HIROSHIMA MATH. J.
24 (1994), 177–220

Bounded solutions with prescribed numbers of zeros for the Emden-Fowler differential equation

Yūki NAITO

(Received September 18, 1992)

1. Introduction

This paper is concerned with the Emden-Fowler differential equation

$$y'' + p(t)|y|^{\alpha} \operatorname{sgn} y = 0, \qquad t \geq t_0. \tag{1.1}$$

For equation (1.1), we always assume that $\alpha > 0$, $\alpha \neq 1$ and that

$$p \in C^1[t_0, \infty) \text{ and } p(t) > 0 \qquad \text{for } t \geq t_0. \tag{1.2}$$

We are interested in the problem of the existence of infinitely many bounded solutions $y(t)$ of (1.1) which satisfy $y(t_0) = 0$ and have prescribed numbers of zeros in (t_0, ∞). Our main result is stated as follows: Suppose that

$$\limsup_{t \to \infty} \frac{tp'(t)}{p(t)} < -\frac{\alpha + 3}{2} \tag{1.3}$$

for the superlinear case $\alpha > 1$, and suppose that

$$\int_{t_0}^{\infty} tp(t)\,dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{[p'(t)]_+}{p(t)}\,dt < \infty \tag{1.4}$$

for the sublinear case $0 < \alpha < 1$, where $[u]_+ = \max\{0, u\}$. Then there exists an infinite sequence of bounded solutions $y_k(t)$, $k = 1, 2, \cdots$, of (1.1) such that $y_k(t)$ has precisely $k - 1$ zeros in (t_0, ∞) and satisfies $y_k(t_0) = 0$. Moreover the sequence $\{y_k(t)\}$ can be constructed so that

$$0 < \lim_{t \to \infty} y_k(t) < \lim_{t \to \infty} y_{k+1}(t) < \infty \qquad \text{for } \alpha > 1$$

and

$$0 < \lim_{t \to \infty} y_{k+1}(t) < \lim_{t \to \infty} y_k(t) < \infty \qquad \text{for } 0 < \alpha < 1.$$

We note that, under condition (1.3) for the case $\alpha > 1$, and under condition (1.4) for the case $0 < \alpha < 1$, all nontrivial solutions of (1.1) are nonoscillatory.

To prove the above results, we first show that, for each $\lambda > 0$, equation (1.1) has a unique solution $y(t; \lambda)$ existing on $[t_0, \infty)$ and satisfying

$$\lim_{t \to \infty} y(t; \lambda) = \lambda. \tag{1.5}$$

Then, we make a detailed analysis of how the qualitative behavior of $y(t; \lambda)$ changes as $\lambda \to 0$ or as $\lambda \to \infty$, and make effective use of the information thus obtained to count the number of zeros of $y(t; \lambda)$ through the standard Prüfer transformation

$$\begin{cases} y(t; \lambda) = \rho(t; \lambda) \sin \varphi(t; \lambda), \\ y'(t; \lambda) = \rho(t; \lambda) \cos \varphi(t; \lambda). \end{cases}$$

In this procedure we need the fundamental results about the continuability and the uniqueness of solutions of the initial value problem for (1.1) as well as the results about the asymptotic properties of nonoscillatory solutions.

For boundary value problems in the compact interval $[t_0, t_1]$, the existence of solutions with prescribed numbers of zeros is studied by many authors (see, e.g., [7, 9, 18, 19, 21]). As an example, consider the boundary condition

$$y(t_0) = y(t_1) = 0. \tag{1.6}$$

It is known that, for any positive integer k, the boundary value problem (1.1)–(1.6) has a nontrivial solution $y(t)$ which has exactly $k-1$ zeros in (t_0, t_1). The superlinear case of this result is due to Nehari [19] and Tal [21], while the sublinear case has recently been proved by Naito-Naito [18]. For a more general equation under the boundary value condition (1.6), we refer to [7, 9, 18]. It seems to the author, however, that very little is known about the existence of solutions of (1.1) with prescribed numbers of zeros in an *infinite* interval. The study of the present paper was motivated by this observation.

Our results for the ordinary differential equation (1.1) can easily be applied to radial solutions of the elliptic differential equation

$$\Delta u + q(|x|)|u|^{\alpha} \operatorname{sgn} u = 0, \qquad x \in \Omega, \tag{1.7}$$

where Δ is the n-dimensional Laplace operator, $n \geq 3$, $|x|$ is the Euclidean length of $x \in R^n$, and Ω is either the unit ball

$$\Omega = \{x \in R^n : |x| < 1\} \tag{1.8}$$

or the exterior domain

$$\Omega = \{x \in R^n : |x| > 1\}. \tag{1.9}$$

For the case where Ω is given by (1.8) we consider the boundary condition

$$(1.10) \qquad u=0 \quad \text{on } |x|=1.$$

Then it is easily seen that the problem of finding radial solutions $u=u(r)$, $r=|x|$, of (1.7)–(1.10) is converted to the following problem:

$$(1.11) \qquad \begin{cases} (r^{n-1}u')' + r^{n-1}q(r)|u|^{\alpha}\operatorname{sgn} u = 0, & 0<r<1, \\ u'(0)=0 \quad \text{and} \quad u(1)=0. \end{cases}$$

For the case where Ω is given by (1.9) we consider the boundary and the asymptotic conditions

$$(1.12) \qquad u=0 \text{ on } |x|=1 \text{ and } \lim_{|x|\to\infty} |x|^{n-2}u(x)=\lambda>0 \quad \text{for some } \lambda.$$

Then, radial solutions $u=u(r)$ of the problem (1.7)–(1.12) can be obtained by solving

$$(1.13) \qquad \begin{cases} (r^{n-1}u')' + r^{n-1}q(r)|u|^{\alpha}\operatorname{sgn} u = 0, & r>1, \\ u(1)=0 \text{ and } \lim_{r\to\infty} r^{n-2}u(r)=\lambda>0 & \text{for some } \lambda. \end{cases}$$

By the change of variables, equations in (1.11) and (1.13) are reduced to equations of the form (1.1). Then, using the results for (1.1), we can conclude that each of the problems (1.11) and (1.13) has an infinite sequence of solutions $u_k(r)$, $k=1, 2,\cdots$, such that $u_k(r)$ has exactly $k-1$ zeros in the interval in question.

We also consider equation (1.7) for the case $\Omega=R^n$, that is, on the entire space R^n. In this case we study entire solutions u satisfying

$$(1.14) \qquad \lim_{|x|\to\infty} |x|^{n-2}u(x)=\mu\neq 0 \quad \text{for some } \mu.$$

As in the above, the problem of finding radial entire solutions $u=u(r)$, $r=|x|$, of (1.7)–(1.14) is equivalent to the problem

$$(1.15) \qquad \begin{cases} (r^{n-1}u')' + r^{n-1}q(r)|u|^{\alpha}\operatorname{sgn} u = 0, & r>0, \\ u'(0)=0 \text{ and } \lim_{r\to\infty} r^{n-2}u(r)=\mu\neq 0 & \text{for some } \mu. \end{cases}$$

To investigate the problem (1.15), we use the technique of Yanagida-Yotsutani [23]. Their idea is to divide (1.15) into the two problems

$$(1.16) \qquad \begin{cases} (r^{n-1}u')' + r^{n-1}q(r)|u|^{\alpha}\operatorname{sgn} u = 0, & 0<r<1, \\ u'(0)=0 \quad \text{and} \quad u(0)=\lambda>0, \end{cases}$$

and

$$(1.17)\qquad \begin{cases} (r^{n-1}v')' + r^{n-1}q(r)|v|^{\alpha}\operatorname{sgn} v = 0, & r > 1, \\ \lim_{r\to\infty} r^{n-2}v(r) = \mu > 0, \end{cases}$$

and to glue $u(r)$ and $v(r)$ at $r = 1$ by choosing λ and μ appropriately. For this method, we also refer to Bianchi-Egnell [1] and Cheng-Chern [3]. The results about bounded solutions $y(t; \lambda)$ of (1.1) satisfying (1.5) play an importnat role and are effectively used in studying the problems (1.16) and (1.17).

The problems (1.11) and (1.15) are discussed by several authors. For the superlinear case of the problem (1.11) with $q(r) \equiv 1$, Castro-Kurepa [2] and Struwe [20] showed the existence of an infinite sequence of solutions with prescribed numbers of zeros. In [10], Kajikiya obtained a similar result for the sublinear case as well as the superlinear case. For the superlinear case of (1.11) with $q(r) = r^l$, $l \geq 0$, Nagasaki [15] showed the existence and the uniqueness of an infinite sequence of solutions having the same properties as above.

Recently the problem (1.15) was discussed by Naito-Naito [17] for the sublinear case, and by Yanagida-Yotsutani [23] for the superlinear case. In both papers, it is shown that, under certain conditions on α and $q(t)$, the problem (1.15) has an infinite sequence of solutions $u_k(r)$, $k = 1, 2, \cdots$, such that $u_k(r)$ has exactly $k - 1$ zeros in $(0, \infty)$.

The outline of this paper is as follows. The main results concerning the ordinary differential equation (1.1) are stated in Section 2. The results for the problems (1.11), (1.13) and (1.15) are also stated in Section 2. Sections 3, 4 and 5 are devoted to the study of equation (1.1). Section 3 contains the basic result which ensures the existence and uniqueness of solution $y(t; \lambda)$ of (1.1) satisfying (1.5). The superlinear case and the sublinear case of (1.1) are studied in Sections 4 and 5, respectively. Section 6 is concerned with the problems (1.11), (1.13) and (1.15). The results for (1.11) and (1.13) are proved in the subsection 6.1, and the results for the superlinear case and the sublinear case of (1.15) are proved in the subsections 6.2 and 6.3, respectively.

2. Main results

First we state the results concerning the ordinary differential equation (1.1).

Theorem 1. *Consider equation* (1.1). *Let* $\alpha > 1$ *and suppose that condition* (1.3) *holds. Then, for each* $k = 1, 2, \cdots$, *there exists a bounded solution* $y_k(t)$ *of* (1.1) *such that* $y_k(t)$ *has exactly* $k - 1$ *zeros in* (t_0, ∞) *and satisfies* $y_k(t_0) = 0$ *and*

(2.1) $$0 < \lim_{t\to\infty} y_k(t) < \lim_{t\to\infty} y_{k+1}(t) < \infty, \qquad k = 1, 2, \cdots.$$

REMARK 2.1. Condition (1.3) is equivalent to the following condition: There is $\varepsilon > 0$ such that

$$\frac{d}{dt}(t^{(\alpha+3)/2+\varepsilon}p(t)) \leq 0 \qquad \text{for all large } t.$$

Therefore if (1.3) is satisfied, then all nontrivial solutions of (1.1) are nonoscillatory (see Kiguradze [11]). It is also known that if

$$\frac{tp'(t)}{p(t)} \geq -\frac{\alpha+3}{2}, \qquad t \geq t_0,$$

then a solution $y(t)$ of (1.1) satisfying $y(t_0) = 0$ has an infinite number of zeros in (t_0, ∞) (see Coffman-Wong [4] and Heidel-Hinton [8]).

THEOREM 2. *Consider equation* (1.1). *Let* $0 < \alpha < 1$ *and suppose that condition* (1.4) *holds. Then, for each* $k = 1, 2, \cdots$, *there exists a bounded solution* $y_k(t)$ *of* (1.1) *such that* $y_k(t)$ *has exactly* $k-1$ *zeros in* (t_0, ∞) *and satisfies* $y_k(t_0) = 0$ *and*

(2.2) $$0 < \lim_{t\to\infty} y_{k+1}(t) < \lim_{t\to\infty} y_k(t) < \infty, \qquad k = 1, 2, \cdots.$$

REMARK 2.2. It is known that if (1.4) is satisfied, then all nontrivial solutions of (1.1) are nonoscillatory (see Gollwitzer [6] and Kwong-Wong [14]).

Next we consider the problem (1.11) which arises in the search of radial solutions of the elliptic problem (1.7)–(1.10).

THEOREM 3. *Consider the problem* (1.11). *Suppose that* $\alpha > 1$ *and that*

(2.3) $$q \in C[0, 1] \cap C^1(0, 1], \; q(r) > 0 \qquad \text{for } r \in (0, 1].$$

Suppose further that

(2.4) $$\liminf_{r\to 0} \frac{rq'(r)}{q(r)} > -\frac{n+2-\alpha(n-2)}{2}.$$

Then, for each $k = 1, 2, \cdots$, *there exists a solution* $u_k(r)$ *of* (1.11) *such that* $u_k(r)$ *has exactly* $k-1$ *zeros in* $(0, 1)$ *and*

(2.5) $$0 < u_1(0) < u_2(0) < \cdots < u_k(0) < u_{k+1}(0) < \cdots.$$

REMARK 2.3. For the case $q(r) = r^l$ $(0 \leq r \leq 1)$ where $l \geq 0$, condition

(2.4) becomes $\alpha < (n+2+2l)/(n-2)$. In this case, the existence and uniqueness of the sequence of solutions $u_k(r)$ are proved by Nagasaki [15]. By a result of Kusano-Naito [12, 13] we see that, under the condition

$$\frac{rq'(r)}{q(r)} \leq -\frac{n+2-\alpha(n-2)}{2}, \qquad 0<r<1,$$

any solution u of

$$(r^{n-1}u')' + r^{n-1}q(r)|u|^{\alpha}\operatorname{sgn} u = 0, \qquad 0<r<1,$$

satisfying $u'(0)=0$ and $u(0)\neq 0$ has no zeros in $[0, 1]$.

THEOREM 4. *Consider the problem* (1.11). *Suppose that* $0<\alpha<1$ *and that* (2.3) *holds. Suppose further that*

$$\int_0^1 \frac{[(r^{2n-2}q(r))']_-}{r^{2n-2}q(r)}\,dr < \infty, \tag{2.6}$$

where $[u]_- = \max\{0, -u\}$. *Then, for each* $k=1, 2,\cdots$, *there exists a solution* $u_k(r)$ *of* (1.11) *such that* $u_k(r)$ *has exactly* $k-1$ *zeros in* $(0, 1)$ *and*

$$0<\cdots<u_{k+1}(0)<u_k(0)<\cdots<u_2(0)<u_1(0). \tag{2.7}$$

Next we state the results concerning the problem (1.13).

