_{l\in\mathbf{Z}} X_l t^l;\ X_l \in \mathfrak{gl}(n, F_l) \quad \text{for} \quad l \in \mathbf{Z}\}.$$

It is obvious from the filtration that the Lie bracket $[X, Y]$ for any two elements X and Y of $\mathscr{F}L\mathfrak{g}$ is well-defined in the formal power series category. Therefore $\mathscr{F}L\mathfrak{g}$ becomes a formal loop Lie algebra over F.

Let $\mathscr{F}L\mathscr{G}$ be the formal loop group defined by (1.1.3). Then, following to the usual matrix exponential, we introduce the exponential map $\mathscr{F}L\mathfrak{g} \to \mathscr{F}L\mathscr{G}$ by defining

$$e^X = 1 + X + \frac{X^2}{2!} \cdots + \frac{X^l}{l!} \cdots \tag{1.2.3}$$

for $X \in \mathscr{F}L\mathfrak{g}$. It is obvious that e^X is well-defined and the exponential gives a map of $\mathscr{F}L\mathfrak{g}$ into the formal loop group $\mathscr{F}L\mathscr{G}$.
Also the formal log map from an open neighborhood of $\mathscr{F}L\mathscr{G}$ around the identity to $\mathscr{F}L\mathfrak{g}$ is defined.

Suppose $\mathscr{F}L\mathfrak{g}$ is any formal loop Lie algebra over F. For any $X \in \mathscr{F}L\mathfrak{g}$, let $\operatorname{ad} X$ denote the endomorphism of $\mathscr{F}L\mathfrak{g}$ given by

$$\operatorname{ad} X\colon Y \longmapsto [X, Y] \qquad (Y \in \mathscr{F}L\mathfrak{g}). \tag{1.2.4}$$

Since the exponential map is a formal map around $0 \in \mathscr{F}L\mathfrak{g}$, so an open neighborhood $\mathscr{F}L\mathfrak{a}$ of 0 and a formal map $C\colon \mathscr{F}L\mathfrak{a} \times \mathscr{F}L\mathfrak{a} \to \mathscr{F}L\mathfrak{g}$ can be found such that

$$\exp X \exp Y = \exp C(X\colon Y) \qquad (X, Y \in \mathscr{F}L\mathfrak{a}).$$

Let $C(vX\colon vY) = \sum_{l=0}^{\infty} c_l(X\colon Y) v^l$.
Then c_l's are given by the recursion formula:

(1.2.5)

$$(l+1)c_{l+1}(X\colon Y) = \frac{1}{2}[X - Y, c_l(X\colon Y)]$$

$$+ \sum_{p\geq 1, 2p\leq l} K_{2p} \sum_{\substack{k_1,\ldots,k_{2p}>0 \\ k_1+\ldots+k_{2p}=l}} [c_{k_1}(X\colon Y), [\cdots, [c_{k_{2p}}(X\colon Y), X + Y]\cdots]]$$

($l \geq 1$, X, $Y \in \mathscr{F}L\mathfrak{a}$) and by the condition $c_1(X: Y) = X + Y$, where K_{2p}'s are defined by

$$f(x) = \frac{x}{1 - e^{-x}} - \frac{1}{2}x = 1 + \sum_{p=1}^{\infty} K_{2p} x^{2p}. \tag{1.2.6}$$

The expression for C is called the Baker-Campbell-Hausdorff (BCH) formula.

It is easily seen from the BCH formula that

$$C(X: vY) = X + \frac{\operatorname{ad} X}{1 - e^{-\operatorname{ad} X}} vY + O(v^2). \tag{1.2.7}$$

Also, there exists a formal loop Lie algebra valued function L: $\mathscr{F}L\mathfrak{a} \times \mathscr{F}L\mathfrak{a} \to \mathscr{F}L\mathfrak{g}$ such that

$$C(X: Y) = X + Y + [X, L(X, Y)] + [Y, L(-Y, -X)], \tag{1.2.8}$$

where we fix L in order to remove the ambiguity, coming from the Jacobi identity.

The existence of L follows from the recursion formula (1.2.5), by letting

$$L(vX, vY) = \sum_{l=0}^{\infty} L_l(X, Y) v^l.$$

From (1.2.7) we get

$$L(X, vY) = \left(\frac{e^{-\operatorname{ad} X} - 1 + \operatorname{ad} X}{\operatorname{ad} X (1 - e^{-\operatorname{ad} X})} - \frac{1}{4} \right) vY + O(v^2). \tag{1.2.9}$$

The first few of c_l and L_l are given by

$$c_2(X: Y) = \frac{1}{2}[X, Y],$$

$$c_3(X: Y) = \frac{1}{12}[[X, Y], Y] - \frac{1}{12}[[X, Y], X]$$

and

$$L_1(X, Y) = \frac{1}{4} Y,$$

$$L_2(X, Y) = \frac{1}{12}[X, Y],$$

$$L_3(X, Y) = \frac{1}{48}[[X, Y], Y].$$

REMARK. *Let $F^{(+)}$ be an associative filtered algebra with the filtration $\{F_l^{(+)}\}_{l\in\mathbf{Z}}$. Then we have another associative filtered algebra F with the opposite filtration defined by $\{F_l^{(-)} = F_{-l}^{(+)}\}_{l\in\mathbf{Z}}$. Also we have the formal loop group defined by this opposite filtration, that is,*

$$\mathscr{F}L\mathscr{G}^{(-)} = \{\sum_{l\in\mathbf{Z}} g_l t^l;\ g_l \in \mathfrak{gl}(n, F_l^{(-)}) \ \ for \ \ l\in\mathbf{Z}\} \tag{1.2.5}$$

The above discussions and results, especially Lemma 1.2, *hold.*

1.3. Central extensions

In the rest of the section we discuss the central extensions of a formal loop Lie algebra and a formal loop group. Fix notations such that F is an associative filtered algebra of formal power series with respect to some variables over $\mathbf{R}$ and $\mathscr{F}\mathscr{G}$ is a formal loop group, and $\mathscr{F}\mathfrak{g}$ is its formal loop Lie algebra. In addition, let us assume that the exponential map: $\mathscr{F}\mathfrak{g} \to \mathscr{F}\mathscr{G}$ is surjective.

Let us begin with a central extension of a formal loop Lie algebra $\mathscr{F}\mathfrak{g}$ by F. To do this we take the Lie algebra 2-cocycle $\omega\colon \mathscr{F}\mathfrak{g} \times \mathscr{F}\mathfrak{g} \to F$ which is as usual defined by

$$\omega(X(t),\ Y(t)) = \frac{1}{2\pi\sqrt{-1}} \oint \operatorname{tr}\left(X(t)\frac{d}{dt}Y(t)\right)dt. \tag{1.3.1}$$

The integration path in (1.3.1) is a closed contour around the origin. Note that the Lie algebra 2-cocycle is well-defined due to the filtration. It should be noticed that the integral of the right hand side of (1.3.1) picks up the coefficient of t^{-1} in the integrand.

By the definition of the Lie algebra 2-cocycle ω we have the following properties:

$$\omega(X,\ Y) = -\,\omega(Y,\ X), \tag{1.3.2}$$

$$\omega([X,\ Y],\ Z) + \omega([Y,\ Z],\ X) + \omega([Z,\ X],\ Y) = 0, \tag{1.3.3}$$

where the relations (1.3.2) and (1.3.3) are called skew symmetricity and Jacobi identity, respectively. Let $\mathscr{F}\mathfrak{g} \oplus_{\mathbf{R}} F$ be the direct sum as a vector space over $\mathbf{R}$. Then defining its Lie bracket in the following manner:

$$[(X,\ \mu),\ (Y,\ \nu)] = ([X,\ Y],\ \omega(X,\ Y)) \quad \text{for} \quad (X,\ \mu),\ (Y,\ \nu) \in \mathscr{F}\mathfrak{g} \oplus_{\mathbf{R}} F,$$

we get the central extension of $\mathscr{F}\mathfrak{g}$ by F.

We now proceed to the central extension of the formal loop group $\mathscr{F}\mathscr{G}$ by the formal ablelian group F.

The central extension is diagramatically expressed by

$$0 \longrightarrow F \longrightarrow \mathscr{F}\mathscr{G}_{ce} \longrightarrow \mathscr{F}\mathscr{G} \longrightarrow 0.$$

Following [1], let us define a group 2-cocycle $\Xi: \mathscr{F}\mathscr{G} \times \mathscr{F}\mathscr{G} \to F$ by

$$\Xi(e^X, e^Y) = \omega(X, L(X, Y)) + \omega(-L(-Y, -X), Y). \tag{1.3.5}$$

Then, the group 2-cocycle Ξ satisfies the following identity

$$\Xi(e^Y, e^Z) - \Xi(e^X e^Y, e^Z) + \Xi(e^X, e^Y e^Z) - \Xi(e^X, e^Y) = 0, \tag{1.3.6}$$

for $X, Y, Z \in \mathscr{F}L\mathfrak{g}$.

Furthermore the group 2-cocycle Ξ has the following relations:

$$\begin{aligned} \Xi(e^X, e^Y) &= -\Xi(e^{-X}, e^X e^Y) \\ &= -\Xi(e^X e^Y, e^{-Y}) \\ &= -\Xi(e^{-Y}, e^{-X}). \end{aligned} \tag{1.3.7}$$

The relations in (1.3.7) are called the anti-symmetric conditions.

By use of the group 2-cocycle Ξ defined above, it is possible to define a central extension $\mathscr{G}_{ce}^{(\infty)}$ of the group $\mathscr{G}^{(\infty)}$ taking pairs $(g_1, e^a), (g_2, e^b) \in \mathscr{G}^{(\infty)} \times F^+$ with the group multiplication

$$(g_1, e^a) \circ (g_2, e^b) = (g_1 g_2, e^{a+b+\Xi(g_1, g_2)}) \tag{1.3.8}$$

which is associative due to the identity (1.3.6).

So far we have defined the formal loop group 2-cocycle Ξ and the Lie algebra 2-cocycle ω. In the rest of this subsection, we prepare two lemmas about a mixed form which play a crucial role for proving our main theorem. The mixed form $\Xi': \mathscr{F}\mathscr{G} \times \mathscr{F}\mathfrak{g} \to F$ is defined by

$$\Xi'(e^X, Y) = \frac{d}{dv}\bigg|_{v=0} \Xi(e^X, e^{vY}) \quad \text{for} \quad X, Y \in \mathscr{F}\mathfrak{g}. \tag{1.3.9}$$

It is noticed that Ξ' is linear with respect to the 2-nd variable.

LEMMA 1.3. (*see* [1]) *Then the mixed form Ξ' has the following expression*:

$$\Xi'(e^X, Y) = -\frac{1}{2\pi\sqrt{-1}} \oint \operatorname{tr}\left(Y\left(\frac{1}{2} + \chi(\operatorname{ad} X)\right)\left(e^{-X}\frac{d}{dt}e^X\right)\right) dt \tag{1.3.10}$$

with the odd function $\chi(x) = \frac{1}{2} \times \frac{\sin h(x) - x}{\cos h(x) - 1}$.

PROOF. The mixed form is calculated as follows:

$$\Xi'(e^X, Y) = \frac{d}{dv}\bigg|_{v=0} (\omega(X, L(X, vY)) + \omega(-L(-vY, -X), vY))$$

using the relation (1.2.9),

$$= -\omega\left(\frac{e^{-\operatorname{ad} X} - 1 + \operatorname{ad} X}{\operatorname{ad} X(1 - e^{-\operatorname{ad} X})}, Y\right)$$

$$= -\frac{1}{2\pi\sqrt{-1}} \oint \operatorname{tr}\left(\left(\frac{e^{-\operatorname{ad} X} - 1 + \operatorname{ad} X}{\operatorname{ad} X(1 - e^{-\operatorname{ad} X})} Y\right) \frac{d}{dt} X\right) dt.$$

Moving the adjoint action to $\frac{d}{dt}X$ in place of Y and using the identity

$$\frac{d}{dt}X = \frac{\operatorname{ad} X}{1 - e^{-\operatorname{ad} X}} e^{-X} \frac{d}{dt} e^X, \tag{1.3.11}$$

we get the following result for the mixed form:

$$\Xi'(e^X, Y) = -\frac{1}{2\pi\sqrt{-1}} \oint \operatorname{tr}\left(Y\left(\frac{e^{\operatorname{ad} X} - 1 - \operatorname{ad} X}{e^{\operatorname{ad} X} + e^{-\operatorname{ad} X} - 2}\right)\left(e^{-X}\frac{d}{dt}e^X\right)\right) dt.$$

Therefore the lemma is proved. □

It is easy to see from the anti-symmetric conditions (1.3.7) that another type of the mixed form is given by

$$\frac{d}{dv}\bigg|_{v=0} \Xi(e^{vX}, e^Y) = \Xi'(e^{-Y}, X). \tag{1.3.12}$$

Finally in the following lemma we have an expression for the derivative of a formal group 2-cocycle with respect to a variable in the associative filtered algebra F.

LEMMA 1.4. *Let F be an associative filtered algebra, whose underlying associative algebra consists of formal power series with respect to some variables over* $\mathbf{R}$, *with a filtration $\{F_l\}_{l\in\mathbf{Z}}$ and let X and Y be the elements of the formal loop Lie algebra $\mathscr{F}L\mathfrak{g}$ corresponding to the filtration. Then the derivative of the 2-cocycle Ξ on the formal loop group $\mathscr{F}L\mathscr{G}$ with respect to one of the variables is described by the following relation*:

$$\begin{aligned}\partial\Xi(e^X, e^Y) &= \Xi'(e^{-Y}e^{-X}, \partial e^X e^{-X}) - \Xi'(e^{-X}, \partial e^X e^{-X}) \\ &\quad + \Xi'(e^X e^Y, e^{-Y}\partial e^Y) - \Xi'(e^Y, e^{-Y}\partial e^Y).\end{aligned} \tag{1.3.13}$$

PROOF. The left hand side of the above relation is calculated as follows:

$$\partial\Xi(e^X, e^Y) = \frac{d}{dv}\bigg|_{v=0} \Xi(e^{X+v\partial X}, e^{Y+v\partial Y})$$

$$= \frac{d}{dv}\bigg|_{v=0} \Xi(e^{X+v\partial X}, e^Y) + \frac{d}{dv}\bigg|_{v=0} \Xi(e^X, e^{Y+v\partial Y}).$$

We show that the first term of the right hand side of the above relation

$$\frac{d}{dv}\bigg|_{v=0} \Xi(e^{X+v\partial X}, e^Y) \tag{1.3.14}$$

is equal to $\Xi'(e^{-Y}e^{-X}, \partial e^X e^{-X}) - \Xi'(e^{-X}, \partial e^X e^{-X})$.

Using the identity

$$\partial X = \frac{1 - e^{\mathrm{ad}\, X}}{\mathrm{ad}\, X} \partial e^X e^{-X},$$

it follows from (1.2.9), (1.3.6) and (1.3.12) that (1.3.14) is calculated as follows:

$$\frac{d}{dv}\bigg|_{v=0} \Xi(e^{X+v\partial X}, e^Y)$$

$$= \frac{d}{dv}\bigg|_{v=0} \Xi(\exp(v\partial e^X e^{-X})e^X, e^Y)$$

$$= \frac{d}{dv}\bigg|_{v=0} \Xi(\exp(v\partial e^X e^{-X}), e^X e^Y) - \Xi(\exp(v\partial e^X e^{-X}), e^X)$$

$$= \Xi'(e^{-Y}e^{-X}, \partial e^X e^{-X}) - \Xi'(e^{-X}, \partial e^X e^{-X}).$$

By the same way we have

$$\frac{d}{dv}\bigg|_{v=0} \Xi(e^X, e^{Y+v\partial Y}) = \Xi'(e^X e^Y, e^{-Y}\partial e^Y) - \Xi'(e^Y, e^{-Y}\partial e^Y).$$

Therefore we have the relation of the lemma for the derivative of Ξ. □

2. 2-dimensional reduction of the Einstein vacuum field equations

In this section we give a starting point of our mathematical discussions in this paper, which comes from a physical motivation. Namely, the starting point consists of definitions of the solution space of the Einstein vacuum field equations in stationary axisymmetric space-times and the conformal factor.

Our space-time manifold is considered to be locally a Lorentzian manifold which has the signature (1, -1, -1, -1). Let $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ be a space-time metric on the manifold. The coordinate x^0 always indicates the time axis t.

Then the Einstein vacuum filed equations (see [13]) for couplings of the gravitational fields without other fields are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \qquad (\mu, \nu = 0, 1, 2, 3), \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor for a 4-dimensional space-time manifold and R is the scalar curvature in the Riemannian geometry, as usual, given by:

$$\begin{aligned} \Gamma^{\beta}_{\mu\nu} &= \frac{1}{2} g^{\beta\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}), \\ R_{\mu\nu} &= \partial_\beta \Gamma^{\beta}_{\mu\nu} - \partial_\nu \Gamma^{\beta}_{\mu\beta} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\kappa}_{\mu\beta} - \Gamma^{\kappa}_{\mu\beta}\Gamma^{\beta}_{\nu\kappa}, \\ R &= g^{\mu\nu} R_{\mu\nu}. \end{aligned}$$

Taking the contraction about the indices μ, ν, we get $R = 0$. So the Einstein vacuum equations become

$$R_{\mu\nu} = 0 \qquad (\mu, \nu = 0, 1, 2, 3). \tag{2.2}$$

In physics, stationary and axially symmetric space-times are characterized by the existence of two independent Killing vector fields, where one is time-like and the other is space-like corresponding to the time translations and the axial rotations, respectively. Suppose that a space-time metric is given. Adapting coordinates t and φ by integrating the Killing vector fields and expressing other two coordinates by z, ρ, we have a metric dependent on z and ρ (independent of t and φ). This is called a 2-dimensional reduction. Since a 2-dimensional space is conformally flat, we can choose coordinates z, ρ so that the metric with respect to z, ρ is diagonal, which is called the conformal gauge.

Thus we assume that the stationary and axially symmetric space-times have the following metric form in cylindrical polar coordinates

$$ds^2 = h_{pq} dx^p \otimes dx^q - \lambda^2 (dz \otimes dz + d\rho \otimes d\rho), \tag{2.3}$$

where the indices p, q run over 0 and 1, and h_{pq} and λ are functions of the variables z, ρ. λ in (2.3) is called the conformal factor, which is assumed to be a positive function (see Definition 2.6).

