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Classification and uniqueness of positive solutions of $\Delta u + f(u) = 0$

Dedicated to Professor Takaŝi Kusano on his 60th birthday

Fu-Hsiang WONG and Cheh-Chih YEH*

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Abstract

Given $0 \leq \theta < \infty$. In this paper, we classify the solutions of the initial value problem

$$(*)\qquad \begin{cases} u''(r) + \dfrac{m}{r}u'(r) + f(u(r)) = 0 \text{ on } (\theta, R(\xi)), \\ u(\theta) = \xi > 0 \text{ and } u'(\theta) = 0, \end{cases}$$

where f is locally Lipschitz on $(0, \infty)$ and there exist two positive constants α, β such that $f(u) < 0$ on $(0, \alpha)$, $f(u) > 0$ on (α, ∞) and $F(\beta) > 0$. Here $R(\xi) := \sup\{r \in (\theta, \infty) | u(s) > 0$ for all $s \in [\theta, r)\}$ and $F(u) := \int_0^u f(s)ds$ for $u \geq 0$. Moreover, we establish an existence-uniqueness theorem of a solution for equation $(*)$ satisfying $u'(0) = 0$ and $\lim_{r \to \infty} u(r) = 0$.

1. Introduction

Let $\theta \in [0, \infty)$ be given and $\boldsymbol{R}^n (n \geq 2)$ denote the usual n-dimensional Euclidean space. Consider the following two problems:

$$(\mathrm{I}_1)\qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(R(\xi)), \\ \dfrac{\partial u}{\partial n} = 0 & \text{if } |x| = \theta, \\ u(x) = \xi > 0 & \text{if } |x| = \theta; \end{cases}$$

$$(\mathrm{I}_2)\qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(\infty), \\ \dfrac{\partial u}{\partial n} = 0 & \text{if } |x| = \theta, \\ \lim\limits_{|x| \to \infty} u(x) = 0, & \end{cases}$$

where $R(\xi) := \sup\{r \in (\theta, \infty) | u(x) > 0$ for $\theta \leq |x| < r\}$ and

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$$\Omega(\lambda) := \begin{cases} \{x \in \boldsymbol{R}^n \mid \theta < |x| < \lambda\} & \text{if } \theta > 0, \\ \{x \in \boldsymbol{R}^n \mid |x| < \lambda\} & \text{if } \theta = 0 \end{cases}$$

for every $\lambda \in (\theta, \infty]$.

When we confine ourselves to positive radial solutions, it is well-known that the above two problems (I_1) and (I_2) can be reduced to the following equivalent problems

$$(I_3) \qquad \begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) + f(u(r)) = 0 \quad \text{on} \quad (\theta, R(\xi)), \\ u(\theta) = \xi > 0 \quad \text{and} \quad u'(\theta) = 0 \end{cases}$$

$$(I_4) \qquad \begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) + f(u(r)) = 0 \quad \text{on} \quad (\theta, \infty), \\ u'(\theta) = 0, \\ \lim\limits_{r \to \infty} u(r) = 0, \end{cases}$$

respectively, where r is the radial variable. See, for example, Kaper and Kwong [2, 3], Kwong [4], Kwong and Zhang [5]. Recently, Kwong and Zhang [5] and Kwong [4] separate the set of solutions of (I_3) into the following three subsets under suitable conditions on f:

$$N := \{\xi \in (0, \infty) \mid R(\xi) < \infty\},$$

$$G := \{\xi \in (0, \infty) \mid R(\xi) = \infty \text{ and } \lim_{r \to \infty} u(r, \xi) = 0\},$$

$$P := (0, \infty) - N - G,$$

where $u(r, \xi)$ denotes the solution of (I_3).

