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An estimate on the codimension of local isometric imbeddings of compact Lie groups

Dedicated to the memory of Professor Masahisa Adachi

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Introduction

In the previous paper [3], we gave an estimate on the codimension of the Euclidean space into which a Riemannian manifold (M, g) can be locally isometrically or conformally immersed, by using some quantity which is naturally associated with (M, g). In the present paper, we introduce another new quantities of (M, g), and improve the estimate on the codimension based on these newly introduced quantities. The principle of our new method is explained as follows.

Let (M, g) be an n-dimensional Riemannian manifold. We assume that (M, g) is isometrically (or conformally) immersed into the $(n+r)$-dimensional Euclidean space $\boldsymbol{R}^{n+r}$. Let x be a point of M and X be a tangent vector in T_xM. We denote by $\mathcal{N}(X)$ the family of linear subspaces W of T_xM satisfying

$$R(Y, Z)X = 0 \qquad \text{for all } Y, Z \in W,$$

where R denotes the curvature tensor field of type (1, 3) at x. We denote by $d(X)$ the maximum dimension of $W \in \mathcal{N}(X)$ and set $p_M(x) = \min d(X)$ $(X \in T_xM)$. Then, by the Gauss equation, or its modified equation for conformal immersions, we have the following inequalities on the codimension r;

$$(*) \qquad \begin{aligned} r &\geq n - p_M(x) && \text{(the isometric case)}, \\ r &\geq n - p_M(x) - 2 && \text{(the conformal case)} \end{aligned}$$

(Proposition 1.1). And using these inequalities, we obtain an estimate on the codimention of isometric or conformal immersions. In fact, we may assert that any open neighborhood of x in M cannot be isometrically (resp. conformally) immersed into the Euclidean space $\boldsymbol{R}^{n+r}$ with $r < n - p_M(x)$ (resp.

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$r < n - p_M(x) - 2$). The isometric case of the above inequalities is essentially equivalent to the condition stated in [2; Theorem 3.1], which the first named author obtained by introducing the notion of "generalized Gauss equation". (For details, see Theorem 3.2.)

Let us now assume that (M, g) is a Riemannian symmetric space and consider the problem to determine an actual estimate by the principle stated above. Because of homogeneity, it suffices to calculate the number $p_{G/K}(o)$ at the origin o of G/K. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$, and let $\mathfrak{k}_0$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then the integer $p_{G/K}(= p_{G/K}(o))$ equals the maximum dimension of the subspaces W in $\mathfrak{m}$ satisfying

$$[W, W] \subset \mathfrak{k}_0.$$

In particular, in the special case where $\mathfrak{k}_0 = \{0\}$ (i.e., the Satake diagram does not contain any black circles nor any arrows), the equality $p_{G/K} = \operatorname{rank} G/K$ holds (Theorem 2.4). From these results, it follows that the canonical imbedding of the space $SU(m)/Sp(m)$ (cf. [7]) gives the least dimensional local isometric imbedding into the Euclidean spaces (Corollary 2.5). However, for general spaces, it is difficult to determine the exact value $p_{G/K}$, even in the case G/K is a compact Lie group.

We now introduce another quantity p_M^c, which is just the complex version of p_M. For $x \in M$ and $X \in T_xM$, we denote by $\mathcal{N}^c(X)$ the family of complex linear subspaces of $(T_xM)^c$ satisfying

$$R^c(Y, Z)X = 0 \qquad \text{for all} \quad Y, Z \in W,$$

where $(T_xM)^c$ and R^c mean the complexifications of T_xM and R, respectively. We denote by $d^c(X)$ the maximum dimension of the complex vector space $W \in \mathcal{N}^c(X)$, and set $p_M^c(x) = \min d^c(X)$ $(X \in T_xM)$. Then, in the same way as before, we have the following inequalities on r;

$$(**) \qquad \begin{aligned} & r \geq n - p_M^c(x) && \text{(the isometric case)}, \\ & r \geq n - p_M^c(x) - 2 && \text{(the conformal case)}. \end{aligned}$$

Therefore, the same statement after $(*)$ holds if we replace $p_M(x)$ by $p_M^c(x)$. Since $p_M^c(x) \geq p_M(x)$, which follows directly from the definition, the estimate obtained by $p_M^c(x)$ is in general inferior to the one obtained by $p_M(x)$. However, by using the value p_M^c for compact Lie groups, we can improve the results in [3] on the codimension of isometric or conformal immersions, and it is the main purpose of this paper to determine the value p_M^c for all compact Lie groups.

Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra, and we fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Then the integer p_G^c equals the maximum dimension of complex linear subspaces W of $\mathfrak{g}^c$ satisfying

$$[W, W] \subset \mathfrak{t}^c.$$

Then, our problem is completely reduced to a problem concerning the root system of $\mathfrak{g}^c$. Let Δ be the set of all non-zero roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$. We say that a subset Γ of Δ is *non-additive* if $\alpha + \beta \notin \Delta$ for any $\alpha, \beta \in \Gamma$. Then, the integer p_G^c is equal to the maximum of the value ${}^{\#}\Gamma + \operatorname{rank} \mathfrak{g} - \dim \boldsymbol{R}\Gamma$, where Γ runs over the set of all non-additive set in Δ (Proposition 3.4 and Corollary 3.5). Our main results are summarized in Theorem 3.1. In particular, for compact classical Lie groups G, the order of p_G^c is about $1/4 \cdot \dim G$, and therefore, G cannot be locally isometrically (or conformally) immersed in codimension about $3/4 \cdot \dim G$. This improves the previous results in [3], where we showed the non-existence of isometric (or conformal) immersions in codimension about $1/2 \cdot \dim G$.

Now, we explain the contents of this paper. In §1, we first define two functions $p_M(x)$, $p_M^c(x)$, and prove the inequalities (*) and (**). Next, we state some fundamental properties of these functions (Proposition 1.2). In §2, after reformulating these results adapted to Riemannian symmetric spaces, we prove Theorem 2.4. In §3 $\sim$ §5, we determine the value p_G^c for all compact simple Lie groups G. First, in §3, we state the main results on the value p_G^c (Theorem 3.1), and to prove this theorem, prepare some notions on the root systems. Using these results, we prove Theorem 3.1 in §4 and §5 for the classical and the exceptional Lie groups, respectively. For the classical case, we divide the non-additive sets Γ into five types, and after evaluating the maximum of ${}^{\#}\Gamma + \operatorname{rank} \mathfrak{g} - \dim \boldsymbol{R}\Gamma$ inductively for each type, we determine the value p_G^c. Since each type possesses its own feature, we must prepare several lemmas to obtain the final results. For the exceptional Lie groups, we determine p_G^c by applying the results of Malcev [10] on the maximum dimension of abelian subsalgebras of complex simple Lie algebras. (See also Appendix.) Finally, in §6, we state a result on the value p_G for compact Lie groups with small rank. We also give some lower bound of p_G for general compact simple Lie groups, in terms of a set of roots satisfying some conditions.

