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Oscillation of parabolic equations with oscillating coefficients

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1. Introduction

We shall be concerned with the oscillatory behavior of solutions of the parabolic equation with oscillating coefficients

(1)
$$u_t(x, t) - (a(t)\Delta u(x, t) + \sum_{i=1}^{k} b_i(t)\Delta u(x, t - \sigma_i))$$
$$+ c(x, t, u(x, t), u(x, \tau_1(t)),\dots, u(x, \tau_m(t))) = f(x, t), \quad (x, t) \in \Omega \equiv G \times (0, \infty),$$

where G is a bounded domain of $\mathbf{R}^n$ with piecewise smooth boundary ∂G and Δ is the Laplacian in $\mathbf{R}^n$. We assume throughout this paper that:

(H_1) $a(t) \in C([0, \infty); [0, \infty))$, $b_i(t) \in C([0, \infty); \mathbf{R}^1) (i = 1, 2,\dots, k)$, $f(x, t) \in C(\bar{\Omega}; \mathbf{R}^1)$ and $c(x, t, \xi, \eta_1,\dots, \eta_m) \in C(\bar{\Omega} \times \mathbf{R}^1 \times \mathbf{R}^m; \mathbf{R}^1)$;

(H_2) $c(x, t, \xi, \eta_1,\dots, \eta_m) \geq 0$ for $(x, t) \in \Omega$, $\xi \geq 0$, $\eta_i \geq 0$ $(i = 1, 2,\dots, m)$, and $c(x, t, \xi, \eta_1,\dots, \eta_m) \leq 0$ for $(x, t) \in \Omega$, $\xi \leq 0$, $\eta_i \leq 0$ $(i = 1, 2,\dots, m)$;

(H_3) σ_i $(i = 1, 2,\dots, k)$ are nonnegative constants, $\tau_i(t) \in C([0, \infty); \mathbf{R}^1)$ and $\lim_{t \to \infty} \tau_i(t) = \infty$ $(i = 1, 2,\dots, m)$.

We consider two kinds of boundary conditions:

$$(B_1) \quad u = \psi \quad \text{on} \quad \partial G \times (0, \infty),$$

$$(B_2) \quad \frac{\partial u}{\partial \nu} = \tilde{\psi} \quad \text{on} \quad \partial G \times (0, \infty),$$

where ψ, $\tilde{\psi}$ are continuous functions on $\partial G \times [0, \infty)$ and ν denotes the unit exterior normal vector to ∂G.

There has been much current interest in studying the oscillation of solutions of parabolic equations with deviating arguments. We refer the reader to [1, 3, 5] for linear parabolic equations, and to [1, 2, 4, 6–8] for nonlinear parabolic equations. Parabolic equations of neutral type were considered in the papers [2, 4, 5, 8]. All of them, however, assume that the coefficients $b_i(t)$

are nonnegative in $[0, \infty)$.

The purpose of this paper is to present conditions which imply that every solution u of some boundary value problems is oscillatory in Ω in the sense that u has a zero in $G \times [t, \infty)$ for any $t > 0$. We note that $b_i(t)$ $(i = 1, 2, \ldots, k)$ are not required to have a constant sign, that is, $b_i(t)$ are allowed to be oscillatory. In Section 2 we reduce the multi-dimensional oscillation problem to a one-dimensional problem for delay differential inequalities. Sufficient conditions are given in Section 3 that a delay differential inequality has no eventually positive solution. In Section 4 we derive oscillation criteria for the boundary value problems for (1) by combining the results obtained in Sections 2 and 3.