Let the indices p, q and r take values in 0 or 1, and let a, b be 2 or 3. Then the Einstein vacuum equations are calculated by use of the above metric form as follows:

$$\partial^a(\sqrt{-\det h}\, h^{pr} \partial_a h_{rq}) = 0, \tag{2.4.a}$$

$$(2.4.b) \qquad -\delta_{ab} \Delta \log \lambda - \partial_a \partial_b \log \sqrt{-\det h} + \frac{1}{\lambda}(\partial_a \lambda \partial_b \log \sqrt{-\det h}$$

$$+ \partial_b\lambda\partial_a \log\sqrt{-\det h} - \delta_{ab}\nabla\lambda\cdot\nabla \log\sqrt{-\det h}) + \frac{1}{4}\partial_a h^{pq}\partial_b h_{pq} = 0,$$

where δ_{ab} is the Kronecker's delta and $\partial_2 = \partial/\partial z$, $\partial_3 = \partial/\partial\rho$, $\Delta = \partial_z^2 + \partial_\rho^2$, $\nabla = (\partial_z, \partial_\rho)$.

Only for a technical reason we put h_{pq} to the following matrix form:

$$h = \begin{pmatrix} h_{11} & h_{10} \\ h_{01} & h_{00} \end{pmatrix}.$$

It is noticed that h_{00} is assumed to be positive by a physical requirement, and that $\det h \leq 0$ because of the indefinitness of the space-time metric. Hereafter we put $f = h_{00}$.

Taking the trace of the equation (2.4.a), we get

$$(\partial_z^2 + \partial_\rho^2)(\sqrt{-\det h})^2 = 0.$$

A canonical coordinatization allows to us that $\rho = \sqrt{-\det h}$ can be taken as long as $d\sqrt{-\det h} \neq 0$ (Weyl's canonical coordinates).

Finally we put

$$\tau = \sqrt{f}\,\lambda, \tag{2.5}$$

which is considered to be our τ-function.

From the equations (2.4.b) we get the following two equations (2.6.b) and (2.6.c) by taking the equation (2.6.a) into account.

In summary we have the following equations for the 2-dimensional reduction of the Einstein vacuum field equations:

$$d(\rho^{-1}h\varepsilon * dh) = 0, \tag{2.6.a}$$

$$\tau^{-1}\partial_z\tau = \frac{\partial_z f}{2f} - \frac{\rho}{4}\operatorname{tr}(\partial_z h^{-1}\partial_\rho h), \tag{2.6.b}$$

$$\tau^{-1}\partial_\rho\tau = \frac{\partial_\rho f}{2f} - \frac{1}{2\rho} + \frac{\rho}{8}\operatorname{tr}(\partial_z h^{-1}\partial_z h - \partial_\rho h^{-1}\partial_\rho h), \tag{2.6.c}$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $* =$ Hodge operator for the metric $dz^2 + d\rho^2$.

Since h is symmetric, $\det h = -\rho^2$ and $f > 0$, so we can parametrize h by introducing a new function γ as

$$h = \begin{pmatrix} f\gamma^2 - \rho^2/f & f\gamma \\ f\gamma & f \end{pmatrix}. \tag{2.7}$$

It follows from $\det h = -\rho^2$ and $f > 0$ that

$$\gamma(z, 0) = 0, \tag{2.8.a}$$

$$h(z, 0) = \begin{pmatrix} 0 & 0 \\ 0 & f(z, 0) \end{pmatrix}. \tag{2.8.b}$$

By use of the parametrization of h, the matrix form equation (2.6.a) becomes the following equations:

$$d(\rho^{-1} f^2 * d\gamma) = 0, \tag{2.9.a}$$

$$d(\rho f^{-1} * df + \rho^{-1} f^2 \gamma * d\gamma) = 0. \tag{2.9.b}$$

Now we define our solution space of the Einstein vacuum field equations.

DEFINITION 2.1. *Let $\mathcal{S}E$ denote the set of all formal solutions h of the equation* (2.6.a) *which satisfy the following conditions*:

(i) $h \in \mathrm{gl}(2, \mathbf{R}[[z, \rho]])$,

(ii) $h = {}^t h$, $\det h = -\rho^2$, $f > 0$.

For any given h of the solution space $\mathcal{S}E$ there exists a conformal factor λ, equivalently τ, which is obtained by solving the equations (2.6.b) and (2.6.c). But we postpone this discussion till rewriting these by use of the Ernst potential.

Because thus obtained non-linear differential equations describing the stationary and axially symmetric space-times are not appropriate for applying group theoretical methods, we have to change these equations into another equivalent matrix equation.

First we consider the so-called Ernst potential ψ, which was first introduced by F. J. Ernst (for example see [5]), defined by

$$d\psi = \rho^{-1} f^2 * d\gamma, \tag{2.10}$$

setting $\psi(0, 0) = 0$. The existence of ψ is trivial from the equation (2.9.a), and then we deduce the following equations from the equations (2.9.a) and (2.9.b):

$$d(\rho f^{-2} * d\psi) = 0, \tag{2.11.a}$$

$$d(\rho f^{-1} * df + \rho f^{-2} \psi * d\psi) = 0, \tag{2.11.b}$$

which are called the Ernst equations.

Conversely, we can get the equations (2.9.a) and (2.9.b) from the Ernst equations (see Lemma 2.3).

Let $M(\mathbf{R}[[z, \rho]])$ be s follows:

(2.12)

$$\{m \in \mathrm{gl}(2, \mathbf{R}[[z, \rho]]);\ {}^t m = m,\ \det m = 1,\ \text{the } (2, 2) \text{ component of } m > 0\}.$$

Then, we fix the parametrization of $m\in M(\mathbf{R}[[z,\rho]])$, which is widely used (for example see [1]) by

$$m=\begin{pmatrix} f+\dfrac{\psi^2}{f} & \dfrac{\psi}{f} \\ \dfrac{\psi}{f} & \dfrac{1}{f} \end{pmatrix}. \tag{2.13}$$

Now we describe the Ernst equations (2.11.a) and (2.11.b) in a matrix form.

DEFINITION 2.2. *Let* $M(\mathbf{R}[[z,\rho]])$ *be as above. Then we define the solution space* $\mathscr{S}M$ *by*:

(i) $m\in M(\mathbf{R}[[z,\rho]])$,
(ii) $d(\rho m^{-1}*dm)=0$ *and*
(iii) $\psi(z,0)=0$ *in the parametrization* (2.12).

The defining differential equation of $\mathscr{S}M$ is equal to the differential equations (2.11.a) and (2.11.b). This is easily verified as follows. Taking the (2, 1), (2, 2) and (2, 1) components of the defining differential equation of $\mathscr{S}M$ by use of the parametrization (2.13), we actually get these equations. Its (2, 1) component becomes

$$d(\rho*d\psi-2\rho f^{-1}\psi*df-\rho f^{-2}\psi^2*d\psi)=0 \tag{2.14}$$

It follows from the equations (2.11.a) and (2.11.b) that the equation (2.14) always vanishes.

The lemma below is well-known (for example see [1] [5] [15]).

LEMMA 2.3. *Let* $\mathscr{S}E$ *be the solution space of the Einstein equations in the stationary and axisymmetric space-times and let* $\mathscr{S}M$ *be the solution space of the Ernst equation in the matrix form. Then we have a standard isomorphism*

$$\varepsilon:\mathscr{S}M\xrightarrow{\cong}\mathscr{S}E. \tag{2.15}$$

PROOF. Let m be any element of $\mathscr{S}M$. From the parametrization (2.13) of m we have the equations (2.11.a) and (2.11.b). Let us consider the differential equation

$$d\gamma=-\rho f^{-2}d\psi \tag{2.16}$$

Then it follows from the equations (2.11.a) and (2.11.b) that there exists a unique $\gamma\in\mathbf{R}[[z,\rho]]$ such that $\gamma(z,0)=0$. Also it is easy to show that f and γ satisfies the differential equations (2.9.a) and (2.9.b). So using the parametrization (2.7) we have $h\in\mathscr{S}E$. Hence we put

(2.17) $$\varepsilon(m) = h.$$

It follows from the discussions so far that ε is a bijective map. □

Now we return to the discussion of the conformal factor.
By use of the above introduced Ernst potential ψ, the equations (2.6.*b*) and (2.6.*c*) of τ become

(2.18.*a*) $$\tau^{-1}\partial_z\tau = \frac{\rho}{2f^2}(\partial_z f\partial_\rho f + \partial_z\psi\partial_\rho\psi),$$

(2.18.*b*) $$\tau^{-1}\partial_\rho\tau = \frac{\rho}{4f^2}((\partial_\rho f)^2 - (\partial_z f)^2 + (\partial_\rho\psi)^2 - (\partial_z\psi)^2).$$

And using the matrix m we have a more elegant expression as follows:

(2.19.a) $$\tau^{-1}\partial_z\tau = -\frac{\rho}{4}\operatorname{tr}(\partial_z m^{-1}\partial_\rho m),$$

(2.19.b) $$\tau^{-1}\partial_\rho\tau = -\frac{\rho}{8}\operatorname{tr}(\partial_\rho m^{-1}\partial_\rho m - \partial_z m^{-1}\partial_z m).$$

This is one of the reason why τ is considered to be the τ-function.

As stated above, the existence of the conformal factor is referred to the lemma below.

LEMMA 2.4. *For any element m of the solution space $\mathscr{S}M$ there exists a unique conformal factor τ up to a multiplicative ositive constant, which satisfies the equations* (2.19.*a*) *and* (2.19.*b*).

PROOF. In order to prove the existence of τ, we have only to show that the 1-form

$$-\frac{\rho}{4}\operatorname{tr}(\partial_z m^{-1}\partial_\rho m)\,dz - \frac{\rho}{8}\operatorname{tr}(\partial_\rho m^{-1}\partial_\rho m - \partial_z m^{-1}\partial_z m)d\rho$$

is a closed form under the assumption that m satisfies the equations (2.19.*a*) and (2.19.*b*). So taking the exterior differentiation of the above 1-form, we immediately conclude that

$$\partial_\rho(2\operatorname{tr}(m^{-1}\partial_z m\rho m^{-1}\partial_\rho m))$$
$$- \partial_z(\rho\operatorname{tr}(m^{-1}\partial_\rho m m^{-1}\partial_\rho m - m^{-1}\partial_z m m^{-1}\partial_z m))$$

is equal to zero by means of the equations.

Therefore the existence of the conformal factor τ is proved. The remained is obvious by expressing $\tau^{-1}d\tau = d\log\tau$. □

DEFINITION 2.5. *Let* Λ *be* $\{\tau = \sum_{a,b=0}^{\infty} \tau_{ab} z^a \rho^b \in \mathbf{R}[[z, \rho]]; \tau_{00} = 1\}$. *Then for any given solution* $m \in \mathscr{S}M$ *we define the mapping*

$$\eta : \mathscr{S}M \longrightarrow \Lambda \tag{2.20}$$

and call $\lambda \in \Lambda$ *the conformal factor in a strictly meaning.*

Hereafter whenever the conformal factor is referred to, it is an element of Λ.

Finally we remark that the Minkowski space-time, which has the metric in the cylindrical polar coordinate

$$ds^2 = dt \otimes dt - \rho^2 d\varphi \otimes d\varphi - dz \otimes dz - d\rho \otimes d\rho, \tag{2.21}$$

is explicitly expressed by

$$\begin{aligned} h_e &= \begin{pmatrix} -\rho^2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{S}E, \\ m_e &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \in \mathscr{S}M, \\ \tau_e &= 1 \qquad\quad \in \Lambda. \end{aligned} \tag{2.22}$$

3. Linearization and Potentials

In this section, we discuss a linearization, and define $\mathscr{S}\mathscr{P}$ and $\mathscr{S}P$ in the category of formal power series which are our main concerns.

Let $SL(2, \mathbf{R})$ denote the group of all real 2×2 matrices with determinant 1 and let I_2 its identity element. The center of $SL(2, \mathbf{R})$ consists of matrices $\pm I_2$. So we take $G = PSL(2, \mathbf{R}) \equiv SL(2, \mathbf{R})/\{\pm I_2\}$ as a target group, which is isomorphic, as a Lie group, to $SO_o(1, 2)$. And let θ be the Cartan involution defined by $\theta(g) = {}^t g^{-1}$ for $g \in G$. Then a maximal compact subgroup of G is given by $K = \{g \in G; \theta(g) = g\}$.

Hence we have an Iwasawa decomposition (for example see [12]) of G:

$G = KAN$ with

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}; a > 0\right\},$$

$$N = \left\{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}; x \in \mathbf{R}\right\}$$

and $K =$ the above defined maximal compact subgroup of G. For the convenience of our discussions we rewrite this decomposition as $G \cong K \times AN$, and put α to use the mapping from G onto AN through it.

Let $G(\mathbf{R}[[z, \rho]]) = PSL(2, \mathbf{R}[[z, \rho]])$ be the formal group of all 2×2 matrices with determinant 1 modulo center, whose entries lie in the formal power series $\mathbf{R}[[z, \rho]]$ with respect to the the variables z and ρ. Taking the same one as the Cartan involution, we can go to the parallel discussions in the above finite dimensional case by the properties of $\mathbf{R}[[z, \rho]]$. So, corresponding to the Iwasawa decomposition, we have the decomposition

$$G(\mathbf{R}[[z, \rho]]) \cong K(\mathbf{R}[[z, \rho]]) \times AN(\mathbf{R}[[z, \rho]]) \tag{3.1}$$

where $K(\mathbf{R}[[z, \rho]])$ and $AN(\mathbf{R}[[z, \rho]])$ are the formal groups with values in K and AN, respectively. Hereafter we consider G, K and AN to be naturally embedded into $G(\mathbf{R}[[z, \rho]])$, $K(\mathbf{R}[[z, \rho]])$ and $AN(\mathbf{R}[[z, \rho]])$, respectively. And we as usual employ the following parametrization for the element P in $AN(\mathbf{R}[[z, \rho]])$ (for example see [15]):

$$P = \begin{pmatrix} \sqrt{f} & 0 \\ \dfrac{\psi}{\sqrt{f}} & \dfrac{1}{\sqrt{f}} \end{pmatrix}, \tag{3.2}$$

which is called the triangular representation.

Now we give a correspondence between $AN(\mathbf{R}[[z, \rho]])$ and $M(\mathbf{R}[[z, \rho]])$ in which the latter was defined in Section 2. First for any element m of $M(\mathbf{R}[[z, \rho]])$ we can construct an element P of $AN(\mathbf{R}[[z, \rho]])$ by means of the following manner:

$$M(\mathbf{R}[[z, \rho]]) \ni m = \begin{pmatrix} f + \dfrac{\psi^2}{f} & \dfrac{\psi}{f} \\ \dfrac{\psi}{f} & \dfrac{1}{f} \end{pmatrix} \longmapsto P = \begin{pmatrix} \sqrt{f} & 0 \\ \dfrac{\psi}{\sqrt{f}} & \dfrac{1}{\sqrt{f}} \end{pmatrix} \in AN(\mathbf{R}[[z, \rho]])$$

Conversely, a map

$$\bar{\theta}\colon AN(\mathbf{R}[[z, \rho]]) \longrightarrow M(\mathbf{R}[[z, \rho]])$$

is given by defining $\bar{\theta}(P) = \theta(P^{-1})P$ for $P \in AN(\mathbf{R}[[z, \rho]])$. It is clear that $\bar{\theta}$ is a bijective map.

Hence the solution space $\mathscr{S}M$ of the Ernst equation (see Definition 2.2) is equivalently translated into the equation of $AN(\mathbf{R}[[z, \rho]])$.

DEFINITION 3.1. *Fix the above paramerlization of* $AN(\mathbf{R}[[z, \rho]])$. *Then we define the solution space* $\mathscr{S}P$, *which is equivalent to* $\mathscr{S}M$, *by*

(3.3) $\mathscr{S}P = \{P \in AN(\mathbf{R}[[z, \rho]]);\ d(\rho P^{-1}\theta(P) * d(\theta(P^{-1})P)) = 0,\ \psi(z, 0) = 0\}.$

Therefore we have

$$\bar{\theta}|_{\mathscr{S}P}\colon \mathscr{S}P \longrightarrow \mathscr{S}M.$$

Hereafter a restriction of a map is, for simplicity, written by the same symbol of the map. That is, $\bar{\theta}|_{\mathscr{S}P}$ is written as $\bar{\theta}$.
We remark that it is easily seen from

$$d(\rho P^{-1}\theta(P) * d(\theta(P^{-1})P)) = d(\rho P^{-1} * dP) - d(\rho P^{-1}\theta(* dPP^{-1})P),$$

that the defining equation of $\mathscr{S}P$ is equivalent to

$$\begin{aligned} &d(\rho * dPP^{-1}) - d(\rho\theta(* dPP^{-1})) \\ &\qquad + \rho(dPP^{-1} \wedge \theta(* dPP^{-1}) - \theta(dPP^{-1}) \wedge * dPP^{-1}) = 0. \end{aligned} \tag{3.4}$$

Let α be the map from $G(\mathbf{R}[[z, \rho]])$ to $AN(\mathbf{R}[[z, \rho]])$ through the decomposition (3.1). We denote by $\bar{\alpha}$ the map $K(\mathbf{R}[[z, \rho]]) \backslash G(\mathbf{R}[[z, \rho]]) \to AN(\mathbf{R}[[z, \rho]])$ induced from α. Then an action of $G(\mathbf{R}[[z, \rho]])$ on $AN(\mathbf{R}[[z, \rho]])$ is defined such that for $g \in G(\mathbf{R}[[z, \rho]])$ the following diagram is commutative:

$$\begin{array}{ccc} K(\mathbf{R}[[z, \rho]]) \backslash G(\mathbf{R}[[z, \rho]]) & \xrightarrow{g} & K(\mathbf{R}[[z, \rho]]) \backslash G(\mathbf{R}[[z, \rho]]) \\ \bar{\alpha}\downarrow & & \downarrow\bar{\alpha} \\ AN(\mathbf{R}[[z, \rho]]) & \longrightarrow & AN(\mathbf{R}[[z, \rho]]). \end{array} \tag{3.5}$$

Now it is easy to see from the defining equation (3.4) of $\mathscr{S}P$ that for any element $g \in G$ we have the following commutative diagram:

$$\begin{array}{ccc} K(\mathbf{R}[[z, \rho]]) \backslash K(\mathbf{R}[[z, \rho]])\mathscr{S}P & \xrightarrow{g} & K(\mathbf{R}[[z, \rho]]) \backslash K(\mathbf{R}[[z, \rho]])\mathscr{S}P \\ \bar{\alpha}\downarrow & & \downarrow\bar{\alpha} \\ \mathscr{S}P & \longrightarrow & \mathscr{S}P. \end{array} \tag{3.6}$$

This action of G, which gives no intrinsic change for the metric, is called the gauge transformation.