THEOREM 5. *Consider the problem* (1.13). *Assume that* $\alpha>1$ *and that*

$$q\in C^1[1,\infty),\ q(r)>0 \qquad \text{for } r\in[1,\infty). \tag{2.8}$$

Suppose that

$$\limsup_{r\to\infty}\frac{rq'(r)}{q(r)} < -\frac{n+2-\alpha(n-2)}{2}. \tag{2.9}$$

Then, for each $k=1, 2,\cdots$, *there exists a solution* $u_k(r)$ *of* (1.13) *such that* $u_k(r)$ *has exactly* $k-1$ *zeros in* $(1,\infty)$ *and*

$$0<\lim_{r\to\infty} r^{n-2}u_k(r) < \lim_{r\to\infty} r^{n-2}u_{k+1}(r) < \infty, \qquad k=1, 2,\cdots. \tag{2.10}$$

THEOREM 6. *Consider the problem* (1.13). *Assume that* $0<\alpha<1$ *and that* (2.8) *holds. Suppose that*

$$\begin{cases} \displaystyle\int_1^\infty r^{n-1-\alpha(n-2)}q(r)\,dr < \infty \qquad \text{and} \\ \displaystyle\int_1^\infty \frac{[(r^{-n+4-\alpha(n-2)}q(r))']_+}{r^{-n+4-\alpha(n-2)}q(r)}\,dr < \infty, \end{cases} \tag{2.11}$$

where $[u]_+ = \max\{u, 0\}$. *Then, for each* $k = 1, 2, \cdots$, *there exists a solution* $u_k(r)$ *of* (1.13) *such that* $u_k(r)$ *has exactly* $k-1$ *zeros in* $(1, \infty)$ *and*

$$(2.12) \qquad 0 < \lim_{r\to\infty} r^{n-2}u_{k+1}(r) < \lim_{r\to\infty} r^{n-2}u_k(r) < \infty, \qquad k = 1, 2, \cdots.$$

Next we consider the existence of solutions of the problem (1.15) which arises in the search of radial entire solutions of the elliptic problem (1.7)–(1.14).

THEOREM 7. *Consider the problem* (1.15). *Assume that* $\alpha > 1$ *and that*

$$(2.13) \qquad q \in C[0, \infty) \cap C^1(0, \infty), \ q(r) > 0 \qquad \text{for } r \in (0, \infty).$$

Suppose that (2.4) *and* (2.9) *hold. Then, for each* $k = 1, 2, \cdots$, *there exists a solution* $u_k(r)$ *of* (1.15) *such that* $u_k(r)$ *has exactly* $k-1$ *zeros in* $(0, \infty)$ *and satisfies* (2.5).

REMARK 2.4. It is known (Kusano-Naito [12, 13]) that, under the condition

$$\frac{rq'(r)}{q(r)} \geq -\frac{n+2-\alpha(n-2)}{2}, \qquad r > 0,$$

any solution u of

$$(2.14) \qquad (r^{n-1}u')' + r^{n-1}q(r)|u|^\alpha \operatorname{sgn} u = 0, \qquad r > 0,$$

satisfying $u'(0) = 0$ and $u(0) \neq 0$ has no zeros in $[0, \infty)$. It is also known [12, 13] that, under the condition

$$\frac{rq'(r)}{q(r)} < -\frac{n+2-\alpha(n-2)}{2}, \qquad r > 0,$$

any solution u of (2.14) satisfying $u'(0) = 0$ and $u(0) \neq 0$ has an infinite number of zeros in $[0, \infty)$. Recently the following theorem has been proved by Yanagida-Yotsutani [23]: If there is a number $a > -\dfrac{n+2-\alpha(n-2)}{2}$ satisfying either

$$q(r) = Ar^a + o(r^a) \text{ as } r \longrightarrow 0 \qquad \text{for some } A > 0$$

or

$$(2.15) \qquad \lim_{r\to 0} \frac{rq'(r)}{q(r)} = a$$

and if there is a number $b < -\dfrac{n+2-\alpha(n-2)}{2}$ satisfying either

$$q(r) = Br^b + o(r^b) \text{ as } r \longrightarrow \infty \qquad \text{for some } B > 0$$

or

$$\lim_{r\to\infty} \frac{rq'(r)}{q(r)} = b, \tag{2.16}$$

then the problem (1.15) has solutions $u_k(r)$, $k = 1, 2, \cdots$, such that $u_k(r)$ has exactly $k - 1$ zeros in $(0, \infty)$. Theorem 7 means that the limit as $r \to 0$ in (2.15) and the limit as $r \to \infty$ in (2.16) can be replaced by the lower limit as $r \to 0$ and the upper limit as $r \to \infty$, respectively.

THEOREM 8. *Consider the problem* (1.15). *Assume that* $0 < \alpha < 1$ *and that* (2.13) *holds. Suppose that* (2.6) *and* (2.11) *hold. Then, for each* $k = 1, 2, \cdots$, *there exists a solution* $u_k(r)$ *of* (1.15) *such that* $u_k(r)$ *has exactly* $k - 1$ *zeros in* $(0, \infty)$ *and satisfies* (2.7).

REMARK 2.5. Recently, Naito-Naito [17] have showed that if

$$\int_0^\infty r^{n-1} q(r)\, dr < \infty,$$

then there exist solutions $u_k(r)$, $k = 1, 2, \cdots$, of (1.15) such that $u_k(r)$ has exactly $k - 1$ zeros in $(0, \infty)$.

3. Preliminaries

In this section we consider the ordinary differential equation (1.1). First of all, we remark the following fundamental fact: For any initial condition

$$y(t_1) = a, \quad y'(t_1) = b, \tag{3.1}$$

where $t_1 \in [t_0, \infty)$ and $a, b \in R$, the solution $y(t)$ of (1.1) satisfying (3.1) exists and is unique on the whole interval $[t_0, \infty)$. In particular, any solution of (1.1) is uniquely continuable to $[t_0, \infty)$. This fact can be concluded under the hypothesis (1.2) by using a theory on the uniqueness and continuability of solutions of (1.1)–(3.1). The results concerning the uniqueness problem and the continuability problem are found in the survey papers of Naito [16] and Wong [22]. For a more general equation, we refer to Coffman-Wong [4] and Naito-Naito [18].

Hereafter we assume the integral condition

$$\int_{t_0}^\infty sp(s)\, ds < \infty. \tag{3.2}$$

It is well known (see, e.g., Coffman-Wong [5]) that, under (3.2), equation (1.1) has a bounded and eventually positive solution $y(t)$ such that $\lim_{t\to\infty} y(t)$ exists and is a positive finite value. The purpose of this section is to show that, for each $\lambda > 0$, equation (1.1) has a unique solution $y(t)$ satisfying

$$\lim_{t\to\infty} y(t) = \lambda \tag{3.3}$$

and existing on the whole interval $[t_0, \infty)$.

It is to be noted that if $y(t)$ is a solution of (1.1) satisfying (3.3), then we have

$$y(t) = \lambda - \int_t^\infty (s-t)p(s)|y(s)|^\alpha \operatorname{sgn} y(s)\,ds, \qquad t \geq t_0, \tag{3.4}$$

and

$$y'(t) = \int_t^\infty p(s)|y(s)|^\alpha \operatorname{sgn} y(s)\,ds, \qquad t \geq t_0. \tag{3.5}$$

Let $y(t)$ be a solution of (1.1) satisfying (3.3) with $\lambda > 0$, and let (T, ∞) $(\subset [t_0, \infty))$ be the largest interval such that $y(t) > 0$ on (T, ∞). Then, by (3.5) we see that $y(t)$ is strictly increasing on (T, ∞), and $0 < y(t) < \lambda$ for $t > T$.

We state the following proposition.

PROPOSITION 3.1. *Let* $\alpha > 0$, $\alpha \neq 1$. *Suppose that* (3.2) *holds. Then, for each* $\lambda > 0$, *there exists a unique solution* $y(t)$ *of* (1.1) *which exists on* $[t_0, \infty)$ *and satisfies* (3.3).

PROOF. First we show the existence of a solution of (1.1) satisfying (3.3). Choose $T_0 \geq t_0$ so large that

$$\int_{T_0}^\infty sp(s)\,ds \leq \frac{1}{2}\lambda^{1-\alpha}.$$

Let $C[T_0, \infty)$ denote the Fréchet space of all continuous functions on $[T_0, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_0, \infty)$. Consider the set

$$Y = \left\{y \in C[T_0, \infty) : \frac{1}{2}\lambda \leq y(t) \leq \lambda \qquad \text{for } t \geq T_0\right\}$$

which is a closed subset of $C[T_0, \infty)$. We define an operator F on Y by

$$Fy(t) = \lambda - \int_t^\infty (s-t)p(s)|y(s)|^\alpha\,ds, \qquad t \geq T_0.$$

If $y \in Y$, then $Fy(t) \leq \lambda$, $t \geq T_0$, and

$$Fy(t) \geq \lambda - \int_t^\infty sp(s)|y(s)|^\alpha ds$$

$$\geq \lambda - \lambda^\alpha \int_{T_0}^\infty sp(s)ds \geq \frac{1}{2}\lambda, \qquad t \geq T_0.$$

Thus, the operator F maps Y into itself. It is easy to see that the operator F is continuous on Y and FY is relatively compact in the topology of $C[T_0, \infty)$. By the Schauder-Tychonoff fixed point theorem, F has an element $y \in Y$ such that $y = Fy$, i.e., $y(t) = Fy(t)$ for $t \geq T_0$. Then, $y(t)$ is a solution on $[T_0, \infty)$ of (1.1) and satisfies the asymptotic condition (3.3). As mentioned in the first part of this section, any solution of (1.1) is continuable to $[t_0, \infty)$. Thus, $y(t)$ exists on the interval $[t_0, \infty)$.

Next we verify that the solution of (1.1) satisfying (3.3) is unique. Let $y_1(t)$ and $y_2(t)$ be solutions of (1.1) satisfying (3.3). Then, we have (3.4) with $y = y_1$ and $y = y_2$. Take $T_1 \geq t_0$ so that

$$y_1(t) \geq \frac{1}{2}\lambda \text{ and } y_2(t) \geq \frac{1}{2}\lambda \qquad \text{for } t \geq T_1.$$

As stated above, we obtain

$$y_1(t) < \lambda \text{ and } y_2(t) < \lambda \qquad \text{for } t \geq T_1.$$

Then it is easy to see that

$$|y_1(t) - y_2(t)| \leq \int_t^\infty (s-t)p(s)|[y_1(s)]^\alpha - [y_2(s)]^\alpha| ds$$

$$\leq L \int_t^\infty sp(s)|y_1(s) - y_2(s)| ds, \qquad t \geq T_1,$$

where $L = \alpha\lambda^{\alpha-1}$ if $\alpha > 1$ and $L = \alpha[\lambda/2]^{\alpha-1}$ if $0 < \alpha < 1$. Define the function $Y(t)$ by

$$Y(t) = L \int_t^\infty sp(s)|y_1(s) - y_2(s)| ds, \qquad t \geq T_1.$$

Observe that

$$Y'(t) = -Ltp(t)|y_1(t) - y_2(t)| \geq -Ltp(t)Y(t), \qquad t \geq T_1,$$

so that

$$\frac{d}{dt}\left[Y(t)\exp\left(-L\int_t^\infty sp(s)\,ds\right)\right] \geq 0, \qquad t \geq T_1.$$

Integrating the above on $[t, \tau]$, $\tau > t \geq T_1$, we obtain

$$Y(t) \leq Y(\tau)\exp\left(L\int_t^\tau sp(s)\,ds\right) \leq Y(\tau)\exp\left(L\int_{T_1}^\infty sp(s)\,ds\right). \tag{3.6}$$

Note here that $Y(\tau) \to 0$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (3.6), we get $Y(t) \equiv 0$ for $t \geq T_1$, which implies that $y_1(t) \equiv y_2(t)$ for $t \geq T_1$. Since any solution of (1.1) is uniquely continuable to $[t_0, \infty)$, we can conclude that the solution of (1.1) satisfying (3.3) is unique on $[t_0, \infty)$. This completes the proof of Proposition 3.1.

We denote by $y(t; \lambda)$ the solution of (1.1) satisfying (3.3). By Proposition 3.1, $y(t; \lambda)$ is a function of $(t, \lambda) \in [t_0, \infty) \times (0, \infty)$. Equations (3.4) and (3.5) imply

$$y(t; \lambda) = \lambda - \int_t^\infty (s - t)p(s)|y(s; \lambda)|^\alpha \operatorname{sgn} y(s; \lambda)\,ds, \qquad t \geq t_0, \tag{3.7}$$

and

$$y'(t; \lambda) = \int_t^\infty p(s)|y(s; \lambda)|^\alpha \operatorname{sgn} y(s; \lambda)\,ds, \qquad t \geq t_0, \tag{3.8}$$

respectively. Since the initial value problem (1.1)–(3.1) has a unique solution for each $t_1 \in [t_0, \infty)$ and each $a, b \in R$, $y(t; \lambda)$ and $y'(t; \lambda)$ cannot vanish simultaneously. Hence the zeros of $y(t; \lambda)$ are all simple and cannot have a cluster point in a finite interval $J \subset [t_0, \infty)$. In view of (3.3), it is clear that $y(t; \lambda) > 0$ for all sufficiently large t. Therefore, we see that, for each $\lambda > 0$, the number of zeros of $y(t; \lambda)$ in (t_0, ∞) is finite.

