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Borel-Weil theory and Feynman path integrals on flag manifolds

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0 Introduction

In [6] we computed path integrals on coadjoint orbits of the Heisenberg group, $SU(1, 1)$ and $SU(2)$ etc.. As to the Heisenberg group, we succeeded in computing the path integrals for complex polarizations as well as real polarizations.

For the complex polarizations of $SU(1, 1)$ and $SU(2)$, however, we found it difficult to carry out the computation of path integrals, so that we computed the path integrals without Hamiltonians. Soon after we encountered difficulty of divergence of the path integrals along the method in [6].

For the complex polarizations of $SU(1, 1)$ and $SU(2)$, by taking the operator ordering into account and then regularizing the path integrals by use of the explicit form of the integrand, we computed the path integrals with Hamiltonians in [7].

In this paper, we shall give an idea how to regularize the path integrals for complex polarizations of any connected semisimple Lie group G which contains a compact Cartan subgroup T and shall show, along this idea, that the path integral gives the kernel function of the irreducible unitary representation of G realized by Borel-Weil theory.

Our idea is roughly explained as follows. Let $\mathfrak{h}$ be the Lie algebra of T and $\mathfrak{h}^{\mathbf{C}}$ the complexification of $\mathfrak{h}$. Denote by $\mathfrak{n}^+$ and $\mathfrak{n}^-$ the Lie algebras spanned by the positive root vectors and the negative root vectors, respectively. For any integral form Λ on $\mathfrak{h}^{\mathbf{C}}$ we denote by ξ_Λ the holomorphic character of $T^{\mathbf{C}}$ defined by Λ and by L_Λ the associated holomorphic line bundle on the flag manifold G/T. Let π_Λ be the irrducible unitary representation of G on the Hilbert space of all square integrable holomorphic sections of L_Λ which is realized by the Borel-Weil theorem.

Put $\lambda = \sqrt{-1}\Lambda$. Then for any element Y of the Lie algebra of G, the Hamiltonian on the flag manifold G/T is defind by

$$\begin{aligned} H_Y(g) &= \langle \mathrm{Ad}^*(g)\lambda, Y \rangle \\ &= \sqrt{-1}\Lambda(\mathrm{Ad}(g^{-1})Y). \end{aligned}$$

Since the path integral of this Hamiltonian is divergent we regularize it by replacing

$$e^{-\sqrt{-1}H_Y(g)} = e^{\Lambda(H(\mathrm{Ad}(g^{-1})Y))}$$
$$= \xi_\Lambda(\exp(H(\mathrm{Ad}(g^{-1})Y)))$$

by

$$\xi_\Lambda(h(\exp(\mathrm{Ad}(g^{-1})Y))),$$

where H and h denote the projection operators:

$$H\colon \mathfrak{n}^+ + \mathfrak{h}^{\mathbf{C}} + \mathfrak{n}^- \longrightarrow \mathfrak{h}^{\mathbf{C}},$$
$$h\colon \exp\mathfrak{n}^+ \exp\mathfrak{h}^{\mathbf{C}} \exp\mathfrak{n}^- \longrightarrow \exp\mathfrak{h}^{\mathbf{C}} = T^{\mathbf{C}}.$$

There are plenty of references on path integrals. For further references see, e.g., references in [6].

1 Preliminaries

Let G be a connected semisimple Lie group such that there exists a complexification $G^{\mathbf{C}}$ with $\pi_1(G^{\mathbf{C}}) = \{1\}$ and such that rank $G = \dim T$, T a maximal torus of G. Let K be a maximal compact subgroup of G which contains T, and $\mathfrak{k}$ the Lie algebra of K. Note that G can be realized as a matrix group. We denote the conjugation of $G^{\mathbf{C}}$ with respect to G, and that of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$, both by $^-$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and T. We denote complexifications of $\mathfrak{g}$ and $\mathfrak{h}$ by $\mathfrak{g}^{\mathbf{C}}$ and $\mathfrak{h}^{\mathbf{C}}$, respectively. Then $\mathfrak{h}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$.

Let Δ denote the set of all nonzero roots and Δ^+ the set of all positive roots. Then we have the root space decomposition

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha.$$

Define

$$\mathfrak{n}^{\pm} = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{\pm\alpha}, \quad \mathfrak{b} = \mathfrak{h}^{\mathbf{C}} + \mathfrak{n}^-.$$

Let N, N^-, B and $T^{\mathbf{C}}$ be the analytic subgroups corresponding to $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{b}$ and $\mathfrak{h}^{\mathbf{C}}$, respectively.

We fix an integral form Λ on $\mathfrak{h}^{\mathbf{C}}$.

Let

$$\xi_\Lambda\colon T \longrightarrow U(1), \qquad \exp H \longmapsto e^{\Lambda(H)}$$

be the corresponding unitary character of T, and

$$\xi_\Lambda\colon T^{\mathbf{C}} \longrightarrow \mathbf{C}^\times, \qquad \exp H \longmapsto e^{\Lambda(H)}$$

the corresponding holomorphic character of $T^{\mathbf{C}}$. Then ξ_Λ extends uniquely to a holomorphic one-dimensional representation of B:

$$\xi_\Lambda\colon B = T^{\mathbf{C}}N^- \longrightarrow \mathbf{C}^\times, \qquad \exp H\cdot n^- \longmapsto e^{\Lambda(H)}.$$