Recently, Kwong and Zhang [5], Kaper and Kwong [3] also established some existence theorems for solutions of (I_3) as follows:

THEOREM A. *Assume that*

(F_1) *f is continuous on $[0, \infty)$ and locally Lipschitz on $(0, \infty)$,*

(F_2) *there exists a $u_0 > 0$ such that $F(u) < 0$ for $0 < u < u_0$, $F(u_0) = 0$, $f(u) > 0$ for $u \geq u_0$, where $F(u) := \int_0^u f(s)ds$,*

(F_3) *with u_0 as in (F_2), $\int_0^{u_0} (-F)^{-1/2}(u)du < \infty$,*

(F_4) $\liminf_{u \to \infty} f(u) > 0$,

(G_1) *$g(r) \geq 0$ for all $r \geq 0$,*

(G_2) $\lim_{t \to \infty} g(r) = 0$,

(G_3) *g is continuous on $[0, \infty)$.*

Then, the boundary value problem

$$(\mathrm{I}_5) \qquad \begin{cases} u''(r) + g(r)u'(r) + f(u(r)) = 0 \quad on \quad (0, p), \\ u'(0) = 0;\ u(r) > 0 \quad on \quad (0, p);\ u(p) = u'(p) = 0 \end{cases}$$

has a solution on $(0, p)$, *where* $p \in (0, \infty)$.

THEOREM B. *Let* $(\mathrm{F}_1) - (\mathrm{F}_4)$, $(\mathrm{G}_1) - (\mathrm{G}_2)$ *hold. If*
(F_5) $u \to f(u)/(u - u_0)$ *is nonicreasing for* $u > u_0$,
(F_4) g *is continuous on* $(0, \infty)$ *and* $g(r) = 0(r^{-1})$ *as* $r \to 0^+$,
then, (I_5) *has a solution on* $(0, p)$.

The purpose of this paper is to classify the solutions of (I_3) under fewer assumptions than those of Kaper and Kwong [3] and Kwong and Zhang [5]. We also establish a uniqueness theorem of a solution for (I_4).

2. Main results

Let $m > 0$ and $\theta \geq 0$ be given constants. Consider the following intial value problem

$$(\mathrm{IVP}) \qquad \begin{cases} u''(r) + \dfrac{m}{r} u'(r) + f(u(r)) = 0, \quad r > 0 \\ u(\theta) = \xi \quad \text{and} \quad u'(\theta) = 0, \end{cases}$$

where f satisfies the following two assumptions:
(A_1) f is locally Lipschitz continuous on $(0, \infty)$,
(A_2) there exist two constants α, β such that $f(u) < 0$ on $(0, \alpha)$, $f(u) > 0$ on (α, ∞) and $F(\beta) > 0$, where $F(u) := \int_0^u f(s)ds$ for $u \geq 0$.

Clearly, if $f(u) = u^p - u^q$, where $p > q \geq 0$, then (A_1) and (A_2) hold.

Throughout this paper, $u(r, \xi)$ denotes the solution of (IVP) and $R(\xi) := \sup\{r \in (\theta, \infty) \mid u(s, \xi) > 0 \text{ for } s \in (\theta, r)\}$.

In order to discuss our main results, we need the following two well-known theorems.

THEOREM C. *For any given* $\xi > 0$, *the initial value problem* (IVP) *has a unique positive solution* $u(r) := u(r, \xi)$ *on the interval* $[\theta, R(\xi))$.

THEOREM D. *For any given* $\xi > 0$, *the positive solution* $u(r, \xi)$ *of* (IVP) *on* $[\theta, R(\xi))$ *satisfies the following two identities*:

(E_1)

$$\left\{\frac{1}{2} u'^2(r, \xi) + F(u(r, \xi))\right\}_{r=a}^{r=b} = -\int_a^b \frac{m}{s} u'^2(s, \xi)ds \qquad \textit{for any } a, b \in [\theta, R(\xi)),$$

$$(\mathrm{E}_2) \qquad u'(r, \xi) = -\frac{1}{r^m}\int_\theta^r s^m f(u(s, \xi))ds \qquad \text{for all } r \in [\theta, R(\xi)).$$

LEMMA 1. *Let* $\xi \in (0, \infty) - \{\alpha\}$. *If there exists* $a \in [\theta, R(\xi))$ *such that* $u'(a, \xi) = 0$, *then* $u(r, \xi) \neq u(a, \xi)$ *for all* $r \in (a, R(\xi))$.