It is proved that $y(t; \lambda)$ is a continuous function of $(t, \lambda) \in [t_0, \infty) \times (0, \infty)$. It is also proved that $\lim_{\lambda \to +0} N(\lambda) = 0$ and $\lim_{\lambda \to \infty} N(\lambda) = \infty$ for the superlinear case $\alpha > 1$, and that $\lim_{\lambda \to +0} N(\lambda) = \infty$ and $\lim_{\lambda \to \infty} N(\lambda) = 0$ for the sublinear case $0 < \alpha < 1$, where $N(\lambda)$ denotes the number of zeros of $y(t; \lambda)$ in $[t_0, \infty)$. The proofs of these facts are given in Section 4 for the superlinear case and in Section 5 for the sublinear case. These facts play an essential part for the proofs of Theorems 1 and 2.

4. The superlinear equation

In this section we consider equation (1.1) in the superlinear case $\alpha > 1$. Associated with every solution $y(t)$ of (1.1), we define the function $V[y](t)$ by

(4.1) $$V[y](t)=\frac{1}{2}[y'(t)]^2+\frac{1}{\alpha+1}p(t)|y(t)|^{\alpha+1}, \qquad t\geq t_0.$$

It is clear that

(4.2) $$|y(t)|\leq\left(\frac{(\alpha+1)V[y](t)}{p(t)}\right)^{1/(\alpha+1)}, \qquad t\geq t_0,$$

and

(4.3) $$|y'(t)|\leq(2V[y](t))^{1/2}, \qquad t\geq t_0.$$

Moreover we see that, for $\tau>t\geq t_0$,

(4.4)
$$V[y](t)\exp\left(-\int_t^\tau\frac{[p'(s)]_-}{p(s)}ds\right)\leq V[y](\tau)\leq V[y](t)\exp\left(\int_t^\tau\frac{[p'(s)]_+}{p(s)}ds\right)$$

(see, e.g., [18]). Inequalities (4.2)–(4.4) will be effectively used in this section.

Hereafter we assume in (1.1) that $\alpha>1$ and that (3.2) holds. These assumptions are used without further mention. Later, it is shown that condition (3.2) is satisfied under condition (1.3) in Theorem 1.

For $\lambda>0$, let $y(t;\lambda)$ be the solution of (1.1) satisfying (3.3). The existence and uniqueness of $y(t;\lambda)$ are ensured by Proposition 3.1. Note that $y(t;\lambda)$ satisfies (3.7) and (3.8).

PROPOSITION 4.1. *If $\lambda>0$, $\lambda(i)>0$ $(i=1,2,\cdots)$ and $\lim_{i\to\infty}\lambda(i)=\lambda$, then $\lim_{i\to\infty}y(t:\lambda(i))=y(t;\lambda)$ and $\lim_{i\to\infty}y'(t;\lambda(i))=y'(t;\lambda)$ uniformly on $[t_0,\infty)$. In particular, $y(t;\lambda)$ and $y'(t;\lambda)$ are continuous in $(t,\lambda)\in[t_0,\infty)\times(0,\infty)$.*

To prove Proposition 4.1, we prepare the next lemma.

LEMMA 4.1. *Let $\lambda_0>0$. Then there exists a positive constant $M(\lambda_0)$ such that, for any $\lambda\in(0,\lambda_0]$,*

(4.5) $$|y(t;\lambda)|\leq M(\lambda_0), \qquad t\geq t_0.$$

PROOF. Choose $T_0=T_0(\lambda_0)$ large enough so that

(4.6) $$\int_{T_0}^\infty sp(s)ds<\lambda_0^{1-\alpha}.$$

We claim that, for any $\lambda\in(0,\lambda_0]$, $y(t;\lambda)>0$ for $t\geq T_0$. Assume to the contrary that $y(t;\lambda)$ has a zero in $[T_0,\infty)$. Let $T_1\in[T_0,\infty)$ be the largest zero of $y(t;\lambda)$. Then we have

$$0\leq y(t;\lambda)<\lambda, \qquad t\geq T_1.$$

From (3.7) we see that

$$\lambda = \int_{T_1}^{\infty} (s - T_1)p(s)|y(s;\lambda)|^{\alpha} ds \\ \leq \lambda^{\alpha} \int_{T_0}^{\infty} sp(s)\, ds.$$

Then we have

$$\lambda_0^{1-\alpha} \leq \lambda^{1-\alpha} \leq \int_{T_0}^{\infty} sp(s)\, ds,$$

which contradicts (4.6). Thus we conclude that, for any $\lambda \in (0, \lambda_0]$, $y(t;\lambda) > 0$ for $t \geq T_0$. This implies that

$$|y(t;\lambda)| < \lambda_0, \qquad t \geq T_0, \tag{4.7}$$

for any $\lambda \in (0, \lambda_0]$. From (3.8) and (4.7) we have

$$|y'(t;\lambda)| \leq \lambda_0^{\alpha} \int_{T_0}^{\infty} p(s)\, ds \equiv m_1, \qquad t \geq T_0, \tag{4.8}$$

for any $\lambda \in (0, \lambda_0]$. By virtue of (4.7) and (4.8) we see that

$$V[y(\cdot\,;\lambda)](T_0) \leq \frac{1}{2} m_1^2 + \frac{1}{\alpha+1} p(T_0)\lambda_0^{\alpha+1} \equiv m_2,$$

where $V[y]$ is defined by (4.1). From (4.4) with $y = y(\cdot\,;\lambda)$ we easily see that

$$V[y(\cdot\,;\lambda)](t) \leq V[y(\cdot\,;\lambda)](T_0) \exp\left(\int_t^{T_0} \frac{[p'(s)]_-}{p(s)} ds\right) \\ \leq m_2 \exp\left(\int_{t_0}^{T_0} \frac{[p'(s)]_-}{p(s)} ds\right), \qquad t_0 \leq t \leq T_0.$$

Then, on account of (4.2) with $y = y(\cdot\,;\lambda)$, we obtain

$$|y(t;\lambda)| \leq m_3, \qquad t_0 \leq t \leq T_0, \tag{4.9}$$

where

$$m_3 = \left[\frac{m_2(\alpha+1)}{\min\{p(t)\colon t_0 \leq t \leq T_0\}}\right]^{1/(\alpha+1)} \exp\left(\frac{1}{\alpha+1} \int_{t_0}^{T_0} \frac{[p'(s)]_-}{p(s)} ds\right).$$

Let $M(\lambda_0) = \max\{\lambda_0, m_3\}$. Then, from (4.7) and (4.9), we have (4.5). This completes the proof of Lemma 4.1.

Proof of Proposition 4.1. There exists a λ_0 such that $\lambda(i) \leq \lambda_0$ for

$i = 1, 2, \cdots$. From Lemma 4.1, there exists a positive constant $M = M(\lambda_0)$ such that, for $i = 1, 2, \cdots$,

$$(4.10) \qquad |y(t; \lambda(i))| \leq M \text{ and } |y(t; \lambda)| \leq M, \qquad t \geq t_0.$$

Then, from (3.7), we see that

$$(4.11) \qquad |y(t; \lambda(i)) - y(t; \lambda)| \leq |\lambda(i) - \lambda| + \alpha M^{\alpha-1} \int_t^{\infty} sp(s) |y(s; \lambda(i)) - y(s; \lambda)| \, ds, \qquad t \geq t_0.$$

Define the function $Y(t)$ by

$$Y(t) = |\lambda(i) - \lambda| + \alpha M^{\alpha-1} \int_t^{\infty} sp(s) |y(s; \lambda(i)) - y(s; \lambda)| \, ds, \qquad t \geq t_0.$$

We easily see that

$$\frac{d}{dt}\left[Y(t) \exp\left(-\alpha M^{\alpha-1} \int_t^{\infty} sp(s) \, ds \right) \right] \geq 0, \qquad t \geq t_0.$$

Integrating the above on $[t, \tau]$, we have

$$Y(t) \leq Y(\tau) \exp\left(\alpha M^{\alpha-1} \int_t^{\tau} sp(s) \, ds \right), \qquad \tau > t \geq t_0.$$

Thus,

$$(4.12) \qquad Y(t) \leq Y(\tau) \exp\left(\alpha M^{\alpha-1} \int_{t_0}^{\infty} sp(s) \, ds \right), \qquad \tau > t \geq t_0.$$

Note here that $Y(\tau) \to |\lambda(i) - \lambda|$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (4.12) and using (4.11), we get

$$(4.13) \qquad |y(t; \lambda(i)) - y(t; \lambda)| \leq |\lambda(i) - \lambda| \exp\left(\alpha M^{\alpha-1} \int_{t_0}^{\infty} sp(s) \, ds \right), \qquad t \geq t_0,$$

which implies that $y(t; \lambda(i))$ converges to $y(t; \lambda)$ as $i \to \infty$ uniformly on $[t_0, \infty)$. From (3.8) and (4.10) we have

$$|y'(t; \lambda(i)) - y'(t; \lambda)| \leq \alpha M^{\alpha-1} \int_t^{\infty} p(s) |y(s; \lambda(i)) - y(s; \lambda)| \, ds, \qquad t \geq t_0.$$

Then it follows from (4.13) that

$$|y'(t; \lambda(i)) - y'(t; \lambda)| \leq \alpha M^{\alpha-1} |\lambda(i) - \lambda| \exp\left(\alpha M^{\alpha-1} \int_{t_0}^{\infty} sp(s) \, ds \right) \int_{t_0}^{\infty} p(s) \, ds, \qquad t \geq t_0,$$

which means that $y'(t; \lambda(i))$ converges to $y'(t; \lambda)$ as $i \to \infty$ uniformly on $[t_0, \infty)$. This completes the proof of Proposition 4.1.

PROPOSITION 4.2. *For sufficiently small* $\lambda > 0$, $y(t; \lambda) > 0$ *on* $[t_0, \infty)$. *Moreover, the solution* $y(t; \lambda)$ *has the following properties*:

(i) $\lim_{\lambda \to 0} y(t; \lambda) = 0$ *uniformly on* $[t_0, \infty)$; *and*

(ii) $\lim_{\lambda \to 0} \dfrac{y'(t; \lambda)}{y(t; \lambda)} = 0$ *uniformly on* $[t_0, \infty)$.

PROOF. Let $\lambda > 0$ be small enough so that

$$\lambda < \left[\int_{t_0}^{\infty} sp(s)\, ds \right]^{1/(1-\alpha)}. \tag{4.14}$$

We claim that $y(t; \lambda) > 0$ on $[t_0, \infty)$. Assume to the contrary that $y(t; \lambda)$ has a zero in $[t_0, \infty)$. Let t_1 be the largest zero of $y(t; \lambda)$ in $[t_0, \infty)$. Then we have

$$0 \leq y(t; \lambda) < \lambda, \qquad t \geq t_1.$$

From (3.7) we see that

$$\begin{aligned} \lambda &= \int_{t_1}^{\infty} (s - t_1) p(s) |y(s; \lambda)|^{\alpha}\, ds \\ &\leq \lambda^{\alpha} \int_{t_0}^{\infty} sp(s)\, ds. \end{aligned}$$

Then we have

$$\lambda^{1-\alpha} \leq \int_{t_0}^{\infty} sp(s)\, ds,$$

which is a contradiction to (4.14). Thus we conclude that, for sufficiently small $\lambda > 0$, $y(t; \lambda)$ has no zeros in $[t_0, \infty)$.

Let $\lambda > 0$ be small enough so that $y(t; \lambda) > 0$, $t \geq t_0$. Then we have

$$0 < y(t; \lambda) < \lambda, \qquad t \geq t_0, \tag{4.15}$$

which implies (i). From (3.7) and (4.15) we see that

$$\begin{aligned} \left| \frac{y(t; \lambda)}{\lambda} - 1 \right| &= \frac{1}{\lambda} \int_{t}^{\infty} (s - t) p(s) |y(s; \lambda)|^{\alpha}\, ds \\ &\leq \lambda^{\alpha - 1} \int_{t_0}^{\infty} sp(s)\, ds, \qquad t \geq t_0. \end{aligned}$$

Thus we obtain

$$\lim_{\lambda \to 0} \frac{y(t; \lambda)}{\lambda} = 1 \quad \text{uniformly on} \quad [t_0, \infty). \tag{4.16}$$

From (3.8) and (4.15), we have

$$\left|\frac{y'(t; \lambda)}{y(t; \lambda)}\right| = \frac{1}{y(t; \lambda)} \int_t^\infty p(s)|y(s; \lambda)|^\alpha ds$$
$$\leq \frac{\lambda}{y(t; \lambda)} \lambda^{\alpha-1} \int_{t_0}^\infty p(s) ds, \qquad t \geq t_0.$$

Then, by virtue of (4.16), we obtain (ii). This completes the proof of Proposition 4.2.

Proposition 4.3. *Suppose that* (1.3) *holds. Then the number of zeros of* $y(t; \lambda)$ *in* $[t_0, \infty)$ *tends to* ∞ *as* $\lambda \to \infty$.

First we show that condition (1.3) implies the integral condition (3.2). Condition (1.3) guarantees the existence of $t_1 > t_0$ and $l > \frac{\alpha + 3}{2}$ such that

$$\frac{tp'(t)}{p(t)} \leq -l, \qquad t \geq t_1, \tag{4.17}$$

which is equivalent to

$$(t^l p(t))' \leq 0, \qquad t \geq t_1. \tag{4.18}$$

Then we have

$$p(t) \leq C_1 t^{-l}, \qquad t \geq t_1, \tag{4.19}$$

for some positive constant C_1. Hence, by virtue of $l > 2$, condition (1.3) implies (3.2). In Proposition 4.3, it is to be noted that condition (1.3) ensures the well-definedness of $y(t; \lambda)$, and that, as is shown in Section 3, the number of zeros of $y(t; \lambda)$ is finite for each $\lambda > 0$.

To prove Proposition 4.3, we need a series of lemmas.

Lemma 4.2. *Suppose that* (1.3) *holds. Then, there exists a positive constant* C_2 *such that*

$$\int_t^\infty sp(s) ds \leq C_2 t \int_t^\infty p(s) ds, \qquad t \geq t_1, \tag{4.20}$$

where t_1 *is a constant appearing in* (4.17).