Let $\tilde{L}_\Lambda$ be the holomorphic line bundle over $G^{\mathbf{C}}/B$ associated to the holomorphic one-dimensional representation ξ_Λ of B. We denote by L_Λ the restriction of $\tilde{L}_\Lambda$ to the open submanifold G/T of $G^{\mathbf{C}}/B$:

$$\begin{array}{ccccc} G & \hookrightarrow & GB & \hookrightarrow & G^{\mathbf{C}} \\ \downarrow & & \downarrow & & \downarrow \\ G/T & = & GB/B & \hookrightarrow & G^{\mathbf{C}}/B \end{array}$$

and

$$\begin{array}{ccc} L_\Lambda & \hookrightarrow & \tilde{L}_\Lambda \\ \downarrow & & \downarrow \\ G/T & \hookrightarrow & G^{\mathbf{C}}/B. \end{array}$$

Then we can identify the space of all holomorphic sections of L_Λ with

$$\Gamma(L_\Lambda) = \{f\colon GB \xrightarrow{\text{hol.}} \mathbf{C}; f(xb) = \xi_\Lambda(b)^{-1}f(x),\ x\in GB,\ b\in B\}.$$

Let π_Λ be a representation of G on $\Gamma(L_\Lambda)$ defined by

$$\pi_\Lambda(g)f(x) = f(g^{-1}x) \qquad \text{for } g\in G,\ x\in GB \quad \text{and} \quad f\in\Gamma(L_\Lambda).$$

For any $f\in\Gamma(L_\Lambda)$ we define

$$\|f\|^2 = \int_G |f(g)|^2 dg,$$

where dg is the Haar measure on G. We put

$$\Gamma_2(L_\Lambda) = \{f\in\Gamma(L_\Lambda);\ \|f\| < +\infty\}.$$

Then the Borel-Weil theorem asserts that $(\pi_\Lambda, \Gamma_2(L_\Lambda))$ is an irreducible unitary representation of G (Bott [1], Kostant [8] and Harish-Chandra [3][4][5]).

For the moment we assume that G is noncompact.

We fix a Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

We denote complexifications of $\mathfrak{k}$ and $\mathfrak{p}$ by $\mathfrak{k}^{\mathbf{C}}$ and $\mathfrak{p}^{\mathbf{C}}$, respectively.

Let Δ_c and Δ_n denote the set of all compact roots and noncompact roots, respectively.

Now we assume that $\Gamma_2(L_\Lambda) \neq 0$. Then there exists an ordering in the dual space of $\mathfrak{h}_{\mathbf{R}} = i\mathfrak{h}$ so that every noncompact positive root is larger than every compact positive root. The ordering determines sets of compact positive roots Δ_c^+ and noncompact positive roots Δ_n^+. Furthermore Λ satisfies the following two conditions:

$$\langle \Lambda, \alpha \rangle \geqslant 0 \qquad \text{for } \alpha \in \Delta_c^+, \tag{1.1a}$$

$$\langle \Lambda + \rho, \alpha \rangle < 0 \qquad \text{for } \alpha \in \Delta_n^+, \tag{1.1b}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

Then (1.1a) assures the existence of a unique element ψ_Λ in $\Gamma(L_\Lambda)$ which satisfies the following conditions:

$$\pi_\Lambda(h)\psi = \xi_\Lambda(h)\psi_\Lambda \qquad \text{for } h \in T, \tag{1.2a}$$

$$d\pi_\Lambda(X)\psi_\Lambda = 0 \qquad \text{for } X \in \mathfrak{n}^+, \tag{1.2b}$$

$$\psi_\Lambda(e) = 1, \tag{1.2c}$$

where $d\pi_\Lambda$ is the complexification of the differential representation of π_Λ. And (1.1b) implies that ψ_Λ is an element of $\Gamma_2(L_\Lambda)$. We normalize dg so that $\int_G |\psi_\Lambda(g)|^2 dg = 1$.

Define D to be an open subset of $\mathfrak{n}^+$ which satisfies $\exp D \cdot B = GB \cap NB$, where exp is the exponential map of $\mathfrak{n}^+$ onto N:

$$\begin{array}{ccc} \exp: \mathfrak{n}^+ & \xrightarrow{\sim} & N \\ \cup & & \cup \\ D & \longrightarrow & \exp D. \end{array}$$

For each $\alpha \in \Delta$, we choose an E_α of $\mathfrak{g}^\alpha$ such that

$$B(E_\alpha, E_{-\alpha}) = 1$$

and

$$E_\alpha - E_{-\alpha}, \quad \sqrt{-1}(E_\alpha + E_{-\alpha}) \in \mathfrak{g}_u,$$

where $B(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}^{\mathbf{C}}$ and $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\,\mathfrak{p}$, the compact real form of $\mathfrak{g}^{\mathbf{C}}$. Note that

$$\bar{E}_\alpha = \begin{cases} -E_{-\alpha} & \text{for } \alpha \in \Delta_c, \\ E_{-\alpha} & \text{for } \alpha \in \Delta_n. \end{cases}$$

We put $m = \dim \mathfrak{n}^+$ and introduce holomorphic coordinates on $\mathfrak{n}^+$ and $\mathfrak{n}^-$ by

$$\mathbf{C}^m \longrightarrow \mathfrak{n}^+, \quad (z_\alpha)_{\alpha\in\Delta} \longmapsto z = \sum_{\alpha\in\Delta^+} z_\alpha E_\alpha,$$

$$\mathbf{C}^m \longrightarrow \mathfrak{n}^-, \quad (w_\alpha)_{\alpha\in\Delta^+} \longmapsto w = \sum_{\alpha\in\Delta^+} w_\alpha E_{-\alpha}.$$