PROOF. Assume, on the contrary, that there exists $r_0 \in (a, R(\xi))$ such that $u(r_0, \xi) = u(a, \xi)$. It follows from (E_1) that

$$0 \le \frac{1}{2}u'^2(r_0, \xi) = -\int_a^{r_0} \frac{m}{s} u'^2(s, \xi)ds \le 0,$$

which implies $u'(r, \xi) = 0$ on $[a, r_0]$. Hence, $u''(r, \xi) = 0$ on $[a, r_0]$. It follows from (IVP) that $f(u(r, \xi)) = 0$ on $[a, r_0]$. By (A_2) and $r_0 < R(\xi)$, we see that $u(r, \xi) = \alpha$ on $[a, r_0]$. Thus, $u(r, \xi) = \alpha$ on $[\theta, R(\xi))$ by Theorem C, which contradicts $\xi \neq \alpha$. Hence, the proof is complete.

For any given $\xi > \alpha$, it follows from $u(\theta, \xi) = \xi > \alpha$ that there exists $r_1 \in (\theta, R(\xi))$ such that $u(r, \xi) > \alpha$ on $[\theta, r_1)$. It follows from (A_2) and (E_2) that $u'(r, \xi) < 0$ on (θ, r_1). Hence,

$$(1) \qquad R_1 := \sup\{r \in (\theta, R(\xi)) \mid u'(s, \xi) < 0 \quad \text{for} \quad s \in (\theta, r)\}$$

exists and satisfies $r_1 \le R_1 \le R(\xi)$. Seeing such a fact, we have the following lemma.

LEMMA 2. *For any given* $\xi > \alpha$, $u(r, \xi)$ *must satisfy one of the following properties*:

(P_1) *If* $R_1 = R(\xi) < \infty$, *then* $u(r, \xi)$ *is strictly decreasing to* 0 *as* $r \to R(\xi)$,

(P_2) *If* $R_1 = R(\xi) = \infty$, *then* $u(r, \xi)$ *is strictly decreasing to* 0 *or* α *as* $r \to R(\xi) = \infty$,

(P_3) *If* $R_1 < R(\xi)$, *then* $u(R_1, \xi) < \alpha$ *is the absolute minimum of* $u(r, \xi)$ *on* $[\theta, R(\xi))$.

Moreover, $u(r, \xi)$ *monotonically converges to* α *eventually as* $r \to \infty$ *or* $u(r, \xi)$ *is oscillatory about* α, *that is, there exists an increasing sequence* $\{R_k\}_{k=1}^{\infty}$ *satisfying* $\lim_{k\to\infty} R_k = \infty$,

$$0 < u(R_1, \xi) < u(R_3, \xi) < u(R_5, \xi) < \cdots < \alpha,$$

$$\xi > u(R_2, \xi) > u(R_4, \xi) > u(R_6, \xi) > \cdots > \alpha$$

and $u'(r, \xi) > 0$ *on* (R_{2k-1}, R_{2k}), $u'(r, \xi) < 0$ *on* (R_{2k}, R_{2k+1}) *for* $k = 1, 2, 3, \cdots$, *where* R_1 *is defined as in* (1).

PROOF. *Case* (1). Since $R_1 = R(\xi)$, it follows from the definitions of R_1 and $R(\xi)$ that $u(r, \xi)$ is strictly decreasing and bounded below by 0 on $(\theta, R(\xi))$.

Hence, $\lim_{r \to R(\xi)} u(r, \xi) = u(R(\xi), \xi) \geq 0$. It is clear that $u(R(\xi), \xi) = 0$. In fact, if $u(R(\xi), \xi) > 0$, then $u(R(\xi) + \varepsilon, \xi) > 0$ for any sufficiently small $\varepsilon > 0$. This contradicts the definition of $R(\xi)$. Thus, $u(R(\xi), \xi) = 0$.