§1. A condition derived from the Gauss equation

Let (M, g) be an n-dimensional Riemannian manifold. In this section, we first state some necessary conditions in order that (M, g) may be locally isometrically (or conformally) imbedded into $\boldsymbol{R}^{n+r}$ in terms of some quantity associated with (M, g).

Let $x \in M$ and for each tangent vector $X \in T_x M$, we define two sets $\mathcal{N}(X)$ and $\mathcal{N}^c(X)$ consisting of subspaces of $T_x M$ and its complexification $(T_x M)^c$ by

$$\mathcal{N}(X) = \{W \subset T_x M \mid R(Y, Z)X = 0, \quad \text{for all} \quad Y, Z \in W\},$$

$$\mathcal{N}^c(X) = \{W \subset (T_x M)^c \mid R^c(Y, Z)X = 0, \quad \text{for all} \quad Y, Z \in W\},$$

where $R: T_x M \times T_x M \times T_x M \to T_x M$ is the curvature tensor of type $(1, 3)$ at x, and $R^c: (T_x M)^c \times (T_x M)^c \times (T_x M)^c \to (T_x M)^c$ is the complexification of R. For a real tangent vector $X \in T_x M$, we put

$$d(X) = \max_{W \in \mathcal{N}(X)} \dim W,$$

$$d^c(X) = \max_{W \in \mathcal{N}^c(X)} \dim W.$$

If the element $X \in T_x M$ is sufficiently generic, the integers $d(X)$ and $d^c(X)$ take the minimum value and we denote them by $p_M(x)$ and $p_M^c(x)$. Namely, p_M and p_M^c are $\boldsymbol{Z}$-valued functions on M defined by

$$p_M(x) = \min_{X \in T_x M} d(X),$$

$$p_M^c(x) = \min_{X \in T_x M} d^c(X).$$

Since there is a canonical inclusion $\mathcal{N}(X) \subset \mathcal{N}^c(X)$ for each $X \in T_x M$, the inequality $p_M(x) \leq p_M^c(x)$ holds for $x \in M$. The importance of these functions are explained in the following proposition.

PROPOSITION 1.1. *Assume that an n-dimensional Riemannian manifold (M, g) is isometrically (resp. conformally) immersed into $\boldsymbol{R}^{n+r}$. Then the following inequalities hold for any $x \in M$.*

$$r \geq n - p_M(x) \qquad (\textit{resp. } r \geq n - p_M(x) - 2)$$

$$r \geq n - p_M^c(x) \qquad (\textit{resp. } r \geq n - p_M^c(x) - 2).$$

Consequently, any open submanifold of M containing x can not be isometrically (resp. conformally) immersed into the Euclidean space with codimension $r = n - p_M(x) - 1$, $n - p_M^c(x) - 1$ (resp. $r = n - p_M(x) - 3$, $n - p_M^c(x) - 3$).

PROOF. We prove only "real" part of this proposition because the second inequality follows immediately from $p_M(x) \leq p_M^c(x)$ and the first inequality.

First, we treat the "isometric" case. We have only to show that the inequality $d(X) \geq n - r$ holds for any $X \in T_x M$ because $p_M(x) = d(X)$ for some $X \in T_x M$. We denote by $T_x^{\perp} M$ the normal space of the isometric immersion at x and let $\alpha: T_xM \times T_xM \to T_x^\perp M$ be the second fundamental form associated with the immersion. For $X \in T_xM$, we define a linear map $\varphi_X: T_xM \to T_x^\perp M$ by $\varphi_X(Y) = \alpha(X, Y)$. If $Y, Z \in \operatorname{Ker} \varphi_X$, then for any $W \in T_xM$, we have, from the Gauss equation,

$$-g(R(Y, Z)X, W) = \langle \alpha(X, Y), \alpha(Z, W) \rangle - \langle \alpha(X, Z), \alpha(Y, W) \rangle = 0$$

because $\alpha(X, Y) = \alpha(X, Z) = 0$. (We denote by $\langle\ ,\ \rangle$ the inner product of $T_x^\perp M$.) Therefore, we have $R(Y, Z)X = 0$, which implies $\operatorname{Ker} \varphi_X \in \mathcal{N}(X)$. Since $\dim \operatorname{Ker} \varphi_X \geq \dim T_xM - \dim T_x^\perp M = n - r$, we obtain the desired inequality $d(X) \geq \dim \operatorname{Ker} \varphi_X \geq n - r$.

Next, we treat the "conformal" case. In our previous paper [3; p. 110], we constructed symmetric tensors

$$\alpha: T_xM \times T_xM \longrightarrow T_x^\perp M$$
$$\beta: T_xM \times T_xM \longrightarrow \boldsymbol{R}$$

associated with the conformal immersion of (M, g), and showed that they satisfy the modified Gauss equation for conformal immersions:

$$\begin{aligned}\langle \alpha(X, Y), \alpha(W, Z) \rangle - \langle \alpha(X, Z), \alpha(W, Y) \rangle + \beta(X, Y)g(W, Z) + g(X, Y)\beta(W, Z) \\ - \beta(X, Z)g(W, Y) - g(X, Z)\beta(W, Y) = -\rho g(R(X, W)Y, Z),\end{aligned}$$

where ρ is a positive function on M (see [3; Lemma 1.1]). In terms of these tensors, we define a linear map $\psi_X: T_xM \to T_x^\perp M \oplus \boldsymbol{R}^2$ $(X \in T_xM)$ by $\psi_X(Y) = (\alpha(X, Y), \beta(Y), g(X, Y))$. Then, by using the modified Gauss equation for conformal immersions, we can easily show that $\operatorname{Ker} \psi_X \in \mathcal{N}(X)$ in the same way as above. Hence, we have $d(X) \geq \dim \operatorname{Ker} \psi_X \geq n - (r + 2)$, which implies $p_M(x) \geq n - r - 2$. q.e.d.

As seen in the above proposition, we may say that the functions p_M and p_M^c are fundamental quantities associated with (M, g). Therefore, it is an interesting problem to determine p_M and p_M^c for a given Riemannian manifold (M, g).