We put

$$n_z = \exp \sum_{\alpha\in\Delta^+} z_\alpha E_\alpha \in N,$$

$$n_w^- = \exp \sum_{\alpha\in\Delta^+} z_\alpha E_{-\alpha} \in N.$$

Let $\Gamma(D)$ be the space of all holomorphic fucntions on D. The following correspondence gives an isomorphism of $\Gamma(L_\Lambda)$ into $\Gamma(D)$:

$$\Phi\colon \Gamma(L_\Lambda) \longrightarrow \Gamma(D), \quad f \longmapsto F, \tag{1.3}$$

where

$$F(z) = f(n_z) \qquad \text{for } z \in D.$$

We put $\mathscr{H}_\Lambda = \Phi(\Gamma_2(L_\Lambda))$. Let us denote by $U_\Lambda(g)$ the representation of G on $\mathscr{H}_\Lambda$ such that the diagram

$$\begin{array}{ccc} \Gamma_2(L_\Lambda) & \longrightarrow & \mathscr{H}_\Lambda \\ {\scriptstyle \pi_\Lambda(g)}\big\downarrow & & \big\downarrow{\scriptstyle U_\Lambda(g)} \\ \Gamma_2(L_\Lambda) & \longrightarrow & \mathscr{H}_\Lambda \end{array}$$

is commutative for all $g \in G$.

We normalize the invariant measure μ on G/T such that

$$\int_G f(g)\,dg = \int_{G/T} \left(\int_T f(gh)\,dh \right) d\mu(gT) \qquad \text{for any } f \in C_c^\infty(G),$$

where dh is the Haar measure on T such that $\int_T dh = 1$.

We denote the measure on D also by μ which is induced by the complex analytic isomorphism:

$$\phi: D \hookrightarrow G/T.$$

By the definition of D, $\phi(D)$ is open dense in G/T. For any $x \in NT^{\mathbf{C}}N^-$ we denote the N-, $T^{\mathbf{C}}$- and N^-- component by $n(x)$, $h(x)$ and $n^-(x)$, respectively. Then, for any $f \in \Gamma(L_\Lambda)$, $g \in G$ and $h \in T$ we have

$$|f(gh)| = |f(g)| \quad \text{and} \quad |\xi_\Lambda(h(gh))| = |\xi_\Lambda(h(g))|.$$

This shows that $|f(g)|$ and $|\xi_\Lambda(h(g))|$ can be regarded as functions on G/T.

We put

$$J_\Lambda(z) = |\xi_\Lambda(h(\phi(z)))|^{-2}.$$

Then we have

$$\begin{aligned}\int_G |f(g)|^2 dg &= \int_{G/T} |f(g)|^2 d\mu(gT) \\ &= \int_D |F(z)|^2 J_\Lambda(z) d\mu(z).\end{aligned}$$

We define

$$\Gamma_2(D) = \{F \in \Gamma(D);\ \|F\| < +\infty\},$$

where

$$\|F\|^2 = \int_D |F(z)|^2 J_\Lambda(z) d\mu(z).$$

In case that G is compact, we remark in the above that

$$G = K, \quad GB = G^{\mathbf{C}}, \quad D = \mathfrak{n}^+, \quad \mathfrak{g} = \mathfrak{k}, \quad \mathfrak{p} = 0,$$

$$\Delta_c = \Delta, \quad \Delta_n = \emptyset, \quad \Gamma_2(L_\Lambda) = (L_\Lambda),$$

and $\Gamma(L_\Lambda) \neq \{0\}$ if and only if Λ is dominant.

There are various proofs of the Borel-Weil theorem. Since ψ_Λ plays a crucial role in this paper, we give a proof which exhibits the important properties of ψ_Λ.

The unitarity of $(\pi_\Lambda, \Gamma_2(L_\Lambda))$ is obvious. Suppose that there exists a nonzero subspace V of $\Gamma_2(L_\Lambda)$ which is closed and invariant under the action of G. Let f_0 be a nonzero element of V. Then there exists $g \in G$ such that $f_0(g) \neq 0$. We put $f = \pi_\Lambda(g^{-1})f_0$. Since V is G-invariant, $f \in V$. We note that

$$f(e) = (\pi_\Lambda(g^{-1})f_0)(e) = f_0(g) \neq 0.$$

For any $h \in T$,

$$\frac{1}{f(e)}\ \overline{\xi_\Lambda(h)}\ \pi_\Lambda(h) f \in V,$$

by the invariance of V.

We put

$$\hat{f}(x) = \frac{1}{f(e)} \int_T \overline{\xi_\Lambda(h)}\, (\pi_\Lambda(h) f)(x) dh \qquad \text{for } x \in GB.$$

Since V is closed, $\hat{f} \in V$. In particular

$$\hat{f}(e) = \frac{1}{f(e)} \int_T \overline{\xi_\Lambda(h)}\, (\pi_\Lambda(h) f)(e) dh = 1.$$

For any $h_1 \in T$, since dh is the Haar measure, we have

$$\begin{aligned}
(\pi_\Lambda(h_1)\hat{f})(x) &= \int_T \overline{\xi_\Lambda(h)}\, (\pi_\Lambda(h) f)(h_1^{-1} x) dh \\
&= \int_T \overline{\xi_\Lambda(h)}\, f(h^{-1} h_1^{-1} x) dh \\
&= \int_T \overline{\xi_\Lambda(h_1 h)}\, (\pi_\Lambda(h) f)(x) dh \\
&= \xi_\Lambda(h_1)^{-1} \hat{f}(x).
\end{aligned}$$

Using the fact that the weight space with the weight Λ is one dimensional, we obtain that $\hat{f} = \psi_\Lambda$. Thus $\psi_\Lambda \in V$.