PROOF. Integrating (4.18) on $[\tau, s]$, $t_1 \le \tau < s$, we obtain

$$\tau^l p(\tau) s^{1-l} \ge sp(s), \qquad s > \tau \ge t_1. \tag{4.21}$$

Let $t \ge t_1$. From (4.21) we see that

$$\begin{aligned} \int_{2t}^{\infty} sp(s)\,ds &\le (2t)^l p(2t) \int_{2t}^{\infty} s^{1-l}\,ds \\ &= \frac{4}{l-2} t^2 p(2t). \end{aligned} \tag{4.22}$$

Since $sp(s)$ is nonincreasing on $[t_1, \infty)$, we have

$$\int_{t}^{2t} sp(s)\,ds \ge 2tp(2t)(2t-t) = 2t^2 p(2t). \tag{4.23}$$

By virtue of (4.22) and (4.23) we obtain

$$\int_{2t}^{\infty} sp(s)\,ds \le \frac{2}{l-2} \int_{t}^{2t} sp(s)\,ds.$$

Then we have

$$\begin{aligned} \int_{t}^{\infty} sp(s)\,ds &= \int_{t}^{2t} sp(s)\,ds + \int_{2t}^{\infty} sp(s)\,ds \\ &\le \frac{l}{l-2} \int_{t}^{2t} sp(s)\,ds \\ &\le \frac{2l}{l-2} t \int_{t}^{2t} p(s)\,ds \\ &\le \frac{2l}{l-2} t \int_{t}^{\infty} p(s)\,ds, \qquad t \ge t_1. \end{aligned}$$

This completes the proof of Lemma 4.2.

LEMMA 4.3. *Suppose that* (1.3) *holds. Then, there exist positive constants* C_3 *and* C_4 *such that, for any* $\lambda > 0$,

$$V[y(\cdot\,;\lambda)](t_1) + C_3 \ge C_4 \int_{t_1}^{\infty} p(s)\,|y(s;\lambda)|^{\alpha+1}\,ds, \tag{4.24}$$

where t_1 *is a constant appearing in* (4.17).

PROOF. We note that $y(t;\lambda)$ satisfies

$$\frac{d}{dt}\left(\frac{1}{2}t[y'(t;\lambda)]^2+\frac{1}{\alpha+1}tp(t)|y(t;\lambda)|^{\alpha+1}-\frac{1}{2}y(t;\lambda)y'(t;\lambda)\right)$$
$$=\frac{1}{\alpha+1}\left[\frac{\alpha+3}{2}+\frac{tp'(t)}{p(t)}\right]p(t)|y(t;\lambda)|^{\alpha+1},\qquad t\geq t_0. \tag{4.25}$$

Using (3.8), we get

$$|ty'(t;\lambda)|\leq\lambda^{\alpha}t\int_t^{\infty}p(s)\,ds\leq\lambda^{\alpha}\int_t^{\infty}sp(s)\,ds$$

for all sufficiently large t; and hence

$$\lim_{t\to\infty}ty'(t;\lambda)=0.$$

Further, by (4.19) we have

$$\lim_{t\to\infty}tp(t)=0.$$

Then, integrating (4.25) on $[t, \tau]$ and letting $\tau\to\infty$, we find that

$$\frac{1}{2}t[y'(t;\lambda)]^2+\frac{1}{\alpha+1}tp(t)|y(t;\lambda)|^{\alpha+1}-\frac{1}{2}y(t;\lambda)y'(t;\lambda)$$
$$=-\frac{1}{\alpha+1}\int_t^{\infty}\left[\frac{\alpha+3}{2}+\frac{sp'(s)}{p(s)}\right]p(s)|y(s;\lambda)|^{\alpha+1}\,ds,\qquad t\geq t_0.$$

From the definition of $V[y]$ and (4.17) we see that

$$t_1V[y(\cdot\,;\lambda)](t_1)-\frac{1}{2}y(t_1;\lambda)y'(t_1;\lambda)$$
$$\geq\frac{1}{\alpha+1}\left[l-\frac{\alpha+3}{2}\right]\int_{t_1}^{\infty}p(s)|y(s;\lambda)|^{\alpha+1}\,ds. \tag{4.26}$$

By (4.2) and(4.3) with $y=y(\cdot\,;\lambda)$ we have

$$\frac{1}{2}|y(t_1;\lambda)y'(t_1;\lambda)|$$
$$\leq\frac{1}{2^{1/2}}\left(\frac{\alpha+1}{p(t_1)}\right)^{1/(\alpha+1)}[V[y(\cdot\,;\lambda)](t_1)]^{(\alpha+3)/[2(\alpha+1)]}. \tag{4.27}$$

Now remember Young's inequality

$$AB\leq\frac{A^a}{a}+\frac{B^b}{b}, \tag{4.28}$$

where A, B, a and b are positive numbers and $(1/a)+(1/b)=1$. We apply (4.28) to the case $A=[V[y(\cdot\,;\lambda)](t_1)]^{(\alpha+3)/[2(\alpha+1)]}$, $B=1$, $a=2(\alpha+1)/(\alpha+3)$ and $b=2(\alpha+1)/(\alpha-1)$. Then we have

$$\text{(4.29)}\qquad [V[y(\cdot\,;\lambda)](t_1)]^{(\alpha+3)/[2(\alpha+1)]} \le \frac{\alpha+3}{2(\alpha+1)}V[y(\cdot\,;\lambda)](t_1)+\frac{\alpha-1}{2(\alpha+1)}.$$

Combining (4.27) with (4.29), we obtain

$$\text{(4.30)}\qquad \frac{1}{2}|y(t_1;\lambda)y'(t_1;\lambda)|\le C_5V[y(\cdot\,;\lambda)](t_1)+C_6,$$

where

$$C_5=\frac{\alpha+3}{2^{3/2}(\alpha+1)}\left(\frac{\alpha+1}{p(t_1)}\right)^{1/(\alpha+1)}$$

and

$$C_6=\frac{\alpha-1}{2^{3/2}(\alpha+1)}\left(\frac{\alpha+1}{p(t_1)}\right)^{1/(\alpha+1)}.$$

From (4.26) and (4.30) it follows that

$$[t_1+C_5]V[y(\cdot\,;\lambda)](t_1)+C_6 \ge\frac{1}{\alpha+1}\left[l-\frac{\alpha+3}{2}\right]\int_{t_1}^{\infty}p(s)|y(s;\lambda)|^{\alpha+1}ds,$$

which implies (4.24) with

$$C_3=\frac{C_6}{t_1+C_5}\quad\text{and}\quad C_4=\frac{1}{(\alpha+1)(t_1+C_5)}\left[l-\frac{\alpha+3}{2}\right].$$

LEMMA 4.4. *Suppose that* (1.3) *holds. Then,*

$$\text{(4.31)}\qquad \lim_{\lambda\to\infty}V[y(\cdot\,;\lambda)](t_1)=\infty,$$

where t_1 *is a constant appearing in* (4.17).

PROOF. First we show the following: Let λ be large enough so that

$$\text{(4.32)}\qquad \lambda\ge 2\left[\int_{t_1}^{\infty}(s-t_1)p(s)\,ds\right]^{1/(1-\alpha)}.$$

Then there exists $t_2=t_2(\lambda)\ge t_1$ such that

(4.33) $$y(t_2; \lambda) = \frac{1}{2}\lambda \quad \text{and} \quad y(t; \lambda) > \frac{1}{2}\lambda, \qquad t > t_2.$$

To prove this, assume to the contrary that

$$y(t; \lambda) > \frac{1}{2}\lambda, \qquad t \geq t_1.$$

From (3.7) we see that

$$\frac{1}{2}\lambda < \lambda - \left[\frac{1}{2}\lambda\right]^\alpha \int_{t_1}^\infty (s - t_1)p(s)\,ds.$$

Then we have

$$\int_{t_1}^\infty (s - t_1)p(s)\,ds < \left[\frac{1}{2}\lambda\right]^{1-\alpha},$$

which contradicts (4.32). Thus there exists $t_2 \geq t_1$ satisfying (4.33). From (3.7) we have

$$\frac{1}{2}\lambda = \int_{t_2}^\infty (s - t_2)p(s)\,|y(s; \lambda)|^\alpha\,ds \leq \lambda^\alpha \int_{t_2}^\infty sp(s)\,ds,$$

which implies

(4.34) $$\frac{1}{2}\lambda^{1-\alpha} \leq \int_{t_2}^\infty sp(s)\,ds.$$

By virtue of (4.19), we see that

$$\int_{t_2}^\infty sp(s)\,ds \leq C_1 \int_{t_2}^\infty s^{1-l}\,ds = \frac{C_1}{l-2}\,t_2^{2-l}.$$

From (4.34) it follows that

$$\frac{1}{2}\lambda^{1-\alpha} \leq \frac{C_1}{l-2}\,t_2^{2-l},$$

that is,

(4.35) $$t_2 \leq C_7 \lambda^{(\alpha-1)/(l-2)},$$

where

$$C_7 = \left(\frac{2C_1}{l-2}\right)^{1/(l-2)}.$$

From (4.20), (4.34) and (4.35) we obtain

$$\int_{t_2}^{\infty} p(s)\,ds \geq \frac{1}{C_2 t_2}\int_{t_2}^{\infty} sp(s)\,ds \geq \frac{1}{2C_2 t_2}\lambda^{1-\alpha}$$
$$\geq \frac{1}{2C_2 C_7}\lambda^{(l+\alpha-\alpha l-1)/(l-2)}.$$

Then we see that

$$\int_{t_1}^{\infty} p(s)|y(s;\lambda)|^{\alpha+1}\,ds \geq \left[\frac{1}{2}\lambda\right]^{\alpha+1}\int_{t_2}^{\infty} p(s)\,ds \tag{4.36}$$
$$\geq \frac{1}{2^{\alpha+2}C_2 C_7}\lambda^{(2l-\alpha-3)/(l-2)}.$$

We note here that

$$\frac{2l-\alpha-3}{l-2} = \frac{2}{l-2}\left[l - \frac{\alpha+3}{2}\right] > 0.$$

Then, letting $\lambda \to \infty$ in (4.36), we obtain

$$\lim_{\lambda\to\infty}\int_{t_1}^{\infty} p(s)|y(s;\lambda)|^{\alpha+1}\,ds = \infty.$$

By virtue of Lemma 4.3, we obtain (4.31). This completes the proof of Lemma 4.4.

LEMMA 4.5. *Consider equation* (1.1) *on a compact interval* $[t_0, t_1]$. *For any integer* $k = 1, 2, \cdots$, *there exists a positive constant* $V_0 = V_0(k)$ *such that if* $y(t)$ *is a solution of* (1.1) *satisfying*

$$V[y](t) \geq V_0, \qquad t_0 \leq t \leq t_1, \tag{4.37}$$

then $y(t)$ *has at least* k *zeros in* $[t_0, t_1]$.

PROOF. We define p_* and p^* by

$$p_* = \min\{p(t)\colon t_0 \leq t \leq t_1\}$$

and

$$p^* = \max\{p(t)\colon t_0 \leq t \leq t_1\},$$

respectively. Choose $\tau > 0$ so small that

$$\tau < \frac{t_1 - t_0}{3k}. \tag{4.38}$$

Take $\mu > 0$ so large that

$$\frac{\pi}{(\mu p_*)^{1/2}} \leq \tau, \tag{4.39}$$

and put

$$\mu_0 = \mu^{1/(\alpha-1)}. \tag{4.40}$$

We define $V_0 = V_0(k)$ as follows:

$$V_0 = \frac{1}{2}\left(\frac{\mu_0}{\tau}\right)^2 + \frac{1}{\alpha+1} p^* \mu_0^{\alpha+1}. \tag{4.41}$$

We verify that a solution $y(t)$ of (1.1) satisfying (4.37) with this V_0 has at least k zeros in $[t_0, t_1]$.

Let $y(t)$ be a solution of (1.1) satisfying (4.37) in which V_0 is given by (4.41). Notice here that if $|y(t)| \leq \mu_0$ at a point $t \in [t_0, t_1]$, then $|y'(t)| \geq \mu_0/\tau$. Indeed, if $|y(t)| \leq \mu_0$, then from (4.1), (4.37) and (4.41) we have

$$\frac{1}{2}[y'(t)]^2 + \frac{1}{\alpha+1} p(t)|y(t)|^{\alpha+1} \geq \frac{1}{2}\left(\frac{\mu_0}{\tau}\right)^2 + \frac{1}{\alpha+1} p^* \mu_0^{\alpha+1},$$

which implies $|y'(t)| \geq \mu_0/\tau$.

First we show that $y(t)$ has at least one zero in an arbitrary subinterval $[T_0, T_1]$ of $[t_0, t_1]$ such that $T_1 - T_0 = 3\tau$. We may assume that $y(T_0) > 0$ without loss of generality. It is convenient to distinguish the following three cases:

(i) $y(T_0) \leq \mu_0$ and $y'(T_0) \leq 0$;

(ii) $y(T_0) > \mu_0$;

(iii) $y(T_0) \leq \mu_0$ and $y'(T_0) > 0$.

(i) *The case where* $y(T_0) \leq \mu_0$ *and* $y'(T_0) \leq 0$. We show that $y(t)$ has at least one zero in $[T_0, T_0 + \tau]$. Assume to the contrary that $y(t) > 0$ on $[T_0, T_0 + \tau]$. By equation (1.1) we have $y''(t) < 0$ on $[T_0, T_0 + \tau]$. Then it is easy to see that $y(t) \leq y(T_0) + y'(T_0)(t - T_0)$ on $[T_0, T_0 + \tau]$. Thus we have

$$0 < y(t) \leq \mu_0, \qquad T_0 \leq t \leq T_0 + \tau.$$

By the above notice we obtain

$$y'(t) \leq -\mu_0/\tau, \qquad T_0 \leq t \leq T_0 + \tau.$$

This implies that

$$y(T_0 + \tau) = y(T_0) + \int_{T_0}^{T_0+\tau} y'(s)\,ds \le y(T_0) - \mu_0 \le 0,$$

which is a contradiction to $y(T_0 + \tau) > 0$. Thus $y(t)$ has at least one zero in $[T_0, T_0 + \tau]$.