Case (2). If follows from $R_1 = R(\xi) = \infty$ and the definitions of R_1, $R(\xi)$ that $u(r, \xi)$ is strictly decreasing on (θ, ∞) and bounded below by 0 on (θ, ∞). This implies $\lim_{r \to \infty} u(r, \xi) = u_1$ exists. Hence, $\lim_{r \to \infty} u'(r, \xi) = \lim_{r \to \infty} u''(r, \xi) = 0$, which and (IVP) imply $f(u_1) = 0$. By (A_2), we see that $u_1 = 0$ or α.

Case (3). We claim that there exists $r_2 \in (R_1, R(\xi))$ such that $u'(r, \xi) > 0$ on (R_1, r_2). Assume, on the contrary, that there exists $r_3 \in (R_1, R(\xi))$ such that $u'(r, \xi) = 0$ on (R_1, r_3) or $u'(r, \xi) < 0$ on (R_1, r_3). Thus, $u''(R_1, \xi) = 0$ by the C^2-continuity of u at R_1. This and $u'(r, \xi) = 0$ imply $f(u(R_1, \xi)) = 0$. It follows from (A_2) that $u(R_1, \xi) = \alpha$. By Theorem C, we see that $u(r, \xi) = \alpha$ is a constant solution of (IVP) on $[\theta, R(\xi))$, which contradicts $u(\theta, \xi) = \xi > \alpha$. Hence, there exists $r_2 \in (R_1, R(\xi))$ such that $u'(r, \xi) > 0$ on (R_1, r_2). Thus, $u''(R_1, r) > 0$ and

$$R_2 := \sup \{ r \in (R_1, R(\xi)) \mid u'(s, \xi) > 0 \quad \text{on} \quad (R_1, r) \}$$

exists. It follows from $u'(R_1, \xi) = 0$, $u''(R_1, \xi) > 0$ and (IVP) that $u''(R_1, \xi) = -f(u(R_1, \xi)) > 0$, which and ($A_2$) imply $u(R_1, \xi) < \alpha$. By Lemma 1, $u''(R_1, \xi) > 0$ and $u'(R_1, \xi) = 0$, we see that $u(R_1, \xi)$ is the absolute minimum of $u(r, \xi)$ on $[\theta, R(\xi))$. Furthermore, if $R_2 = \infty$, then $R(\xi) = R_2 = \infty$. Hence, $u(r, \xi)$ is strictly increasing on (R_1, ∞). It follows form Lemma 1 that $u(r, \xi)$ is bounded above by ξ on $[R_1, \infty)$. Thus, $\lim_{r \to \infty} u(r, \xi) := u_2$ exists and $\lim_{r \to \infty} u'(r, \xi) = \lim_{r \to \infty} u''(r, \xi) = 0$. By (IVP), we see that $f(u_2) = 0$. This and (A_2) imply $u_2 = 0$ or α. Since

$$u_2 = \lim_{r \to \infty} u(r, \xi) > u(R_1, \xi) > 0,$$

we see that $u_2 = \alpha$. On the other hand, if $R_2 < \infty$, then it follows from $u(R_2, \xi) > 0$ and the definition of $R(\xi)$ that $R_2 < R(\xi)$. As discussed at the beginning of this case, we see that there exists $r_4 \in (R_2, R(\xi))$ such that $u'(r, \xi) < 0$ on (R_2, r_4). Thus, $u''(R_2, \xi) < 0$ and

$$R_3 := \sup \{ r \in (R_2, R(\xi)) \mid u'(s, \xi) < 0 \quad \text{on} \quad (R_2, r) \}$$