2. Reduction to a one-dimensional problem

The object of this section is to reduce the boundary value problems (1), (B_i) $(i = 1, 2)$ to functional differential inequalities with delays.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\Delta w + \lambda w = 0 \quad \text{in} \quad G,$$
$$w = 0 \quad \text{on} \quad \partial G$$

is positive and the corresponding eigenfunction $\Phi(x)$ is positive in G. Associated with every function $u \in \mathscr{D}(\Omega) \equiv C^2(\Omega) \cap C^1(\bar{\Omega})$, we define

$$U(t) = \int_G u(x, t)\Phi(x)dx, \qquad t \geq 0,$$
$$\tilde{U}(t) = \int_G u(x, t)dx, \qquad t \geq 0.$$

The following notation will be used:

$$F(t) = \int_G f(x, t)\Phi(x)dx, \qquad t \geq 0,$$
$$\tilde{F}(t) = \int_G f(x, t)dx, \qquad t \geq 0,$$
$$\Psi(t) = \int_{\partial G} \psi(x, t)\frac{\partial \Phi}{\partial \nu}(x)dS, \qquad t \geq 0,$$
$$\tilde{\Psi}(t) = \int_{\partial G} \tilde{\psi}(x, t)dS, \qquad t \geq 0.$$

THEOREM 1. *Assume that* (H_1)-(H_3) *hold. Every solution* $u \in \mathscr{D}(\Omega)$ *of the* *problem* (1), (B_1) *is oscillatory in* Ω *if the delay differential inequalities*

$$y'(t) + \lambda_1 \sum_{i=1}^{k} b_i(t) \exp\left(\lambda_1 \int_{t-\sigma_i}^{t} a(s)ds\right) y(t-\sigma_i) \leq Q(t), \tag{2}$$

$$y'(t) + \lambda_1 \sum_{i=1}^{k} b_i(t) \exp\left(\lambda_1 \int_{t-\sigma_i}^{t} a(s)ds\right) y(t-\sigma_i) \leq -Q(t) \tag{3}$$

are oscillatory at $t = \infty$ *in the sense that neither* (2) *nor* (3) *has a solution which is eventually positive, where*

$$Q(t) = \exp\left(\lambda_1 \int_0^t a(s)ds\right)\left(F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(t-\sigma_i)\right).$$

PROOF. Suppose to the contrary that there is a solution u of the problem (1), (B_1) which is nonoscillatory in Ω. First we assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. There exists a number $T > t_0$ such that $u(x, \tau_i(t)) > 0$ in $G \times [T, \infty)$ $(i = 1, 2, \ldots, m)$. The hypothesis (H_2) implies that

$$c(x, t, u(x, t), u(x, \tau_1(t)), \ldots, u(x, \tau_m(t))) \geq 0 \quad \text{in} \quad G \times [T, \infty)$$

and hence

$$u_t(x, t) - [a(t)\Delta u(x, t) + \sum_{i=1}^{k} b_i(t)\Delta u(x, t-\sigma_i)] \leq f(x, t) \quad \text{in} \quad G \times [T, \infty). \tag{4}$$

Multiplying (4) by $\Phi(x)$ and integrating over G yield

$$\frac{d}{dt}\int_G u\Phi\, dx - a(t)\int_G \Delta u(x, t)\Phi\, dx - \sum_{i=1}^{k} b_i(t)\int_G \Delta u(x, t-\sigma_i)\Phi\, dx \leq \int_G f(x, t)\Phi\, dx, \qquad t \geq T. \tag{5}$$

From Green's formula it follows that

$$\begin{aligned} \int_G \Delta u(x, t)\Phi\, dx &= \int_{\partial G}\left(\frac{\partial u}{\partial \nu}\Phi - u\frac{\partial \Phi}{\partial \nu}\right)dS + \int_G u\Delta\Phi\, dx \\ &= -\int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu} dS - \lambda_1 \int_G u\Phi\, dx \\ &= -\Psi(t) - \lambda_1 U(t), \qquad t \geq T. \end{aligned} \tag{6}$$

Analogously we obtain

$$\int_G \Delta u(x, t-\sigma_i)\Phi\, dx = -\Psi(t-\sigma_i) - \lambda_1 U(t-\sigma_i), \qquad t \geq T. \tag{7}$$