Finally, we state some properties of p_M and p_M^c.

PROPOSITION 1.2. (1) *Let $\pi: \tilde{M} \to M$ be a Riemannian covering. Then, $\pi^* p_M = p_{\tilde{M}}$ and $\pi^* p_M^c = p_{\tilde{M}}^c$.*

(2) *Let $M = M_1 \times \cdots \times M_k$ be a product of Riemannian manifolds. Then, for $x_i \in M_i$, the following equalities hold.*

$$p_M(x_1, \ldots, x_k) = p_{M_1}(x_1) + \cdots + p_{M_k}(x_k),$$
$$p_M^c(x_1, \ldots, x_k) = p_{M_1}^c(x_1) + \cdots + p_{M_k}^c(x_k).$$

(3) *Let M be a Riemannian symmetric space. Then the functions* p_M and p_M^c are constant on M. *In addition,*

(a) *If M is of Euclidean type, then* $p_M = p_M^c = \dim M$.

(b) *If M is of compact type and M^* is its non-compact dual, then* $p_M = p_{M^*}$ *and* $p_M^c = p_{M^*}^c$.

PROOF. The assertion (1) is clear. If M is a Riemannian symmetric space, then, since the isometry group acts transitively on M, both the functions p_M and p_M^c are constant. If M is of Euclidean type, M is locally isometric to $\boldsymbol{R}^n$ and hence we have clearly $p_M = p_M^c = \dim M$. If M is of compact type, then the curvatures of M and M^* differ only in sign, and therefore, we have $p_M = p_{M^*}$ and $p_M^c = p_{M^*}^c$. This proves the assertion (3).

Finally, we prove (2) in the case $k = 2$. The general case can be treated in the same way. For $x = (x_1, x_2) \in M_1 \times M_2$, we take a tangent vector $X = (X_1, X_2) \in T_xM = T_{x_1}M_1 \oplus T_{x_2}M_2$ such that $p_M(x) = d(X)$. Then there exist subspaces $W_i \subset T_{x_i}M_i$ satisfying $W_i \in \mathcal{N}(X_i)$ and $\dim W_i = d(X_i)$ $(i = 1, 2)$. We put $W = W_1 \oplus W_2 \subset T_xM$. For tangent vectors $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2) \in W$, we have $R(Y, Z)X = (R_1(Y_1, Z_1)X_1, R_2(Y_2, Z_2)X_2)$ where R_i is the curvature of M_i. Using the conditions $Y_i, Z_i \in W_i$ and $W_i \in \mathcal{N}(X_i)$, it follows that $R(Y, Z)X = 0$, and hence $W \in \mathcal{N}(X)$. Therefore, we have $p_M(x) = d(X) \geq \dim W = \dim W_1 + \dim W_2 = d(X_1) + d(X_2) \geq p_{M_1}(x_1) + p_{M_2}(x_2)$.

Next, for $x = (x_1, x_2) \in M$, we take $X_i \in T_{x_i}M_i$ such that $p_{M_i}(x_i) = d(X_i)$, and put $X = (X_1, X_2)$. Then there exists a subspace $W \subset T_xM$ satisfying $W \in \mathcal{N}(X)$ and $\dim W = d(X)$. We denote by $W_i \subset T_{x_i}M_i$ the image of the space W with respect to the orthogonal projection $T_xM = T_{x_1}M_1 \oplus T_{x_2}M_2 \to T_{x_i}M_i$. Then we have $W_i \in \mathcal{N}(X_i)$. In fact, for $Y_1, Z_1 \in W_1$, we can take $Y_2, Z_2 \in T_{x_2}M_2$ such that $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2) \in W$. Then we have $0 = R(Y, Z)X = (R_1(Y_1, Z_1)X_1, R_2(Y_2, Z_2)X_2)$, and from the first component, it follows that $W_1 \in \mathcal{N}(X_1)$. The property $W_2 \in \mathcal{N}(X_2)$ can be proved in the same way. Since $W \subset W_1 \oplus W_2$, we have $p_M(x) \leq d(X) = \dim W \leq \dim W_1 + \dim W_2 \leq d(X_1) + d(X_2) = p_{M_1}(x_1) + p_{M_2}(x_2)$. Thus, combining with the first inequality, we obtain the desired result. q.e.d.

In particular, from this proof, it follows that the subspace $W \subset T_xM$ realizing the equality $p_M(x) = \dim W$ is expressed as a direct sum of subspaces $W_i \in \mathcal{N}(X_i)$ such that $d(X_i) = \dim W_i$.

§ 2. Riemannian symmetric spaces

In this section, we consider the problem to determine the quantities p_M and p_M^c for Riemannian symmetric spaces. By Proposition 1.2, we may assume that M is irreducible and of compact type.

Let $M = G/K$ be an irreducible Riemannian symmetric space of compact type. Since the isometry group of M acts transitively on M, we have only to determine p_M and p_M^c at the origin o of M. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and B the Killing form of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition. As usual, we identify $\mathfrak{m}$ with the tangent space of M at o. We define the $\mathrm{Ad}(K)$-invariant inner product $\langle\, , \rangle$ of $\mathfrak{m}$ by $\langle X, Y\rangle = -B(X, Y)$ for $X, Y \in \mathfrak{m}$. We may assume that the Riemannian metric g on M coincides with $\langle\, , \rangle$ at o. Then the curvature tensor R of (M, g) is given by

$$R(X, Y)Z = -[[X, Y], Z] \quad \text{for } X, Y, Z \in \mathfrak{m}.$$

Now let us fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and set

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0\}.$$

We define two sets $\mathscr{N}_M$ and $\mathscr{N}_M^c$ consisting of subspaces of $\mathfrak{m}$ and $\mathfrak{m}^c$ as follows:

$$\mathscr{N}_M = \{W \subset \mathfrak{m} \mid [W, W] \subset \mathfrak{k}_0\},$$
$$\mathscr{N}_M^c = \{W \subset \mathfrak{m}^c \mid [W, W] \subset \mathfrak{k}_0^c\}.$$

Then we have

PROPOSITION 2.1. *Let $M = G/K$ be an irreducible Riemannian symmetric space of compact type. Then*:

(1) $p_M = \max\limits_{W \in \mathscr{N}_M} \dim W$

(2) $p_M^c = \max\limits_{W \in \mathscr{N}_M^c} \dim_{\mathbf{C}} W.$

To prove this proposition, we first prepare the following lemma.

LEMMA 2.2. *Let $H \in \mathfrak{a}$. Then*:

(1) $\mathscr{N}_M \subset \mathscr{N}(H)$ *and* $\mathscr{N}_M^c \subset \mathscr{N}^c(H)$.