(ii) *The case where* $y(T_0) > \mu_0$. Let us show that $y(t)$ has at least one zero in $[T_0, T_0 + 2\tau]$. By the argument as in Case (i), it is enough to show that

$$y(t_*) \le \mu_0 \quad \text{and} \quad y'(t_*) \le 0 \tag{4.42}$$

for some $t_* \in [T_0, T_0 + \tau]$. Assume that

$$y(t) > \mu_0, \qquad T_0 \le t \le T_0 + \tau. \tag{4.43}$$

We consider the linear differential equation

$$z'' + \mu p_* z = 0. \tag{4.44}$$

Equation (4.44) has a solution $z(t) = \sin((\mu p_*)^{1/2}(t - T_0))$ which vanishes at $t = T_0$ and $t = T_0 + \pi/(\mu p_*)^{1/2}$. We see that $y(t)$ is a solution of the linear equation

$$w'' + p(t)|y(t)|^{\alpha-1} w = 0, \qquad T_0 \le t \le T_0 + \tau.$$

From (4.40) and (4.43) we have

$$p(t)|y(t)|^{\alpha-1} > \mu p_*, \qquad T_0 \le t \le T_0 + \tau.$$

By using Sturm's comparison theorem, we find that $y(t)$ has at least one zero in $[T_0, T_0 + \pi/(\mu p_*)^{1/2}]$. Then, by virtue of (4.39), we see that $y(t)$ has at least one zero in $[T_0, T_0 + \tau]$. However this contradicts (4.43). Thus (4.43) does not hold. Then it is clear that there is $t_* \in [T_0, T_0 + \tau]$ satisfying (4.42).

(iii) *The case where* $y(T_0) \le \mu_0$ *and* $y'(T_0) > 0$. Let us show that $y(t)$ has at least one zero in $[T_0, T_0 + 3\tau]$. First we claim that $y(t^*) > \mu_0$ for some $t^* \in [T_0, T_0 + \tau]$. Assume to the contrary that $0 < y(t) \le \mu_0$ for $T_0 \le t \le T_0 + \tau$. Then, by the above notice, we see that $y'(t) \ge \mu_0/\tau$ for $T_0 \le t \le T_0 + \tau$, and hence

$$y(T_0 + \tau) = y(T_0) + \int_{T_0}^{T_0+\tau} y'(s)\,ds > \mu_0.$$

This contradicts our assumption. Thus, $y(t^*) > \mu_0$ for some $t^* \in [T_0, T_0 + \tau]$. By the argument similar to Case (ii), it is seen that $y(t)$ has at least one zero in $[t^*, t^* + 2\tau] \subset [T_0, T_0 + 3\tau]$.

We have showed that $y(t)$ has at least one zero in every subinterval

$[T_0, T_1]$ of $[t_0, t_1]$ such that $|T_0 - T_1| = 3\tau$. Then, by virtue of (4.38), we conclude that $y(t)$ has at least k zeros in $[t_0, t_1]$. The proof of Lemma 4.5 is complete.

We are now ready to prove Proposition 4.3.

PROOF OF PROPOSITION 4.3. We consider equation (1.1) on the compact interval $[t_0, t_1]$, where t_1 is a constant appearing in (4.17). Let k be any positive integer. By Lemma 4.5, there exists a positive constant $V_0 = V_0(k)$ such that a solution $y(t)$ of (1.1) satisfying (4.37) has at least k zeros in $[t_0, t_1]$. From (4.4) we see that

$$V[y(\cdot\,;\lambda)](t) \geq V[y(\cdot\,;\lambda)](t_1)\exp\left(-\int_{t_0}^{t_1}\frac{[p'(s)]_+}{p(s)}\,ds\right), \qquad t_0 \leq t \leq t_1. \tag{4.45}$$

By virtue of Lemma 4.4 and (4.45) we have

$$\lim_{\lambda\to\infty} V[y(\cdot\,;\lambda)](t) = \infty \quad \text{uniformly on} \quad [t_0, t_1].$$

Then, for all sufficiently large λ, $y(t;\lambda)$ satisfies

$$V[y(t;\lambda)](t) \geq V_0, \qquad t_0 \leq t \leq t_1,$$

which implies that $y(t;\lambda)$ has at least k zeros in $[t_0, t_1] \subset [t_0, \infty)$. Since k is arbitrary, this means that the number of zeros of $y(t;\lambda)$ tends to ∞ as $\lambda \to \infty$. The proof of Proposition 4.3 is complete.

To prove Theorem 1, we employ the Prüfer transformation. For the solution $y(t;\lambda)$ of (1.1) satisfying (3.3), we define the functions $\rho(t;\lambda)$ and $\varphi(t;\lambda)$ by

$$y(t;\lambda) = \rho(t;\lambda)\sin\varphi(t;\lambda), \tag{4.46}$$

$$y'(t;\lambda) = \rho(t;\lambda)\cos\varphi(t;\lambda). \tag{4.47}$$

Note that $y(t;\lambda)$ and $y'(t;\lambda)$ cannot vanish at the same point $t \in [t_0, \infty)$. Then we see that $\rho(t;\lambda)$ and $\varphi(t;\lambda)$ are given by

$$\rho(t;\lambda) = ([y'(t;\lambda)]^2 + [y(t;\lambda)]^2)^{1/2}$$

and

$$\varphi(t;\lambda) = \arctan\frac{y(t;\lambda)}{y'(t;\lambda)},$$

respectively, and that $\rho(t; \lambda)$ and $\varphi(t; \lambda)$ are determined as continuously differentiable functions with respect to t. Moreover, by Proposition 4.1, $\rho(t; \lambda)$ and $\varphi(t; \lambda)$ are continuous in $(t, \lambda) \in [t_0, \infty) \times (0, \infty)$. By a simple caluculation we see that

$$\varphi'(t; \lambda) = [\cos \varphi(t; \lambda)]^2 + p(t)[\rho(t; \lambda)]^{\alpha-1} |\sin \varphi(t; \lambda)|^{\alpha+1}, \qquad t \geq t_0,$$

which implies that $\varphi(t; \lambda)$ is increasing in $t \geq t_0$. Since $\lim_{t\to\infty} y(t; \lambda) = \lambda$ and $\lim_{t\to\infty} y'(t; \lambda) = 0$, we have $\lim_{t\to\infty} \rho(t; \lambda) = \lambda$ and $\lim_{t\to\infty} \varphi(t; \lambda) = \frac{\pi}{2}$ (mod 2π). For simplicity, we take

$$\lim_{t\to\infty} \varphi(t; \lambda) = \frac{\pi}{2}. \tag{4.48}$$

Then it is easy to see that $y(t; \lambda)$ has exactly $k-1$ zeros in (t_0, ∞) if and only if

$$-(k-1)\pi \leq \varphi(t_0; \lambda) < -(k-2)\pi. \tag{4.49}$$

Proof of Theorem 1. From Propositions 4.2 and 4.3, we see that

$$\lim_{\lambda\to 0} \varphi(t_0; \lambda) = \frac{1}{2}\pi$$

and

$$\lim_{\lambda\to\infty} \varphi(t_0; \lambda) = -\infty.$$

By the continuity of $\varphi(t_0; \lambda)$ with respect to λ, for each $k = 1, 2, \cdots$, there exists a $\lambda > 0$ such that

$$\varphi(t_0; \lambda) = -(k-1)\pi. \tag{4.50}$$

Let λ_k denote the smallest $\lambda > 0$ satisfying (4.50). Then we easily see that $\{\lambda_k\}$ must be a strictly increasing sequence and that $y(t; \lambda_k)$ has exactly $k-1$ zeros in (t_0, ∞) and satisfies $y(t_0; \lambda_k) = 0$. Let $y_k(t) \equiv y(t; \lambda_k)$. Then, $y_k(t)$ is a solution of (1.1) such that $y_k(t)$ has exactly $k-1$ zeros in (t_0, ∞) and satisfies $y_k(t_0) = 0$ and

$$\lim_{t\to\infty} y_k(t) = \lambda_k. \tag{4.51}$$

Thus, $y_k(t)$ is a desired solution of (1.1). The proof of Theorem 1 is complete.

5. The sublinear equation

Let us now consider the case where equation (1.1) is sublinear. Throughout this section we assume that $0 < \alpha < 1$ and that (3.2) holds. As is shown in Section 3, for each $\lambda > 0$, there exists a unique solution $y(t; \lambda)$ of (1.1) which exists on $[t_0, \infty)$ and satisfies (3.3). Moreover, $y(t; \lambda)$ satisfies (3.7) and (3.8).

PROPOSITION 5.1. *If* $\lambda > 0$, $\lambda(i) > 0$ $(i = 1, 2 \cdots)$ *and* $\lim_{i\to\infty} \lambda(i) = \lambda$, *then* $\lim_{i\to\infty} y(t; \lambda(i)) = y(t; \lambda)$ *and* $\lim_{i\to\infty} y'(t; \lambda(i)) = y'(t; \lambda)$ *uniformly on* $[t_0, \infty)$. *In particular,* $y(t; \lambda)$ *and* $y'(t; \lambda)$ *are continuous in* $(t, \lambda) \in [t_0, \infty) \times (0, \infty)$.

The truth is that the statements of Propositions 4.1 and 5.1 are identical. However the proof of Proposition 4.1 is of no use for the proof of Proposition 5.1. Indeed, the superlinearity is essentially used in the proof of Proposition 4.1. To prove Proposition 5.1, we prepare the next lemma.

LEMMA 5.1. *For each* $\lambda > 0$, $y(t; \lambda)$ *is estimated as follows*:

$$(5.1) \qquad |y(t; \lambda)| \le \left[\lambda^{1-\alpha} + (1-\alpha)\int_{t_0}^{\infty} sp(s)\,ds\right]^{1/(1-\alpha)}, \qquad t \ge t_0.$$

PROOF. From (3.7) we see that

$$(5.2) \qquad |y(t; \lambda)| \le \lambda + \int_t^{\infty} sp(s)|y(s; \lambda)|^{\alpha}\,ds, \qquad t \ge t_0.$$

Define the function $Y(t)$ by

$$(5.3) \qquad Y(t) = \lambda + \int_t^{\infty} sp(s)|y(s; \lambda)|^{\alpha}\,ds, \qquad t \ge t_0.$$

We easily see that

$$-\frac{Y'(t)}{[Y(t)]^{\alpha}} \le tp(t), \qquad t \ge t_0.$$

Integrating the above on $[t, \tau]$, $t_0 \le t < \tau$, and letting $\tau \to \infty$, we obtain

$$[Y(t)]^{1-\alpha} - \lambda^{1-\alpha} \le (1-\alpha)\int_t^{\infty} sp(s)\,ds, \qquad t \ge t_0,$$

so that

$$(5.4) \qquad Y(t) \le \left[\lambda^{1-\alpha} + (1-\alpha)\int_t^{\infty} sp(s)\,ds\right]^{1/(1-\alpha)}, \qquad t \ge t_0.$$

Then, from (5.2)–(5.4), we conclude that (5.1) holds. This completes the proof of Lemma 5.1.

PROOF OF PROPOSITION 5.1. We shall prove that

$$\lim_{i\to\infty} [\sup_{t\geq t_0} |y(t;\lambda(i)) - y(t;\lambda)| + \sup_{t\geq t_0} |y'(t;\lambda(i)) - y'(t;\lambda)|] = 0. \tag{5.5}$$

Assume that (5.5) is false. Then there exist an $\varepsilon > 0$ and a subsequence (again denoted by $\{\lambda(i)\}$) of $\{\lambda(i)\}$ such that

$$\sup_{t\geq t_0} |y(t;\lambda(i)) - y(t;\lambda)| + \sup_{t\geq t_0} |y'(t;\lambda(i)) - y'(t;\lambda)| \geq \varepsilon \tag{5.6}$$

for $i = 1, 2, \cdots$. By Lemma 5.1, there exists a certain constant $M > 0$ such that

$$|y(t;\lambda(i))| \leq M, \qquad t \geq t_0,\ i = 1, 2, \cdots. \tag{5.7}$$

From (3.8) and (5.7) we see that

$$|y'(t;\lambda(i))| \leq M^\alpha \int_{t_0}^{\infty} p(s)\,ds, \qquad t \geq t_0,\ i = 1, 2, \cdots.$$

Therefore $\{y(t;\lambda(i))\}$ is uniformly bounded and is equicontinuous on $[t_0, \infty)$. According to Ascoli-Arzelà's theorem there exists a subsequence $\{\lambda(i_j)\}$ of $\{\lambda(i)\}$ and a continuous function $z(t)$ on $[t_0, \infty)$ such that $\{y(t;\lambda(i_j))\}$ converges to $z(t)$ uniformly on any compact subinterval of $[t_0, \infty)$. Note here that $\{y(t;\lambda(i_j))\}$ satisfies the equality

$$y(t;\lambda(i_j)) = \lambda(i_j) - \int_t^{\infty} (s-t)p(s)f(y(s;\lambda(i_j)))\,ds, \qquad t \geq t_0, \tag{5.8}$$

where $f(u) = |u|^\alpha \operatorname{sgn} u$. Let $j \to \infty$ in (5.8). Then, by the Lebesgue dominated convergence theorem, we see that

$$z(t) = \lambda - \int_t^{\infty} (s-t)p(s)f(z(s))\,ds, \qquad t \geq t_0.$$

Thus $z(t)$ is a solution of (1.1) satisfying the asymptotic property (3.3). By the uniqueness of $y(t;\lambda)$ we find that $y(t;\lambda) \equiv z(t)$ for $t \geq t_0$. Therefore $\{y(t;\lambda(i_j))\}$ converges to $y(t;\lambda)$ uniformly on any compact subinterval of $[t_0, \infty)$. Observe that

$$\begin{aligned}
&|y(t;\lambda(i_j)) - y(t;\lambda)| \\
&\leq |\lambda(i_j) - \lambda| + \int_t^{\infty} (s-t)p(s)|f(y(s;\lambda(i_j))) - f(y(s;\lambda))|\,ds \\
&\leq |\lambda(i_j) - \lambda| + \int_{t_0}^{\infty} sp(s)|f(y(s;\lambda(i_j))) - f(y(s;\lambda))|\,ds, \qquad t \geq t_0,
\end{aligned} \tag{5.9}$$

and

$$|y'(t;\lambda(i_j)) - y'(t;\lambda)| \le \int_t^\infty p(s)|f(y(s;\lambda(i_j)) - f(y(s;\lambda))|\,ds \le \int_{t_0}^\infty p(s)|f(y(s;\lambda(i_j)) - f(y(s;\lambda))|\,ds, \qquad t \ge t_0. \tag{5.10}$$

Letting $j \to \infty$ in (5.9) and (5.10), and using the Lebesgue dominated convergence theorem, we conclude that $\{y(t;\lambda(i_j))\}$ and $\{y'(t;\lambda(i_j))\}$ converge to $y(t;\lambda)$ and $y'(t;\lambda)$ as $j \to \infty$ uniformly on $[t_0, \infty)$. This contradicts (5.6). Hence, (5.5) holds. The proof of Proposition 5.1 is complete.