Combining (5)–(7), we have

$$U'(t) + \lambda_1 a(t)U(t) + \sum_{i=1}^{k} \lambda_1 b_i(t)U(t-\sigma_i) \tag{8}$$
$$\leq F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(t-\sigma_i), \qquad t \geq T,$$

which is equivalent to

$$y'(t) + \lambda_1 \sum_{i=1}^{k} b_i(t) \exp\left(\lambda_1 \int_{t-\sigma_i}^{t} a(s)\,ds\right) y(t-\sigma_i) \leq Q(t), \qquad t \geq T, \tag{9}$$

where

$$y(t) = \exp\left(\lambda_1 \int_0^t a(s)\,ds\right) U(t).$$

Hence, $y(t)$ is an eventually positive solution of (9), which contradicts the hypothesis. If $u < 0$ in $G \times [t_0, \infty)$, $v \equiv -u$ satisfies the problem

$$v_t(x, t) - [a(t)\Delta v(x, t) + \sum_{i=1}^{k} b_i(t)\Delta v(x, t-\sigma_i)] \leq -f(x, t) \quad \text{in} \quad G \times [T, \infty),$$

$$v = -\psi \quad \text{on} \quad \partial G \times (0, \infty).$$

Proceeding as in the case where $u > 0$, we are led to a contradiction. The proof is complete.

THEOREM 2. *Assume that* (H_1), (H_3) *hold. Assume, moreover, that the following hypothesis holds*:

(H_2') *there is a number* $j \in \{1, 2, \dots, m\}$ *such that* $c(x, t, \xi, \eta_1, \dots, \eta_m) \geq p(t)\eta_j$ *for* $(x, t) \in \Omega$, $\xi \geq 0$, $\eta_i \geq 0$ $(i \neq j)$, *and* $c(x, t, \xi, \eta_1, \dots, \eta_m) \leq p(t)\eta_j$ *for* $(x, t) \in \Omega$, $\xi \leq 0$, $\eta_i \leq 0$ $(i \neq j)$.

Let $\tau_j(t) = t - \tau_j$, *where* τ_j *is a nonnegative constant. Every solution* $u \in \mathscr{D}(\Omega)$ *of the problem* (1), (B_2) *is oscillatory in* Ω *if the delay differential inequalities*

$$y'(t) + p(t)y(t-\tau_j) \leq \tilde{Q}(t), \tag{10}$$

$$y'(t) + p(t)y(t-\tau_j) \leq -\tilde{Q}(t) \tag{11}$$

are oscillatory at $t = \infty$, *where*

$$\tilde{Q}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(t)\tilde{\Psi}(t-\sigma_i).$$

PROOF. Suppose that there is a solution u of the problem (1), (B_2) which has no zero in $G \times [t_0, \infty)$ for some $t_0 > 0$. First we assume tht $u > 0$ in $G \times [t_0, \infty)$. Then we see that $u(x, \tau_i(t)) > 0$ in $G \times [T, \infty)$ $(i \neq j)$ for some $T > t_0$. It follows from the hypothesis (H_2') that

$$c(x, t, u(x, t), u(x, \tau_1(t)), \ldots, u(x, \tau_m(t))) \geq p(t)u(x, t - \tau_j) \quad \text{in} \quad G \times [T, \infty)$$

and therefore

(12)
$$u_t(x, t) - [a(t)\Delta u(x, t) + \sum_{i=1}^{k} b_i(t)\Delta u(x, t - \sigma_i)] + p(t)u(x, t - \tau_j) \leq f(x, t)$$

in $G \times [T, \infty)$. Integrating (12) over G and using Green's formula, we obtain

$$\begin{aligned}(13) \qquad & \frac{d}{dt}\int_G u\,dx - \left[a(t)\int_{\partial G} \frac{\partial u}{\partial \nu}(x, t)\,dS + \sum_{i=1}^{k} b_i(t)\int_{\partial G} \frac{\partial u}{\partial \nu}(x, t - \sigma_i)\,dS\right] \\ & + p(t)\int_G u(x, t - \tau_j)\,dx \leq \int_G f(x, t)\,dx, \qquad t \geq T.\end{aligned}$$