(2) *In case H is regular, i.e., the centralizer of H in $\mathfrak{m}$ coincides with $\mathfrak{a}$, then*:

$\mathscr{N}_M = \mathscr{N}(H)$ *and* $\mathscr{N}_M^c = \mathscr{N}^c(H)$.

PROOF. We prove only the real case, because the complex case can be proved in the same way.

Let $W \in \mathscr{N}_M$ and let $Y, Z \in W$. Then, since $[Y, Z] \in \mathfrak{k}_0$ and $[\mathfrak{k}_0, H] = 0$, we have

$$R(Y, Z)H = -[[Y, Z], H] = 0.$$

This implies $W \in \mathcal{N}(H)$. Therefore we have $\mathcal{N}_M \subset \mathcal{N}(H)$.

We now assume that H is a regular element of $\mathfrak{a}$ and prove $\mathcal{N}(H) = \mathcal{N}_M$. Let $W \in \mathcal{N}(H)$. We put $V = [W, W]$. Then by the very definition, we have $[V, H] = 0$. Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, we have $V \subset \mathfrak{k}$. Moreover we can show that $[V, \mathfrak{a}] = 0$. We first note that $[V, \mathfrak{a}] \subset \mathfrak{m}$, because $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. By the Jacobi identity, we have

$$[H, [V, \mathfrak{a}]] \subset [[H, V], \mathfrak{a}] + [V, [H, \mathfrak{a}]] = \{0\}.$$

This proves that $[V, \mathfrak{a}]$ is contained in the centralizer of H in $\mathfrak{m}$. Since H is regular in $\mathfrak{a}$, we have $[V, \mathfrak{a}] \subset \mathfrak{a}$. Moreover, by the ad$(\mathfrak{g})$-invariance of the Killing form, we have

$$\langle A_1, [X, A_2] \rangle = -B(A_1, [X, A_2]) = B([A_1, A_2], X) = 0$$

for $A_1, A_2 \in \mathfrak{a}$, $X \in V$. Since $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{a}$, we have $[V, \mathfrak{a}] = 0$. Consequently, we have $V \subset \mathfrak{k}_0$, which shows that $W \in \mathcal{N}_M$. Therefore, we have $\mathcal{N}(H) \subset \mathcal{N}_M$. This together with the first assertion implies $\mathcal{N}(H) = \mathcal{N}_M$.

q.e.d.

PROOF OF PROPOSITION 2.1. We prove the assertion (1). Let $X \in \mathfrak{m}$. Then there exists $g \in K$ and $H \in \mathfrak{a}$ such that $X = \mathrm{Ad}(g)H$. Then we have $\mathcal{N}(X) = \mathrm{Ad}(g)\mathcal{N}(H) = \{\mathrm{Ad}(g)W \mid W \in \mathcal{N}(H)\}$, and hence we have $d(X) = d(H)$. Therefore, to determine the integer p_M, we may assume that $X = H \in \mathfrak{a}$. By (1) of Lemma 2.2, it follows that $d(H) \geq \max_{W \in \mathcal{N}_M} \dim W$. On the other hand, if H is a regular element of $\mathfrak{a}$, we have the equality $d(H) = \max_{W \in \mathcal{N}_M} \dim W$ from (2) of Lemma 2.2. This proves the assertion (1).

The assertion (2) can be proved in the same way, and we omit the proof.

q.e.d.

As an immediate consequence of Proposition 2.1, the quantity p_M can be determined for a special class of Riemannian symmetric spaces.

PROPOSITION 2.3. *Let $M = G/K$ be an irreducible Riemannian symmetric space of compact type satisfying* $\operatorname{rank} M = \operatorname{rank} G$. *Then the equality* $p_M = \operatorname{rank} M$ *holds.*

PROOF. Since $\operatorname{rank} M = \operatorname{rank} G$ and $\dim \mathfrak{a} = \operatorname{rank} M$, $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$. Hence the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ coincides with $\mathfrak{a}$ itself. Therefore we have $\mathfrak{k}_0 = \{0\}$, because $\mathfrak{a} \cap \mathfrak{k} = \{0\}$. Consequently, it is clear that a subspace W of $\mathfrak{m}$ is contained in $\mathcal{N}_M$ if and only if W is abelian, i.e., $[W, W] = 0$. Since the dimension of an abelian subspace of $\mathfrak{m}$ does not exceed $\operatorname{rank} M$, we have $\dim W \leq \operatorname{rank} M$ for any $W \in \mathcal{N}_M$. On the other hand, since $\mathfrak{a} \in \mathcal{N}_M$ and $\dim \mathfrak{a} = \operatorname{rank} M$, we have $p_M = \operatorname{rank} M$. q.e.d.

In terms of the Satake diagram, an irreducible Riemannian symmetric space $M = G/K$ with rank M = rank G corresponds to a diagram without any black circles nor any arrows. Viewing the classification table of irreducible Riemannian symmetric spaces of compact type, we have

THEOREM 2.4. *Let $M = G/K$ be one of the following Riemannian symmetric spaces of compact type and let M^* be the non-compact dual of M. Then the equality $p_M = p_{M^*} = \operatorname{rank} M$ holds.*

AI	$SU(m)/SO(m)$,	***BI***	$SO(2m+1)/SO(m+1) \times SO(m)$,
CI	$Sp(m)/U(m)$,	***DI***	$SO(2m)/SO(m) \times SO(m)$,
EI	$E_6/Sp(4)$,	***EV***	$E_7/SU(8)$,
EVIII	$E_8/Spin\ (16)$,	***FI***	$F_4/Sp(3) \cdot SU(2)$,
G	$G_2/SO(4)$.		

For the spaces listed in Theorem 2.4, we can conclude the non-existence of isometric (resp. conformal) immersions in codimension $\dim M - \operatorname{rank} M - 1$ (resp. $\dim M - \operatorname{rank} M - 3$). These results improve our previous estimates in [3], where the non-existence of isometric (or conformal) immersions in codimension about $1/2 \cdot \dim M$ is proved.