PROPOSITION 5.2. *For sufficiently large $\lambda > 0$, $y(t;\lambda) > 0$ on $[t_0, \infty)$. Moreover, the solution $y(t;\lambda)$ has the following properties*:

(i) $\lim_{\lambda\to\infty} y(t;\lambda) = \infty$ *uniformly on* $[t_0, \infty)$; *and*

(ii) $\lim_{\lambda\to\infty} \dfrac{y'(t;\lambda)}{y(t;\lambda)} = 0$ *uniformly on* $[t_0, \infty)$.

PROOF. Let λ be large enough so that

$$\lambda > \left[\int_{t_0}^\infty sp(s)\,ds\right]^{1/(1-\alpha)} \tag{5.11}$$

We claim that $y(t;\lambda) > 0$ on $[t_0, \infty)$. Assume to the contrary that $y(t;\lambda)$ has a zero in $[t_0, \infty)$. Then, exactly as in the proof of Proposition 4.2, we have

$$\lambda^{1-\alpha} \le \int_{t_0}^\infty sp(s)\,ds.$$

This contradicts (5.11). Thus we conclude that, for sufficiently large $\lambda > 0$, $y(t;\lambda)$ has no zeros in $[t_0, \infty)$.

Let $\lambda > 0$ be large enough so that $y(t;\lambda) > 0$, $t \ge t_0$. We have $0 < y(t;\lambda) < \lambda$ for $t \ge t_0$. Then, by (3.7), we easily see that

$$\left|\frac{y(t;\lambda)}{\lambda} - 1\right| \le \lambda^{\alpha-1}\int_{t_0}^\infty sp(s)\,ds, \qquad t \ge t_0.$$

Thus

$$\lim_{\lambda\to\infty} \frac{y(t;\lambda)}{\lambda} = 1 \quad \text{uniformly on} \quad [t_0, \infty), \tag{5.12}$$

which implies (i). From (3.8) we have

$$\left|\frac{y'(t;\lambda)}{y(t;\lambda)}\right| \le \frac{\lambda}{y(t;\lambda)}\lambda^{\alpha-1}\int_{t_0}^{\infty} p(s)\,ds, \qquad t \ge t_0.$$

By virtue of (5.12), we obtain (ii). This completes the proof of Proposition 5.2.

PROPOSITION 5.3. *Suppose that* (1.4) *holds. Then the number of zeros of* $y(t;\lambda)$ *in* $[t_0, \infty)$ *tends to* ∞ *as* $\lambda \to 0$.

The next lemmas are necessary for the proof of Proposition 5.3.

LEMMA 5.2. *Suppose that* (1.4) *holds. Then*

$$\lim_{\lambda\to 0} y(t;\lambda) = 0 \quad \textit{uniformly in} \quad [t_0, \infty). \tag{5.13}$$

PROOF. Let $\{\lambda(i)\}$ be an arbitrary sequence such that $\lambda(i) > 0$ $(i = 1, 2, \cdots)$ and $\lim_{i\to\infty}\lambda(i) = 0$. It is enough to show that $\{\lambda(i)\}$ has a subsequence $\{\lambda(i_j)\}$ such that

$$\lim_{j\to\infty} y(t;\lambda(i_j)) = 0 \quad \text{uniformly in} \quad [t_0, \infty). \tag{5.14}$$

Exactly as in the proof of Proposition 5.1, we can conclude that there exist a subsequence $\{\lambda(i_j)\}$ of $\{\lambda(i)\}$ and a continuous function $z(t)$ on $[t_0, \infty)$ such that $\{y(t;\lambda(i_j)\}$ converges to $z(t)$ as $j \to \infty$ uniformly on $[t_0, \infty)$ and such that $z(t)$ satisfies

$$z(t) = -\int_t^{\infty} (s-t)p(s)|z(s)|^{\alpha}\operatorname{sgn} z(s)\,ds, \qquad t \ge t_0.$$

Note that $z(t)$ is a bounded solution of (1.1) satisfying $\lim_{t\to\infty} z(t) = 0$.

By the first condition in (1.4), we have

$$\lim_{t\to\infty} t\int_t^{\infty} p(s)\,ds = 0. \tag{5.15}$$

According to the results of Gollwitzer [6] and Kwong-Wong [14], the second condition in (1.4) and (5.15) together imply that all nontrivial solutions of (1.1) are nonoscillatory. Further it is well known that a bounded nonoscillatory solution of (1.1) has a nonzero finite limit as $t \to \infty$. This means that $z(t)$ cannot be a nontrivial solution of (1.1). Thus, $z(t) \equiv 0$ for $t \ge t_0$. Consequently we get (5.14). This completes the proof of Lemma 5.2.

LEMMA 5.3. *Consider equation* (1.1) *on a compact subinterval* $[t_0, t_1]$. *For any integer* $k = 1, 2, \cdots$, *there exists a positive constant* $Y_0 = Y_0(k)$ *such that if*

$y(t)$ is a nontrivial solution of (1.1) *satisfying*

$$|y(t)| \le Y_0, \qquad t_0 \le t \le t_1, \tag{5.16}$$

then $y(t)$ has at least k zeros in $[t_0, t_1]$.

PROOF. Take $\mu > 0$ so large that a nontrivial solution $z(t)$ of

$$z'' + \mu p(t) z = 0, \qquad t_0 \le t \le t_1, \tag{5.17}$$

has at least $k+1$ zeros in $[t_0, t_1]$. Take a number Y_0 such that

$$0 < Y_0 < \mu^{1/(\alpha-1)}. \tag{5.18}$$

Let $y(t)$ be a solution of (1.1) satisfying (5.16) in which Y_0 is given by (5.18). We verify that $y(t)$ has at least k zeros in $[t_0, t_1]$. Let s_0 and s_1, $t_0 \le s_0 < s_1 \le t_1$, be any successive zeros of the solution $z(t)$ of (5.17). We claim that $y(t)$ has at least one zero in (s_0, s_1). Assume to the contrary that $y(t)$ has no zeros in (s_0, s_1). There is no loss of generality in supposing that $z(t) > 0$ and $y(t) > 0$ on (s_0, s_1). Multiplying (1.1) by $z(t)$ and (5.17) by $y(t)$, subtracting, and integrating over $[s_0, s_1]$, we obtain

$$\begin{aligned} y(s_1)z'(s_1) - y(s_0)z'(s_0) \\ = \int_{s_0}^{s_1} p(s)y(s)z(s)\{[y(s)]^{\alpha-1} - \mu\}\, ds. \end{aligned} \tag{5.19}$$

Because of $z(t) > 0$ on (s_0, s_1), $z(s_0) = z(s_1) = 0$ and $y(t) \ge 0$ on $[s_0, s_1]$, the left-hand side of (5.19) is nonpositive. On the other hand, it follows from (5.16) and (5.18) that $[y(s)]^{\alpha-1} > \mu$ on (s_0, s_1); and hence the right-hand side of (5.19) is positive. This is a contradiction. Thus $y(t)$ has at least one zero in (s_0, s_1). Since $y(t)$ has at least one zero between each successive zeros of $z(t)$, we conclude that $y(t)$ has at least k zeros in $[t_0, t_1]$. This completes the proof of Lemma 5.3.

PROOF OF PROPOSITION 5.3. Let $t_1 > t_0$ be an arbitrary number. We fix t_1 and consider equation (1.1) on $[t_0, t_1]$. Let k be any positive integer. By Lemma 5.3, there exists a positive constant $Y_0 = Y_0(k)$ such that a solution $y(t)$ of (1.1) satisfying (5.16) has at least k zeros in $[t_0, t_1]$. By virtue of Lemma 5.2 we have $\lim_{\lambda \to 0} y(t; \lambda) = 0$ uniformly on $[t_0, t_1]$. Then, for sufficiently small λ, $y(t; \lambda)$ satisfies

$$|y(t; \lambda)| \le Y_0, \qquad t_0 \le t \le t_1,$$

which implies that $y(t; \lambda)$ has at least k zeros in $[t_0, t_1] \subset [t_0, \infty)$. Since k is arbitrary, we conclude that the number of zeros of $y(t; \lambda)$ in $[t_0, \infty)$ tends to ∞ as $\lambda \to 0$. The proof of Proposition 5.3 is complete.

To prove Theorem 2, we employ the Prüfer transformation. For the solution $y(t;\lambda)$ of (1.1) satisfying (3.3), we define the functions $\rho(t;\lambda)$ and $\varphi(t;\lambda)$ by (4.46)–(4.48). In the same way as in Section 4, $\rho(t;\lambda)$ and $\varphi(t;\lambda)$ are well defined and uniquely determined, and they are continuously differentiable functions with respect to t. Moreover, by Lemma 5.1, $\rho(t;\lambda)$ and $\varphi(t;\lambda)$ are continuous in $(t,\lambda)\in[t_0,\infty)\times(0,\infty)$. We see that $\varphi(t;\lambda)$ is increasing in $t\in(t_0,\infty)$, and that $y(t;\lambda)$ has exactly $k-1$ zeros in (t_0,∞) if and only if (4.49) holds.

PROOF OF THEOREM 2. From Propositions 5.2 and 5.3, we see that

$$\lim_{\lambda\to\infty}\varphi(t_0;\lambda)=\frac{1}{2}\pi$$

and

$$\lim_{\lambda\to\infty}\varphi(t_0;\lambda)=-\infty.$$

By the continuity of $\varphi(t_0;\lambda)$ with respect to λ, for each $k=1,2,\cdots$, there exists a $\lambda>0$ such that (4.50) holds. Let λ_k denote the largest $\lambda>0$ satisfying (4.50). Then we easily see that $\{\lambda_k\}$ must be a strictly decreasing sequence and that $y(t;\lambda_k)$ has exactly $k-1$ zeros in (t_0,∞) and satisfies $y(t_0;\lambda_k)=0$. Let $y_k(t)\equiv y(t;\lambda_k)$. Then, $y_k(t)$ is a solution of (1.1) such that $y_k(t)$ has exactly $k-1$ zeros in (t_0,∞) and satisfies $y_k(t_0)=0$ and (4.51). Thus, $y_k(t)$ is a desired solution of (1.1). The proof of Theorem 2 is complete.

6. The elliptic problems

6.1. In this subsection we consider the problems (1.11) and (1.13), and prove Theorems 3–6.

First we consider the equation

$$(r^{n-1}u')'+r^{n-1}q(r)|u|^{\alpha}\operatorname{sgn}u=0,\qquad 0<r<1,\tag{6.1}$$

where $n\geq 3$, $q\in C[0,1]\cap C^1(0,1]$ and $\alpha>0$, $\alpha\neq 1$. We introduce the change of variables

$$y(t)=u(r)\quad\text{and}\quad t=r^{2-n},\tag{6.2}$$

which reduces (6.1) to the equation

$$\ddot{y}+p(t)|y|^{\alpha}\operatorname{sgn}y=0,\qquad t\geq 1,\tag{6.3}$$

where $\;=d/dt$ and

(6.4) $$p(t) = \frac{1}{(n-2)^2} t^{2(n-1)/(2-n)} q(t^{1/(2-n)}), \qquad t \geq 1.$$

We see that if q satisfies (2.3), then p satisfies (1.2) with $t_0 = 1$. Suppose that, for each $k = 1, 2, \cdots$, equation (6.3) has a bounded solution $y_k(t)$ such that y_k has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $y_k(1) = 0$. We denote by $u_k(r)$ the solution of (6.1) which corresponds to the solution $y_k(t)$. It is clear that $u_k(r)$ has exactly $k-1$ zeros in $(0, 1)$ and satisfies $u_k(1) = 0$. We see that

(6.5) $$\lim_{r\to 0} u_k(r) = \lim_{t\to\infty} y_k(t) < \infty.$$

Moreover, we have

$$\lim_{r\to 0} \frac{1}{2-n} r^{n-1} u_k'(r) = \lim_{t\to\infty} \dot{y}_k(t) = 0.$$

Then, integrating (6.1) on $[r_1, r]$, $0 < r_1 < r$, and letting $r_1 \to 0$, we obtain

$$u_k'(r) = r^{1-n} \int_0^r s^{n-1} q(s) |u_k(s)|^\alpha \operatorname{sgn} u_k(s)\, ds, \qquad 0 < r < 1.$$

Then we have

$$\lim_{r\to 0} u_k'(r) = 0.$$

Therefore, the solution $u_k(r)$ satisfies the boundary condition

$$u_k'(0) = 0 \quad \text{and} \quad u_k(1) = 0$$

and has exactly $k-1$ zeros in $(0, 1)$.

PROOF OF THEOREM 3. Observe that, by (6.2), condition (2.4) is transformed into condition (1.3). Then, from Theorem 1, there exists an infinite sequence of bounded solutions $y_k(t)$, $k = 1, 2, \cdots$, of (6.3) such that $y_k(t)$ has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $y_k(1) = 0$ and (2.1). For each $k = 1, 2, \cdots$, let $u_k(r)$ be the solution of (6.1) which corresponds to the solution $y_k(t)$. As mentioned above, $u_k(r)$ is a solution of the problem (1.11) such that $u_k(r)$ has exactly $k-1$ zeros in $(0, 1)$. By virtue of (2.1) and (6.5), we have (2.5). The proof of Theorem 3 is complete.