Taking account of (B_2), we find that (13) reduces to

$$\begin{aligned}\tilde{U}'(t) + p(t)\tilde{U}(t - \tau_j) &\leq \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(t)\tilde{\Psi}(t - \sigma_i) \\ &= \tilde{Q}(t), \qquad t \geq T,\end{aligned}$$

and hence $\tilde{U}(t)$ is an eventually positive solution of (13). This contradicts the hypothesis. In the case where $u < 0$ in $G \times [t_0, \infty)$, the same arguments as in the case where $u > 0$ lead us to a contradiction. The proof is complete.

3. Delay differential inequalities

We deal with the delay differential inequality

$$(14) \qquad y'(t) + \sum_{i=1}^{k} p_i(t)y(t - \sigma_i) \leq q(t), \qquad t \geq t_0,$$

where t_0 is a positive number. It is assumed that σ_i $(i = 1, 2, \ldots, k)$ are nonnegative constants, $q(t) \in C([t_0, \infty); \mathbf{R}^1)$, $p_i(t) \in C([t_0, \infty); \mathbf{R}^1)$ $(i = 1, 2, \ldots, k)$ and

$$p_i(t) \geq 0 \quad \text{on} \quad \bigcup_{n=1}^{\infty} \mathrm{I}_{n,i},$$

where $\mathrm{I}_{n,i} = (t_n - 2\sigma_i, t_n)$ and the sequence $\{t_n\}_{n=1}^{\infty}$ is chosen so that $\{\mathrm{I}_{n,i}\}_{n=1}^{\infty}$ are disjoint intervals for each $i = 1, 2, \ldots, k$.

THEOREM 3. *Assume that there is a subsequence* $\{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ *with the properties that*:

$$\lim_{k\to\infty} n_k = \infty,$$

$$\int_{t_{n_k}-\sigma_j}^{t_{n_k}} p_j(s)\,ds \geq 1,$$

$$G(t_{n_k}) \leq 0,$$

where $\sigma_j = \min_{1\leq i\leq k} \{\sigma_i\} > 0$ *and*

$$G(t) \equiv \int_{t-\sigma_j}^{t} q(s)\,ds + \int_{t-\sigma_j}^{t} p_j(s)\left(\int_{s-\sigma_j}^{t-\sigma_j} q(r)\,dr\right)ds.$$

Then, (14) *has no eventually positive solution.*

PROOF. Suppose that $y(t)$ is a solution of (14) which is positive on $[t_1, \infty)$ for some $t_1 \geq t_0$. Then $y(t-\sigma_i) > 0$ on $[t_2, \infty)$ for some $t_2 > t_1$. We note that $\lim_{n\to\infty}(t_n - 2\sigma_i) = \infty$, and hence there is an integer $N \in \mathrm{N}$ such that $t_n - 2\sigma_i > t_2$ for any $n \geq N$. Letting $\xi_n = t_n - 2\sigma_j$, we find that $(\xi_n, t_n) \subset (t_n - 2\sigma_i, t_n)$ $(i = 1, 2,\dots, k)$. Therefore, $p_i(t) \geq 0$ in (ξ_n, t_n) and $y(t-\sigma_i) > 0$ in (ξ_n, t_n) for any $n \geq N$. Hence, it follows from (14) that

$$y'(t) \leq q(t) \quad \text{in} \quad (\xi_n, t_n).$$

By continuity we obtain

$$y'(t) \leq q(t) \quad \text{on} \quad [\xi_n, t_n].$$

For any $t \in [t_n - \sigma_j, t_n]$ we see that $[t-\sigma_j, t_n - \sigma_j] \subset [\xi_n, t_n]$, and therefore

$$\int_{t-\sigma_j}^{t_n-\sigma_j} y'(s)\,ds \leq \int_{t-\sigma_j}^{t_n-\sigma_j} q(s)\,ds, \qquad t \in [t_n - \sigma_j, t_n],$$

or

$$(15) \qquad y(t-\sigma_j) \geq y(t_n - \sigma_j) - \int_{t-\sigma_j}^{t_n-\sigma_j} q(s)\,ds, \qquad t \in [t_n - \sigma_j, t_n].$$