Let $F^{(+)}$ be an associative filtered algebra of $\mathbf{R}[[z, \rho]]$ with a filtration

$$\{F_l^{(+)} = \rho^{\max(l, 0)}\mathbf{R}[[z, \rho]]\}_{l \in \mathbf{Z}}. \tag{3.7}$$

Then the formal loop groups $\mathscr{F}\mathscr{G}^{(+)}$ and $\mathscr{F}\mathscr{P}^{(+)}$ are defined, introducing a new parameter t (see Section 1), as follows:

$$\mathscr{F}\mathscr{G}^{(+)} = \{g = \sum_{l \in \mathbf{Z}} g_l t^l;\ g_l \in \mathrm{gl}(2, F_l^{(+)}),\ \det g = 1\}/\{\pm I_2\}, \tag{3.8}$$

and its subgroup

$$\mathscr{F}\mathscr{P}^{(+)} = \{\mathscr{P}(t) = \sum_{l=0}^{\infty} P_l t^l \in \mathscr{F}\mathscr{G}^{(+)};\ P_0 \in AN(\mathbf{R}[[z, \rho]])\}\,. \tag{3.9}$$

We remark that $\mathscr{F}\mathscr{P}^{(+)}$ will be identified with $\mathscr{F}\mathscr{P}$ (see Section 5).

Let *proj* be a map from $\mathscr{F}\mathscr{G}^{(+)}$ to $G(\mathbf{R}[[z, \rho]])$ defined by

$$proj\colon \mathscr{F}\mathscr{G}^{(+)} \ni g = \sum_{l\in\mathbf{Z}} g_l t^l \longmapsto g_0 \in G(\mathbf{R}[[z, \rho]]).$$

Hence we have $proj\colon \mathscr{F}\mathscr{P}^{(+)} \to AN(\mathbf{R}[[z, \rho]])$.

Let $\wedge^l\mathbf{R}[[z, \rho]]$ denote the set of all the exterior differential l-forms in the variables z and ρ whose coefficients lie in $\mathbf{R}[[z, \rho]]$, where l is a non-negative integer. And we also denote by $d\colon \wedge^l\mathbf{R}[[z, \rho]] \to \wedge^{l+1}\mathbf{R}[[z, \rho]]$ the exterior differentiation.

Let us consider $\mathbf{R}[[z, \rho]][\rho^{-1}]$. Then it is clear that $\mathbf{R}[[z, \rho]][\rho^{-1}]$ is an associative algebra over $\mathbf{R}$ with the addition and multiplication in the category of formal power series. And we also denote by $\wedge^l\mathbf{R}[[z, \rho]][\rho^{-1}]$ and d the set of all the l-forms and the exterior differentiation in $\wedge^l\mathbf{R}[[z, \rho]][\rho^{-1}]$.

Then the exterior differentiation

(3.10)

$$d\colon \mathfrak{gl}(2, \wedge^l\mathbf{R}[[z, \rho]][\rho^{-1}] \otimes \mathbf{R}[[t]]) \longrightarrow \mathfrak{gl}(2, \wedge^{l+1}\mathbf{R}[[z, \rho]][\rho^{-1}] \otimes \mathbf{R}[[t]])$$

is defined as follows.

We have only to give a d operation for $1 \otimes t$, which is one of the generators of $\mathbf{R}[[z, \rho]][\rho^{-1}] \otimes \mathbf{R}[[t]]$, that is,

$$dt = \frac{t}{\rho(1 + t^2)}((1 - t^2)d\rho + 2t\,dz). \tag{3.11}$$

It follows from the definition (3.11) that $dd = 0$. Therefore d is well-defined as an exterior differentiation. It should be noticed that

$$d\colon \mathscr{F}\mathscr{P}^{(+)} \longrightarrow \mathfrak{gl}(2, \wedge^1\mathbf{R}[[z, \rho]] \otimes \mathbf{R}[[t]])$$

is a well-defined correspondence from (3.11) and the filtration of $\mathscr{F}\mathscr{P}^{(+)}$.

Let $\mathfrak{g}(\mathbf{R}[[z, \rho]]) = \mathfrak{sl}(2, \mathbf{R}[[z, \rho]])$ be the Lie algebra of $G(\mathbf{R}[[z, \rho]])$, i.e.,

$$\mathfrak{g}(\mathbf{R}[[z, \rho]]) = \{X \in \mathfrak{gl}(2, \mathbf{R}[[z, \rho]]);\ \mathrm{tr}(X) = 0\}.$$

Let θ be the Cartan involution defined in the beginning of this section. We also denote by θ the involution of $\mathfrak{g}(\mathbf{R}[[z, \rho]])$ induced from the Cartan involution of $G(\mathbf{R}[[z, \rho]])$, that is, $\theta(X) = -{}^tX$ for $X \in \mathfrak{g}(\mathbf{R}[[z, \rho]])$. Then we have the Cartan decomposition $\mathfrak{g}(\mathbf{R}[[z,\rho]])=\mathfrak{k}(\mathbf{R}[[z,\rho]])\oplus\mathfrak{p}(\mathbf{R}[[z,\rho]])$ of the formal Lie algebra $\mathfrak{g}(\mathbf{R}[[z,\rho]])$.

Then let us start the discussions of a linearization of the defining equation (3.4) of $\mathscr{S}P$.

First, following [1], we introduce a 1-form with the spectral parameter.

DEFINITION 3.2. *For* $P\in AN(\mathbf{R}[[z,\rho]])$, *let* $\mathscr{A}\in\mathfrak{k}(\wedge^1\mathbf{R}[[z,\rho]])$ *and* $\mathscr{I}\in\mathfrak{p}(\wedge^1\mathbf{R}[[z,\rho]])$ *be the* $\mathfrak{k}$ *and* $\mathfrak{p}$ *valued* 1-*forms defined by*

$$\mathscr{A}=\frac{1}{2}(dPP^{-1}+\theta(dPP^{-1})),\qquad \mathscr{I}=\frac{1}{2}(dPP^{-1}-\theta(dPP^{-1})).$$

A $\mathfrak{g}$-*valued* 1-*form* Ω_P *for* P *is defined by*

$$\Omega_P=\mathscr{A}+\frac{1-t^2}{1+t^2}\mathscr{I}-\frac{2t}{1+t^2}*\mathscr{I}. \tag{3.12}$$

Using the paramerization (3.2) of P, we have an explicit expression of Ω_P as follows:

$$\begin{aligned}\Omega_P&=\frac{1}{1+t^2}((1-t*)dPP^{-1}+t(*+t)\theta(dPP^{-1}))\\ &=\frac{1}{2f(1+t^2)}\begin{pmatrix}(1-t^2-2t*)df & -2(t^2+t*)d\psi\\ 2(1-t*)d\psi & -(1-t^2-2t*)df\end{pmatrix}.\end{aligned}\tag{3.13}$$

Next we introduce the potential space designated by $\mathscr{S}\mathscr{P}$, whose compatibility condition recovers the defining equation of $\mathscr{S}P$ (this will be proved in Proposition 3.4).

DEFINITION 3.3. *Let* $\mathscr{F}\mathscr{P}^{(+)}$ *be the formal loop group defined in* (3.9). *We define* $\mathscr{S}\mathscr{P}$ *to be the set of all elements* $\mathscr{P}(t)=\sum_{m=0}^{\infty}P_mt^m$ *of* $\mathscr{F}\mathscr{P}^{(+)}$ *satisfying the following conditions*:

(*i*) $d\mathscr{P}(t)=\Omega_P\mathscr{P}(t)$,

(*ii*) $\mathscr{P}(t)|_{\rho=0}=\begin{pmatrix}\sqrt{f(z,0)} & 0\\ 0 & 1/\sqrt{f(z,0)}\end{pmatrix}$,

where we put $P=P_0$ *and used the parametrization* (3.2) *of* P.

It is noted that $\Omega_P|_{t=0}=dPP^{-1}$ ensures $\mathscr{P}(0)=P$.

The defining equation (*i*) of $\mathscr{S}\mathscr{P}$ is explicitly expressed by

$$\partial_z \mathscr{P}(t) + \frac{2t^2}{\rho(1+t^2)} \partial_t \mathscr{P}(t) = \Omega_{P,z} \mathscr{P}(t), \tag{3.14.a}$$

$$\partial_\rho \mathscr{P}(t) + \frac{t(1-t^2)}{\rho(1+t^2)} \partial_t \mathscr{P}(t) = \Omega_{P,\rho} \mathscr{P}(t), \tag{3.14.b}$$

where $\Omega_{P,z}$ and $\Omega_{P,\rho}$ are the coefficients of dz and $d\rho$ in Ω_P, respectively.

Remind us that the map *proj* is defined on $\mathscr{S}\mathscr{P}$ as follows:

$$proj\colon \mathscr{S}\mathscr{P} \ni \mathscr{P}(t) = \sum_{l=0}^{\infty} P_l t^l \longmapsto P_0 \in AN(\mathbf{R}[[z, \rho]]).$$

Then we have the proposition below, which states that *proj* is considered to be a map into $\mathscr{S}P$.

PROPOSITION 3.4. *Let $\mathscr{P}(t)$ be any element of the potential space $\mathscr{S}\mathscr{P}$. Then $proj(\mathscr{P}(t))$ is an element of $\mathscr{S}P$.*

PROOF. For a given $\mathscr{P}(t) \in \mathscr{S}\mathscr{P}$, put $P = proj(\mathscr{P}(t))$. Note that 1-form Ω_P satisfies the integrability condition:

$$d\Omega_P - \Omega_P \wedge \Omega_P = 0. \tag{3.15}$$

Using (3.11) and (3.13), we obtain

$$\begin{aligned} d\Omega_P - \Omega_P \wedge \Omega_P = -\frac{t}{(1+t^2)\rho}&(d(\rho * dPP^{-1}) - d(\rho\theta(*dPP^{-1})) \\ &+ \rho(dPP^{-1} \wedge \theta(*dPP^{-1}) - \theta(dPP^{-1}) \wedge *dPP^{-1})). \end{aligned}$$

Therefore P belongs to $\mathscr{S}P$. □

It should be noticed that $proj\colon \mathscr{S}\mathscr{P} \to \mathscr{S}P$ is not surjective. For example, let us consider the simplest Weyl solution (see [20]) given by

$$\begin{aligned} ds_w^2 = \exp\left(-\frac{2}{\sqrt{z^2+\rho^2}}\right) dt \otimes dt - \rho^2 \exp\left(\frac{2}{\sqrt{z^2+\rho^2}}\right) d\varphi \otimes d\varphi \\ - \exp\left(\frac{2}{\sqrt{z^2+\rho^2}} - \frac{\rho^2}{(z^2+\rho^2)^2}\right)(dz \otimes dz + d\rho \otimes d\rho) \end{aligned}$$

Then a simple calculation gives us the following candidate for the potential:

$\mathscr{P}_w(t)$

$$= \begin{pmatrix} \exp\left(\dfrac{-\rho(1+t^2)}{\sqrt{z^2+\rho^2}(\rho-\rho^2 t^2+2zt)}\right) & 0 \\ 0 & \exp\left(\dfrac{\rho(1+t^2)}{\sqrt{z^2+\rho^2}(\rho-\rho^2 t^2+2zt)}\right) \end{pmatrix}.$$

But we obviously find that $\mathscr{P}_w(t)$ is not an element of $\mathscr{S}\mathscr{P}$ because of the mismatch in the filtration, even if we shift the z variable in order to remove the singularity at the origin.

In summary, from Proposition 3.4 and the discussions so far we have the following well-defined diagram:

$$\mathscr{S}\mathscr{P} \xrightarrow{proj} \mathscr{S}P \xrightarrow{\theta} \mathscr{S}M \xrightarrow{\varepsilon} \mathscr{S}E. \tag{3.16}$$

For the sake of later discussions we change the linealized equations into two other equivalet forms. The first one is described the following lemma.

LEMMA 3.5. *The defining equations* (3.14.*a*) *and* (3.14.*b*) *of the potential space* $\mathscr{S}\mathscr{P}$ *are equivalent to the following equations*:

$$\partial_t \mathscr{P}(t) + \rho\left(\partial_z + \frac{1}{t}\partial_\rho\right)\mathscr{P}(t) = \frac{\rho}{2f}\tilde{\Omega}_0 \mathscr{P}(t), \tag{3.17.a}$$

$$\partial_t \mathscr{P}(t) + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z \mathscr{P}(t) = \frac{\rho}{2f}\tilde{\Omega}_1 \mathscr{P}(t), \tag{3.17.b}$$

where

$$\tilde{\Omega}_0 = \begin{pmatrix} -\partial_z f + \dfrac{1}{t}\partial_\rho f & -2\partial_z \psi \\ \dfrac{2}{t}\partial_\rho \psi & \partial_z f - \dfrac{1}{t}\partial_\rho f \end{pmatrix}, \tag{3.18.a}$$

$$\tilde{\Omega}_1 = \begin{pmatrix} -\dfrac{1}{2}\left(1 - \dfrac{1}{t^2}\right)\partial_z f + \dfrac{1}{t}\partial_\rho f & -\partial_z \psi + \dfrac{1}{t}\partial_\rho \psi \\ \dfrac{1}{t^2}\partial_z \psi + \dfrac{1}{t}\partial_\rho \psi & \dfrac{1}{2}\left(1 - \dfrac{1}{t^2}\right)\partial_z f - \dfrac{1}{t}\partial_\rho f \end{pmatrix}. \tag{3.18.b}$$

PROOF. From the equations (3.14.*a*) and (3.14.*b*) we immediately get the above equations. □

The other equivalent form is obtained by the replacement $t \to -\dfrac{1}{t}$, under which the solution space $\mathcal{S}P$ is invariant, for the equations (3.17.*a*) and (3.17.*b*). It is easy to see that

$$(3.19.a)\qquad \partial_t\mathscr{P}\left(-\frac{1}{t}\right)+\rho\left(\frac{1}{t^2}\partial_z-\frac{1}{t}\partial_\rho\right)\mathscr{P}\left(-\frac{1}{t}\right)=\frac{\rho}{2f}\tilde{\Omega}_2\mathscr{P}\left(-\frac{1}{t}\right),$$

$$(3.19.b)\qquad \partial_t\mathscr{P}\left(-\frac{1}{t}\right)+\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z\mathscr{P}\left(-\frac{1}{t}\right)=\frac{\rho}{2f}\tilde{\Omega}_3\mathscr{P}\left(-\frac{1}{t}\right),$$

where

$$(3.20.a)\qquad \tilde{\Omega}_2=\begin{pmatrix} -\frac{1}{t^2}\partial_z f-\frac{1}{t}\partial_\rho f & -\frac{2}{t^2}\partial_z\psi \\ -\frac{2}{t}\partial_\rho\psi & \frac{1}{t^2}\partial_z f+\frac{1}{t}\partial_\rho f \end{pmatrix},$$

$$(3.20.b)\qquad \tilde{\Omega}_3=\begin{pmatrix} \frac{1}{2}\left(1-\frac{1}{t^2}\right)\partial_z f-\frac{1}{t}\partial_\rho f & -\frac{1}{t^2}\partial_z\psi-\frac{1}{t}\partial_\rho\psi \\ \partial_z\psi-\frac{1}{t}\partial_\rho\psi & -\frac{1}{2}\left(1-\frac{1}{t^2}\right)\partial_z f+\frac{1}{t}\partial_\rho f \end{pmatrix}.$$

Let $\mathscr{P}(t)=\sum_{l=0}^{\infty}P_l t^l=\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t)\end{pmatrix}$ with

$$a(t)=\sum_{l=0}^{\infty}a_l t^l,\ b(t)=\sum_{l=1}^{\infty}b_l t^l,\ c(t)=\sum_{l=1}^{\infty}c_l t^l \quad\text{and}\quad d(t)=\sum_{l=0}^{\infty}d_l t^l.$$

Since $\mathscr{P}(t)$ is determined by P_0 as an element of the potential space, it is possible to write down $P_l(l\geq 1)$ by use of P_0. To prove Proposition 5.9 we calculate only P_1. It follows from the parametrization of P_0 that

$$(3.21)\qquad a_0=\sqrt{f},\quad c_0=\psi/\sqrt{f},\quad d_0=1/\sqrt{f}.$$

Simple calculations for (3.17.*a*) tells us that the following differential equations for each component of P_1 are satisfied:

$$(3.22.\text{a})\qquad \partial_z\left(\frac{\rho a_1}{\sqrt{f}}\right)=\frac{\rho}{f^2}(f\partial_\rho f+\psi\partial_\rho\psi),$$

$$(3.22.\text{b})\qquad \partial_z\left(\frac{\rho b_1}{\sqrt{f}}\right)=\frac{\rho}{f^2}\partial_\rho\psi,$$

$$(3.22.\text{c})\qquad \partial_z(\rho\sqrt{f}c_1)=\frac{\rho}{f^2}(f\partial_\rho\psi-\psi\partial_\rho f+\sqrt{f}\partial_z\psi a_1),$$

(3.22.d) $$\partial_z(\rho\sqrt{f}\,d_1) = \frac{\rho}{f^2}(-\partial_\rho f + \sqrt{f}\,\partial_z\psi b_1),$$

and for the equation (3.17.*b*)

(3.23.a) $$\partial_\rho\left(\frac{\rho a_1}{\sqrt{f}}\right) = -\frac{\rho}{f^2}(f\partial_z f + \psi\partial_z\psi),$$

(3.23.b) $$\partial_\rho\left(\frac{\rho b_1}{\sqrt{f}}\right) = -\frac{\rho}{f^2}\partial_z\psi,$$

(3.23.c) $$\partial_\rho(\rho\sqrt{f}\,c_1) = -\frac{\rho}{f^2}(f\partial_z\psi - \psi\partial_z f + \sqrt{f}\,\partial_\rho\psi a_1),$$

(3.23.d) $$\partial_\rho(\rho\sqrt{f}\,d_1) = -\frac{\rho}{f^2}(-\partial_z f + \sqrt{f}\,\partial_\rho\psi b_1).$$

The last condition for the components of P_1 comes from the fact that $\det\mathscr{P}(t) = 1$, which is given by

(3.24) $$a_1 = \psi b_1 - f d_1.$$

Finally we comment that the Minkowski potential in the potential space $\mathscr{SP}$ is

(3.25) $$\mathscr{P}_e = I_2.$$

4. Action of the Hauser group

The aim of this section is to give an action of the Hauser group on the potential space $\mathscr{SP}$ defined in Section 3. The formulation of its action is that of a homogeneous space in the finite dimensional Lie group theory.

DEFINITION 4.1. *Let* $\mathscr{G}^{(\infty)} = PSL(2, \mathbf{R}[[s]])$ *be an infinite dimensional group*

$$\{g(s) \in \mathrm{gl}(2, \mathbf{R}[[s]]);\ \det g(s) = 1\}/\{\pm I_2\},$$

where $\mathbf{R}[[s]]$ *is the associative algebra of formal power series in* s *over* $\mathbf{R}$. *We call* $\mathscr{G}^{(\infty)}$ *the Hauser group.*

It is noticed that the parameter s differs from the one in the papers [1] [7], in which it is the inverse of that in this paper.