exists. It follows from $u'(R_2, \xi) = 0$, $u''(R_2, \xi) < 0$ and (IVP) that $u''(R_2, \xi) = -f(u(R_2, \xi)) < 0$. This and ($A_2$) imply $u(R_2, \xi) > \alpha$. By Lemma 1, we see also that $u(R_2, \xi) < \xi$. Furthermore, if $R_3 = \infty$, then $R(\xi) = R_3 = \infty$. Hence, $u(r, \xi)$ is strictly decreasing on (R_2, ∞). It follows from Lemma 1 that $u(r, \xi)$ is bounded below by $u(R_1, \xi)$ on (R_1, ∞). Thus, $\lim_{r \to \infty} u(r, \xi) := u_3$ exists and $\lim_{r \to \infty} u'(r, \xi) = \lim_{r \to \infty} u''(r, \xi) = 0$. By (IVP), we see that $f(u_3) = 0$, which and (A_2) imply $u_3 = 0$ or α. Since

$$u_3 = \lim_{r\to\infty} u(r, \xi) > u(R_1, \xi) > 0,$$

we see that $u_3 = \alpha$. Continuing in this way, we can obtain the desired result.

By Cases (1), (2) and (3), the proof is complete.

Clearly, if $\xi \in (0, \alpha)$, then

$$r_5 := \sup \{r \in (\theta, R(\xi)) \mid u(s, \xi) < \alpha \text{ on } (\theta, r)\}$$

exists. It follows form (A_2) and (E_2) that $u'(r, \xi) > 0$ on (θ, r_5). Hence,

$$r_6 := \sup \{r \in (\theta, R(\xi)) \mid u'(s, \xi) < \alpha \text{ on } (\theta, r)\}$$

exists and satisfies $r_5 \leq r_6 \leq R(\xi)$. Seeing such a fact, we have the following lemma.

LEMMA 3. *For any given* $\xi \in (0, \alpha)$, $u(r, \xi)$ *must satisfy one of the following properties*:

(P_4) *If* $r_5 = \infty$, *then* $u(r, \xi)$ *converges increasingly to* α *as* $r \to \infty$,

(P_5) *If* $r_5 < \infty$, *then* $r_5 < r_6 < R(\xi)$, $u(r_6, \xi) > \alpha$ *and* $u'(r_6, \xi) = 0$. *Moreover,* $u(r, \xi)$ *satisfies* (P_3).

PROOF. *Case* (1). Since $r_5 = \infty$, we see that $r_6 = R(\xi) = \infty$. It follows from the definitions of r_5 and r_6 that $u(r, \xi)$ is strictly increasing and bounded above by α on (θ, ∞). Hence, $\lim_{r\to\infty} u(r, \xi) = u_4$ exists, which implies $\lim_{r\to\infty} u'(r, \xi) = \lim_{r\to\infty} u''(r, \xi) = 0$. By (IVP), we see that $f(u_4) = 0$. This and (A_2) implies $u_4 = 0$ or α. Since

$$u_4 = \lim_{r\to\infty} u(r, \xi) > u(\theta, \xi) = \xi > 0,$$

we see that $u_4 = \alpha$.

Case (2). It follows from $r_5 < \infty$ that $u(r_5, \xi) = \alpha$ and $u'(r_5, \xi) > 0$. Thus, by the continuity of $u(r, \xi)$, we see that $r_5 < r_6$. We claim that $r_6 < \infty$. Assume, on the contrary, that $r_6 = \infty$. Take $\eta \in (\alpha, \infty)$ satisfying $F(\eta) > 0$. It is clear that $u(r, \xi) < \eta$ on (θ, ∞). In fact, if there exists $r_7 \in (\theta, \infty)$ such that $u(r_7, \xi) = \eta$. It follows from (A_2) and (E_1) that

$$0 < \frac{1}{2} u'^2(r_7, \xi) + F(\eta) - F(\xi) = -\int_\theta^{r_7} \frac{m}{s} u'^2(s, \xi) ds \leq 0,$$