If $V \neq \Gamma_2(L_\Lambda)$, then we can show that $V^\perp \neq \{0\}$ is a closed invariant subspace, which implies that $\psi_\Lambda \in V^\perp$, contradicting that $V \cap V^\perp = \{0\}$. Therefore $V = \Gamma_2(L_\Lambda)$. This shows that π_Λ is irreducible.

2. Kernel Functions

We retain the notation of §1. For the rest of this paper we assume that

$\Gamma_2(L_\Lambda) \neq \{0\}$.

We start with the case that G is noncompact. We put

$$\mathfrak{p}_\pm = \sum_{\alpha \in \Delta_n^\pm} \mathfrak{g}^\alpha.$$

We denote by $K^{\mathbf{C}}$, P_+ and P_- the analytic subgroups of $G^{\mathbf{C}}$ corresponding to $\mathfrak{k}^{\mathbf{C}}$, $\mathfrak{p}_+$ and $\mathfrak{p}_-$, respectively. Then there is a unique open subset Ω of $\mathfrak{p}_+$ such that $GB = GK^{\mathbf{C}}P_- = \exp \Omega K^{\mathbf{C}}P_-$. We put $W = P_+K^{\mathbf{C}}P_-$. Then by (1.2a) and (1.2b), ψ_Λ is uniquely extended to a holomorphic function on W (which we denote also by ψ_Λ) such that

$$\psi_\Lambda(tx) = \xi_\Lambda(t)^{-1}\psi_\Lambda(x) \qquad \text{for all } t \in T^{\mathbf{C}} \quad \text{and} \quad x \in W, \tag{2.1}$$

and

$$\psi_\Lambda(nx) = \psi_\Lambda(x) \qquad \text{for all } n \in N \quad \text{and} \quad x \in W. \tag{2.2}$$

Now we prove that

$$P_+K^{\mathbf{C}}P_- = B^*GB, \tag{2.3}$$

where $(\cdot)^* = (\bar{\cdot})^{-1}$. Since $\exp \Omega K^{\mathbf{C}}P_- = GB$, we have

$$P_+K^{\mathbf{C}}P_- = P_+GB.$$

It follows that

$$P_+K^{\mathbf{C}}P_- = P_+K^{\mathbf{C}}GB = \bar{B}GB$$

because $K^{\mathbf{C}}$ normalizes P_+. Thus

$$W = B^*GB.$$

In case that G is compact, obviously $W = G^{\mathbf{C}}$, and as is easily seen, ψ_Λ satisfies (2.1) and (2.2).

Henceforth, throughout the paper, the discussions are valid for the compact case as well as for the noncompact case.

Define a scalar function $\mathscr{K}_\Lambda$ on $GB \times G\bar{B}$ by

$$\mathscr{K}_\Lambda(g_1, \bar{g}_2) = \psi_\Lambda(g_2^* g_1). \tag{2.4}$$

Then $\mathscr{K}_\Lambda(\cdot, \bar{g}_2)$, with g_2 fixed, can be regarded as an element of $\Gamma_2(L_\Lambda)$.

We define a scalar function K_Λ on $D \times \bar{D}$ by

$$K_\Lambda(z', \bar{z}'') = \mathscr{K}_\Lambda(n_{z'}, \bar{n}_{z''}).$$

Note that $K_\Lambda(z', z'')$ is holomorphic in the first variable and anti-holomorphic in the second and that it can be regarded, with $n_{z''}$ fixed, as an element of $\mathscr{H}_\Lambda$.

Now we define operators $\mathscr{K}_\Lambda$ and K_Λ on $\Gamma_2(L_\Lambda)$ and $\mathscr{H}_\Lambda$ by

$$(\mathscr{K}_\Lambda f)(g'') = \int_G \mathscr{K}_\Lambda(g'', \bar{g}')f(g')dg' \qquad \text{for } f \in \Gamma_2(L_\Lambda)$$

and

$$(K_\Lambda F)(z'') = \int_D K_\Lambda(z'', \bar{z}')F(z')J_\Lambda(z')d\mu(z') \qquad \text{for } F \in \mathscr{H}_\Lambda,$$

where dg' is the Haar measure on G and $d\mu(z')$ is the invariant measure on D given in §1. Then we have the following commutative diagram:

$$\begin{tikzcd} \Gamma_2(L_\Lambda) \arrow[r] \arrow[d, "\mathscr{K}_\Lambda"'] & \mathscr{H}_\Lambda \arrow[d, "K_\Lambda"] \\ \Gamma_2(L_\Lambda) \arrow[r] & \mathscr{H}_\Lambda, \end{tikzcd}$$

where the horizontal maps are given by (1.3).

LEMMA 2.1. *The weight vector $\psi_\Lambda \in \Gamma_2(L_\Lambda)$ has the following property*:

$$\psi_\Lambda(x^*) = \overline{\psi_\Lambda(x)} \qquad \text{for } x \in W,$$

where denotes the complex conjugation in $\mathbf{C}$.

Proof: We set

$$\phi(x) = \overline{\psi_\Lambda(x^*)}.$$

Since the map $x \mapsto x^*$ is anti-holomorphic, ϕ is a holomorphic function on W.