Let $\{F_l^{(+)}\}_{l\in\mathbf{Z}}$, $\{F_l^{(+)}\}_{l\in\mathbf{Z}}$ and $\{F_l\}_{l\in\mathbf{Z}}$ be filtrations of the associative algebra of the formal power series $\mathbf{R}[[z, \rho]]$ over $\mathbf{R}$ given by

$F_l^{(+)} = \rho^{\max(l,0)}\mathbf{R}[[z, \rho]]$ (the filtration introduced in Section 3),

$F_l^{(-)} = F_{-l}^{(+)}$ (the opposite filtation of $F^{(+)}$) and

$F_l = \rho^{|l|}\mathbf{R}[[z, \rho]]$ (a two-sided filtration).

Then $F^{(+)}$, $F^{(-)}$ and F denote the associative filtered algebras with the filtration $\{F_l^{(+)}\}_{l\in\mathbf{Z}}$, $\{F_l^{(-)}\}_{l\in\mathbf{Z}}$ and $\{F_l\}_{l\in\mathbf{Z}}$, respectively.

So we define the following formal loop groups for the associative filtered algebras $F^{(+)}$ and $F^{(-)}$:

$$\mathscr{F}\mathscr{G}^{(\pm)} = \{g^{(\pm)} = \sum_{l\in\mathbf{Z}} g_l^{(\pm)} t^l;\ g_l^{(\pm)} \in \mathrm{gl}(2, F_l^{(\pm)}),\ \det g^{(\pm)} = 1\}/\{\pm I_2\},$$

and its subgroups

$$\mathscr{F}\mathscr{N}^{(\pm)} = \{n^{(\pm)}(1/t) = \sum_{l\in\mathbf{Z}} n_l^{(\pm)} t^l \in \mathscr{F}\mathscr{G}^{(\pm)};\ n_l^{(\pm)} = 0 (l > 0),\ n_0^{(\pm)} = 1\},$$

$$\mathscr{F}\mathscr{B}^{(\pm)} = \{b^{(\pm)}(t) = \sum_{l\in\mathbf{Z}} b_l^{(\pm)} t^l \in \mathscr{F}\mathscr{G}^{(\pm)};\ b_l^{(\pm)} = 0 (l < 0)\}.$$

Moreover we define the formal loop group $\mathscr{F}\mathscr{G} = \mathscr{F}\mathscr{G}^{(+)} \cap \mathscr{F}\mathscr{G}^{(-)}$, which has the two-sided filtration and is explicitly expressed as

$$\mathscr{F}\mathscr{G} = \{g = \sum_{l\in\mathbf{Z}} g_l t^l;\ g_l \in \mathrm{gl}(2, F_l),\ \det g = 1\}/\{\pm I_2\}, \tag{4.1}$$

and define its subgroups $\mathscr{F}\mathscr{N}$ and $\mathscr{F}\mathscr{B}$ with the filtration $\{F_l\}_{l\in\mathbf{Z}}$ as we did $\mathscr{F}\mathscr{N}^{(\pm)}$ and $\mathscr{F}\mathscr{B}^{(\pm)}$.

We apply Lemma 1.2 to $\mathscr{F}\mathscr{G}^{(+)}$ and $\mathscr{F}\mathscr{G}^{(-)}$. Since $\mathscr{F}\mathscr{G}$ is the subgroup of $\mathscr{F}\mathscr{G}^{(+)}$ and $\mathscr{F}\mathscr{G}^{(-)}$, it immediately follows from these that we have the following Birkhoff decomposition.

LEMMA 4.2. *Let the notations be as above. Then, any element g of $\mathscr{F}\mathscr{G}$ can be uniquely decomposed as*

$$g = n^{-1}(1/t)b(t), \quad n(1/t) \in \mathscr{F}\mathscr{N}, \quad b(t) \in \mathscr{F}\mathscr{B}.$$

To formulate an action of the Hauser group $\mathscr{G}^{(\infty)}$, we embed it into the formal loop group $\mathscr{F}\mathscr{G}$ by use of the homomorphism:

$$j\colon \mathscr{G}^{(\infty)} \longrightarrow \mathscr{F}\mathscr{G}, \tag{4.2}$$

given by substituting $\rho\left(\frac{1}{t} - t\right) + 2z$ into the parameter s in elements of $\mathscr{G}^{(\infty)}$. Then, it is easy to see that this formal group homomorphism is well-defined and injective. Hence we identify the Hauser group $\mathscr{G}^{(\infty)}$ with the subgroup $\mathscr{F}\mathscr{H} = \mathrm{Im}\,(j)$ of the formal loop group $\mathscr{F}\mathscr{G}$. From the special substitution for s, we have the following lemma, which characterizes the elements of $\mathscr{F}\mathscr{H}$ in $\mathscr{F}\mathscr{G}$.

LEMMA 4.3. *Let g be an element of the formal loop group $\mathscr{F}\mathscr{G}$. Then g is an element of its subgroup $\mathscr{F}\mathscr{H}$ if and only if the following equations are satisfied*:

$$\partial_t g = -\rho\left(\partial_z + \frac{1}{t}\partial_\rho\right)g, \tag{4.3.a}$$

$$\partial_t g = -\frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z g. \tag{4.3.b}$$

PROOF. Let g be an element of $\mathscr{F}\mathscr{H}$. Then, since g is expressed by a formal power series of the parameter $s = \rho\left(\frac{1}{t} - t\right) + 2z$ over $\mathbf{R}$, we immediately get the equations (4.3.a) and (4.3.b) by appropriately differentiating the generators $s^l (l \geq 0)$ with respect to the variables t, z and ρ.

Conversely, let us assume that $g \in \mathscr{F}\mathscr{G}$ satisfies the equations (4.3.a) and (4.3.b). Then, we can consider g to belong to $\mathbf{R}[[t, t^{-1}, z, \rho]]$ without a loss of generality.

Expanding g as follows:

$$g = \sum_{l \in \mathbf{Z}} g_l(z, \rho)\rho^{|l|}t^l \qquad (g_l \in \mathbf{R}[[z, \rho]]),$$

we make a formal power series $g'(s) = \sum_{l=0}^{\infty} c_l s^l$ such that $c_l = g_l(0, 0)$ for $l \geq 0$.

So we prove that $w \equiv g - g'\left(\rho\left(\frac{1}{t} - t\right) + 2z\right) = 0$, that is to say, if

$$w = \sum_{l \in \mathbf{Z}} w_l(z, \rho)\rho^{|l|}t^l \begin{cases} w_l \in z\mathbf{R}[[z, \rho]] + \rho\mathbf{R}[[z, \rho]] & \text{for } l \geq 0, \\ w_l \in \mathbf{R}[[z, \rho]] & \text{for } l < 0 \end{cases}$$

satisfies the equations (4.3.a) and (4.3.b), then w is equal to zero.

We divide the proof of the above statement into two steps, that is, the first stands for $l \geq 0$ and the last for $l < 0$.

Step 1. We expand w_l for each $l (\geq 0)$ as follows:

$$w_l(z, \rho) = \sum_{i,j=0}^{\infty} w_{l,ij}\rho^i z^j.$$

Then, we express w_l in the matrix form

$$\begin{pmatrix} 0 & w_{l,01} & w_{l,02} & \cdots \\ w_{l,10} & w_{l,11} & w_{l,12} & \cdots \\ w_{l,20} & w_{l,21} & w_{l,22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where the (0, 0) component is zero by the assumption, and the down and right directions are indicated to the powers with respect to ρ and z, respectively.

We now show the following:

(i) Arbitrarily fix $i_0 \geq 0$ and $j_0 > 0$. If $w_{l,ij} = 0$ for $0 \leq i \leq i_0, 0 \leq j \leq j_0$ and $l \geq 0$, then $w_{l,i_0(j_0+1)} = 0$.

(ii) Arbitrarily fix $i_0 > 0$ and $j_0 > 0$. If $w_{l,ij} = 0$ for $0 \leq i \leq i_0, 0 \leq j \leq j_0$ and $l \geq 0$, then $w_{l,(i_0+1)j_0} = 0$.

Taking the coefficients of $t^l (l \geq 0)$ in the equations (4.3.a) and $2 \times$ (4.3.b), we have

(4.4.a) $$\partial_z w_l = -2(l+1)w_{l+1} - \rho\partial_\rho w_{l+1},$$

(4.4.b) $$\partial_z w_l = -2(l+1)w_{l+1} - \rho^2\partial_z w_{l+2}, \text{ respectively.}$$

Let us assume the conditions of (i). Then it is clear that $w_{l,i_0(j_0+1)} = 0$ for $l \geq 0$, by taking the coefficient of $\rho^{i_0} z^{j_0}$ in (4.4.a).

Let us assume the conditions of (ii). Since $\partial_\rho w_{l+1} = \rho\partial_z w_{l+2}$ from the relations (4.4.a) and (4.4.b), we have $w_{l,(i_0+1)j_0} = 0$ for $l \geq 0$ also by taking its coefficient of $\rho^{i_0} z^{j_0}$.

Therefore we can conclude that $w_l(z, \rho) = 0$ for $l \geq 0$ in full chase of the matrices with (i) and (ii).

Step 2. Next we show that $w = \sum_{l=1}^{\infty} w_{-l} t^{-l}$ vanishes under the asumption that w satisfies the equations (4.3.a) and (4.3.b).

Taking the coefficients of $t^{-l} (l \geq 1)$ in he equations (4.3.a) and $2 \times$ (4.3.b), we have also

(4.5.a) $$\rho\partial_z w_{-l} = -\partial_\rho w_{-l+1},$$

(4.5.b) $$2(l-1)w_{-l+1} = \rho^2\partial_z w_{-1} + \partial_z w_{-l+2}, \text{ respectively.}$$

Then it follows from these relations by shifting $l \to l+1$ that

(4.6) $$2lw_{-l} = -\rho\partial_\rho w_{-l} + \partial_z w_{-1+1} \qquad \text{for } l \geq 1.$$

We prove $w_{-l} = 0$ for $l \geq 1$ by induction. For the case $l = 1$ the relation (4.6) becomes

(4.7) $$w_{-1} = -\frac{\rho}{2}\partial_\rho w_{-1}.$$

It is clear from the ρ factor in the right hand side of (4.7) that if w_{-1} had a non-zero solution of this equation, $w_{-1} \in \rho\mathbf{R}[[z, \rho]]$. On the other hand, from the equation (4.5.a) in case of $l = 1$, we have $\rho\partial_z w_{-1} = 0$. This implies $w_{-1} \in \mathbf{R}[[\rho]]$. So we conclude from the equation (4.7) that w_{-1} is equal to zero. Hence the case $l = 1$ is O. K. Suppose $l > 1$. Then it follows from the assumption of induction that we have also $w_{-l} = -\frac{\rho}{2l}\partial_z w_{-l}$. Just in the same way as the case $l = 1$, w_{-1} belongs to $\mathbf{R}[[\rho]]$. Since w_{-l} is a solution of the equation $w_{-l} = -\frac{\rho}{2l}\partial_z w_{-l}$, w_{-l} vanishes. This completes the proof of this lemma. □

In order to construct an infinite dimensional homogeneous space analogous to the finite dimensional Lie group case, we need an involutive automorphism of $\mathscr{F}\mathscr{G}$, a "maximal compact" subgroup and a decomposition of $\mathscr{F}\mathscr{G}$. Now let us start this discussions.

First we introduce an involutive automorphism $\theta^{(\infty)}$ of $\mathscr{F}\mathscr{G}$ by

(4.8) $$\theta^{(\infty)}\colon \mathscr{F}\mathscr{G} \ni g(t) \longmapsto \theta\left(g\left(-\frac{1}{t}\right)\right) \in \mathscr{F}\mathscr{G},$$

which is well-defined and satisfies $\theta^{(\infty)} \circ \theta^{(\infty)} = identity$ as at once checked, and is called the Cartan involution, too.

By use of the Cartan involution we define the subgroup of $\mathscr{F}\mathscr{G}$ such that

(4.9) $$\mathscr{F}\mathscr{K} = \{k \in \mathscr{F}\mathscr{G};\ \theta^{(\infty)}(k) = k\},$$

corresponding to the maximal compact subgroup in case of the finite dimensional theory.

Let $AN([[z, \rho]])$ be the set of the formal power series with values in AN of the Iwasawa decomposition and let

(4.10) $$\mathscr{F}\mathscr{P} = \{\mathscr{P}(t) = \sum_{l=0}^{\infty} P_l t^l \in \mathscr{F}\mathscr{G};\ P_0 \in AN(\mathbf{R}[[z, \rho]])\}$$

as in Section 3.

Then it is easy to see that the following lemma holds.

LEMMA 4.4. *Let the notations be as above.*
Then,

(4.11) $$\mathscr{F}\mathscr{K} \cap \mathscr{F}\mathscr{P} = \{1\}.$$

PROOF. The case in the finite dimensional has no problem. Otherwise any element of $\mathscr{F}\mathscr{K}$ has negative power terms of the spectral parameter t. $\square$

DEFINITION 4.5. *Let* $g = \sum_{l\in\mathbf{Z}} g_l(z, \rho)t^l$ *be an element of the formal loop group* $\mathscr{F}\mathscr{G}$. *If* $g_0(0, 0)$ *is positive definite, then we say that* g *is positive definite. So is the Hauser group* $\mathscr{G}^{(\infty)}$.

We define subspaces of $\mathscr{F}\mathscr{G}$ and $\mathscr{G}^{(\infty)}$ as follows:

$$\mathscr{M}(\mathscr{F}\mathscr{G}) = \{g \in \mathscr{F}\mathscr{G};\ \theta^{(\infty)}(g^{-1}) = g,\ g \text{ is positive definite}\} \tag{4.12.a}$$

and

$$\mathscr{M}(\mathscr{G}^{(\infty)}) = \{g \in \mathscr{G}^{(\infty)};\ \theta(g^{-1}) = g,\ g \text{ is positive definite}\}. \tag{4.12.b}$$

The following lemma and proposition are included in the proof of Theorem 4.1 in [7].

LEMMA 4.6. *Let* g *be any element of* $\mathscr{M}(\mathscr{F}\mathscr{G})$. *Then we have a unique decomposition of* g *as follows*:

$$g = \theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t) \qquad (\mathscr{P}(t) \in \mathscr{F}\mathscr{P}). \tag{4.13}$$

PROOF. Since $g \in \mathscr{M}(\mathscr{F}\mathscr{G})$ belongs to $\mathscr{F}\mathscr{G}$, it follows from applying Lemma 4.2 to g that we have the following unique decomposition:

$$g = n(1/t)b(t) \quad (n(1/t) \in \mathscr{F}\mathscr{N},\ b(t) \in \mathscr{F}\mathscr{B}).$$

Expressed $b(t) = b_0 b_1(t)$, in which b_0 is the leading term of $b(t)$, we calculate

$$\begin{aligned}\theta^{(\infty)}(g^{-1}) &= \theta^{(\infty)}(b_1(t)^{-1} b_0^{-1} n(1/t)^{-1}) \\ &= \theta(b_1(-1/t)^{-1})\theta(b_0^{-1})\theta(n(-t)^{-1}).\end{aligned}$$

Since $\theta(b_1(-1/t)^{-1}) \in \mathscr{F}\mathscr{N}$ and $\theta(b_0^{-1})\theta(n(-t)^{-1}) \in \mathscr{F}\mathscr{B}$, we get the decomposition of $\theta^{(\infty)}(g^{-1})$. Hence, since $\theta^{(\infty)}(g^{-1}) = g$, it follows from the uniqueness of the Birkhoff decomposition that we get the equalities

$$\begin{aligned}n(1/t) &= \theta(b_1(-1/t)^{-1}), \\ b_0 b_1(t) &= \theta(b_0^{-1})\theta(n(-t)^{-1}).\end{aligned}$$

Hence $\theta(b_0^{-1}) = b_0$ holds. In addition, taking it into account that

$$b_0 = \theta^{(\infty)}((b_1(t)^{-1})^{-1}) g b_1(t)^{-1}$$

is also positive definite, b_0 can be uniquely decompose as

$$b_0 = \theta(p_0^{-1})p_0 \quad \text{for} \quad p_0 \in AN(\mathbf{R}[[z, \rho]]).$$

Therefore we have the unique decomposition:

$$g = \theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t) \qquad (\mathscr{P}(t) \in \mathscr{F}\mathscr{P}),$$

where we put $\mathscr{P}(t) = p_0 b_1(t)$. $\square$

PROPOSITION 4.7. *The formal loop group $\mathscr{F}\mathscr{G}$ is uniquely decomposed as*

$$\mathscr{F}\mathscr{G} = \mathscr{F}\mathscr{K}\,\mathscr{F}\mathscr{P}. \tag{4.14}$$

PROOF. First we show that any element $g \in \mathscr{F}\mathscr{G}$ belongs to $\mathscr{F}\mathscr{K}\,\mathscr{F}\mathscr{P}$. To do this, let us consider the following map

$$\delta : \mathscr{F}\mathscr{G} \longrightarrow \mathscr{M}(\mathscr{F}\mathscr{G})$$

defined by $\delta(x) = \theta(x^{-1})x$ for $x \in \mathscr{F}\mathscr{G}$. Put $m = \delta(g)$. Then, since m has the property:

$$\begin{aligned}\theta^{(\infty)}(m^{-1}) &= \theta^{(\infty)}((\theta^{(\infty)}(g^{-1})g)^{-1}) \\ &= \theta^{(\infty)}(g^{-1}\theta^{(\infty)}(g)) \\ &= \theta^{(\infty)}(g^{-1})g \\ &= m\end{aligned}$$

and m is positive definite, it follows that δ is well-defined.

From Lemma 4.6 we have

$$m = \theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t) \qquad (\mathscr{P}(t) \in \mathscr{F}\mathscr{P}). \tag{4.15}$$

Now let $k = g\mathscr{P}(t)^{-1}$. Then, from $k \in \mathscr{F}\mathscr{G}$ and $\theta^{(\infty)}(k) = k$ we conclude that $k \in \mathscr{F}\mathscr{K}$. Hence we get the decomposition of $g \in \mathscr{F}\mathscr{G}$ into $k\mathscr{P}(t) \in \mathscr{F}\mathscr{K}\,\mathscr{F}\mathscr{P}$.