It is easily seen that

$$(16) \qquad y'(t) + p_j(t)y(t-\sigma_j) \leq y'(t) + \sum_{i=1}^{k} p_i(t)y(t-\sigma_i)$$
$$\leq q(t), \quad t \in [t_n - \sigma_j, t_n].$$

Combining (15) with (16) yields

$$y'(t) + p_j(t)y(t_n - \sigma_j) \le q(t) + p_j(t)\int_{t-\sigma_j}^{t_n-\sigma_j} q(s)\,ds, \qquad t\in[t_n - \sigma_j, t_n].$$

Integrating the above inequality on $[t_n - \sigma_j, t_n]$, we obtain

$$y(t_n) - y(t_n - \sigma_j) + y(t_n - \sigma_j)\int_{t_n-\sigma_j}^{t_n} p_j(s)\,ds \le \int_{t_n-\sigma_j}^{t_n}\left[q(s) + p_j(s)\int_{s-\sigma_j}^{t_n-\sigma_j} q(r)\,dr\right]ds,$$

which is equivalent to

$$y(t_n) + y(t_n - \sigma_j)\left(\int_{t_n-\sigma_j}^{t_n} p_j(s)\,ds - 1\right) \le G(t_n), \qquad n \ge N. \tag{17}$$

Since $\lim_{k\to\infty} n_k = \infty$, there exists a $k_0 \in \mathrm{N}$ such that $n_k > N$ for any $k \ge k_0$. Letting $t_n = t_{n_k} (k \ge k_0)$ in (17), we conclude that the left hand side of (17) is positive and the right hand side of (17) is nonpositive. This contradiction establishes the theorem.

4. Oscillation of parabolic equations

We are now ready to state oscillation theorems for the boundary value problems (1), (B_i) $(i = 1, 2)$.

THEOREM 4. *Assume that* (H_1)-(H_3) *hold, and that the following hypothesis* (H_4) *holds*:

(H_4) $b_i(t) \ge 0$ *on* $\bigcup_{n=1}^{\infty} \mathrm{I}_{n,i}$, *where* $\mathrm{I}_{n,i}$ *are defined in Section* 3.

Every solution $u\in\mathscr{D}(\Omega)$ *of the problem* (1), (B_1) *is oscillatory in* Ω *if there is a subsequence* $\{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ *with the properties that*:

$$\lim_{k\to\infty} n_k = \infty,$$

$$\lambda_1 \int_{t_{n_k}-\sigma_j}^{t_{n_k}} b_j(s)\exp\left(\lambda_1\int_{s-\sigma_j}^{s} a(r)\,dr\right)ds \ge 1,$$

$$H_1(t_{n_k}) = 0,$$

where $\sigma_j = \min_{1\le i\le k}\{\sigma_i\} > 0$ *and*

$$H_1(t) \equiv \int_{t-\sigma_j}^{t} Q(s)\,ds + \int_{t-\sigma_j}^{t} \lambda_1 b_j(s)\exp\left(\lambda\int_{s-\sigma_j}^{s} a(r)\,dr\right)\left(\int_{s-\sigma_j}^{t-\sigma_j} Q(r)\,dr\right)ds.$$

PROOF. Theorem 3 implies that the delay differential inequalities (2) and (3) have no eventually positive solutions. Hence, the conclusion follows from Theorem 1.

By combining Theorem 2 with Theorem 3, we can obtain the analogue of Theorem 4.