First we show that ϕ also enjoys the properties (1.2a-c).

For any $h \in T$ and $x \in W$ we have

$$\begin{aligned}(\pi_\Lambda(h)\phi)(x) &= \phi(h^{-1}x) \\ &= \overline{\psi_\Lambda(x^*\bar{h})} \\ &= \overline{\xi_\Lambda(\bar{h})^{-1}\psi_\Lambda(x^*)} \\ &= \xi_\Lambda(h)\phi(x).\end{aligned}$$

For any $n \in N$ and $x \in W$ we have

$$\phi(nx) = \overline{\psi_\Lambda(x^*n^*)}$$

$$= \overline{\xi_\Lambda(n^*)^{-1}\psi_\Lambda(x^*)}$$

$$= \overline{\psi_\Lambda(x^*)}$$

$$= \phi(x).$$

Obviously $\phi(e)=1$. Thus ϕ satisfies (1.2a-c).

Next we show that ϕ is an element of $\Gamma_2(L_\Lambda)$.

Note that $(n^-)^*\in N$ for $n^-\in N^-$. Then for any $x\in W$ and $b=tn^-\in B=T^{\mathbf{C}}N^-$

$$\begin{aligned}\phi(xb) &= \overline{\psi_\Lambda((n^-)^*t^*x^*)}\\ &= \overline{\psi_\Lambda(t^*x^*)}\\ &= \overline{\xi_\Lambda(t^*)}^{-1}\,\overline{\psi_\Lambda(x^*)}\\ &= \xi_\Lambda(t)^{-1}\,\overline{\psi_\Lambda(x^*)}\\ &= \xi_\Lambda(b)^{-1}\phi(x),\end{aligned}$$

because $\overline{\xi_\Lambda(t)}=\xi_\Lambda(t)^{-1}$ and $\psi_\Lambda(t^*x^*)=\xi_\Lambda(t^*)^{-1}\psi_\Lambda(x^*)$ for $t\in T^{\mathbf{C}}$.

Hence the uniqueness of the element of $\Gamma_2(L_\Lambda)$ which satisfies (1.1a-c) implies that $\phi=\psi_\Lambda$. This completes the proof. ∎

Now we can prove that $\mathscr{K}_\Lambda$ and K_Λ are the identity operators.

PROPOSITION 2.2. *$\mathscr{K}_\Lambda$ is the identity operator on the Hilbert space $\Gamma_2(L_\Lambda)$ i.e.*

$$\mathscr{K}_\Lambda f=f \qquad \text{for } f\in\Gamma_2(L_\Lambda).$$

Proof: First we show that $\mathscr{K}_\Lambda$ is a scalar operator. Since $\Gamma_2(L_\Lambda)$ is an irreducible representation of G, it is sufficient, by Schur's lemma, to show that

$$\pi_\Lambda(g)\circ\mathscr{K}_\Lambda=\mathscr{K}_\Lambda\circ\pi_\Lambda(g) \qquad \text{for all } g\in G. \tag{2.5}$$

For any $f\in\Gamma_2(L_\Lambda)$

$$\begin{aligned}(\pi_\Lambda(g)(\mathscr{K}_\Lambda))(g'') &= (\mathscr{K}_\Lambda f)(g^{-1}g'')\\ &= \int_G \mathscr{K}_\Lambda(g^{-1}g'',\bar{g}')f(g')dg'\end{aligned}$$

$$= \int_G \mathscr{K}_\Lambda(g'', \overline{gg'}) f(g') dg'$$

$$= \int_G \mathscr{K}_\Lambda(g'', \bar{g}') f(g^{-1} g') dg',$$

$$= \int_G \mathscr{K}_\Lambda(g'', \bar{g}')(\pi_\Lambda(g) f)(g') dg'$$

$$= (\mathscr{K}_\Lambda(\pi_\Lambda(g) f))(g'').$$

Thus (2.5) implies that $\mathscr{K}_\Lambda = cI (\exists c \in \mathbf{C})$ where I denotes the identity operator.

Now applying $\mathscr{K}_\Lambda$ to ψ_Λ, we have

$$c = \int_G |\psi_\Lambda(g')|^2 dg' = 1,$$

where we used (1.2c) and Lemma 2.1. ■

COROLLARY 2.3. *K_Λ is the identity operator on $\mathscr{H}_\Lambda$ i.e.*

$$K_\Lambda F = F \qquad \text{for } F \in \mathscr{H}_\Lambda.$$

3 Path Integrals

In this section we calculate path integrals on the flag maniforld G/T. Let $\lambda = \sqrt{-1}\Lambda$. We extend λ to an element of the dual space of $\mathfrak{g}^{\mathbf{C}}$ which vanishes on the orthogonal complement of $\mathfrak{h}^{\mathbf{C}}$ in $\mathfrak{g}^{\mathbf{C}}$ with respect to the Killing form.