Finally we prove the uniqueness of the decomposition. Suppose that we have an another pair $k' \in \mathscr{F}\mathscr{K}$ and $\mathscr{P}'(t) \in \mathscr{F}\mathscr{P}$ such that $g = k'\mathscr{P}'(t)$. Then $k'^{-1}k = \mathscr{P}'(t)\mathscr{P}(t)^{-1}$. On the other hand, by Lemma 4.4, we have $k'^{-1}k = \mathscr{P}'(t)\mathscr{P}(t)^{-1} = 1$. Therefore the uniqueness is proved. $\square$

From Proposition 4.7 we have the following decomposition:

$$\mathscr{F}\mathscr{G} \cong \mathscr{F}\mathscr{K} \times \mathscr{F}\mathscr{P}. \tag{4.16}$$

Let α be the map: $\mathscr{F}\mathscr{G} \to \mathscr{F}\mathscr{P}$ through the decomposition (4.16). We denote by $\bar{\alpha}$ the map from $\mathscr{F}\mathscr{K} \backslash \mathscr{F}\mathscr{G}$ to $\mathscr{F}\mathscr{P}$ induced from α. Then for any $g \in \mathscr{F}\mathscr{G}$ we define an action on $\mathscr{F}\mathscr{P}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{F}\mathscr{K} \backslash \mathscr{F}\mathscr{G} & \xrightarrow{\ g\ } & \mathscr{F}\mathscr{K} \backslash \mathscr{F}\mathscr{G} \\ \bar{\alpha}\downarrow & & \downarrow\bar{\alpha} \\ \mathscr{F}\mathscr{P} & \longrightarrow & \mathscr{F}\mathscr{P}. \end{array} \tag{4.17}$$

For the action of $g \in \mathscr{F}\mathscr{G}$ on $\mathscr{F}\mathscr{P}$ we use g as a notation, that is,

$$g: \mathscr{F}\mathscr{P} \longrightarrow \mathscr{F}\mathscr{P}.$$

LEMMA 4.8. *Let $\alpha: \mathscr{F}\mathscr{G} \to \mathscr{F}\mathscr{P}$ be as above and let $\mathscr{P}(t)$ be any element of $\mathscr{F}\mathscr{P}$ such that*

$$\theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t) \in \mathscr{F}\mathscr{H}.$$

Then $\mathscr{P}(t) \in \alpha(\mathscr{F}\mathscr{H})$.

PROOF. Let j be the injective homomorphism defined by (4.2). Then, since $\theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t) \in \mathscr{F}\mathscr{H}$, we have $g_P = j^{-1}(\theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t)) \in \mathscr{G}^{(\infty)}$. Now let us consider the following map

$$\delta: \mathscr{G}^{(\infty)} \longrightarrow \mathscr{M}(\mathscr{G}^{(\infty)}),$$

defined by $\delta(g) = \theta(g^{-1})g$ for $g \in \mathscr{G}^{(\infty)}$. Then it is clear that δ is well-defined. If δ is surjective, there exists g'_P such that $\delta(g'_P) = g_P$. Since $j(g'_P) \in \mathscr{F}\mathscr{H} \subset \mathscr{F}\mathscr{G}$, it follows from Lemma 4.6 that $\alpha(j(g'_P))$ is equal to $\mathscr{P}(t)$. Therefore we have only to prove that δ is surjective. From a direct calculation we find that it is true. □

PROPOSITION 4.9. *For any element $\mathscr{P}(t)$ of the potential space $\mathscr{F}\mathscr{P}$, the product*

$$\theta^{(\infty)}(\mathscr{P}^{-1}(t))\mathscr{P}(t) \tag{4.18}$$

belongs to the Hauser group $\mathscr{F}\mathscr{H}$ which is the subgroup of the formal loop group $\mathscr{F}\mathscr{G}$.

PROOF. Since $\mathscr{P}(t) \in \mathscr{F}\mathscr{G}$, $\theta^{(\infty)}(\mathscr{P}^{-1}(t))\mathscr{P}(t)$ is also an element of $\mathscr{F}\mathscr{G}$. Then, due to Lemma 4.3, in order to prove the proposition we check only to satisfy the following equations:

$$\left(\partial_t + \rho\left(\partial_z + \frac{1}{t}\partial_\rho\right)\right)(\theta^{(\infty)}(\mathscr{P}^{-1}(t))\mathscr{P}(t)) = 0, \tag{4.19.a}$$

$$\left(\partial_t + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z\right)(\theta^{(\infty)}(\mathscr{P}^{-1}(t))\mathscr{P}(t)) = 0, \tag{4.19.b}$$

on the assumption that $\mathscr{P}(t)$ satisfies the equations (3.17.a) and (3.17.b), or equivalently (3.19.a) and (3.19.b).

First we show the equality of (4.19.a). To do this we make a relation about the derivative $\partial_t + \rho\left(\partial_z + \frac{1}{t}\partial_\rho\right)$ of $\theta^{(\infty)}(\mathscr{P}^{-1}(t))$ by subtracting the equation (3.19.b) from the twice of the equation (3.19.a):

$$\partial_t \mathscr{P}\left(-\frac{1}{t}\right)+\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)\mathscr{P}\left(-\frac{1}{t}\right)=\frac{\rho}{2f}(2\tilde{\Omega}_3-\tilde{\Omega}_2)\mathscr{P}\left(-\frac{1}{t}\right). \tag{4.20}$$

Hence, using (3.17.a), the left hand side of (4.19.a) becomes

$$\frac{\rho}{2f}\theta(\mathscr{P}^{-1}(-1/t))\tilde{\Omega}_0\mathscr{P}(t)+\frac{\rho}{2f}\theta(\mathscr{P}^{-1}(-1/t))^t(2\tilde{\Omega}_3-\tilde{\Omega}_2)\mathscr{P}(t).$$

It follows from ${}^t(2\tilde{\Omega}_3-\tilde{\Omega}_2)=-\tilde{\Omega}_0$ that the above equation vanishes.

Next the left hand side of the equation (4.19.b) becomes, by use of the equations (3.17.b), (3.19.a) and (3.19.b),

$$\frac{\rho}{2f}\theta(\mathscr{P}^{-1}(-1/t))\tilde{\Omega}_1\mathscr{P}(t)+\frac{\rho}{2f}\theta(\mathscr{P}^{-1}(-1/t))^t\tilde{\Omega}_3\mathscr{P}(t).$$

Since $\tilde{\Omega}_0=-{}^t\tilde{\Omega}_3$, the above equation also vanishes.

Therefore $\theta^{(\infty)}(\mathscr{P}^{-1}(\mathrm{t}))\mathscr{P}(t)$ is an element of $\mathscr{F}\mathscr{H}$ under the assumption that $\mathscr{P}(t)\in\mathscr{S}\mathscr{P}$. □

It follows from Proposition 4.9 and Lemma 4.8 that any $\mathscr{P}(t)\in\mathscr{S}\mathscr{P}$ is given by the action of some element of $\mathscr{F}\mathscr{H}$ to the Minkowski potential $\mathscr{P}_e$, which is the identity element of $\mathscr{F}\mathscr{G}$, that is, $\mathscr{S}\mathscr{P}$ is included in $\mathscr{F}\mathscr{K}\mathscr{F}\mathscr{H}$.

Conversely, we have the following theorem.

THEOREM 4.10. *Let $\mathscr{F}\mathscr{H}$ be the Hauser group embedded into the formal loop group $\mathscr{F}\mathscr{G}$.*
Then, $\mathscr{S}\mathscr{P}\mathscr{F}\mathscr{H}\subset\mathscr{F}\mathscr{K}\mathscr{S}\mathscr{P}$. Therefore for any element g of $\mathscr{F}\mathscr{H}$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{F}\mathscr{K}\backslash\mathscr{F}\mathscr{K}\mathscr{S}\mathscr{P} & \xrightarrow{g} & \mathscr{F}\mathscr{K}\backslash\mathscr{F}\mathscr{K}\mathscr{S}\mathscr{P} \\ \bar{\alpha}\downarrow & & \downarrow\bar{\alpha} \\ \mathscr{S}\mathscr{P} & \longrightarrow & \mathscr{S}\mathscr{P}. \end{array} \tag{4.21}$$

PROOF. In order to prove the theorem, from the discussion above, it is sufficient to show that for any element g of $\mathscr{F}\mathscr{H}$ $\alpha(g)$ is an element of $\mathscr{S}\mathscr{P}$. Put $m=\theta^{(\infty)}(g^{-1})g$ for an arbitrarily given $g\in\mathscr{F}\mathscr{H}$. Then, since m belongs to $\mathscr{F}\mathscr{H}$, the equations of Lemma 4.3

$$\partial_t m=-\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)m, \tag{4.22.a}$$

$$\partial_t m=-\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z m \tag{4.22.b}$$

are satisfied. It is noticed that applying Lemma 4.6 to $m \in \mathscr{M}(\mathscr{G}^{(\infty)}) \subset \mathscr{M}(\mathscr{F}\mathscr{G})$ is uniquely expressed by $\theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t)$, using $\mathscr{P}(t)=\alpha(g)\in\mathscr{F}\mathscr{P}$.

So, the equations (4.22.a) and (4.22.b) equivalently become

$$\begin{aligned}(4.23.a)\quad &\left(\partial_t\mathscr{P}(t)+\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)\mathscr{P}(t)\right)\mathscr{P}(t)^{-1}\\ &=-\theta(\mathscr{P}(-1/t))\left(\partial_t\theta(\mathscr{P}(-1/t)^{-1})+\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)\theta(\mathscr{P}(-1/t)^{-1})\right)\end{aligned}$$

$$\begin{aligned}(4.23.b)\quad &\left(\partial_t\mathscr{P}(t)+\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z\mathscr{P}(t)\right)\mathscr{P}(t)^{-1}\\ &=-\theta(\mathscr{P}(-1/t))\left(\partial_t\theta(\mathscr{P}(-1/t)^{-1})+\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z\theta(\mathscr{P}(-1/t)^{-1})\right).\end{aligned}$$

To prove the theorem, we have only to show that (4.23.a) and (4.23.b) are the equations (3.17.a) and (3.17.b). Comparing the both sides of the equation (4.23.a), we conclude that its coefficients except t^0 and t^{-1} are equal to zero.

Hence it is obtained that

$$\left(\partial_t\mathscr{P}(t)+\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)\mathscr{P}(t)\right)\mathscr{P}(t)^{-1}=\rho\left(-\theta(P)\partial_z\theta(P^{-1})+\frac{1}{t}\partial_\rho PP^{-1}\right).$$

In fact, this is equal to the equation (3.17.a).

Also comparing the both sides of the equation (4.23.b), we conclude that its coefficients except t^0, t^{-1} and t^{-2} are equal to zero. In order to know the coefficient of t^{-1} in (4.23.b) we write down the coefficient of t^{-1} in the equation (4.23.a) as follows:

$$(4.24)\quad \partial_\rho PP^{-1}=\{-\theta(\mathscr{P}(-1/t))\partial_z\theta(\mathscr{P}(-1/t)^{-1})\}_{t^{-1}}-\theta(P)\partial_\rho\theta(P^{-1}),$$

by dropping the multiplying variable ρ.

Taking it into account that the coefficient of t^{-1} in the right hand side of the equation (4.23) is

$$\frac{\rho}{2}\{-\theta(\mathscr{P}(-1/t))\partial_z\theta(\mathscr{P}(-1/t)^{-1})\}_{t^{-1}},$$

it follows from the equations (4.23.b) and (4.24) that

$$\begin{aligned}&\left(\partial_t\mathscr{P}(t)+\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z\mathscr{P}(t)\right)\mathscr{P}(t)^{-1}\\ &=\frac{\rho}{2}\left(-\theta(P)\partial_z\theta(P^{-1})+\frac{1}{t^2}\partial_z PP^{-1}+\frac{1}{t}(\partial_\rho PP^{-1}+\theta(P)\partial_z\theta(P^{-1}))\right).\end{aligned}$$

We immediately find that the above equation is equal to (3.17.b). Therefore the theorem is proved. □

From Theorem 4.10 and the discussions so far we have the following diagram for $g \in \mathscr{F}\mathscr{H}$:

$$\begin{array}{ccccccc} \mathscr{S}\mathscr{P} & \xrightarrow{proj} & \mathscr{S}\mathscr{P} & \xrightarrow{\bar{\theta}} & \mathscr{S}M & \xrightarrow{\varepsilon} & \mathscr{S}E \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathscr{S}\mathscr{P} & \xrightarrow{proj} & \mathscr{S}\mathscr{P} & \xrightarrow{\bar{\theta}} & \mathscr{S}M & \xrightarrow{\varepsilon} & \mathscr{S}E. \end{array}$$

And, by taking Proposition 4.9 into account, we conclude that the Hauser group $\mathscr{F}\mathscr{H}(\cong \mathscr{G}^{(\infty)})$ acts transitively on the potential space $\mathscr{S}\mathscr{P}$; $\mathscr{S}\mathscr{P}$ is an infinite dimensional homogeneous space.

5. Central extensions

In this section we shall describe the main results of this paper, that is, the facts that the conformal factor τ of a stationary and axially symmetric spacetime metric is related to the central exension of the formal loop group $\mathscr{F}\mathscr{G}$ with values in $PSL(2, \mathbf{R})$ by the additive formal group F, and is expressed by the evaluation of the group 2-cocycle Ξ on the corresponding potential.

Let F be the associative filtered algebra with the filtration $\{F_l\}_{l\in\mathbf{Z}}$, where $F_l = \rho^{|l|}\mathbf{R}[[z, \rho]]$, defined in Section 4. We define the following formal loop Lie algebra:

$$\mathscr{F}\mathfrak{g} = \{X = \sum_{l\in\mathbf{Z}} x_l t^l;\ x_l \in \mathfrak{gl}(2, F_l),\ \mathrm{tr}(X) = 0\}.$$

Then it should be noticed that since the exponential map $\exp: \mathfrak{sl}(2, \mathbf{R}) \to PSL(2, \mathbf{R})$ is surjective, so is the exponential map $\exp: \mathscr{F}\mathfrak{g} \to \mathscr{F}\mathscr{G}$.

At the first place, we consider the central extension of the Hauser group $\mathscr{G}^{(\infty)}$ (see Section 4) by the aditive group $\mathbf{R}$.

Thus the central extension is described by using the following exact sequence

$$0 \longrightarrow \mathbf{R} \longrightarrow \mathscr{G}^{(\infty)}_{ce} \longrightarrow \mathscr{G}^{(\infty)} \longrightarrow 0.$$

Since the cohomology group $H^2(\mathscr{G}^{(\infty)}, \mathbf{R})$ is trivial, so the central extension $\mathscr{G}^{(\infty)}_{ce}$ is always isomorphic to the direct product $\mathscr{G}^{(\infty)} \times \mathbf{R}$ as groups. In order to relate the center with the integral constant of the conformal factor we take the multiplicative group $\mathbf{R}^+ = \{e^v; v \in \mathbf{R}\}$ in place of $\mathbf{R}$. Then the group multiplication is defined by

(5.1) $\quad (g_1, e^v)\cdot(g_2, e^u) = (g_1 g_2, e^{v+u}) \quad$ for $(g_1, e^v), (g_2, e^u) \in \mathscr{G}^{(\infty)} \times \mathbf{R}^+$.

DEFINITION 5.1. *Let* $\mathscr{G}_{ce}^{(\infty)} = \mathscr{G}^{(\infty)} \times \mathbf{R}^+$ *with the group multiplication* (5.1). *We call the subgroup* $\mathscr{G}_{ce0}^{(\infty)} = \mathscr{G}^{(\infty)} \times \{1\}$ *also the Hauser group.*

Next we consider the central extensions of the formal loop group $\mathscr{F}\mathscr{G}$ (in details, see Section 4) by the additive formal group $F = \mathbf{R}[[z, \rho]]$. Then a central extension is equivalent that the following diagram is an exact sequence.

$$0 \longrightarrow F \longrightarrow \mathscr{F}\mathscr{G}_{ce} \longrightarrow \mathscr{F}\mathscr{G} \longrightarrow 0.$$

In this case, as discussed in Section 1, the cohomology group $H^2(\mathscr{F}\mathscr{G}, F)$ is not trivial. Therefore we have the nontrivial central extension by the choice of the representative Ξ in the nontrivial cohomology class. From the same reason in the central extension of the Hauser group, we choose the center $F^+ = \{e^\mu; \mu \in F\}$.

DEFINITION 5.2. *We define the centrally extended formal loop group* $\mathscr{F}\mathscr{G}_{ce}$ *to be the direct product* $\mathscr{F}\mathscr{G} \times F^+$ *with the group multiplication*:

(5.2) $\quad (g_1, e^\mu)\circ(g_2, e^\nu) = (g_1 g_2, e^{\mu+\nu+\Xi(g_1,g_2)}) \quad$ *for* $(g_1, e^\mu), (g_2, e^\nu) \in \mathscr{F}\mathscr{G}_{ce}$.

In the following, the group multiplication $(g_1, e^\mu)\circ(g_2, e^\nu)$ is simply written as $(g_1, e^\mu)(g_2, e^\nu)$.

From the anti-symmetric conditions (1.3.7) of our 2-cocycle Ξ the inverse element of $(g, e^\mu) \in \mathscr{F}\mathscr{G}_{ce}$ becomes $(g^{-1}, e^{-\mu})$.

Here we prepare the lemma below needed for the proof of Proposition 5.10.

LEMMA 5.3. *For any element* $g, g' \in \mathscr{F}\mathscr{H}$, *we have the identity*:

(5.3) $$\Xi(g, g') = 0.$$

PROOF. Express $g = e^X$ and $g' = e^Y$. Since $g(-1/t)$, $g'(-1/t)$ for $g, g' \in \mathscr{F}\mathscr{H}$ are respectively equal to g, g', so are $X(-1/t)$, $Y(-1/t)$. Using the definition (1.3.5) of the group 2-cocycle Ξ, the left hand side of (5.3) becomes

(5.4) $$\Xi(e^X, e^Y) = -\omega(L(X, Y), X) + \omega(-L(-Y, -X), Y).$$

It follows from the definition (1.3.1) of the Lie algebra 2-cocycle ω that the right hand side of (5.4) is

$$\frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}(-L(X, Y)\partial_t X - L(-Y, -X)\partial_t Y)\,dt.$$

Let us consider the replacement $t \to -\dfrac{1}{t}$ in the integrand (cf. Lemma 6.2).

Then, since X, Y are invariant under the replacement, we get $\Xi(e^X, e^Y) = -\Xi(e^X, e^Y)$. Therefore (5.3) identically holds. □

Especially for $g \in \mathscr{F}\mathscr{H}$ we have $\Xi(\theta(g^{-1}), g) = 0$.

From now on, we proceed to discussions theoretically parallel to those in Section 4.

Define the mapping j_{ce} from $\mathscr{G}^{(\infty)}_{ce0}$ to $\mathscr{F}\mathscr{G}_{ce}$ by the mapping product $j \times i_0$, where j is defined in Section 4 (see (4.2)) and i_0 sends 1 to $1 \in F$. Then it is clear from Lemma 5.3 that j_{ce} is an injective homomorphism. And the images of j_{ce} is denoted by $\mathscr{F}\mathscr{H}^0_{ce}$. As in Section 4, we identify them.