PROOF OF THEOREM 4. Observe that, by (6.2), the first condition in (1.4) corresponds to the condition

(6.6) $$\int_0^1 rq(r)\, dr < \infty,$$

and the second condition in (1.4) corresponds to (2.6). Since $q \in C[0, 1]$, condition (6.6) is always satisfied. Thus, under condition (2.6), we have (1.4). Then, from Theorem 2, there exists an infinite sequence of bounded solutions $y_k(t)$, $k = 1, 2, \cdots$, of (6.3) such that $y_k(t)$ has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $y_k(1) = 0$ and (2.2). For each $k = 1, 2, \cdots$, let $u_k(r)$ be the solution of (6.1) which corresponds to the solution $y_k(t)$. As mentioned above, $u_k(r)$ is a solution of the problem (1.11) such that $u_k(r)$ has exactly $k-1$ zeros in $(0, 1)$. By (2.2) and (6.5) we have (2.7). The proof of Theorem 4 is complete.

Next we consider the equation

$$(r^{n-1}u')' + r^{n-1}q(r)|u|^\alpha \operatorname{sgn} u = 0, \qquad r > 1, \tag{6.7}$$

where $n \geq 3$, $q \in C^1[1, \infty)$ and $\alpha > 0$, $\alpha \neq 1$. In this case, we introduce the change of variables

$$y(t) = r^{n-2}u(r) \quad \text{and} \quad t = r^{n-2}, \tag{6.8}$$

which reduces (6.7) to equation (6.3) with

$$p(t) = \frac{1}{(n-2)^2} t^{(-n+4-\alpha n+2\alpha)/(n-2)} q(t^{1/(n-2)}), \qquad t \geq 1. \tag{6.9}$$

We see that, under condition (2.8), p satisfies (1.2) with $t_0 = 1$. Suppose that $y_k(t)$, $k = 1, 2, \cdots$, are bounded solutions of (6.3) such that $y_k(t)$ has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $y_k(1) = 0$. We denote by $u_k(r)$ the solution of (6.7) which corresponds to the solution $y_k(t)$. It is clear that $u_k(r)$ has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $u_k(1) = 0$ and

$$\lim_{r \to \infty} r^{n-2}u_k(r) = \lim_{t \to \infty} y_k(t) < \infty. \tag{6.10}$$

PROOF OF THEOREM 5. Observe that, by (6.8), condition (2.9) is transformed into condition (1.3). Then, from Theorem 1, there exists an infinite sequence of bounded solutions $y_k(t)$, $k = 1, 2, \cdots$, of (6.3) such that $y_k(t)$ has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $y_k(1) = 0$ and (2.1). For each $k = 1, 2, \cdots$, let $u_k(r)$ be the solution of (6.7) which corresponds to the solution $y_k(t)$. As mentioned above, $u_k(r)$ is a solution of the problem (1.13) such that $u_k(r)$ has exactly $k-1$ zeros in $(1, \infty)$. By virtue of (2.1) and (6.10), we have (2.10). This completes the proof of Theorem 5.

PROOF OF THEOREM 6. Observe that, by (6.2), condition (1.4) corresponds to condition (2.11). Then, from Theorem 2, there exists an infinite sequence of bounded solutions $y_k(t)$, $k = 1, 2, \cdots$, of (6.3) such that $y_k(t)$ has exactly $k-1$ zeros in $(1, \infty)$ and satisfies $y_k(1) = 0$ and (2.2). For each $k = 1, 2, \cdots$, let $u_k(r)$ be the solution of (6.7) which corresponds to the solution $y_k(t)$. As mentioned above, $u_k(r)$ is a solution of the problem (1.13) such that $u_k(r)$ has exactly $k-1$ zeros in $(1, \infty)$. By virtue of (2.2) and (6.10), we have (2.12). The proof of Theorem 6 is complete.

6.2. In this subsection we consider the problem (1.15) in the superlinear case $\alpha > 1$ and prove Theorem 7 by employing the method used in [23]. We devide (1.15) into the two problems (1.16) and (1.17), and glue solutions of (1.16) and (1.17) at $r = 1$.

First we deal with the problem (1.16) and consider equation (6.1). In the subsection 6.1 we have shown that, by the change of variables (6.2), equation (6.1) is reduced to equation (6.3) in which $p(t)$ is given by (6.4). We note that condition (3.2) is always satisfied, and that condition (1.3) corresponds to condition (2.4). From Proposition 3.1, for each $\lambda > 0$, there exists a unique solution $y(t; \lambda)$ of (6.3) satisfying

$$\lim_{t\to\infty} y(t; \lambda) = \lambda. \tag{6.11}$$

We denote by $u(r; \lambda)$ the solution of (6.1) which corresponds to the solution $y(t; \lambda)$. In the same way as in the subsection 6.1, we have

$$u(0; \lambda) = \lambda \quad \text{and} \quad u'(0; \lambda) = 0.$$

Thus $u(r; \lambda)$ is a solution of the problem (1.16). From Propositions 4.1–4.3, we obtain the following lemmas.

Lemma 6.1. *$u(r; \lambda)$ and $u'(r; \lambda)$ are continuous in $(r, \lambda) \in [0, 1] \times (0, \infty)$.*

Lemma 6.2. *For sufficiently small $\lambda > 0$, $u(r; \lambda) > 0$ on $[0, 1]$. Moreover, the solution $u(r; \lambda)$ has the properties:*

$$\lim_{\lambda\to 0} u(r; \lambda) = 0 \quad \textit{uniformly in} \quad [0, 1]; \textit{ and} \tag{6.12}$$

$$\lim_{\lambda\to 0} \frac{r^{n-1} u'(r; \lambda)}{u(r; \lambda)} = 0 \quad \textit{uniformly in} \quad [0, 1]. \tag{6.13}$$

Lemma 6.3. *Suppose that (2.4) holds. Then the number of zeros of $u(r; \lambda)$ in $[0, 1]$ tends to ∞ as $\lambda \to \infty$.*

We employ the Prüfer transformation. For the solution $u(r; \lambda)$, we define the functions $\rho(r; \lambda)$ and $\varphi(r; \lambda)$ by

$$u(r; \lambda) = \rho(r; \lambda) \sin \varphi(r; \lambda), \tag{6.14}$$

$$r^{n-1} u'(r; \lambda) = \rho(r; \lambda) \cos \varphi(r; \lambda). \tag{6.15}$$

Note that $u(r; \lambda)$ and $u'(r; \lambda)$ cannot vanish at the same point $r \in [0, 1]$. Then we see that $\rho(r; \lambda)$ and $\varphi(r; \lambda)$ are given by

$$\rho(r; \lambda) = ([r^{n-1}u'(r; \lambda)]^2 + [u(r; \lambda)]^2)^{1/2} \tag{6.16}$$

and

$$\varphi(r; \lambda) = \arctan \frac{u(r; \lambda)}{r^{n-1}u'(r; \lambda)}, \tag{6.17}$$

respectively, and that $\rho(r; \lambda)$ and $\varphi(r; \lambda)$ are determined as continuously differentiable functions with respect to r. Moreover, by Lemma 6.1, $\rho(r; \lambda)$ and $\varphi(r; \lambda)$ are continuous in $(r, \lambda) \in [0, 1] \times (0, \infty)$. By a simple calculation we see that

$$\begin{aligned} \varphi'(r; \lambda) = {} & r^{1-n}[\cos \varphi(r; \lambda)]^2 \\ & + r^{n-1}q(r)[\rho(r; \lambda)]^{\alpha-1}|\sin \varphi(r; \lambda)|^{\alpha+1}, \qquad 0 < r < 1, \end{aligned}$$

which implies that $\varphi(r; \lambda)$ is increasing in $r \in (0, 1)$. Since $u(0; \lambda) = \lambda$ and $u'(0; \lambda) = 0$, we get $\varphi(0; \lambda) = \frac{1}{2}\pi \pmod{2\pi}$. For simplicity, we take

$$\varphi(0; \lambda) = \frac{1}{2}\pi. \tag{6.18}$$

Then it is easy to see that $u(r; \lambda)$ has exactly $k - 1$ zeros in $(0, 1)$ if and only if

$$(k-1)\pi < \varphi(1; \lambda) \leq k\pi. \tag{6.19}$$

We have the next lemma.

LEMMA 6.4. *Suppose that* (2.4) *holds. Then,* $\rho(r; \lambda)$ *and* $\varphi(r; \lambda)$ *have the following properties*:

$$\lim_{\lambda \to 0} \varphi(1; \lambda) = \frac{1}{2}\pi; \tag{6.20}$$

$$\lim_{\lambda \to \infty} \varphi(1; \lambda) = \infty; \quad \textit{and} \tag{6.21}$$

$$\lim_{\lambda \to 0} \rho(1; \lambda) = 0. \tag{6.22}$$

PROOF. From Lemma 6.2, for sufficiently small $\lambda > 0$, $u(r; \lambda)$ is positive on $[0, 1]$, which implies that

$$\frac{1}{2}\pi < \varphi(1; \lambda) < \pi. \tag{6.23}$$

From (6.13) we obtain

$$\lim_{\lambda \to 0} \frac{u'(1;\lambda)}{u(1;\lambda)} = 0.$$

Then, by virtue of (6.17) and (6.23), we have (6.20). Note that $u(r;\lambda)$ has exactly $k-1$ zeros in (0, 1) if and only if (6.19) is satisfied. Then, using Lemma 6.3, we see that (6.21) holds. By (6.12) and (6.13), we have

$$\lim_{\lambda \to 0} u(1;\lambda) = \lim_{\lambda \to 0} u'(1;\lambda) = 0.$$

Then, by virtue of (6.16), we obtain (6.22). The proof of Lemma 6.4 is complete.

Next we deal with the problem (1.17) and consider equation (6.7). By the change of variables (6.8), equation (6.7) is reduced to equation (6.3) in which $p(t)$ is given by (6.9). We note that conditions (3.2) and (1.3) correspond to conditions

$$\int_1^\infty r^{n-1-\alpha(n-2)} q(r)\,dr < \infty \tag{6.24}$$

and (2.9), respectively. Assume that condition (6.24) holds, that is, condition (3.2) holds. Then Proposition 3.1 ensures that, for each $\mu > 0$, there exists a unique solution $y(t;\mu)$ of (6.3) satisfying

$$\lim_{t \to \infty} y(t;\mu) = \mu. \tag{6.25}$$

We denote by $v(r;\mu)$ the solution of (6.7) which corresponds to the solution $y(t;\mu)$. We have

$$\lim_{r \to \infty} r^{n-2} v(r;\mu) = \mu, \tag{6.26}$$

which implies that $v(r;\mu)$ is a solution of the problem (1.17). We note that $v(r;\mu)$ has the properties

$$\lim_{r \to \infty} r^{n-1} v'(r;\mu) = -(n-2)\mu, \quad \text{and} \tag{6.27}$$

$$\frac{r^{n-1} v'(r;\mu)}{v(r;\mu)} = (n-2)t^2 \frac{\dot{y}(t;\mu)}{y(t;\mu)} - (n-2)t. \tag{6.28}$$

From Propositions 4.1–4.3, we obtain the following lemmas.

LEMMA 6.5. *Suppose that* (6.24) *holds. Then,* $v(r;\mu)$ *and* $v'(r;\mu)$ *are continuous in* $(r, \mu) \in [1, \infty) \times (0, \infty)$.

LEMMA 6.6. *Suppose that* (6.24) *holds. Then, for sufficiently small* $\mu > 0$, $v(r; \mu) > 0$ *on* $[1, \infty)$. *Moreover, the solution* $v(r; \mu)$ *has the following properties*:

(6.29) $$\lim_{\mu \to 0} v(r; \mu) = 0 \quad \textit{uniformly in} \quad [1, \infty); \textit{ and}$$

(6.30) $$\lim_{\mu \to 0} \frac{r^{n-1}v'(r; \mu)}{v(r; \mu)} = -(n-2)r^{n-2} \quad \textit{for each fixed} \quad r \geq 1.$$

LEMMA 6.7. *Suppose that* (2.9) *holds. Then the number of zeros of* $v(r; \mu)$ *in* $[1, \infty)$ *tends to* ∞ *as* $\mu \to \infty$.

We employ the Prüfer transformation. For the solution $v(r; \mu)$, we define the functions $\sigma(r; \mu)$ and $\psi(r; \mu)$ by

(6.31) $$v(r; \mu) = \sigma(r; \mu) \sin \psi(r; \mu),$$

(6.32) $$r^{n-1}v'(r; \mu) = \sigma(r; \mu) \cos \psi(r; \mu).$$

We see that $\sigma(r; \mu)$ and $\psi(r; \mu)$ are given by

(6.33) $$\sigma(r; \mu) = ([r^{n-1}v'(r; \mu)]^2 + [v(r; \mu)]^2)^{1/2}$$

and

(6.34) $$\psi(r; \mu) = \arctan \frac{v(r; \mu)}{r^{n-1}v'(r; \mu)},$$

respectively, and that $\sigma(r; \mu)$ and $\psi(r; \mu)$ are determined as continuously differentiable functions with respect to r. Moreover, by Lemma 6.5, $\sigma(r; \mu)$ and $\psi(r; \mu)$ are continuous in $(r, \mu) \in [1, \infty) \times (0, \infty)$. By a simple calculation we see that

$$\psi'(r; \mu) = r^{1-n}[\cos \psi(r; \mu)]^2 + r^{n-1}q(r)[\sigma(r; \mu)]^{\alpha-1}|\sin \psi(r; \mu)|^{\alpha+1}, \qquad r > 1,$$

which implies that $\psi(r; \mu)$ is increasing in $r \in [1, \infty)$. By virtue of (6.26) and (6.27), we have $\lim_{r \to \infty} \sigma(r; \mu) = (n-2)\mu$ and $\lim_{r \to \infty} \psi(r; \mu) = \pi \pmod{2\pi}$. For simplicity, we take

(6.35) $$\lim_{r \to \infty} \psi(r; \mu) = \pi.$$

Then it is easy to see that $v(r; \mu)$ has exactly $k-1$ zeros in$(1, \infty)$ if and only if

(6.36) $$-(k-1)\pi \leq \psi(1; \mu) < -(k-2)\pi.$$

We have the following lemma.