For any $g \in NB$ we can decompose it as

$$g = n_z n_w^- t \qquad \text{where} \quad n_z \in N,\, n_w^- \in N^-,\, t \in T^{\mathbf{C}}. \tag{3.1}$$

Recall that in §1 we have parametrized n_z and n_w^- as

$$n_z = \exp \sum_{\alpha \in \Delta^+} z_\alpha E_\alpha,$$

$$n_w^- = \exp \sum_{\alpha \in \Delta^+} w_\alpha E_{-\alpha}.$$

LEMMA 3.1. *For any $g \in G \cap NB$, if we decompose it as $g = n_z n_w^- t$(see (3.1)), then we can express w in terms of z and $\bar{z}$.*

Proof: For any $g = n_z n_w^- t$, since $g^* g = e$,

$$t^* (n_w^-)^* n_z^* n_z n_w^- t = e. \tag{3.2}$$

Decomposing $n_z^* n_z = n_1 n_2 t_0$, we substiture it into (3.2).

$$\begin{aligned} e &= t^*(n_w^-)^* n_1 n_2 t_0 n_w^- t \\ &= t^*(n_w^-)^* n_1 t^{*-1} \cdot t^* n_2 t^{*-1} \cdot t^* t_0 n_w^- (t^* t_0)^{-1} \cdot t^* t_0 t. \end{aligned}$$

Thus

$$n_w^- = \bar{n}_1 = \overline{n(n_z^* n_z)} = \overline{n(n_{\bar{z}}^{-1} n_z)}$$

by the uniqueness of the decomposition (3.1). ■

We denote w in the above lemma by $w(z, \bar{z})$. For any $z \in D$, we put $g(z, \bar{z}) = n_z n_{w(z,\bar{z})}^-$.

Let d denote the exterior derivative on D. We decompose it as $d = \partial + \bar{\partial}$, where ∂ and $\bar{\partial}$ are holomorphic part and anti-holomorphic part of d, respectively.

Define

$$\theta = \lambda(g^{-1} dg) = \lambda(n_w^{-\,-1} n_z^{-1} \partial n_z n_w^-) + \lambda(t^{-1} dt),$$

where $g = n_z n_w^- t \in G$. And we choose

$$\alpha = \lambda(n_w^{-\,-1} n_z^{-1} \partial n_z n_w^-).$$

For any $Y \in \mathfrak{g}$, the Hamiltonian function is given by

$$H_Y(g) = \langle \mathrm{Ad}^*(g)\lambda, Y \rangle = \langle \mathrm{Ad}^*(g(z, \bar{z}))\lambda, Y \rangle,$$

where $g = g(z, \bar{z}) t \in G$.

We need some lemmas to compute the path integrals.

LEEMA 3.2. *For any $g \in G \cap NB$, if we decompose it as* (3.1):

$$g = n_z n_w^- t \qquad \text{where} \quad n_z \in N,\ n_w^- \in N^-,\ t \in T^{\mathbf{C}},$$

then

$$K_\Lambda(z, \bar{z}) = \xi_\Lambda(t^* t).$$

Therefore we have

$$J_\Lambda(z) = K_\Lambda(z, \bar{z})^{-1}.$$

Proof: Clearly

$$\psi_\Lambda(n_z^* n_z) = \xi_\Lambda(h(n_z^* n_z))^{-1}. \tag{3.3}$$

The fact that $g \in G$ implies that

$$(n_w^-)^* n_z^* n_z n_w^- = (t^* t)^{-1}, \tag{3.4}$$

hence

$$\psi_\Lambda((n_w^-)^* n_z^* n_z n_w^-) = \xi_\Lambda(t^* t).$$

Since $(n_w^-)^* \in N$ and $\xi_\Lambda(n_w^-) = 1$, $\psi_\Lambda((n_w^-)^* n_z^* n_z n_w^-) = \psi_\Lambda(n_z^* n_z)$. Therefore

$$\psi_\Lambda(n_z^* n_z) = \xi_\Lambda(t^* t). \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$\begin{aligned} \xi_\Lambda(h(n_z^* n_z)) &= \xi_\Lambda(t^* t)^{-1} \\ &= |\xi_\Lambda(t)|^{-2} \\ &= |\xi_\Lambda(h(g))|^{-2} \\ &= J_\Lambda(z). \end{aligned}$$

From (3.5) we have $K_\Lambda(z, \bar{z}) = \xi_\Lambda(t^* t)$. ■

PROPOSITION 3.3. *If we decompose* $g = n_z n_w^- t \in G \cap NB$ *as* (3.1), *then*

$$\begin{aligned} \Lambda(n_w^{-\,-1} n_z^{-1} \partial n_z n_w^-) &= -\Lambda((tt^*)^{-1} \partial(tt^*)) \\ &= -\partial \log K_\Lambda(z, \bar{z}). \end{aligned}$$

Proof: From (3.4) we obtain

$$\begin{aligned} (tt^*)^{-1}\partial(tt^*) = \partial(tt^*)(tt^*)^{-1} \\ = -\{ & n_w^{-\,-1} \partial n_w^- n_w^{-\,-1} n_z^{-1} n_z^{*\,-1} (n_w^-)^{*\,-1} \\ & + n_w^{-\,-1} n_z^{-1} \partial n_z n_z^{-1} n_z^{*\,-1} (n_w^-)^{*\,-1} \\ & + n_w^{-\,-1} n_z^{-1} n_z^{*\,-1} \partial n_z^* n_z^{*\,-1} (n_w^-)^{*\,-1} \\ & + n_w^{-\,-1} n_z^{-1} n_z^{*\,-1} (n_w^-)^{*\,-1} \partial (n_w^-)^* (n_w^-)^{*\,-1} \} (n_w^-)^* n_z^* n_z n_w^-, \\ = - & n_w^{-\,-1} \partial n_w^- - n_w^{-\,-1} n_z^{-1} \partial n_z n_w^- - tt^* \partial (n_w^-)^* (n_w^-)^{*\,-1} (tt^*)^{-1}, \end{aligned}$$

because $\partial n_z^* = 0$ and $(n_w^-)^* n_z^* n_z n_w^- = (tt^*)^{-1}$. Since $\langle \Lambda, X \rangle = 0$ for $X \in \mathfrak{n}^+$ or $\mathfrak{n}^-$, the statement follows immediately from Lemma 3.2. ■

It follows from Proposition 3.3 that $\alpha = -\sqrt{-1}\,\partial \log K_\Lambda(z, \bar{z})$.