We introduce an involutive automorphism $\theta^{(\infty)}_{ce}$ of $\mathscr{F}\mathscr{G}_{ce}$ defined by

$$\theta^{(\infty)}_{ce}: \mathscr{F}\mathscr{G}_{ce} \ni (g, e^{\mu}) \longmapsto (\theta^{(\infty)}(g), e^{-\mu}) \in \mathscr{F}\mathscr{G}_{ce}, \tag{5.5}$$

which is well-defined and satisfies $\theta^{(\infty)}_{ce} \circ \theta^{(\infty)}_{ce} = identity$ as required, and is called the Cartan involution.

By use of the above Cartan involution we define the subgroup of $\mathscr{F}\mathscr{G}_{ce}$ by

$$\mathscr{F}\mathscr{K}_{ce} = \{k_{ce} \in \mathscr{F}\mathscr{G}_{ce};\ \theta^{(\infty)}_{ce}(k_{ce}) = k_{ce}\}, \tag{5.6}$$

which turns out to be

$$\mathscr{F}\mathscr{K}_{ce} = \mathscr{F}\mathscr{K} \times \{1\}.$$

Let $\mathscr{F}\mathscr{P}$ denote the subgroup of $\mathscr{F}\mathscr{G}$ defined in Section 4. Then, we define the subgroup of $\mathscr{F}\mathscr{G}_{ce}$ as follows:

$$\mathscr{F}\mathscr{P}_{ce} = \{\mathscr{P}_{ce}(t) = (\mathscr{P}(t), e^{\mu}) \in \mathscr{F}\mathscr{G}_{ce};\ \mathscr{P}(t) \in \mathscr{F}\mathscr{P},\ \mu \in F\} \tag{5.7}$$

LEMMA 5.4. *Let the notations be as above. Then,*

$$\mathscr{F}\mathscr{K}_{ce} \cap \mathscr{F}\mathscr{P}_{ce} = \{1\}$$

PROOF. The proof of this lemma is obvious. □

DEFINITION 5.5. *Let $\mathscr{F}\mathscr{G}_{ce}$ be the centrally extended formal loop group of $\mathscr{F}\mathscr{G}$. We say that $g_{ce} = (g, e^{\mu}) \in \mathscr{F}\mathscr{G}_{ce}$ is positive definite if g in (g, e^{μ}) is positive definite (cf. Definition 4.5).*

Let

$$\mathscr{M}(\mathscr{F}\mathscr{G}_{ce}) = \{g_{ce} \in \mathscr{F}\mathscr{G}_{ce};\ \theta^{(\infty)}_{ce}(g^{-1}_{ce}) = g_{ce},\ g_{ce} \text{ is positive definite}\}. \tag{5.8}$$

LEMMA 5.6. *Let $g_{ce} = (g, e^{\mu})$ be any element of $\mathscr{M}(\mathscr{F}\mathscr{G}_{ce})$. Then g is uniquely decomposed as*

$$\theta^{(\infty)}_{ce}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t) \quad \textit{for} \quad \mathscr{P}_{ce}(t) \in \mathscr{F}\mathscr{P}_{ce}. \tag{5.9}$$

PROOF. Applying Lemma 5.6 to $g \in \mathscr{F}\mathscr{G}$ in (g, e^{μ}), we have the following unique decomposition:

$$g = \theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t) \qquad (\mathscr{P}(t) \in \mathscr{F}\mathscr{P}).$$

So, let

$$\mathscr{P}_{ce}(t) = (\mathscr{P}(t), \exp(\mu - \Xi(\theta^{(\infty)}(\mathscr{P}(t)^{-1}), \mathscr{P}(t)))).$$

Then, we have $\mathscr{P}_{ce}(t) \in \mathscr{F}\mathscr{P}_{ce}$ and $g_{ce} = \theta_{ce}^{(\infty)}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t)$. It is obvious that the decomposition is unique. □

PROPOSITION 5.7. *Let $\mathscr{F}\mathscr{G}_{ce}$ be the centrally extended formal loop group of $\mathscr{F}\mathscr{G}$. Then $\mathscr{F}\mathscr{G}_{ce}$ is uniquely decomposed as*

$$\mathscr{F}\mathscr{G}_{ce} = \mathscr{F}\mathscr{K}_{ce}\mathscr{F}\mathscr{P}_{ce}. \tag{5.10}$$

PROOF. Let us consider the map $\delta: \mathscr{F}\mathscr{G}_{ce} \to \mathscr{M}(\mathscr{F}\mathscr{G}_{ce})$, defined by $\delta(g_{ce}) = \theta_{ce}^{(\infty)}(g_{ce}^{-1})g_{ce}$ for $g_{ce} \in \mathscr{F}\mathscr{G}_{ce}$. Then it is obvious that δ is well-defined.

Let $g_{ce} = (g, \mu)$ be an arbitrarily given element of $\mathscr{F}\mathscr{G}_{ce}$. Then from Lemma 5.6 we have the following decomposition:

$$\delta(g_{ce}) = \theta^{(\infty)}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t) \qquad (\mathscr{P}_{ce}(t) \in \mathscr{F}\mathscr{P}_{ce}).$$

Put $k_{ce} = g_{ce}\mathscr{P}_{ce}(t)^{-1}$. Then k_{ce} belongs to $\mathscr{F}\mathscr{K}_{ce}$. Therefore we have a decomposition of g in $\mathscr{F}\mathscr{K}_{ce}\mathscr{F}\mathscr{P}_{ce}$.

The uniqueness of this decomposition is proved by the same way in Proposition 4.4. □

From Proposition 5.7 we have the following decomposition:

$$\mathscr{F}\mathscr{G}_{ce} \cong \mathscr{F}\mathscr{K}_{ce} \times \mathscr{F}\mathscr{P}_{ce}. \tag{5.11}$$

Let α be the map: $\mathscr{F}\mathscr{G}_{ce} \to \mathscr{F}\mathscr{P}_{ce}$ through the decomposition (5.11). We denote by $\bar{\alpha}$ the map from $\mathscr{F}\mathscr{K}_{ce}\backslash\mathscr{F}\mathscr{G}_{ce}$ to $\mathscr{F}\mathscr{P}_{ce}$ induced from α. Then for any $g_{ce} \in \mathscr{F}\mathscr{G}_{ce}$ we define the action on $\mathscr{F}\mathscr{P}_{ce}$ such that the following deagram is commutative:

$$\begin{array}{ccc} \mathscr{F}\mathscr{K}_{ce}\backslash\mathscr{F}\mathscr{G}_{ce} & \xrightarrow{g_{ce}} & \mathscr{F}\mathscr{K}_{ce}\backslash\mathscr{F}\mathscr{G}_{ce} \\ \bar{\alpha}\downarrow & & \downarrow\bar{\alpha} \\ \mathscr{F}\mathscr{P}_{ce} & \longrightarrow & \mathscr{F}\mathscr{P}_{ce}. \end{array} \tag{5.12}$$

For the action of $g_{ce} \in \mathscr{F}\mathscr{G}_{ce}$ on $\mathscr{F}\mathscr{P}_{ce}$ we use g_{ce} as a notation, that is,

$$g_{ce}: \mathscr{F}\mathscr{P}_{ce} \longrightarrow \mathscr{F}\mathscr{P}_{ce}.$$

DEFINITION 5.8. *Let $\mathscr{S}\mathscr{P}$ be the potential space defined in* Section 3 *and let*

Λ be the formal power series with respect to z, ρ over $\mathbf{R}$, *whose constant term is equal to* 1. *The latter is defined in* Section 2.

Then we define

$$\Gamma(\mathscr{S}\mathscr{P}) = \{(\mathscr{P}(t), \bar{\eta}(\mathscr{P}(t))^{-1}) \in \mathscr{S}\mathscr{P} \times \Lambda;\ \mathscr{P}(t) \in \mathscr{S}\mathscr{P}\},$$

where $\bar{\eta}\colon \mathscr{S}\mathscr{P} \to \Lambda$ is given by the following diagram:

$$\begin{array}{ccccccc} \mathscr{S}\mathscr{P} & \xrightarrow{proj} & \mathscr{S}P & \xrightarrow{\bar{\theta}} & \mathscr{S}M & \longrightarrow & \mathscr{S}E \\ & & & & \Big\downarrow{\scriptstyle\eta} & & \\ & & & & \Lambda. & & \end{array}$$

We call $\Gamma(\mathscr{S}\mathscr{P})$ the centrally extended potential space.

It is noticed that the centrally extended potential $\mathscr{P}^{e}_{ce} \in \Gamma(\mathscr{S}\mathscr{P})$ corresponding to the Minkowski space-time is $(I_2, 1)$ from (2.20) and (3.25).

Let $g_{ce} = (g, 1)$ be any element of $\mathscr{F}\mathscr{H}^{0}_{ce}$. Since g in $(g, 1)$ is an element of $\mathscr{F}\mathscr{H}$, due to Theorem 4.6 we have $g^{-1} \in \mathscr{F}\mathscr{K}\,\mathscr{S}\mathscr{P}$. So we write $\mathscr{P}(t) = kg^{-1}(\mathscr{P}(t) \in \mathscr{S}\mathscr{P}, k \in \mathscr{F}\mathscr{K})$. Let P denote $proj(\mathscr{P}(t))$, that is, $P(0)$.

PROPOSITION 5.9. *Let the notations be as above. For the derivative of the group 2-cocycle Ξ with respect to z and ρ we have*

$$\partial_z \Xi(\mathscr{P}(t), g) = \frac{\rho}{2f^2}(\partial_z f \partial_\rho f + \partial_z \psi \partial_\rho \psi), \tag{5.13.a}$$

$$\partial_\rho \Xi(\mathscr{P}(t), g) = \frac{\rho}{4f^2}((\partial_\rho f)^2 - (\partial_z f)^2 + (\partial_\rho \psi)^2 - (\partial_z \psi)^2), \tag{5.13.b}$$

where f, ψ are given by the parametrization (3.2) *of P.*

This proposition will be proved in Section 6.

PROPOSITION 5.10. *For any element $\mathscr{P}_{ce}(t)$ of the centrally extended potential $\Gamma(\mathscr{S}\mathscr{P})$ defined above, the product*

$$\theta^{(\infty)}_{ce}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t)$$

belongs to the Hauser group $\mathscr{F}\mathscr{H}^{0}_{ce}$ which is the subgroup of the formal loop group $\mathscr{F}\mathscr{G}_{ce}$.

PROOF. Let $\mathscr{P}_{ce}(t) = (\mathscr{P}(t), \tau^{-1})$ be any element of $\Gamma(\mathscr{S}\mathscr{P})$. Since $\mathscr{P}(t)$ in $(\mathscr{P}(t), \tau^{-1})$ belongs to $\mathscr{F}\mathscr{P}$, it follows from the results of Section 4 that we can write

$$\mathscr{P}(t) = kg^{-1}(k \in \mathscr{F}\mathscr{K},\ g \in \mathscr{F}\mathscr{H}).$$

We consider $k_{ce} = (k, 1)$ and $g_{ce} = (g, 1)$, which respectively belong to $\mathscr{F}\mathscr{K}_{ce}$ and $\mathscr{F}\mathscr{H}^0_{ce}$. So the product of these becomes as follows:

$$(k, 1)(g, 1)^{-1} = (kg^{-1}, e^{\Xi(k, g^{-1})}) = (\mathscr{P}(t), e^{-\Xi(\mathscr{P}(t), g)})$$

Put $\tau' = e^{\Xi(\mathscr{P}(t), g)}$. Then it follows from Proposition 5.9 that τ' satisfies the same differential equations (2.18.a) and (2.18.b). Hence, we conclude from $\Xi(\mathscr{P}(t), g)|_{\rho=0} = 0$ that

$$\tau' = \tau.$$

Therefore

$$\mathscr{P}_{ce}(t) = k_{ce} g_{ce}. \tag{5.14}$$

Now we put $m_{ce} = \theta^{(\infty)}_{ce}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t)$. Using (5.14) we calculate m_{ce} as follows:

$$\begin{aligned} m_{ce} &= \theta^{(\infty)}_{ce}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t) \\ &= \theta^{(\infty)}_{ce}(g_{ce}^{-1})(\theta^{(\infty)}_{ce}(k_{ce}^{-1})k_{ce})g_{ce} \\ &= (\theta^{(\infty)}(g^{-1}), 1)(g, 1) \\ &= (\theta(g^{-1})g, e^{\Xi(\theta(g^{-1}), g)}) \end{aligned}$$

Then it follows from $\Xi(\theta(g^{-1}), g) = 0$ (see Lemma 5.3) that

$$m_{ce} = (\theta(g^{-1})g, 1).$$

This completes the proof of the proposition. □

Now we have our main

THEOREM 5.11. *Let $\mathscr{F}\mathscr{H}^0_{ce}$ and $\Gamma(\mathscr{S}\mathscr{P})$ be the Hauser group and the centrally extended potential space.*
Then, $\Gamma(\mathscr{S}\mathscr{P})\mathscr{F}\mathscr{H}^0_{ce} \subset \mathscr{F}\mathscr{K}_{ce}\Gamma(\mathscr{S}\mathscr{P})$. Therefore for any $g_{ce} \in \mathscr{F}\mathscr{H}^0_{ce}$, the following diagram is well-defined:

$$\begin{array}{ccc} \mathscr{F}\mathscr{K}_{ce}\backslash\mathscr{F}\mathscr{K}_{ce}\Gamma(\mathscr{S}\mathscr{P}) & \xrightarrow{g_{ce}} & \mathscr{F}\mathscr{K}_{ce}\backslash\mathscr{F}\mathscr{K}_{ce}\Gamma(\mathscr{S}\mathscr{P}) \\ \bar{\alpha}\downarrow & & \downarrow\bar{\alpha} \\ \Gamma(\mathscr{S}\mathscr{P}) & \xrightarrow{g_{ce}} & \Gamma(\mathscr{S}\mathscr{P}) \end{array} \tag{5.15}$$

PROOF. Since any element $\mathscr{P}_{ce}(t) \in \Gamma(\mathscr{S}\mathscr{P})$ is obtained from the action of some $g_{ce} \in \mathscr{F}\mathscr{H}^0_{ce}$ to the centrally extended Minkowski potential $\mathscr{P}^e_{ce}$, we have only to prove that $\mathscr{F}\mathscr{H}^0_{ce} \subset \mathscr{F}\mathscr{K}_{ce}\Gamma(\mathscr{S}\mathscr{P})$.

Let $g_{ce} = (g, 1)$ be any element of $\mathscr{F}\mathscr{H}^0_{ce}$. Then from Proposition 5.7 we have the following decomposition:

$$g_{ce}^{-1} = k_{ce}^{-1}\mathscr{P}_{ce}(t) \quad (k_{ce} \in \mathscr{F}\mathscr{K}_{ce}, \mathscr{P}_{ce}(t) \in \mathscr{F}\mathscr{P}_{ce}).$$

Using $g_{ce} = (g, 1)$, $k_{ce} = (k, 1)$ and $\mathscr{P}_{ce}(t) = (\mathscr{P}(t), e^{\mu})$, the above decomposition becomes

$$(kg^{-1}, e^{\Xi(k, g^{-1})}) = (\mathscr{P}(t), e^{\mu}).$$

Then it follows from the results of Section 4 and the proof of Proposition 5.10 that $\mathscr{P}_{ce}(t) \in \Gamma(\mathscr{S}\mathscr{P})$. □

COROLLARY 5.12. *For any centrally extended potential* $\mathscr{P}_{ce}(t) = (\mathscr{P}(t), \tau^{-1}) \in \Gamma(\mathscr{S}\mathscr{P})$, *we have the following relation*:

$$\tau = \exp\left\{\frac{1}{2}\Xi(\theta^{(\infty)}(\mathscr{P}(t)^{-1}), \mathscr{P}(t))\right\}. \tag{5.16}$$

PROOF. For a given $\mathscr{P}_{ce}(t) = (\mathscr{P}(t), \tau^{-1}) \in \Gamma(\mathscr{S}\mathscr{P})$ we put

$$m_{ce} = \theta_{ce}^{(\infty)}(\mathscr{P}_{ce}(t)^{-1})\mathscr{P}_{ce}(t). \tag{5.17}$$

Then we get the following result for m_{ce} with use of the properties of the group 2-cocycle Ξ:

$$\begin{aligned}
&\theta_{ce}^{(\infty)}((\mathscr{P}(t), \tau^{-1})^{-1})(\mathscr{P}(t), \tau^{-1}) \\
&= (\theta^{(\infty)}(\mathscr{P}(t)^{-1}), \tau^{-1})(\mathscr{P}(t), \tau^{-1}) \\
&= (\theta^{(\infty)}(\mathscr{P}(t)^{-1})\mathscr{P}(t), \tau^{-2}\exp\{\Xi(\theta^{(\infty)}(\mathscr{P}(t)^{-1}), \mathscr{P}(t))\})
\end{aligned}$$

On the other hand, since $\mathscr{P}_{ce} \in \Gamma(\mathscr{S}\mathscr{P})$, it follows from Proposition 5.10 that m_{ce} belongs to $\mathscr{F}\mathscr{H}_{ce}^{0}$. That is to say, m_{ce} can be written as

$$m_{ce} = (\theta^{(\infty)}(g^{-1})g, 1) \qquad \text{for some } g \in \mathscr{F}\mathscr{H}. \tag{5.18}$$

Comparing the center components of the relations (5.17) and (5.18) for m_{ce}, we get the desired identity. □

Let $\Gamma(\mathscr{S}P)$, $\Gamma(\mathscr{S}M)$ and $\Gamma(\mathscr{S}E)$ be subspaces of $\mathscr{S}P \times \Lambda$, $\mathscr{S}M \times \Lambda$ and $\mathscr{S}E \times \Lambda$ defined by the same way in $\Gamma(\mathscr{S}\mathscr{P})$ (see Definition 5.8). And, let $i: \Lambda \to \Lambda$ be the identity map.

Then from Theorem 5.11 and the discussions so far we have the following diagram for $g_{ce} \in \mathscr{F}\mathscr{H}_{ce}^{0}$:

$$\begin{array}{ccccccc}
\Gamma(\mathscr{S}\mathscr{P}) & \xrightarrow{proj \times i} & \Gamma(\mathscr{S}P) & \xrightarrow{\bar{\theta} \times i} & \Gamma(\mathscr{S}M) & \xrightarrow{\varepsilon \times i} & \Gamma(\mathscr{S}E) \\
{\scriptstyle g_{ce}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(\mathscr{S}\mathscr{P}) & \xrightarrow{proj \times i} & \Gamma(\mathscr{S}P) & \xrightarrow{\bar{\theta} \times i} & \Gamma(\mathscr{S}M) & \xrightarrow{\varepsilon \times i} & \Gamma(\mathscr{S}E).
\end{array}$$

Furthermore the Hauser group $\mathscr{F}\mathscr{H}^0_{ce}(\cong \mathscr{G}^{(\infty)})$ acts transitively on the potential space $\Gamma(\mathscr{S}\mathscr{P})$; $\Gamma(\mathscr{S}\mathscr{P})$ is an infinite dimensional homogeneous space.