LEMMA 6.8. *Suppose that* (2.9) *holds. Then,* $\sigma(r;\mu)$ *and* $\psi(r;\mu)$ *have the properties*

(6.37) $$\lim_{\mu\to 0}\psi(1;\mu) = \mathrm{Arctan}\left(-\frac{1}{n-2}\right)+\pi,$$

(6.38) $$\lim_{\mu\to\infty}\psi(1;\mu) = -\infty, \textit{ and}$$

(6.39) $$\lim_{\mu\to 0}\sigma(1;\mu) = 0.$$

Here, Arctan x *denotes the principal value of* arctan x: $-\frac{\pi}{2} < \mathrm{Arctan}\, x < \frac{\pi}{2}$ *for all* $x \in R$.

PROOF. From Lemma 6.6, for sufficiently small $\mu > 0$, $v(r;\mu)$ is positive on $[1,\infty)$, which implies that

(6.40) $$0 < \psi(1;\mu) < \pi.$$

From (6.30) we obtain

$$\lim_{\mu\to 0}\frac{v'(1;\mu)}{v(1;\mu)} = -(n-2).$$

Then, by virtue of (6.34) and (6.40), we have (6.37). Noticing that $v(r;\mu)$ has exactly $k-1$ zeros in $(1,\infty)$ if and only if (6.36) holds, and using Lemma 6.7, we get (6.38). By (6.29) and (6.30), we have

$$\lim_{\mu\to 0} v(1;\mu) = \lim_{\mu\to 0} v'(1;\mu) = 0.$$

Then, by virtue of (6.33), we have (6.39). The proof of Lemma 6.8 is complete.

We are now in a position to prove Theorem 7.

PROOF OF THEOREM 7. Let Γ be a continuous curve defined by

(6.41) $$\Gamma = \{(\rho(1;\lambda), \varphi(1;\lambda)) : \lambda \in (0,\infty)\}.$$

Then, because of the uniqueness of solutions of initial value problems for (6.1), Γ does not intersect itself. Further, let Γ_k, $k = 1, 2, \cdots$, be continuous curves defined by

(6.42) $$\Gamma_k = \{(\sigma(1;\mu), \psi(1;\mu) + (k-1)\pi) : \mu \in (0,\infty)\}.$$

Then, by the uniqueness of solutions of initial value problems for (6.7), Γ_k and Γ_j do not intersect for any k and j. By Lemma 6.4, we have

$$(\rho(1;\lambda),\ \varphi(1;\lambda)) \longrightarrow \left(0,\ \frac{1}{2}\pi\right) \qquad \text{as } \lambda \longrightarrow 0$$

and

$$\varphi(1;\lambda) \longrightarrow \infty \qquad \text{as } \lambda \longrightarrow \infty.$$

By Lemma 6.8, we have

$$(\sigma(1;\mu),\ \psi(1;\mu) + (k-1)\pi) \longrightarrow \left(0,\ \text{Arctan}\left(-\frac{1}{n-2}\right) + k\pi\right) \qquad \text{as } \mu \longrightarrow 0$$

and

$$\psi(1;\mu) \longrightarrow -\infty \qquad \text{as } \mu \longrightarrow \infty.$$

It follows from these facts that Γ intersects Γ_k for each $k = 1, 2, \cdots$.

For $k = 1, 2, \cdots$, let λ_k denote the smallest λ such that $(\rho(1;\lambda),\ \varphi(1;\lambda))$ is on the curve Γ_k. We easily see that $\{\lambda_k\}$ must be a strictly increasing sequence. Further, since $(\rho(1;\lambda_k),\ \varphi(1;\lambda_k))$ is on the curve Γ_k, there exists a positive constant μ_k such that

(6.43) $$\rho(1;\lambda_k) = \sigma(1;\mu_k),$$

(6.44) $$\varphi(1;\lambda_k) = \varphi(1;\mu_k) + (k-1)\pi,$$

that is,

(6.45) $$u(1;\lambda_k) = (-1)^{k-1} v(1;\mu_k),$$

(6.46) $$u'(1;\lambda_k) = (-1)^{k-1} v'(1;\mu_k).$$

Let

(6.47) $$u_k(r) = \begin{cases} u(r;\lambda_k) & \text{for } 0 \le r \le 1, \\ (-1)^{k-1} v(r;\mu_k) & \text{for } r \ge 1. \end{cases}$$

Then $u_k(r)$ is a solution of the equation in (1.15). The solution $u_k(r)$ satisfies the following properties:

$$u_k(0) = \lambda_k,\ u_k'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{n-2} u_k(r) = (-1)^{k-1}\mu_k.$$

Therefore, $u_k(r)$ is a solution of the problem (1.15) and satisfies (2.5). Moreover, the solution $u_k(r)$ can be written as

$$u_k(r) = \rho_k(r) \sin \varphi_k(r) \qquad \text{for } 0 \le r < \infty,$$

where

$$\rho_k(r) = \begin{cases} \rho(r; \lambda_k) & \text{for } 0 \le r \le 1, \\ \sigma(r; \mu_k) & \text{for } r \ge 1, \end{cases}$$

and

$$\varphi_k(r) = \begin{cases} \varphi(r; \lambda_k) & \text{for } 0 \le r \le 1, \\ \psi(r; \mu_k) + (k-1)\pi & \text{for } r \ge 1. \end{cases}$$

We see that $\varphi_k(r)$ is increasing in $r \in (0, \infty)$. By virtue of (6.18) and (6.35), we have

$$\varphi_k(0) = \frac{1}{2}\pi \quad \text{and} \quad \lim_{r \to \infty} \varphi_k(r) = k\pi.$$

Then, $u_k(r)$ has exactly $k-1$ zeros in $(0, \infty)$. The proof of Theorem 7 is complete.

6.3. In this subsection we consider the problem (1.15) in the sublinear case $0 < \alpha < 1$, and prove Theorem 8 by using the same arguments as in the subsection 6.2.

First we deal with the problem (1.16) and consider equation (6.1). We make the change of variables (6.2). Then, equation (6.1) is reduced to equation (6.3), where $p(t)$ is given by (6.4). We note that condition (3.2) is always satisfied, and that if (2.6) holds, then (1.4) holds. From Proposition 3.1, for each $\lambda > 0$, there exists a unique solution $y(t; \lambda)$ of (6.3) satisfying (6.11). We denote by $u(r; \lambda)$ the solution of (6.1) which corresponds to the solution $y(t; \lambda)$. We see that $u(r; \lambda)$ is a solution of the problem (1.16). From Propositions 5.1–5.3, we obtain the following lemmas.

LEMMA 6.9. *$u(r; \lambda)$ and $u'(r; \lambda)$ are continuous in $(r, \lambda) \in [0, 1] \times (0, \infty)$.*

LEMMA 6.10. *For sufficiently large $\lambda > 0$, $u(r; \lambda) > 0$ on $[0, 1]$. Moreover, the solution $u(r; \lambda)$ has the properties:*

$$\lim_{\lambda \to \infty} u(r; \lambda) = \infty \qquad \textit{uniformly in } [0, 1]; \textit{ and}$$

$$\lim_{\lambda \to \infty} \frac{r^{n-1} u'(r; \lambda)}{u(r; \lambda)} = 0 \qquad \textit{uniformly in } [0, 1].$$

LEMMA 6.11. *Suppose that (2.6) holds. Then the number of zeros of $u(r; \lambda)$ in $[0, 1]$ tends to ∞ as $\lambda \to 0$.*

We employ the Prüfer transformation. For the solution $u(r; \lambda)$, we define the functions $\rho(r; \lambda)$ and $\varphi(r; \lambda)$ by (6.14), (6.15) and (6.18). As in the subsection 6.2, we see that $\rho(r; \lambda)$ and $\varphi(r; \lambda)$ are well defined and uniquely determined, and that they are continuous in $(r, \lambda) \in [0, 1] \times (0, \infty)$. Moreover, $\varphi(r; \lambda)$ is increasing in $r \in (0, 1)$, and $u(r; \lambda)$ has exactly $k-1$ zeros in $(0, 1)$ if and only if (6.19) holds. Using Lemmas 6.10 and 6.11, we have the following lemma.

LEMMA 6.12. *Suppose that* (2.6) *holds. Then,* $\rho(r; \lambda)$ *and* $\varphi(r; \lambda)$ *have the following properties*:

$$\lim_{\lambda \to \infty} \varphi(1; \lambda) = \frac{1}{2}\pi;$$

$$\lim_{\lambda \to 0} \varphi(1; \lambda) = \infty; \text{ and}$$

$$\lim_{\lambda \to \infty} \rho(1; \lambda) = \infty.$$

Next we deal with the problem (1.17) and consider equation (6.7). By the change of variables (6.8), equation (6.7) is transformed to equation (6.3) in which $p(t)$ is given by (6.9). We note that condition (1.4) corresponds to condition (2.11), and that condition (3.2) corresponds to (6.24). Assume that condition (6.24) holds, that is, condition (3.2) holds. Then, it follows from Proposition 3.1 that, for each $\mu > 0$, equation (6.3) has a unique solution $y(t; \mu)$ satisfying (6.25). We denote by $v(r; \mu)$ the solution of (6.7) which corresponds to the solution $y(t; \mu)$. Then we find that $v(r; \mu)$ is a solution of the problem (1.17) and satisfies (6.27) and (6.28). From Propositions 5.1–5.3, we obtain the following lemmas.

LEMMA 6.13. *Suppose that* (6.24) *holds. Then,* $v(r; \mu)$ *and* $v'(r; \mu)$ *are continuous in* $(r, \mu) \in [1, \infty) \times (0, \infty)$.

LEMMA 6.14. *Suppose that* (6.24) *holds. Then, for sufficiently large* $\mu > 0$, $v(r; \mu) > 0$ *on* $[1, \infty)$. *Moreover, the solution* $v(r; \mu)$ *has the properties*:

$$\lim_{\mu \to \infty} v(r; \mu) = \infty \qquad \text{uniformly in } [1, \infty); \text{ and}$$

$$\lim_{\mu \to \infty} \frac{r^{n-1} v'(r; \mu)}{v(r; \mu)} = -(n-2) r^{n-2} \qquad \text{for each fixed } r \geq 1.$$

LEMMA 6.15. *Suppose that* (2.11) *holds. Then the nomber of zeros of* $v(r; \mu)$ *in* $[1, \infty)$ *tends to* ∞ *as* $\mu \to 0$.

We employ the Prüfer transformation. For the solution $v(r; \mu)$, we define the functions $\sigma(r; \mu)$ and $\psi(r; \mu)$ by (6.31), (6.32) and (6.35). We find that $\sigma(r; \mu)$ and $\psi(r; \mu)$ are well defined and uniquely determined, and they are continuous in $(r, \mu) \in [1, \infty) \times (0, \infty)$. Moreover, $\psi(r; \mu)$ is increasing in

$r \in [1, \infty)$, and $v(r; \mu)$ has exactly $k-1$ zeros in $(1, \infty)$ if and only if (6.36) holds. The following lemma is obtained from Lemmas 6.14 and 6.15.

LEMMA 6.16. *Suppose that* (2.11) *holds. Then,* $\sigma(r; \mu)$ *and* $\psi(r; \mu)$ *satisfy the properties*:

$$\lim_{\mu \to \infty} \psi(1; \mu) = \operatorname{Arctan}\left(-\frac{1}{n-2}\right) + \pi;$$

$$\lim_{\mu \to 0} \psi(1; \mu) = -\infty; \quad \textit{and}$$

$$\lim_{\mu \to \infty} \sigma(1; \mu) = \infty.$$

Here, Arctan x *denotes the principal value of* arctan x.

We are now in a position to prove Theorem 8.

PROOF OF THEOREM 8. Let Γ be a continuous curve defined by (6.41). Then Γ does not intersect itself. Similarly, let Γ_k, $k = 1, 2, \cdots$, be continuous curves defined by (6.42). Then, Γ_k and Γ_j do not intersect for any k and j. By Lemma 6.12, we have

$$(\rho(1; \lambda), \varphi(1; \lambda)) \longrightarrow \left(\infty, \frac{1}{2}\pi\right) \qquad \text{as } \lambda \longrightarrow \infty$$

and

$$\varphi(1; \lambda) \longrightarrow \infty \qquad \text{as } \lambda \longrightarrow 0.$$

By Lemma 6.16, we have

$$(\sigma(1; \mu), \psi(1; \mu) + (k-1)\pi) \longrightarrow \left(\infty, \operatorname{Arctan}\left(-\frac{1}{n-2}\right) + k\pi\right) \qquad \text{as } \mu \longrightarrow \infty$$

and

$$\psi(1; \mu) \longrightarrow -\infty \qquad \text{as } \mu \longrightarrow 0.$$

Then these facts imply that Γ intersects Γ_k for each $k = 1, 2, \cdots$.

Let λ_k denote the largest λ such that $(\rho(1; \lambda), \varphi(1; \lambda))$ is on the curve Γ_k. We easily see that $\{\lambda_k\}$ must be a strictly decreasing sequence. Moreove, since $(\rho(1; \lambda_k), \varphi(1; \lambda_k))$ is on the curve Γ_k, there exists a positive constant μ_k such that (6.43) and (6.44) hold, that is, (6.45) and (6.46) hold. Define $u_k(r)$ by (6.47). As in the subsection 6.2, we can conclude that $u_k(r)$ is a solution of the problem (1.15) such that $u_k(r)$ has exactly $k-1$ zeros in $(0, \infty)$ and satisfies (2.7). The proof of Theorem 8 is complete.

ACKNOWLEDGMENT. The author would like to express his sincere gratitude to Professor Manabu Naito for his constant counsel and encouragement during the preparation of this paper. The author also wishes to thank Professor Takaŝi Kusano for his interest in this work and many helpful suggestions.

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Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 724, *Japan*