Now we consider the Hamiltonian part of the action. Let

$$K_{\bar{w}}(z) = K_\Lambda(z, \bar{w})$$

and regard it as an element of $\mathscr{H}_\Lambda$.

For any $X \in \mathfrak{g}^{\mathbf{C}}$, we decompose it as $X = X_+ + H + X_-$ with $X_\pm \in \mathfrak{n}^\pm$ and $H \in \mathfrak{h}^{\mathbf{C}}$. Then we put $H(X) = H$.

LEMMA 3.4. *For any $X \in \mathfrak{g}^{\mathbf{C}}$, using the above notation, we have*

$$\xi_\Lambda(h(\exp \varepsilon X)) = \xi_\Lambda(\exp \varepsilon H(X)) + O(\varepsilon^2)$$

for sufficiently small ε.

Proof: Decompose X as $X = X_+ + H + X_-$. Then

$$\exp(\varepsilon X) = \exp(\varepsilon X_+)\exp(\varepsilon H)\exp(\varepsilon X_-) + O(\varepsilon^2).$$

Since $\xi_\Lambda(h(\cdot))$ is holomorphic at e, the statement follows. ■

LEMMA 3.5. *For any $X \in \mathfrak{g}^{\mathbf{C}}$ and $g = n_z n_w^- t \in G \cap NB$, using the above notation, we have*

$$(U_\Lambda(\exp \varepsilon X)K_{\bar{z}})(z) = K_{\bar{z}}(z)\xi_\Lambda(h(g^{-1}\exp \varepsilon X g)), \tag{3.6}$$

for sufficiently small ε.

Proof: The left hand side of (3.6) is equal to

$$\begin{aligned}
\psi_\Lambda(n_z^* \exp(-\varepsilon X)n_z) &= \psi_\Lambda(t^*(n_w^-)^* n_z^* \exp(-\varepsilon X)n_z n_w^- t t^{-1}(t^*)^{-1}) \\
&= \psi_\Lambda(g^{-1}\exp(-\varepsilon X)g t^{-1}(t^*)^{-1}) \\
&= \xi_\Lambda(t^* t)\psi_\Lambda(g^{-1}\exp(-\varepsilon X)g) \\
&= K_{\bar{z}}(z)\psi_\Lambda(g^{-1}\exp(-\varepsilon X)g) \\
&= K_{\bar{z}}(z)\xi_\Lambda(h(g^{-1}\exp(\varepsilon X)g)).
\end{aligned}$$

This completes the proof. ■

We put $z_0 = z$ and $z_N = z'$. First we compute the path integrals without Hamiltonians. Taking the same paths as in [6], we generalize Propositions 6.1 and 6.2 in [6] as follows:

$$\begin{aligned}
&\int \mathscr{D}(z, \bar{z})\exp\left(\sqrt{-1}\int_0^T \gamma^*\alpha\right) \\
&= \lim_{N\to\infty}\int_D \cdots \int_D \prod_{i=1}^{N-1} d\mu(z_i)\exp\left(\sum_{k=1}^N \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \partial \log K_\Lambda(z(t), \bar{z}_{k-1})\right) \\
&= \lim_{N\to\infty}\int_D \cdots \int_D \prod_{i=1}^{N-1} d\mu(z_i)\exp\left(\sum_{k=1}^N \log\frac{K_\Lambda(z_k, \bar{z}_{k-1})}{K_\Lambda(z_{k-1}, \bar{z}_{k-1})}\right)
\end{aligned}$$

$$= \lim_{N\to\infty} \int_D \cdots \int_D \prod_{i=1}^{N-1} d\mu(z_i) \prod_{k=1}^{N} \frac{K_\Lambda(z_k, \bar{z}_{k-1})}{K_\Lambda(z_{k-1}, \bar{z}_{k-1})}$$

$$= \lim_{N\to\infty} \int_D \cdots \int_D J_\Lambda(z_0) \prod_{i=1}^{N-1} J_\Lambda(z_i) d\mu(z_i) \prod_{k=1}^{N} K_\Lambda(z_k, \bar{z}_{k-1})$$

$$= J_\Lambda(z) K_\Lambda(z', \bar{z}),$$

where we used Lemma 3.2 and Corollary 2.3.

Next, for any $Y \in \mathfrak{g}$, we quantize the Hamilton H_Y by choosing the following ordering:

$$z \longrightarrow z_k, \qquad \bar{z} \longrightarrow \bar{z}_{k-1}. \tag{$*$}$$

In [6] we proposed to compute the path integral in the following way:

$$\int \mathscr{D}(z, \bar{z}) \exp\left(\sqrt{-1} \int_0^T \gamma^* \alpha - H_Y(g(z, \bar{z})) dt \right)$$

$$= \lim_{N\to\infty} \int_D \cdots \int_D \prod_{i=1}^{N-1} d\mu(z_i) \exp\left(\sum_{k=1}^{N} \log \frac{K_\Lambda(z_k, \bar{z}_{k-1})}{K_\Lambda(z_{k-1}, \bar{z}_{k-1})} \right)$$

$$\times \exp\left(\sum_{k=1}^{N} \Lambda(\mathrm{Ad}(g(z_k, \bar{z}_{k-1}))^{-1} Y) \frac{T}{N} \right).$$