which is a contradiction. Hence $u(r, \xi) < \eta$ on (θ, ∞). Since $r_6 = \infty$, we see that $u(r, \xi)$ is strictly increasing and bounded above by η on (θ, ∞). Thus, $\lim_{r\to\infty} u(r, \xi) = u_5$ exists and $\lim_{r\to\infty} u'(r, \xi) = \lim_{r\to\infty} u''(r, \xi) = 0$. By (IVP), we see that $f(u_5) = 0$, which and (A_2) imply $u_5 = 0$ or α. Since $u(r, \xi)$ is strictly increasing on (θ, ∞), we see that

$$u_5 = \lim_{r \to \infty} u(r, \xi) > u(r_5, \xi) = \alpha,$$

which gives a contradiction. Thus, $r_6 < \infty$, and hence $u'(r_6, \xi) = 0$. It follows from $u(r_6, \xi) > u(r_5, \xi) = \alpha > 0$ that $r_6 < R(\xi) \leq \infty$. Using Lemma 1, we see that $u(r, \xi)$ is bounded below by $\xi > 0$ on $[r_6, R(\xi))$. This and Lemma 2 imply $u(r, \xi)$ satisfies (P_3).

LEMMA 4. *Let* $\tau := \inf\{u \in [\alpha, \infty) \mid F(u) > 0\}$. *Then, for any* $\xi \in (0, \tau]$, $\lim_{r \to R(\xi)} u(r, \xi) > 0$ *and* $R(\xi) = \infty$.

PROOF. Assume, on the contrary, that there exists $\xi \in (0, \tau]$ such that $\lim_{r \to R(\xi)} u(r, \xi) = 0$. It follows from ($E_1$) that

$$0 \leq \lim_{r \to R(\xi)} \frac{1}{2} u'^2(r, \xi) - F(\xi) = - \lim_{r \to R(\xi)} \int_\theta^r \frac{m}{s} u'^2(s, \xi) ds \leq 0,$$

which implies $u'(r, \xi) = 0$ on $[\theta, R(\xi))$. Hence, $u(r, \xi) = \xi$ on $[\theta, R(\xi))$. It follows from $\lim_{r \to R(\xi)} u(r, \xi) = 0$ and $u(r, \xi) = \xi$ on $[\theta, R(\xi))$ that $\xi = 0$, which contradicts $\xi > 0$.
Thus, the proof is complete.

It follows from Lemmas 2, 3 and 4 that we can decomposite the set of solutions of (IVP) into the following three disjoint subsets:

$N^* := \{\xi \in (0, \infty) \mid R(\xi) < \infty$ and $u(r, \xi)$ is decreasing to 0 as $r \to R(\xi)\}$,

$G^* := \{\xi \in (0, \infty) \mid R(\xi) = \infty$ and $u(r, \xi)$ is decreasing to 0 as $r \to \infty\}$,

$$P^* := (0, \infty) - N^* - G^* = \left\{ \begin{array}{l} \xi \in (0, \infty) \mid R(\xi) = \infty \text{ and } u(r, \xi) \text{ strictly monotonically} \\ \text{converges to } \alpha \text{ eventually as } r \to \infty \text{ or } u(r, \xi) \text{ is oscillatory} \\ \text{about } \alpha \text{ and has an absolute minimum on } (\theta, \infty) \end{array} \right\}.$$

In particular, by Lemma 4 and the property of continuous dependence on initial value, we see that $(0, \tau] \subset P^*$ and N^* is an open subset of $(0, \infty)$, where τ is defined as in Lemma 4.

THEOREM 5 (Existence). *If* $f'(\alpha) > 0$, *then for every* $\xi \in P^*$, $\lim_{r \to \infty} u(r, \xi) = \alpha$. *In particular, the boundary value problem*

$$\text{(BVP1)} \qquad \begin{cases} u''(r) + \dfrac{m}{r} u'(r) + f(u(r)) = 0 \text{ on } (\theta, \infty), \\ u'(\theta) = 0 \\ \lim_{r \to \infty} u(r) = \alpha > 0 \end{cases}$$

possesses infinitely many solutions.