Since it is already known that the space ***CI*** $Sp(m)/U(m)$ can be globally isometrically imbedded into the Euclidean space with codimension $m^2 = \dim M - \operatorname{rank} M$ (cf. [7]), we have

COROLLARY 2.5. *For the space $Sp(m)/U(m)$, the canonical isometric imbedding gives the least dimensional* (*local*) *isometric imbedding into the Euclidean spaces.*

Finally, we consider the case of compact simple Lie groups. Let G be a compact simple Lie group and $\mathfrak{g}$ be its Lie algebra. As is known, G endowed with a bi-invariant metric can be regarded as a Riemannian symmetric space $G = \hat{G}/\hat{K}$ where $\hat{G} = G \times G$ and $\hat{K}$ denotes the diagonal subgroup of $\hat{G}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. We define two sets $\mathcal{N}_G$ and $\mathcal{N}_G^c$ by

$$\mathcal{N}_G = \{W \subset \mathfrak{g} \mid [W, W] \subset \mathfrak{t}\},$$
$$\mathcal{N}_G^c = \{W \subset \mathfrak{g}^c \mid [W, W] \subset \mathfrak{t}^c\}.$$

Then the statements in Proposition 2.1 can be reformulated as follows.

PROPOSITION 2.6. *Let G be a compact simple Lie group. Then*:

(1) $p_G = \max\limits_{W \in \mathcal{N}_G} \dim W.$

(2) $p_G^c = \max\limits_{W \in \mathcal{N}_G^c} \dim_{\mathbf{C}} W.$

PROOF. Let $\hat{\mathfrak{g}}$ (resp. $\hat{\mathfrak{k}}$) be the Lie algebra of $\hat{G}$ (resp. $\hat{K}$). Then we have $\hat{\mathfrak{g}} = \mathfrak{g} + \mathfrak{g}$ and $\hat{\mathfrak{k}} = \{(X, X)|X \in \mathfrak{g}\}$. If we put $\hat{\mathfrak{m}} = \{(X, -X)|X \in \mathfrak{g}\}$, then $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{m}}$ gives the canonical decomposition of $\hat{\mathfrak{g}}$ associated with $\hat{G}/\hat{K}$. We note that $\hat{\mathfrak{a}} = \{(H, -H)|H \in \mathfrak{t}\}$ is a maximal abelian subspace of $\hat{\mathfrak{m}}$ and that the centralizer $\hat{\mathfrak{k}}_0$ of $\hat{\mathfrak{a}}$ in $\hat{\mathfrak{k}}$ is given by $\hat{\mathfrak{k}}_0 = \{(H, H)|H \in \mathfrak{t}\}$. Let W be a subspace of $\mathfrak{g}$ and set $\hat{W} = \{(X, -X)|X \in W\}$. Then we have $\hat{W} \subset \hat{\mathfrak{m}}$ and $\dim \hat{W} = \dim W$. Conversely, any subspace of $\hat{\mathfrak{m}}$ can be expressed in this form. We can easily show that $[\hat{W}, \hat{W}] \subset \hat{\mathfrak{k}}_0$ if and only if $[W, W] \subset \mathfrak{t}$. This proves the assertion (1). The assertion (2) can be obtained in the same way.

q.e.d.

§3. The value p_G^c for compact Lie groups

In this and subsequent sections, we determine the quantity p_G^c for compact Lie groups G. On account of Proposition 1.2, we have only to determine p_G^c for compact simple Lie groups. Our main results are summarized in the following theorem.

THEOREM 3.1. *The values p_G^c for compact simple Lie groups are given in the following tables*:

G \ m		1	2	3	4		
A_{m-1}	$SU(m)$	0	2	3	5	$[m^2/4]$	$(m \geq 5)$
B_m	$SO(2m+1)$	2	4	6	8	$1/2 \cdot m(m-1)+1$	$(m \geq 5)$
C_m	$Sp(m)$	2	4			$1/2 \cdot m(m+1)$	$(m \geq 3)$
D_m	$SO(2m)$	1	4	5	8	$1/2 \cdot m(m-1)$	$(m \geq 5)$

G	
E_6	16
E_7	27
E_8	36

G	
F_4	9
G_2	4

Before proceeding to the proof, we first state several remarks on this theorem.

REMARK 1. By Proposition 1.1, it follows that G cannot be locally isometrically (resp. conformally) immersed into the Euclidean space with codimension $= \dim G - p_G^c - 1$ (resp. $\dim G - p_G^c - 3$). The isometric part of this statement is essentially equivalent to the following theorem, which the first named author proved in the previous paper [2] by applying the theory of generalized Gauss equations.

THEOREM 3.2. (cf. [2; Theorem 3.1]). *Assume that an n-dimensional compact semi-simple Lie group G is locally isometrically immersed into* $\boldsymbol{R}^{n+r}$. *Then, there exists a non-zero decomposable r-form* $\Phi \in \wedge^r \mathfrak{g}^{c*}$ *such that* $\Phi \wedge d\omega_\alpha = 0$, *where* $d\omega_\alpha$ is the exterior derivative of the $\mathfrak{g}_\alpha$-*component of the complexified Maurer-Cartan form of G.* ($\mathfrak{g}_\alpha$ *is the root subspace of* $\mathfrak{g}^c$ *corresponding to the root* α.)

In fact, a non-zero decomposable element $\Phi \in \wedge^r \mathfrak{g}^{c*}$ determines the $(n-r)$-dimensional subspace $W \subset \mathfrak{g}^c$, and it is easy to see that the condition $\Phi \wedge d\omega_\alpha = 0$ is equivalent to $[W, W] \subset \mathfrak{t}^c$. Hence, we have $p_G^c \geq n - r$, i.e., $r \geq \dim G - p_G^c$ by this theorem. In addition, in the paper [2], we determined the value p_G^c for the groups $SU(3), SO(4), SO(5)$ by using the exterior calculus. Thus, Theorem 3.1 may be considered as a generalization of these results.

REMARK 2. For each compact classical group G, the order of the value p_G^c is about $1/4 \cdot \dim G$, and hence, G cannot be locally isometrically or conformally immersed into the Euclidean space with codimension about $3/4 \cdot \dim G$. This improves the previous results in [3], where we proved the non-existence of the immersion in codimension $\sim 1/2 \cdot \dim G$.

Theorem 3.1 also improves the estimates for exceptional Lie groups. In fact, we showed in [3] that E_6, E_7, E_8, F_4 and G_2 cannot be locally isometrically immersed into the Euclidean space with codimension 35, 62, 119, 23 and 5, respectively, while Theorem 3.1 indicates the impossibility in codimension 61, 105, 211, 42 and 9.

REMARK 3. It is known that the symplectic group $Sp(m)$ can be globally isometrically imbedded in codimension $2m^2 - m$ (cf. [7]). Hence, as an immediate consequence of Theorem 3.1, we have

THEOREM 3.3. *For the group* $Sp(1), Sp(2)$ *and* $Sp(3)$, *the canonical imbeddings give the least dimensional local isometric imbeddings into the Euclidean spaces.*

For $G = Sp(1)$ and $Sp(2)$, these results are already known because $Sp(1) \simeq S^3$, and $Sp(2)$ is locally isometric to $SO(5)$ (cf. [1]).