However, this integral diverges. Therefore we replace

$$e^{\Lambda(\mathrm{Ad}(g(z_k, \bar{z}_{k-1}))^{-1} Y))(T/N)} = \xi_\Lambda\left(\exp H\left(\frac{T}{N} \mathrm{Ad}(g(z_k, \bar{z}_{k-1}))^{-1} Y \right) \right)$$

by

$$\xi_\Lambda\left(h\left(\exp\left(\frac{T}{N} \mathrm{Ad}(g(z_k, \bar{z}_{k-1}))^{-1} Y \right)\right)\right). \tag{$**$}$$

Then our path integral, which generalizes the path integral given in [7], becomes

$$\lim_{N\to\infty} \int_D \cdots \int_D J_\Lambda(z_0) \prod_{i=1}^{N-1} J_\Lambda(z_i) d\mu(z_i) \prod_{k=1}^{N} K_\Lambda(z_k, \bar{z}_{k-1})$$

$$\times \xi_\Lambda\left(h\left(\exp\left(\frac{T}{N} \mathrm{Ad}(g(z_k, \bar{z}_{k-1})^{-1}) Y \right)\right)\right). \tag{3.7}$$

By Lemma 3.5, we see that

$$K_{\Lambda}(z', \bar{z}'')\xi_{\Lambda}\left(h(g(z', \bar{z}'')^{-1} \exp \frac{T}{N} Y g(z', \bar{z}''))\right)$$

is extended to the function

$$\psi_{\Lambda}\left(n_{z''}^{*} \exp\left(-\frac{T}{N} Y\right) n_{z'}\right)$$

defined on $D \times \bar{D}$ which is holomorphic in z' and anti-holomorphic in z''. To proceed further, we need the following

PROPOSITION 3.6. *For any* $X \in \mathfrak{g}$ *and* $g', g'' \in GB$,

$$\int_{D} \psi_{\Lambda}(n_{z}^{*} g'')\psi_{\Lambda}(g'^{*} \exp X n_{z}) J_{\Lambda}(z) d\mu(z)$$

$$= \int_{D} \psi_{\Lambda}(n_{z}^{*} \exp X g'')\psi_{\Lambda}(g'^{*} n_{z}) J_{\Lambda}(z) d\mu(z).$$

Proof: Since $\mathscr{K}_{\Lambda} \circ \pi_{\Lambda} = \pi_{\Lambda} \circ \mathscr{K}_{\Lambda}$,

$$\int_{G/T} \mathscr{K}_{\Lambda}(g'', \bar{g})\mathscr{K}_{\Lambda}(\exp X g, \bar{g}')dg = \int_{G/T} \mathscr{K}_{\Lambda}(\exp X g'', \bar{g})\mathscr{K}_{\Lambda}(g, \bar{g}')dg. \tag{3.8}$$

Rewriting both sides of (3.8) in terms of ψ_{Λ},

$$\int_{D} \psi_{\Lambda}(t^{*}(n_{w}^{-})^{*} n_{z}^{*} g'')\psi_{\Lambda}(g'^{*} \exp X n_{z} n_{w}^{-} t) d\mu(z)$$

$$= \int_{D} \psi_{\Lambda}(t^{*}(n_{w}^{-})^{*} n_{z}^{*} \exp X g'')\psi_{\Lambda}(g'^{*} n_{z} n_{w}^{-} t) d\mu(z).$$

It follows from (2.1) and (2.2) that

$$\int_{D} \psi_{\Lambda}(n_{z}^{*} g'')\psi_{\Lambda}(g'^{*} \exp X n_{z})\xi_{\Lambda}(t^{*} t)^{-1} d\mu(z)$$

$$= \int_{D} \psi_{\Lambda}(n_{z}^{*} \exp X g'')\psi_{\Lambda}(g'^{*} n_{z})\xi_{\Lambda}(t^{*} t)^{-1} d\mu(z).$$

The proposition now follows from Lemma 3.2. ■

Now, applying Proposition 3.6 to the path integral by taking $-(T/N)Y$, $\exp((T/N)Y)n_{z_{k-1}}$ as X, g'' in the proposition, for each k, respectively, we get that (3.7) equals

$$J_\Lambda(z)\psi_\Lambda(n_z^* \exp(-TY)n_{z'})$$
$$= J_\Lambda(z)\mathscr{K}_\Lambda(\exp(-TY)n_{z'}, \bar{n}_z).$$

Furthermore, by Corollary 2.3, we have

$$\int_D J_\Lambda(z)\mathscr{K}_\Lambda(\exp(-TY)n_{z'}, n_{\bar{z}})F(z)d\mu(z) = (U_\Lambda(\exp(TY))F)(z')$$

for any $F \in \mathscr{H}_\Lambda$.

Thus we have obtained the following theorem.

THEOREM 3.7. *For any* $Y \in \mathfrak{g}$, *choosing the ordering* (*) *and taking the regularization* (**), *the path integral of the Hamiltonian* H_Y *gives the kernel function of the operator* $U_\Lambda(\exp(TY))$.

Remark. In the case that G is compact, Theorem 3.7 is valid for any $Y \in \mathfrak{g}^{\mathbf{C}}$, because Proposition 3.6 holds for any $X \in \mathfrak{g}^{\mathbf{C}}$ in this case.

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