PROOF. It suffices to show that $\lim_{r\to\infty} u(r, \xi) = \alpha$ if $u(r, \xi)$ is oscillatory about α. It follows from Lemma 2 that

$$\lim_{k\to\infty} u(R_{2k-1}, \xi) = \beta_1 \in (0, \alpha],$$

$$\lim_{k\to\infty} u(R_{2k}, \xi) = \beta_2 \in [\alpha, \xi)$$

and

$$u'(R_k, \xi) = 0 \qquad \text{for } k = 1, 2, 3, \cdots$$

we can obtain $\beta_1 = \beta_2 = \alpha$. In fact, it follows from (E_1) that

$$(2) \quad F(u(R_{2k-1})) - F(u(R_{2k})) = \int_{R_{2k-1}}^{R_{2k}} \frac{m}{s} u'^2(s, \xi) ds > 0 \qquad \text{for } k = 1, 2, 3, \cdots.$$

Hence,

$$F(u(R_{2k})) < F(u(R_{2k-1})) \le F(u(R_1)) < 0 \qquad \text{for } k = 1, 2, 3, \cdots,$$

which implies $u(R_2) < \tau$. Using the same technique of Ni [7], we can prove that $\beta_1 = \beta_2 = \alpha$. Hence, $\lim_{r\to\infty} u(r, \xi) = \alpha$. Since $(0, \tau] \subset P^*$, we see that (BVP1) possesses infinitely many solutions.

LEMMA 6. *If* $f'(\alpha) > 0$, *then*

$P^* = \{\xi \in (0, \infty) \mid u(r, \xi)$
is oscillatory about α *and has an absolute minimum on* $(\theta, \infty)\}$,

P^* *is a nonempty open subset of* $(0, \infty)$, *and hence,* G^* *is a closed subset of* $(0, \infty)$.

PROOF. Assume, on the contrary, that there exists $\xi \in P^*$ such that $u(r, \xi)$ is not oscillatory about α, that is, $u(r, \xi)$ strictly monotonically converges to α eventually. Since $f'(\alpha) > 0$, there exists $r_8 \in (\theta, \infty)$ such that

$$\frac{f(u(r, \xi))}{u(r, \xi) - \alpha} \ge \frac{1}{2} f'(\alpha) > 0 \quad \text{on} \quad [r_8, \infty).$$

Clearly, the differential equation

$$(\mathrm{I}_6) \qquad v''(r) + \frac{m}{2} v'(r) + \frac{1}{2} f'(\alpha) = 0$$

is oscillatory and $w(r) := u(r, \xi) - \alpha$ is a negative or positive solution of

$$(\mathrm{I}_7) \qquad w''(r) + \frac{m}{r} w'(r) + \frac{f(u(r, \xi))}{u(r, \xi) - \alpha} w(r) = 0, \; r \in [r_8, \infty).$$

By Sturm's comparison theorem, we see that $w(r)$ is oscillatory, which contradicts $w(r)$ being negative or positive on $[r_8, \infty)$. Hence, we see that

$$P^* = \{\xi \in (0, \infty) \mid u(r, \xi) \text{ is oscillatory about } \alpha \text{ and has an absolute minimum on } (\theta, \infty)\}.$$

Moreover, it follows from the property of continuous dependence on initial value, $u(r, \xi)$ has an absolute minimum on (θ, ∞) and Lemma 4 that P^* is a nonempty open subset of $(0, \infty)$.

REMARK 7. The condition "$f'(\alpha) > 0$" is better than (F_5) (see, Theorem B in Section 1). For example let $f(u) := u^2 - u^{1/2}$. We see easily that $u_0 = 2^{2/3} > 0$ and $u \to f(u)/(u - u_0)$ is increasing for $u > 2^{5/3} > u_0$. Thus, $f(u) = u^2 - u^{1/2}$ does not satisfy consition (F_5). But, for any given p, q *with* $p > q \geq 0$, the function $u^p - u^q$ satisfies the condition "$f'(\alpha) > 0$".