Recall that any element $\tau(z, \rho)$ of our conformal factor space Λ is normalized by $\tau(0, 0) = 1$ (see Definition 2.5). This normalization is ensured from Lemma 2.4.

Now we consider the following total solution space $E(\mathscr{S}\mathscr{P})$:

$$
\begin{aligned}
E(\mathscr{S}\mathscr{P}) = \{(\mathscr{P}(t), e^{\mu}) \in \mathscr{S}\mathscr{P} \times F^+;& \\
& \mathscr{P}(t) \in \mathscr{S}\mathscr{P}, \text{ (put } m = \bar{\theta}(proj(\mathscr{P}(t))) \\
& \partial_z \mu = -\frac{\rho}{4} \operatorname{tr}(\partial_z m^{-1} \partial_\rho m), \\
& \partial_\rho \mu = -\frac{\rho}{8} \operatorname{tr}(\partial_\rho m^{-1} \partial_\rho m - \partial_z m^{-1} \partial_z m)\}
\end{aligned}
$$

and denote by $\pi\colon E(\mathscr{S}\mathscr{P}) \to \mathscr{S}\mathscr{P}$ the surjective map defined by

$$\pi((\mathscr{P}(t), e^{\mu})) = \mathscr{P}(t) \quad \text{for} \quad (\mathscr{P}(t), e^{\mu}) \in E(\mathscr{S}\mathscr{P}).$$

Then a triplet $(E(\mathscr{S}\mathscr{P}), \pi, \mathscr{S}\mathscr{P})$ is considered to be a fiber space with fiber $\mathbf{R}^+$, in fact a principal bundle. Since $\Gamma(\mathscr{S}\mathscr{P})$ is a global section of the fiber space, the fiber space is trivial (cf. the above discussions). It is noticed that the relation (5.16) in Corollary 5.12 is satisfied only for the elements of $\Gamma(\mathscr{S}\mathscr{P})$.

Let $\mathscr{G}^{(\infty)}_{ce} = \mathscr{G}^{(\infty)} \times \mathbf{R}^+$ be the centrally extended Hauser group (see Definition 5.1). We define an action of $\mathscr{G}^{(\infty)}_{ce}$ on $E(\mathscr{S}\mathscr{P})$ by imbedding $\mathscr{G}^{(\infty)}_{ce}$ into $\mathscr{F}\mathscr{G}_{ce}$, using the map j_{ce}.

Then from the discussions of this section we have the following commutative diagram for $g_{ce} = (g, e^a) \in \mathscr{G}^{(\infty)}_{ce}$:

$$
\begin{array}{ccc}
E(\mathscr{S}\mathscr{P}) & \xrightarrow{g_{ce}} & E(\mathscr{S}\mathscr{P}) \\
\pi \downarrow & & \pi \downarrow \\
\mathscr{S}\mathscr{P} & \xrightarrow{g} & \mathscr{S}\mathscr{P}.
\end{array}
$$

It is clear that the center $\mathbf{R}^+$ of $\mathscr{G}^{(\infty)}_{ce}$ corresponds to the fiber $\mathbf{R}^+$ of $E(\mathscr{S}\mathscr{P})$ and $\mathscr{G}^{(\infty)}_{ce}$ acts transitively on $E(\mathscr{S}\mathscr{P})$.

6. Proof of Proposition 5.9

In this section we prove the relations $(5.13.a)$ and $(5.13.b)$ in Proposition 5.9. The way of the proof is very elementary and tedious one (there should exist more elegant proof).

Before starting the proof of Proposition 5.9, we prepare three lemmas.

LEMMA 6.1. *For* $w = \mathbf{R}[[t, z, \rho]]$ *or* $\mathbf{R}[[t^{-1}, z, \rho]]$ *we have the following relations*:

$$(6.1.a)\qquad \frac{1}{2\pi\sqrt{-1}}\oint \frac{1}{t}(\partial_z w(t)\partial_\rho w(-1/t) - \partial_\rho w(t)\partial_z w(-1/t))\,dt = 0,$$

$$(6.1.b)\qquad \frac{1}{2\pi\sqrt{-1}}\oint \left(1+\frac{1}{t^2}\right)\left(t-\frac{1}{t}\right)^l (\partial_z w(t)\partial_\rho w(-1/t) + \partial_\rho w(t)\partial_z w(-1/t))\,dt = 0,$$

where l *is a non-negative integer.*

PROOF. Taking it into account that the residue is to pick up the coefficient of t^{-1} in the integrand, we have the same result for the residue when we divide the integrand by $-t^2$ after replacing $t \to -\frac{1}{t}$. As executing this procedure for the left hand side of the relations, we find that the results become the negative sign of the original ones, respectively.

Therefore these should be equal to zero. □

In order to fix the notations, we return to the setting of Proposition 5.9. Let g be any element of $\mathcal{FH}$. If we take the Minkowski potential $\mathcal{P}_e \in \mathcal{SP}$ in Theorem 4.10, $\mathcal{P}_e g^{-1}$ belongs to $\mathcal{FK}\,\mathcal{SP}$. Hence we write

$$(6.2)\qquad g^{-1} = k^{-1}\mathcal{P}(t) \qquad (k \in \mathcal{FK},\ \mathcal{P}(t) \in \mathcal{SP}).$$

Although there is a simpler proof for the following lemma, we insist on a rather complicated proof. The reason is that we need other relations (6.6.a) and (6.6.b).

LEMMA 6.2. *Let the notations be as above. Let* Ξ' *be the mixed form defined by* (1.3.9), *and let* ∂ *denote the partial derivatives* ∂_z *or* ∂_ρ.

Then, the following relations hold:

$$(6.3.a)\qquad \Xi'(g, g^{-1}\partial g) = 0,$$

$$(6.3.b)\qquad \Xi'(k, k^{-1}\partial k) = 0.$$

PROOF. First we show that (6.3.a) holds. It follows from the formula (1.3.10) that the right hand side of (6.3.a) is expressed by

$$(6.4)\qquad -\frac{1}{2\pi\sqrt{-1}}\oint \operatorname{tr}\left(g^{-1}\partial g\left(\frac{1}{2} + \chi(\operatorname{ad}\log g)\right)g^{-1}\partial_t g\right)dt.$$

Since $g \in \mathscr{F}\mathscr{H}$, g is invariant under the replacement $t \to -\frac{1}{t}$. And so is $\log g$. Taking it into consideration that the same replacement of the integrand of (6.4) yields a change of sign (pay attention to the existence of the derivative with respect to t in the integrand), we find that the execution of the procedure for (6.4) yields the opposite sign. Therefore (6.4) should be zero.

Lastly, (6.3.b) is expressed by

$$(6.5)\qquad -\frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}\left(k^{-1}\partial k\left(\frac{1}{2}+\chi(\mathrm{ad}\log k)\right)k^{-1}\partial_t k\right)dt.$$

Since $\theta^{(\infty)}(k) = k$ for $k \in \mathscr{F}\mathscr{K}$, we conclude from the analogous way in (6.3.a) that (6.5) should vanish. □

Note that from the process of the proof in the above lemma we immediately find the following relations:

$$(6.6.\mathrm{a})\qquad \frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}\,(g^{-1}\partial g g^{-1}\partial_t g)\,dt = 0,$$

$$(6.6.\mathrm{b})\qquad \frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}\,(k^{-1}\partial k\, k^{-1}\partial_t k)\,dt = 0.$$

LEMMA 6.3. *For $\mathscr{P}(t) \in \mathscr{F}\mathscr{P}$, $g \in \mathscr{F}\mathscr{H}$ and $k \in \mathscr{F}\mathscr{K}$ as above, we have*

(6.7)

$$\partial \Xi(\mathscr{P}(t), g) = \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}\,(\mathscr{P}^{-1}(t)\partial\mathscr{P}(t)\partial_t g g^{-1} - \mathscr{P}^{-1}(t)\partial_t\mathscr{P}(t)\partial g g^{-1})\,dt,$$

where ∂ denotes the partial derivatives ∂_z or ∂_ρ.

PROOF. From Lemma 1.4, we have the following relation:

$$\begin{aligned}\partial \Xi(\mathscr{P}(t), g) = {} & \Xi'(k^{-1}, \partial\mathscr{P}(t)\mathscr{P}^{-1}(t)) - \Xi'(\mathscr{P}^{-1}(t), \partial\mathscr{P}(t)\mathscr{P}^{-1}(t)) \\ & + \Xi'(k, g^{-1}\partial g) - \Xi'(g, g^{-1}\partial g).\end{aligned}$$

The second term of the right hand side obviously vanishes, and so is the fourth term by Lemma 6.2.

Inserting $\mathscr{P}(t) = kg^{-1}$ and using the expression of Ξ' in Lemma 1.3, we get the result below for the above equation:

$$-\frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}\left(g^{-1}\partial g\left(\frac{1}{2}+\chi(\mathrm{ad}\log k^{-1})\right)k^{-1}\partial_t k\right.$$

$$+ g^{-1}\partial g\left(\frac{1}{2} + \chi(\mathrm{ad}\log k)\right)k^{-1}\partial_t k\Big)dt.$$

It immediately follows from the oddness of the function χ that

$$\partial\Xi(\mathscr{P}(t), g) = -\frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}(g^{-1}\partial g k^{-1}\partial_t k)\,dt.$$

Then, expressing the above relation by use of $\mathscr{P}(t)$ and g in consideration of (6.6.a) and (6.6.b), we obtain two relations:

(6.8)
$$\begin{aligned}\partial\Xi(\mathscr{P}(t), g) &= \frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}(\mathscr{P}^{-1}(t)\partial\mathscr{P}(t)\partial_t g g^{-1})\,dt \\ &= -\frac{1}{2\pi\sqrt{-1}}\oint \mathrm{tr}(\mathscr{P}^{-1}(t)\partial_t\mathscr{P}(t)\partial g g^{-1})\,dt.\end{aligned}$$

Thus the relation (6.7) is satisfied. □

Let

(6.9)
$$\mathscr{P}(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t)\end{pmatrix}.$$

Then, taking the components of the equation (3.17.b), we get the following equations:

(6.10.a)
$$\partial_t b(t) + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z b(t) = o_1 b(t) + o_2 d(t),$$

(6.10.b)
$$\partial_t d(t) + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z d(t) = o_3 b(t) - o_1 d(t),$$

(6.10.c)
$$\partial_t a(t) + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z a(t) = o_1 a(t) + o_2 c(t),$$

(6.10.d)
$$\partial_t c(t) + \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z c(t) = o_3 a(t) - o_1 c(t),$$

where

(6.11.a)
$$o_1 = \frac{\rho}{4f}\left(-\left(1 - \frac{1}{t^2}\right)\partial_z f + \frac{2}{t}\partial_\rho f\right),$$

(6.11.b)
$$o_2 = \frac{\rho}{2f}\left(-\partial_z\psi + \frac{1}{t}\partial_\rho\psi\right),$$

(6.11.c) $$o_3 = \frac{\rho}{2f}\left(\frac{1}{t^2}\partial_z\psi + \frac{1}{t}\partial_\rho\psi\right).$$

And it follows from the equations (3.17.a) and (3.17.b) that

(6.12) $$\left(1 - \frac{1}{t^2}\right)\partial_z\mathscr{P}(t) + \frac{2}{t}\partial_\rho\mathscr{P}(t) = \frac{1}{f}(\tilde{\Omega}_0 - \tilde{\Omega}_1)\mathscr{P}(t).$$

Also taking the (1, 2) and (2, 2) components of the equation (6.12), we get the following equations:

(6.13.a) $$\left(1 - \frac{1}{t^2}\right)\partial_z b(t) + \frac{2}{t}\partial_\rho b(t) = q_1 b(t) + q_2 d(t),$$

(6.13.b) $$\left(1 - \frac{1}{t^2}\right)\partial_z d(t) + \frac{2}{t}\partial_\rho d(t) = q_3 b(t) - q_1 d(t),$$

where

(6.14) $$q_1 = -\frac{1}{2f}\left(1 + \frac{1}{t^2}\right)\partial_z f,\quad q_2 = -\frac{1}{f}\left(\partial_z\psi + \frac{1}{t}\partial_\rho\psi\right) \quad \text{and} \quad q_3 = -\frac{1}{f}\left(\frac{1}{t^2}\partial_z\psi - \frac{1}{t}\partial_\rho\psi\right).$$

It follows from the (1, 1) and (1, 2) components of (6.12) that

(6.15.a) $$\left(1 - \frac{1}{t^2}\right)\partial_z a(t) + \frac{2}{t}\partial_\rho a(t) = q_1 a(t) + q_2 c(t),$$

(6.15.b) $$\left(1 - \frac{1}{t^2}\right)\partial_z c(t) + \frac{2}{t}\partial_\rho c(t) = q_3 a(t) - q_1 c(t).$$

It should be noticed that the above o_1, o_2, o_3, q_1, q_2 and q_3 have the following properties for dividing $-t^2$ after the replacement $t \to -\frac{1}{t}$:

(6.16.a) $$o_1 \longmapsto o_1,\quad o_2 \longmapsto o_3,\quad o_3 \longmapsto o_2,$$

(6.16.b) $$q_1 \longmapsto -q_1,\ q_2 \longmapsto -q_3,\ q_3 \longmapsto -q_2.$$

Let us consider parametrizations of g and k.

For $g \in \mathscr{F}\mathscr{H}$, without loss of generality, we can assume that

$$g^{-1} = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

So we have

$$g^{-1}\partial g=\begin{pmatrix} r^{-1}\partial r & 0 \\ \partial n-2nr^{-1}\partial r & -r^{-1}\partial r \end{pmatrix}.$$

Since r and n come from the formal power series with respect to s, so these are invariant under the replacement $t\to -\frac{1}{t}$ and satisfy the following differential equations:

$$\partial_t r=-\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)r, \tag{6.17.a}$$

$$\partial_t r=-\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z r, \tag{6.17.b}$$

$$\partial_t n=-\rho\left(\partial_z+\frac{1}{t}\partial_\rho\right)n, \tag{6.17.c}$$

$$\partial_t n=-\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\partial_z n. \tag{6.17.d}$$

As to $k\in\mathscr{F}\mathscr{K}$, since $\theta^{(\infty)}(k)=k$, we can employ the following parametrization:

$$k=\begin{pmatrix} k_1(t) & k_2(t) \\ -k_2(-1/t) & k_1(-1/t) \end{pmatrix},$$

where $k_1(t)k_1(-1/t)+k_2(t)k_2(-1/t)=1$.
It should be noticed that the above entries k_1 and k_2 have both of negative and positive powers with respect to the spectral parameter t (this is the exception of the convention to maps, see Section 1).

By $\mathscr{P}(t)=kg^{-1}$, we have the relations:

$$a(t)=k_1(t)r+k_2(t)r^{-1}n, \qquad b(t)=k_2(t)r^{-1},$$
$$c(t)=-k_2(-1/t)r+k_1(-1/t)r^{-1}n, \quad d(t)=k_1(-1/t)r^{-1}.$$

It follows from the above relations, eliminating k_1 and k_2, that

$$b(t)n=a(t)-d(-1/t)r^2, \tag{6.18.a}$$

$$d(t)n=c(t)+b(-1/t)r^2. \tag{6.18.b}$$

By the parametrization of g and $\mathscr{P}(t)$, (6.7) and det $\mathscr{P}(t)=1$, we have the following relation:

$$\partial \Xi(\mathscr{P}(t), g) = \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}} \oint ((d(t)\partial_t b(t) - b(t)\partial_t d(t))\partial n$$

$$\text{(6.19)} \qquad - (b(-1/t)\partial_t b(t) + d(-1/t)\partial_t d(t))\partial r^2 - (d(t)\partial b(t) - b(t)\partial d(t))\partial_t n$$
$$+ (b(-1/t)\partial b(t) + d(-1/t)\partial d(t))\partial_t r^2)\, dt.$$

Now, let us start the direct calculation for the proof of Proposition 5.9. First we show that

$$\text{(6.20)} \qquad \partial_z \Xi(\mathscr{P}(t), g) = \frac{\rho}{2f^2}(\partial_z f \partial_\rho f + \partial_z \psi \partial_\rho \psi).$$

It follows from the equations (6.10.a), (6.10.b), (6.17.a) and (6.17.d) that the right hand side of (6.19) with changing ∂ into ∂_z becomes

$$\partial_z \Xi(\mathscr{P}(t), g) = \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}} \oint (d(t)(o_1 b(t) + o_2 d(t))\partial_z n - b(t)(o_3 b(t)$$
$$- o_1 d(t))\partial_z n - b(-1/t)(o_1 b(t) + o_2 b(t))\partial_z r^2 - d(-1/t)(o_3 b(t)$$
$$- o_1 d(t))\partial_z r^2)\, dt.$$

Using the following relations:

$$d(t)\partial_z n = \partial_z(d(t)n) - n\partial_z d(t) = \partial_z(c(t) + b(-1/t)r^2) - n\partial_z d(t),$$
$$b(t)\partial_z n = \partial_z(b(t)n) - n\partial_z b(t) = \partial_z(a(t) - d(-1/t)r^2) - n\partial_z b(t),$$

it immediately follows that the right hand side of the above relation is given by

$$\frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}} \oint ((o_1 b(t) + o_2 d(t))\partial_z c(t) - (o_3 b(t) - o_1 d(t))\partial_z a(t)$$

$$\text{(6.21)} \qquad - (o_1 b(t) + o_2 d(t))n\partial_z d(t) + (o_3 b(t) - o_1 d(t))n\partial_z b(t)$$
$$+ (o_1 b(t) + o_2 d(t))r^2 \partial_z b(-1/t) + (o_3 b(t) - o_1 d(t))r^2 \partial_z d(-1/t))\, dt.$$

Collecting the terms containing n in the integrand of (6.21), we obtain

$$\text{(6.22)} \qquad \frac{1}{2\pi\sqrt{-1}} \oint (-(o_1 b(t) + o_2 d(t))n\partial_z d(t) + (o_3 b(t) - o_1 d(t))n\partial_z b(t))\, dt.$$

By eliminating n, (6.22) becomes

$$\frac{1}{2\pi\sqrt{-1}} \oint (-(o_1 a(t) + o_2 c(t))\partial_z d(t) + (o_3 a(t) - o_1 c(t))\partial_z b(t)$$

$$\text{(6.23)} \qquad - (-o_1 d(-1/t) + o_2 b(-1/t))r^2 \partial_z d(t)$$
$$+ (-o_3 d(-1/t) - o_1 b(-1/t))r^2 \partial_z b(t))\, dt.$$

Executing the procedure, which was used in the proof of Lemma 6.1, for the terms containing r in (6.23) and using the properties (6.16.a), we find that (6.22) is equal to

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\oint(&-(o_1a(t)+o_2c(t))\partial_z b(t)+(o_3a(t)-o_1c(t))\partial_z d(t)\\ &-(-o_1d(t)+o_3b(t))r^2\partial_z d(-1/t)\\ &+(-o_2d(t)-o_1b(t))r^2\partial_z b(-1/t))\,dt.\end{aligned} \tag{6.24}$$

Inserting the result (6.24) into (6.21), we reach the following relation:

$$\begin{aligned}\partial_z\Xi(\mathscr{P}(t),g)=\frac{1}{2}\times\frac{1}{2\pi\sqrt{-1}}\oint&((o_1b(t)+o_2d(t))\partial_z c(t)-(o_3b(t)\\ &-o_1d(t))\partial_z a(t)-(o_1a(t)+o_2c(t))\partial_z d(t)+(o_3a(t)\\ &-o_1c(t))\partial_z b(t))\,dt.\end{aligned} \tag{6.25}$$

Then, a straightforward calculation by use of (6.11.a), (6.11.b), (6.11.c), (3.21), (3.22.a), (3.22.b), (3.22.d) and (3.24) shows that the right hand side of (6.25) is just equal to that of (6.20).