Now, in the following, we state a systematic method to determine the value p_G^c for general compact Lie groups. For this purpose, we first fix some notations. Let $(\,,\,)$ be an inner product of $\mathfrak{g}$ which is invariant by the adjoint action of G, and for $\alpha \in \mathfrak{t}$, we put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^c \mid [H, X] = \sqrt{-1}(H, \alpha)X, \quad \text{for all} \quad H \in \mathfrak{t}\}.$$

We say that $\alpha \in \mathfrak{t}$ is a root if $\mathfrak{g}_\alpha \neq \{0\}$, and denote by Δ the set of all non-zero roots of $\mathfrak{g}$. It is well-known that $\dim_{\boldsymbol{C}} \mathfrak{g}_\alpha = 1$ for $\alpha \in \Delta$, $\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ (direct sum), and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. We denote by $\tau: \mathfrak{g}^c \to \mathfrak{g}^c$ the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. Then, there exists a basis Z_α of $\mathfrak{g}_\alpha$ satisfying

$$\tau(Z_\alpha) = Z_{-\alpha}$$
$$[Z_\alpha, Z_{-\alpha}] = 2\sqrt{-1}/(\alpha, \alpha) \cdot \alpha,$$

for $\alpha \in \Delta$ (cf. [3; p. 113]). We use these properties in §6. Note that for $\alpha, \beta \in \Delta$, $[Z_\alpha, Z_\beta] \neq 0$ if $\alpha + \beta \in \Delta$. (We consider $\mathfrak{g}_0 = \mathfrak{t}^c$.) In the following, we fix a linear order in $\mathfrak{t}$ and denote by Δ^+ (resp. Δ^-) the set of all positive (resp. negative) roots with respect to this order.

Let Γ be a non-empty subset of Δ. We denote by $\boldsymbol{R}\Gamma$ the subspace of $\mathfrak{t}$ spanned by the elements of Γ, and by $(\boldsymbol{R}\Gamma)^\perp$ the orthogonal complement of $\boldsymbol{R}\Gamma$ in $\mathfrak{t}$. For $\Gamma \subset \Delta$, we define an integer $a(\Gamma)$ by

$$\begin{aligned} a(\Gamma) &= {}^\#\Gamma + \dim (\boldsymbol{R}\Gamma)^\perp \\ &= {}^\#\Gamma + \dim \mathfrak{t} - \dim \boldsymbol{R}\Gamma. \end{aligned}$$

The above definition is naturally applicable to the case $\Gamma = \emptyset$. We then have $\boldsymbol{R}\emptyset = \{0\}$, $(\boldsymbol{R}\emptyset)^\perp = \mathfrak{t}$ and $a(\emptyset) = \dim \mathfrak{t}$.

We say that a subset $\Gamma \subset \Delta$ is *non-additive* if $\alpha + \beta \notin \Delta$ for all $\alpha, \beta \in \Gamma$. We denote by Ω the set of non-additive subsets of Δ. For $\Gamma \in \Omega$, we define a subspace W^Γ of $\mathfrak{g}^c$ by

$$W^\Gamma = (\boldsymbol{R}\Gamma)^{\perp c} + \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha.$$

The following proposition is essential in the proof of Theorem 3.1.

PROPOSITION 3.4. *Under the above notations,*

(1) *$W^\Gamma \in \mathcal{N}_G^c$ for $\Gamma \in \Omega$ (i.e., $[W^\Gamma, W^\Gamma] \subset \mathfrak{t}^c$), and* $\dim_{\boldsymbol{C}} W^\Gamma = a(\Gamma)$.

(2) *Let W be an element of $\mathcal{N}_G^c$. Then, there exists $\Gamma \in \Omega$ such that* $\dim_{\boldsymbol{C}} W \leq a(\Gamma)$.

As an immediate consequence of this proposition, we have

COROLLARY 3.5. *For a compact Lie group G, $p_G^c = \max_{\Gamma \in \Omega} a(\Gamma)$.*

PROOF OF PROPOSITION 3.4. (1) The equality $\dim_{\boldsymbol{C}} W^\Gamma = a(\Gamma)$ clearly holds. We prove the property $[W^\Gamma, W^\Gamma] \subset \mathfrak{t}^c$. First, $[(\boldsymbol{R}\Gamma)^{\perp c}, (\boldsymbol{R}\Gamma)^{\perp c}] = 0$, and for $\alpha, \beta \in \Gamma$ such that $\alpha + \beta \neq 0$, we have $[Z_\alpha, Z_\beta] = 0$ because

$\alpha+\beta\notin\Delta$. In addition, we have $[Z_\alpha, Z_{-\alpha}]\in\mathfrak{t}^c$, and for $H\in(\boldsymbol{R}\Gamma)^{\perp c}$, $[H, Z_\alpha]=\sqrt{-1}(H,\alpha)Z_\alpha=0$. Combining these results, we have $[W^\Gamma, W^\Gamma]\subset\mathfrak{t}^c$.

(2) Let $\{\alpha_1,\cdots,\alpha_m\}$ be the set of all positive roots of $\mathfrak{g}^c$ such that $\alpha_1>\cdots>\alpha_m$, and $\{H_1,\cdots,H_k\}$ be a basis of $\mathfrak{t}^c$. Then, the vectors

$$(*)\qquad Z_{\alpha_1},\cdots,Z_{\alpha_m}, H_1,\cdots,H_k, Z_{-\alpha_m},\cdots,Z_{-\alpha_1}$$

form the basis of $\mathfrak{g}^c$. We take a basis $\{X_1,\cdots,X_l\}$ of $W\in\mathcal{N}_G^c$, and express X_i as a linear combination of (*) according as the above order. Next, we deform X_i such that the top terms of $X_1\sim X_{i-1}$ do not appear in X_i. Then, finally, after multiplying some non-zero constants, we have the following expressions:

$$
\begin{aligned}
X_1 &= Z_{\beta_1}+\sum_{\alpha<\beta_1}A_{1\alpha}Z_\alpha+\hat{H}_1\\
&\cdots\cdots\cdots\\
X_p &= Z_{\beta_p}+\sum_{\alpha<\beta_p}A_{p\alpha}Z_\alpha+\hat{H}_p\\
X_{p+q+1} &= \tilde{H}_1+\sum_{\alpha<0}A_{p+q+1,\alpha}Z_\alpha\\
&\cdots\cdots\cdots\\
X_{p+q+r} &= \tilde{H}_r+\sum_{\alpha<0}A_{p+q+r,\alpha}Z_\alpha\\
X_{p+1} &= Z_{\beta_{p+1}}+\sum_{\alpha<\beta_{p+1}}A_{p+1,\alpha}Z_\alpha\\
&\cdots\cdots\cdots\\
X_{p+q} &= Z_{\beta_{p+q}}+\sum_{\alpha<\beta_{p+q}}A_{p+q,\alpha}Z_\alpha,
\end{aligned}
$$

where $\beta_1>\cdots>\beta_p>0>\beta_{p+1}>\cdots>\beta_{p+q}$ $(\beta_i\in\Delta)$, $\hat{H}_i, \tilde{H}_i\in\mathfrak{t}^c$, $A_{i\alpha}\in\boldsymbol{C}$ and $p+q+r=l$. (Note that $\tilde{H}_1,\cdots,\tilde{H}_r$ are linearly independent.) Namely,

$$X_i=Z_{\beta_i}+\sum_{\alpha<\beta_i}A_{i\alpha}Z_\alpha+\hat{H}_i\qquad(1\leq i\leq p+q)$$

and

$$X_{p+q+i}=\tilde{H}_i+\sum_{\alpha<0}A_{p+q+i,\alpha}Z_\alpha\qquad(1\leq i\leq r).$$

$(\hat{H}_{p+1}=\cdots=\hat{H}_{p+q}=0.)$ Then, for $1\leq i, j\leq p+q$, it is easy to see that the top term of $[X_i, X_j]$ with respect to the order in (*) is equal to $[Z_{\beta_i}, Z_{\beta_j}]$. If

$\beta_i + \beta_j \in \Delta$, then $0 \neq [Z_{\beta_i}, Z_{\beta_j}] \in \mathfrak{g}_{\beta_i+\beta_j}$, which contradicts the assumption $[X_i, X_j] \in \mathfrak{t}^c$. Therefore, $\beta_i + \beta_j \notin \Delta$, i.e., the set $\Gamma = \{\beta_1, \cdots, \beta_{p+q}\}$ is non-additive. Next, for $1 \leq i \leq p+q$, $1 \leq j \leq r$, the top term of $[X_{p+q+j}, X_i]$ is equal to $[\tilde{H}_j, Z_{\beta_i}] = \sqrt{-1}(\tilde{H}_j, \beta_i)Z_{\beta_i}$, and since this element must belong to $\mathfrak{t}^c$, we have $(\tilde{H}_j, \beta_i) = 0$, i.e., $\tilde{H}_j \in (\boldsymbol{R}\Gamma)^{\perp c}$. Then, since $\tilde{H}_1, \cdots, \tilde{H}_r$ are linearly independent, we have $\dim_{\boldsymbol{C}} (\boldsymbol{R}\Gamma)^{\perp c} \geq r$. In particular, we obtain the inequality $a(\Gamma) = {}^{\#}\Gamma + \dim (\boldsymbol{R}\Gamma)^{\perp} \geq p + q + r = l = \dim_{\boldsymbol{C}} W$. q.e.d.

§4. Proof of Theorem 3.1. (The case of the compact classical Lie groups)

4.1. In this section, by applying the results in §3, we give a proof of Theorem 3.1 for compact simple classical Lie groups. For the group $SU(m)$, however, we determine the value p_G^c for $G = U(m)$ instead of $SU(m)$ in order to simplify the arguments. Note that these values are related by $p_{U(m)}^c = p_{SU(m)}^c + 1$ because $U(m)$ is locally a product of $SU(m)$ and $\boldsymbol{R}^1$ (cf. Proposition 1.2).

In the following, we prove Theorem 3.1 for four types of classical groups in parallel. For this purpose, we prepare several notations concerning the roots and the Weyl groups of classical Lie algebras. First, we consider the countable set $\{\lambda_i | i \in \boldsymbol{N}\}$, and for a positive integer m, we denote by V^m the m-dimensional real vector space spanned by $\lambda_1, \cdots, \lambda_m$, i.e.,

$$V^m = \{\sum_{i=1}^{m} a_i \lambda_i \mid a_i \in \boldsymbol{R}\}.$$

Note that there is a natural inclusion

$$\{0\} \subset V^1 \subset V^2 \subset \cdots \subset V^{m-1} \subset V^m \subset \cdots,$$

because $\lambda_i \in V^j$ for $j \geq i$. We introduce an inner product $(\,,)$ on V^m such that $(\lambda_i, \lambda_j) = \delta_{ij}$. Next, we define subsets $\Delta_A^m, \Delta_B^m, \Delta_C^m, \Delta_D^m$ of V^m by

$$\begin{aligned}
\Delta_A^m &= \{\pm(\lambda_i - \lambda_j) \quad (1 \leq i < j \leq m)\}, \\
\Delta_B^m &= \{\pm \lambda_i \quad (1 \leq i \leq m), \ \pm \lambda_i \pm \lambda_j \quad (1 \leq i < j \leq m)\}, \\
\Delta_C^m &= \{\pm 2\lambda_i \quad (1 \leq i \leq m), \ \pm \lambda_i \pm \lambda_j \quad (1 \leq i < j \leq m)\}, \\
\Delta_D^m &= \{\pm \lambda_i \pm \lambda_j \quad (1 \leq i < j \leq m)\}.
\end{aligned}$$

For $X = A, B, C$ or D, we call an element $\alpha \in \Delta_X^m$ a root of type X. Note that in the case $X = A$ or D, the length of the root is always $\sqrt{2}$, and in the case $X = B$ or C, it is equal to 1, 2 or $\sqrt{2}$. We can consider the space V^m and the sets Δ_X^m $(X = A \sim D)$ as a Cartan subalgebra and the set of non-zero roots of the Lie algebras $\mathfrak{u}(m)$, $\mathfrak{o}(2m+1)$, $\mathfrak{sp}(m)$ and $\mathfrak{o}(2m)$, respectively (cf. [4]). We remark that there is a natural inclusion relation of the sets of roots:

$$\Delta_X^1 \subset \Delta_X^2 \subset \cdots \subset \Delta_X^{m-1} \subset \Delta_X^m \subset \cdots.$$

Next, for $\alpha \in \Delta_X^m$, we define a linear transformation S_α of V^m by

$$S_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha, \qquad \lambda \in V^m.$$

Clearly, S_α is an isometry of V^m. The following lemma is easy to check, and we omit the proof.