It follows from Lemma 6 and N^* is an open subset of $(0, \infty)$ that the boundary value problem

$$(\mathrm{BVP2}) \qquad \begin{cases} u''(r) + \dfrac{m}{r} u'(r) + f(u(r)) = 0 \; \textit{on} \; (\theta, \infty) \\ u'(\theta) = 0 \\ \lim\limits_{r \to \infty} u(r) = 0 \end{cases}$$

has one solution on $[\theta, \infty)$ if N^* is a nonempty set or P^* is a bounded set.

THEOREM 8 (Existence-Uniqueness). *Assume that* $\theta = 0$, $\varepsilon > 0$, $0 \leq q < p < (m + 2)/(m - 1)$ *and* $m > 1$. *If* $f(u) := u^p - u^q$ *or* $u^p - \varepsilon$, *then* (BVP2) *has a unique positive solution on* $[\theta, \infty)$.

PROOF. It follows from Wong [8], Wong and Yeh [9] and Wong, Yeh and Yu [10] that (BVP2) has at most one solution on $[\theta, \infty)$. Next, we prove that (BVP2) has a positive solution by the following two cases.

Case (1). Suppose that $f(u) := u^p - u^q$. Let $A \in (0, \infty)$ be given and define

$$r := A^{-q(p-1)/2(p-q)} t \quad \text{and} \quad v(t) := A^{-q/(p-q)} u(r).$$

Then,

$$u''(r) + \frac{m}{r} u'(r) + f(u(r)) = 0$$

can be transformed into

$$v''(t) + \frac{m}{t} v'(t) + v^p(t) - A^{-q} v^q(t) = 0. \tag{I_8}$$

Now, consider a simpler equation

$$w''(t) + \frac{m}{t} w'(t) + w^p(t) = 0. \tag{I_9}$$

The solution $w(t)$ of (I_9) which satisfies the initial condition

$$w(0) = 1 \quad \text{and} \quad w'(0) = 0 \tag{3}$$

has a zero in $(0, \infty)$ (cf. Kaper and Kwong [3], Ni [6]). It is clear that for each solution $u(r, \xi_1)$ of (IVP), there exists a solution $v(t)$ of (I_8) satisfying $v(0) = 1$ and $v'(0) = 0$, where $\xi_1 := A^{q/p-q}$. Because $v(t)$ is bounded by 1, we see that the last term in (I_8) can be small arbitrarily by choosing A large enough. Hence, $v(t)$ has a zero in $(0, \infty)$ which implies $u(r, \xi_1)$ has a zero in $(0, \infty)$. Hence, N^* is nonempty and (BVP2) has a positive solution on $[0, \infty)$.

Case (2). Suppose that $f(u) := u^p - \varepsilon$. It follows from the existence theorem of Castro and Shivaji [1], and Kaper and Kwong [3] that N^* is nonempty. Hence, (BVP2) has a positive solution on $[0, \infty)$.

By Cases (1) and (2), we complete the proof.

REMARK 9. It follows from Uniqueness Theorems in Wong [8], Wong and Yeh [9], Wong, and Yu [10] that

(a) $P^* = (0, \infty)$ and $N^* = \emptyset$ if $G^* = \emptyset$,

(b) $P^* = (0, \xi_0)$ and $N^* = (\xi_0, \infty)$ if G^* is a singleton, say $G^* = \{\xi_0\}$.

Thus, if $f(u)$ satisfies (A_1), (A_2) and the hypotheses of Lemma 6 and f is strictly decreasing near 0 (or, increasing near 0 and $m > 1/2$) and if we can prove that P^* is bounded or $N^* = (\xi_0, \infty)$ is nonempty, then we can show that (BVP2) has exactly one solution.

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Department of Mathematics
National Central University
Chung-Li, Taiwan
Republic of China