THEOREM 5. *Assume that* (H_1), (H_2'), (H_3) *hold. Let* $\tau_j(t) = t - \tau_j$, *where* τ_j *is a positive constant. Assume, moreover, that*:

(H_5) $p(t) \geq 0$ *on* $\bigcup_{n=1}^{\infty} I_n$, *where* $I_n = (t_n - 2\tau_j, t_n)$ *and* $\{I_n\}_{n=1}^{\infty}$ *are disjoint intervals.*

Every solution $u \in \mathscr{D}(\Omega)$ *of the problem* (1), (B_2) *is oscillatory in* Ω *if there is a subsequence* $\{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ *such that*:

$$\lim_{k \to \infty} n_k = \infty,$$

$$\int_{t_{n_k} - \tau_j}^{t_{n_k}} p(s)\, ds \geq 1,$$

$$H_2(t_{n_k}) = 0,$$

where

$$H_2(t) \equiv \int_{t-\tau_j}^{t} \tilde{Q}(s)\, ds + \int_{t-\tau_j}^{t} p(s)\left(\int_{s-\tau_j}^{t-\tau_j} \tilde{Q}(r)\, dr\right) ds.$$

REMARK 1. Let $g_i(s)$ $(i \in \{0, 1,\dots, m\} \setminus \{j\})$ be continuous, odd functions in $\mathbf{R}^1$ which are nonnegative for $s > 0$, and let

$$(18) \qquad c(x, t, u(x, t), u(x, \tau_1(t)),\dots, u(x, \tau_m(t)))$$
$$= c_0(x, t) g_0(u(x, t)) + p(t) u(x, t - \tau_j) + \sum_{\substack{i=1,2,\dots,m \\ i \neq j}} c_i(x, t) g_i(u(x, \tau_i(t))),$$

where $c_i(x, t) \in C(\bar{\Omega})$, $c_i(x, t) \geq 0$ in Ω $(i \in \{0, 1,\dots, m\} \setminus \{j\})$. Then, $c(x, t, \xi, \eta_1,\dots, \eta_m)$ defined by (18) satisfies the hypothesis (H_2').

REMARK 2. The hypothesis (H_4) is satisfied if $b_i(t) = \cos it$, $\sigma_i = \frac{\pi}{4i}$ $(i = 1, 2,\dots, k)$ and $t_n = 2n\pi$ $(n = 1, 2,\dots)$. In the case where $b_i(t) = -\sin it$, $\sigma_i = \frac{\pi}{2i}$ $(i = 1, 2,\dots, k)$ and $t_n = 2n\pi$ $(n = 1, 2,\dots)$, the hypothesis (H_4) is also satisfied.

REMARK 3. Our theorems hold true even if $a(t)$ is not necessarily nonnegative.

EXAMPLE 1. We consider the problem

$$(19)\quad u_t(x, t) - \left[u_{xx}(x, t) + (-\sin t)u_{xx}\left(x, t - \frac{\pi}{2}\right) + (-\sin 2t)u_{xx}\left(x, t - \frac{\pi}{4}\right)\right] + 2u\left(x, t - \frac{\pi}{4}\right) = 4\sin 2x \cdot \sin 2t \cdot (1 + \sin t + \cos 2t), \qquad (x, t) \in (0, \pi) \times (0, \infty),$$

$$(20)\quad u(0, t) = u(\pi, t) = 0, \qquad t > 0.$$

Here $n = 1$, $G = (0, \pi)$, $k = 2$, $m = 1$, $a(t) = 1$, $b_1(t) = -\sin t$, $b_2(t) = -\sin 2t$, $\sigma_1 = \frac{\pi}{2}$, $\sigma_2 = \frac{\pi}{4}$, $\tau_1(t) = t - \frac{\pi}{4}$ and