Finally we show that the ρ derivative of the group 2-cocycle is expressed by

$$\partial_\rho\Xi(\mathscr{P}(t),g)=\frac{\rho}{4f^2}((\partial_\rho f)^2-(\partial_z f)^2+(\partial_\rho\psi)^2-(\partial_z\psi)^2). \tag{6.26}$$

Since

$$\begin{aligned}\partial_\rho\Xi(\mathscr{P}(t),g)=\frac{1}{2}\times\frac{1}{2\pi\sqrt{-1}}\oint&((d(t)\partial_t b(t)-b(t)\partial_t d(t))\partial_\rho n\\ &-(b(-1/t)\partial_t b(t)+d(-1/t)\partial_t d(t))\partial_\rho r^2-(d(t)\partial_\rho b(t)-b(t)\partial_\rho d(t))\partial_t n\\ &+(b(-1/t)\partial_\rho b(t)+d(-1/t)\partial_\rho d(t))\partial_t r^2)\,dt,\end{aligned}$$

it is easy to see from the equations (6.10.a), (6.10.b), (6.17.b) and (6.17.d) that the right hand side of the above equation becomes

$$\begin{aligned}\frac{1}{2}\times\frac{1}{2\pi\sqrt{-1}}\oint&(d(t)(o_1b(t)+o_2d(t))\partial_\rho n-b(t)(o_3b(t)-o_1d(t))\partial_\rho n\\ &-b(-1/t)(o_1b(t)+o_2d(t))\partial_\rho r^2-d(-1/t)(o_3b(t)-o_1d(t))\partial_\rho r^2\\ &+\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\times\{(-d(t)\partial_z b(t)+b(t)\partial_z d(t))\partial_\rho n\end{aligned}$$

$$+ (d(t)\partial_\rho b(t) - b(t)\partial_\rho d(t))\partial_z n + (b(-1/t)\partial_z b(t) + d(-1/t)\partial_z d(t))\partial_\rho r^2$$
$$- (b(-1/t)\partial_\rho b(t) + d(-1/t)\partial_\rho d(t))\partial_z r^2\})\,dt.$$

Put $\Delta = \frac{1}{2\pi\sqrt{-1}}\oint \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right) \times \{(-d(t)\partial_z b(t) + b(t)\partial_z d(t))\partial_\rho n$

$$+ (d(t)\partial_\rho b(t) - b(t)\partial_\rho d(t))\partial_z n + (b(-1/t)\partial_z b(t) + d(-1/t)\partial_z d(t))\partial_\rho r^2$$
$$- (b(-1/t)\partial_\rho b(t) + d(-1/t)\partial_\rho d(t))\partial_z r^2\}\,dt.$$

From the following relations:

(6.27.a) $$d(t)\partial_\rho n = \partial_\rho(c(t) + b(-1/t)r^2) - n\partial_\rho d(t),$$

(6.27.b) $$b(t)\partial_\rho n = \partial_\rho(a(t) - d(-1/t)r^2) - n\partial_\rho b(t),$$

and using the equations (6.18.a) and (6.18.b), we obtain the following result:

(6.28)

$$\partial_\rho \Xi(\mathscr{P}(t), g) = \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}}\oint ((o_1 b(t) + o_2 d(t))\partial_\rho c(t) - (o_3 b(t) - o_1 d(t))\partial_\rho a(t)$$
$$- (o_1 b(t) + o_2 d(t))n\partial_\rho d(t) + (o_3 b(t) - o_1 d(t))n\partial_\rho b(t)$$
$$+ (o_1 b(t) + o_2 d(t))r^2\partial_\rho b(-1/t) + (o_3 b(t) - o_1 d(t))r^2\partial_\rho d(-1/t))\,dt$$
$$+ \frac{1}{2}\Delta$$
$$= \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}}\oint ((o_1 b(t) + o_2 d(t))\partial_\rho c(t) - (o_3 b(t) - o_1 d(t))\partial_\rho a(t)$$
$$- (o_1 a(t) + o_2 c(t))\partial_\rho d(t) + (o_3 a(t) - o_1 c(t))\partial_\rho b(t)$$
$$+ r^2((o_1 d(-1/t) - o_2 b(-1/t))\partial_\rho d(t) - (o_3 d(-1/t) + o_1 b(-1/t))\partial_\rho b(t)$$
$$+ (o_1 b(t) + o_2 d(t))\partial_\rho b(-1/t) + (o_3 b(t) - o_1 d(t))\partial_\rho d(-1/t)))\,dt$$
$$+ \frac{1}{2}\Delta$$
$$= \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}}\oint ((o_1 b(t) + o_2 d(t))\partial_\rho c(t) - (o_3 b(t) - o_1 d(t))\partial_\rho a(t)$$
$$- (o_1 a(t) + o_2 c(t))\partial_\rho d(t) + (o_3 a(t) - o_1 c(t))\partial_\rho b(t))\,dt$$
$$+ \frac{1}{2}\Delta,$$

where we have used the procedure in Lemma 6.1 and the properties (6.16.a) for the equality from the second relation to the third one.

Now, let us start the evaluation of Δ. Using the relations (6.27.a), (6.27.b) and

$$\frac{1}{2\pi\sqrt{-1}}\oint\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\times(\partial_z b(-1/t)\partial_\rho b(t)+\partial_z d(-1/t)\partial_\rho d(t))r^2\,dt$$
$$=-\frac{1}{2\pi\sqrt{-1}}\oint\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\times(\partial_z b(t)\partial_\rho b(-1/t)+\partial_z d(t)\partial_\rho d(-1/t))r^2\,dt,$$

we obtain

$$\begin{aligned}\Delta=\frac{1}{2\pi\sqrt{-1}}\oint\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)&\times\{-\partial_z b(t)\partial_\rho c(t)+\partial_z d(t)\partial_\rho a(t)+\partial_z c(t)\partial_\rho b(t)\\&-\partial_z a(t)\partial_\rho d(t)+(\partial_z b(t)\partial_\rho d(t)-\partial_z d(t)\partial_\rho b(t))2n\\&+(\partial_z b(-1/t)\partial_\rho b(t)+\partial_z d(-1/t)\partial_\rho d(t))2r^2\}\,dt.\end{aligned}\tag{6.29}$$

Note that from the equations (6.13.a) and (6.13.b) the relation below is easily obtained:

$$\begin{aligned}&\partial_z b(t)\partial_\rho d(t)-\partial_z d(t)\partial_\rho b(t)\\&=\frac{t}{2}(\partial_z b(t)(q_3 b(t)-q_1 d(t))-\partial_z d(t)(q_1 b(t)+q_2 d(t))).\end{aligned}\tag{6.30}$$

So, using (6.30), we calculate the terms containing n in (6.29) as follows:

$$\frac{1}{2\pi\sqrt{-1}}\oint\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)(\partial_z b(t)\partial_\rho d(t)-\partial_z d(t)\partial_\rho b(t))2n\,dt$$
$$=\frac{1}{2\pi\sqrt{-1}}\oint\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\frac{t}{2}(\partial_z b(t)(q_3 b(t)-q_1 d(t))$$
$$-\partial_z d(t)(q_1 b(t)+q_2 d(t)))2n\,dt$$

using (6.18.a) and (6.18.b),

$$=\frac{1}{2\pi\sqrt{-1}}\oint\frac{\rho}{2}\left(1+\frac{1}{t^2}\right)\frac{t}{2}(\partial_z b(t)(q_3 a(t)-q_1 c(t))$$
$$-\partial_z d(t)(q_1 a(t)+q_2 c(t))+\partial_z b(t)(-q_3 d(-1/t)-q_1 b(-1/t))r^2$$
$$-\partial_z d(t)(-q_1 d(-1/t)+q_2 b(-1/t))r^2)2\,dt$$

applying the invariant procedure, which is related to the replacement $t\mapsto -\frac{1}{t}$, to the terms containing r in the integrand and using the properties (6.16.b) in that time,

$$= \frac{1}{2\pi\sqrt{-1}} \oint \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\frac{t}{2}(\partial_z b(t)(q_3 a(t) - q_1 c(t))$$

$$- \partial_z d(t)(q_1 a(t) + q_2 c(t)) - \partial_z b(-1/t)(q_2 d(t) + q_1 b(t))r^2$$

$$+ \partial_z d(-1/t)(q_1 d(t) - q_3 b(t))r^2)2dt.$$

Combining this result with the following relation, which is contained in Lemma 6.1,

$$\frac{1}{2\pi\sqrt{-1}} \oint \frac{\rho}{4}\left(1 + \frac{1}{t^2}\right)\left(t - \frac{1}{t}\right)(\partial_z b(-1/t)\partial_z b(t) + \partial_z d(-1/t)\partial_z d(t))2r^2\, dt = 0,$$

we reach the following relation:

(6.31)

$$\varDelta = \frac{1}{2\pi\sqrt{-1}} \oint \frac{\rho}{2}\left(1 + \frac{1}{t^2}\right) \times \{-\partial_z b(t)\partial_\rho c(t) + \partial_z d(t)\partial_\rho a(t) + \partial_z c(t)\partial_\rho b(t)$$

$$- \partial_z a(t)\partial_\rho d(t) + t(\partial_z b(t)(q_3 a(t) - q_1 c(t)) - \partial_z d(t)(q_1 a(t) + q_2 c(t)))\}\, dt.$$

However, since the expression (6.31) for $\varDelta$ has a nonzero coefficient of t^3 in the integrand, we change the relation into another one without it. This can be down as follows. By the equations (6.10.a) and (6.10.b) the last term is changed into

$$\frac{1}{2\pi\sqrt{-1}} \oint t((o_1 b(t) + o_2 d(t) - \partial_t b(t))(q_3 a(t) - q_1 c(t))$$

$$- (o_3 b(t) - o_1 d(t) - \partial_t d(t))(q_1 a(t) + q_2 c(t)))\, dt,$$

since $\det \mathscr{P}(t) = 1$,

$$= \frac{1}{2\pi\sqrt{-1}} \oint t(-\partial_t b(t)(q_3 a(t) - q_1 c(t)) + \partial_t d(t)(q_1 a(t) + q_2 c(t))$$

$$+ w_1 a(t)b(t) + w_2 c(t)d(t) + w_3 b(t)c(t) + w_4)\, dt,$$

where we put

$$w_1 = o_1 q_3 - o_3 q_1,\ w_2 = o_1 q_2 - o_2 q_1,\ w_3 = o_2 q_3 - o_3 q_2,\ w_4 = o_1 q_1 + o_2 q_3.$$

Inserting the above equality into (6.31) we get a desired expression:

(6.32)

$$\Delta = \frac{1}{2\pi\sqrt{-1}} \oint \left(\frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)(-\partial_z b(t)\partial_\rho c(t) + \partial_z d(t)\partial_\rho a(t) + \partial_z c(t)\partial_\rho b(t)\right.$$
$$- \partial_z a(t)\partial_\rho d(t)) + t(-\partial_t b(t)(q_3 a(t) - q_1 c(t)) + \partial_t d(t)(q_1 a(t) + q_2 c(t))$$
$$\left. + w_1 a(t)b(t) + w_2 c(t)d(t) + w_3 b(t)c(t) + w_4)\right) dt.$$

Hence, using the equations (6.10.a), (6.10.b), (6.10.c) and (6.10.d), we have the relation:

(6.33)

$$\partial_\rho \Xi(\mathscr{P}(t), g) = \frac{1}{2} \times \frac{1}{2\pi\sqrt{-1}} \oint (2(o_3 a(t) - o_1 c(t))\partial_\rho b(t) - 2(o_1 a(t) + o_2 c(t))\partial_\rho d(t)$$
$$+ t(-\partial_t b(t)(q_3 a(t) - q_1 c(t)) + \partial_t d(t)(q_1 a(t) + q_2 c(t))$$
$$+ w_1 a(t)b(t) + w_2 c(t)d(t) + w_3 b(t)c(t) + w_4)) dt.$$

Then, a tedious calculation by use of (6.11.a), (6.11.b), (6.14), (6.11.c), (3.21), (3.23.c), (3.23.d) and (3.24) tells us that the proposition is proved.

7. Example

Let $g(s) = \begin{pmatrix} 1 & c_0 + c_1 s \\ 0 & 1 \end{pmatrix}^{-1}$ where c_0, c_1 are arbitrary real numbers. So we consider the action of $j_{ce}(g \times 1) \in \mathscr{F}\mathscr{H}_{ce}$ to the centrally extended Minkowski potential $\mathscr{P}^e_{ce}$, that is to say,

$$k_{ce}\mathscr{P}^e_{ce} j_{ce}(g \times 1)^{-1} = (kj(g^{-1}), e^{\Xi(k, g^{-1})}) \in \mathscr{F}\mathscr{P}_{ce}.$$

Then it follows that

$$k(t) = \begin{pmatrix} a_0 + a_1 t & b_0 + b_1 t^{-1} \\ -b_0 + b_1 t & a_0 - a_1 t^{-1} \end{pmatrix}$$

with

$$a_0 = \frac{\sqrt{1 - c_1^2\rho^2}}{\sigma}, \qquad a_1 = \frac{c_1(c_0 + 2c_1 z)\rho}{\sigma\sqrt{1 - \rho^2}},$$

$$b_0 = \frac{-(c_0 + 2c_1 z)}{\sigma\sqrt{1 - c_1^2\rho^2}}, \quad b_1 = \frac{-c_1\rho\sqrt{1 - c_1^2\rho^2}}{\sigma},$$

where we put $\sigma = \sqrt{(1 - c_1\rho^2)^2 + (c_0 + 2c_1 z)^2}$.

Let $(\mathscr{P}_g, \tau_g) = (kj(g^{-1}), e^{\Xi(k,g^{-1})})$. Then $\mathscr{P}_g(t)$ and τ_g is given by

$\mathscr{P}_g(t) = P_0 + P_1 t + P_2 t^2$

with

$$P_0 = \frac{1}{\sigma\sqrt{1-c_1^2\rho^2}}\begin{pmatrix} 1-c_1^2\rho^2 & 0 \\ c_0+2c_1 z & (1-c_1^2\rho^2)^2+(c_0+2c_1 z)^2 \end{pmatrix},$$

$$P_1 = \frac{c_1\rho}{\sigma\sqrt{1-c_1^2\rho^2}}\begin{pmatrix} c_0+2c_1 z & c_1^2\rho^2-1+(c_0+2c_1 z)^2 \\ -(1-c_1^2\rho^2) & (c_0+2c_1 z)(\rho^2 c_1^2-2) \end{pmatrix},$$

$$P_2 = \frac{c_1^2\rho^2}{\sigma\sqrt{1-c_1^2\rho^2}}\begin{pmatrix} 0 & -(c_0+2c_1 z) \\ 0 & 1-c_1^2\rho^2 \end{pmatrix},$$

and

$\tau_g = \sqrt{1-c_1^2\rho^2}.$

Hence $m_g = \bar{\theta}(proj(\mathscr{P}_g(t))) = \theta(\mathscr{P}_g(0)^{-1})\mathscr{P}_g(0)$ is given by

$$m_g = \frac{1}{1-c_1^2\rho^2}\begin{pmatrix} 1 & c_0+2c_1 z \\ c_0+2c_1 z & (1-c_1^2\rho^2)^2+(c_0+2c_1 z)^2 \end{pmatrix}.$$

It follows from the parametrization (2.12) of m_g that

$$f = \frac{(1-c_1^2\rho^2)^2}{(c_0+2c_1 z)^2+(1-c_1^2\rho^2)^2}$$

and

$$\psi = \frac{c_0+2c_1 z}{(c_0+2c_1 z)^2+(1-c_1^2\rho^2)^2}.$$

Solving the differential equation (2.9), we get

$$\gamma = c_1\rho^2\left(\frac{(c_0+2c_1 z)^2}{(1-c_1^2\rho^2)^2}-1\right).$$

Therefore the metric

$$ds_g^2 = (h_g)_{pq}dx^p \otimes dx^q - \lambda_g^2(dz\otimes dz + d\rho\otimes d\rho)$$

is given by

$$h_g = \begin{pmatrix} f\gamma^2-\rho^2/f & f\gamma \\ f\gamma & f \end{pmatrix}$$

$$
= \frac{1 - c_1^2\rho^2}{(c_0 + 2c_1 z)^2 + (1 - c_1^2\rho^2)^2}
$$

$$
\times \begin{pmatrix} \rho^2\left(\dfrac{1-(1+(c_0+2c_1 z)^2)^2}{1-c_1^2\rho^2} + (c_1^2\rho^2-(1-c_1^2\rho^2)^2)\right) & c_1\rho\left(\dfrac{(c_0+2c_1 z)^2}{1-c_1^2\rho^2}-1\right) \\ c_1\rho\left(\dfrac{(c_0+2c_1 z)^2}{1-c_1^2\rho^2}-1\right) & 1 \end{pmatrix},
$$

and

$$
\lambda_g = \sqrt{\frac{(c_0 + 2c_1 z)^2 + (1 - c_1^2\rho^2)^2}{1 - c_1^2\rho^2}}.
$$

If $c_0 = 0$ and $c_1 = 1$, then this is the first example given in [14] (also see [7]).

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