$$f(x, t) = 4\sin 2x \cdot \sin 2t \cdot (1 + \sin t + \cos 2t).$$

Since

$$b_1(t) = -\sin t \geq 0 \quad \text{on} \quad \bigcup_{n=1}^{\infty} \mathrm{I}_{n,1},$$

$$b_2(t) = -\sin 2t \geq 0 \quad \text{on} \quad \bigcup_{n=1}^{\infty} \mathrm{I}_{n,2},$$

where $\mathrm{I}_{n,1} = (2n\pi - \pi,\ 2n\pi)$ and $\mathrm{I}_{n,2} = \left(2n\pi - \frac{\pi}{2},\ 2n\pi\right)$, we find that the hypothesis (H_4) is satisfied. It is easily seen that $\lambda_1 = 1$, $\Phi(x) = \sin x$, $\Psi(t) \equiv 0$ in $(0, \infty)$ and $\sigma_j = \sigma_2 = \min\{\sigma_1, \sigma_2\} = \frac{\pi}{4} > 0$. An easy computation shows that

$$F(t) = \int_0^{\pi} f(x, t)\sin x\, dx = 0, \qquad t \in (0, \infty).$$

Hence, we see that $Q(t) \equiv 0$ in $(0, \infty)$, and therefore $H_1(t) \equiv 0$ in $(0, \infty)$. For $t_n = 2n\pi$ $(n = 1, 2, \ldots)$, we obtain

$$\int_{t_n - \frac{\pi}{4}}^{t_n} (-\sin 2s)\, e^{\frac{\pi}{4}}\, ds = \frac{1}{2} e^{\frac{\pi}{4}} \geq 1.$$

Hence, Theorem 4 implies that every solution $u \in \mathscr{D}((0, \pi) \times (0, \infty))$ of the problem (19), (20) is oscillatory in $(0, \pi) \times (0, \infty)$. One such solution is

$u = \sin 2x \cdot \sin 2t$.

EXAMPLE 2. We consider the problem

(21)
$$u_t(x, t) - \left[u_{xx}(x, t) + \sin t \cdot u_{xx}\left(x, t - \frac{\pi}{2}\right)\right] + u(x, t - \pi) - \sin t \cdot u\left(x, t - \frac{\pi}{2}\right)$$
$$= \cos x \cdot \cos t, \ (x, t) \in (0, \pi) \times (0, \infty),$$

(22)
$$-u_x(0, t) = u_x(\pi, t) = 0, \qquad t > 0.$$

Here $n = 1$, $G = (0, \pi)$, $k = 1$, $m = 2$, $a(t) = 1$, $b_1(t) = \sin t$, $\tau_1(t) = t - \pi$, $\tau_2(t) = t - \frac{\pi}{2}$, $\tau_j = \tau_2 = \frac{\pi}{2}$, $p(t) = -\sin t$ and $f(x, t) = \cos x \cdot \cos t$. Since

$$p(t) = -\sin t \geq 0 \quad \text{on} \quad \bigcup_{n=1}^{\infty} \mathrm{I}_n,$$

where $\mathrm{I}_n = (2n\pi - \pi, 2n\pi)$, the hypothesis (H_5) is satisfied. We easily see that

$$\int_{t_n - \frac{\pi}{2}}^{t_n} (-\sin s)\, ds = 1 \qquad \text{for} \quad t_n = 2n\pi.$$

Since $\tilde{\Psi}(t) \equiv 0$ in $(0, \infty)$, we observe that

$$\tilde{Q}(t) = \tilde{F}(t) = \int_0^{\pi} \cos x \cdot \cos t\, dx = 0, \qquad t \in (0, \infty),$$

and hence $H_2(t) \equiv 0$ in $(0, \infty)$. Therefore, it follows from Theorem 5 that every solution $u \in \mathscr{D}((0, \pi) \times (0, \infty))$ of the problem (21), (22) is oscillatory in $(0, \pi) \times (0, \infty)$. For example, $u = \cos x \cdot \sin t$ is such a solution.

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