LEMMA 4.1. (1) $S_\alpha = S_{-\alpha}$ $(\alpha \in \Delta_X^m)$, *and* $S_{2\lambda_i} = S_{\lambda_i}$.

(2) *For distinct* i, j, k, *the following equalities hold.*

$$\begin{aligned}
&S_{\lambda_i - \lambda_j}(\lambda_i) = \lambda_j, \quad && S_{\lambda_i - \lambda_j}(\lambda_j) = \lambda_i, \quad && S_{\lambda_i - \lambda_j}(\lambda_k) = \lambda_k,\\
&S_{\lambda_i + \lambda_j}(\lambda_i) = -\lambda_j, && S_{\lambda_i + \lambda_j}(\lambda_j) = -\lambda_i, && S_{\lambda_i + \lambda_j}(\lambda_k) = \lambda_k,\\
&S_{\lambda_i}(\lambda_i) = -\lambda_i, && S_{\lambda_i}(\lambda_k) = \lambda_k. &&
\end{aligned}$$

In particular, applying this lemma, it is easy to see that S_α $(\alpha \in \Delta_X^m)$ preserves the set Δ_X^m. We denote by W_X^m the group generated by the transformations S_α, and call it the Weyl group of Δ_X^m. (It just coincides with the usual Weyl group of the Lie algebras $\mathfrak{u}(m) \sim \mathfrak{o}(2m)$.) The following lemma is also easy to check (cf. [4]).

LEMMA 4.2. (1) *For distinct i and j, there exists $w \in W_X^m$ such that $w(\lambda_i) = \lambda_j$, $w(\lambda_j) = \lambda_i$ and $w(\lambda_k) = \lambda_k$ $(k \neq i, j)$.*

(2) *In the case $X = B$ or C, there exists $w \in W_X^m$ such that $w(\lambda_i) = -\lambda_i$ and $w(\lambda_k) = \lambda_k$ for $k \neq i$. In the case $X = D$, there exists $w \in W_X^m$ such that $w(\lambda_i) = -\lambda_i$, $w(\lambda_j) = -\lambda_j$ $(i \neq j)$ and $w(\lambda_k) = \lambda_k$ $(k \neq i, j)$.*

(3) *Assume $m \geq 2$ for $X = A$, $m \geq 3$ for $X = D$, and $m \geq 1$ otherwise. If $\alpha, \beta \in \Delta_X^m$ satisfy $\|\alpha\| = \|\beta\|$, then there exists $w \in W_X^m$ such that $w(\alpha) = \beta$.*

In the following arguments, we often use this lemma.

4.2. Next, for a subset $\Gamma \subset \Delta_X^m$ and $k \in \boldsymbol{N}$, we define an integer $a_k(\Gamma)$ as in §3 by

$$a_k(\Gamma) = {}^{\#}\Gamma + k - \dim \boldsymbol{R}\Gamma,$$

where $\boldsymbol{R}\Gamma$ is the subspace of V^m spanned by the elements of Γ. (As stated before, in the case $\Gamma = \emptyset$, we consider $\boldsymbol{R}\Gamma = \{0\}$ and $a_k(\Gamma) = k$.) Note that if k is equal to the rank of the Lie algebra, the integer $a_k(\Gamma)$ coincides with $a(\Gamma)$ which we defined in §3. As in §3, we say that a subset $\Gamma \subset \Delta_X^m$ is *non-additive* if $\alpha+\beta\notin\Delta_X^m$ for all $\alpha,\beta\in\Gamma$, and put

$$\Omega_X^m=\{\Gamma\subset\Delta_X^m\mid\Gamma\text{ is non-additive}\}.$$

Then, by Corollary 3.5, our problem is to determine the integer $\max_{\Gamma\in\Omega_X^m} a_m(\Gamma)$ because the integer m is equal to the rank of the Lie algebra in our situation. In the following, we express this integer as p_X^m instead of p_G^c in order to distinguish the rank of the group. By the definition, in the case of $m=0$, we have clearly $p_X^m=0$.

Now, we prepare two lemmas, which play an important role in the proof of Theorem 3.1.

LEMMA 4.3. (1) *Let Γ be an element of Ω_X^{m-1}. Then, $\Gamma\in\Omega_X^m$, and $a_m(\Gamma)=a_{m-1}(\Gamma)+1$.*

(2) *For $\Gamma\in\Omega_X^m$ and $w\in W_X^m$, we put $w(\Gamma)=\{w(\alpha)\mid\alpha\in\Gamma\}$. Then, $w(\Gamma)\in\Omega_X^m$ and the equality $a_m(w(\Gamma))=a_m(\Gamma)$ holds.*

This lemma immediately follows from the definition.

LEMMA 4.4. *Assume $\Gamma\in\Omega_X^m$ and $\Gamma'\subset\Gamma$. Then, $\Gamma'\in\Omega_X^m$, and $a_m(\Gamma)=a_m(\Gamma')+s-t$, where $s={}^\#(\Gamma\setminus\Gamma')$ and $t=\dim(\boldsymbol{R}\Gamma/\boldsymbol{R}\Gamma')$. In particular, the following inequality holds:*

$$a_m(\Gamma')\le a_m(\Gamma)\le a_m(\Gamma')+s.$$

PROOF. We have

$$\begin{aligned}a_m(\Gamma)&={}^\#\Gamma+m-\dim\boldsymbol{R}\Gamma\\&={}^\#\Gamma'+s+m-\dim\boldsymbol{R}\Gamma'-t\\&=a_m(\Gamma')+s-t.\end{aligned}$$

Next, we put $\Gamma\setminus\Gamma'=\{\beta_1,\cdots,\beta_s\}$ $(\beta_i\ne\beta_j)$. Then, we have clearly

$$\boldsymbol{R}\Gamma=\boldsymbol{R}\Gamma'+\boldsymbol{R}\beta_1+\cdots+\boldsymbol{R}\beta_s,$$

and hence $t\le s$. The latter half of this lemma follows immediately from this fact. q.e.d.

4.3. We put $I=\{1,\cdots,m\}$, and for $\Gamma\subset\Delta_X^m$, we define subsets of I by

$$\begin{aligned}I^0(\Gamma)&=\{i\in I\mid(\lambda_i,\alpha)=0\quad\text{for all }\ \alpha\in\Gamma\},\\I^+(\Gamma)&=\{i\in I\mid(\lambda_i,\alpha)>0\quad\text{for some }\ \alpha\in\Gamma\},\\I^-(\Gamma)&=\{i\in I\mid(\lambda_i,\alpha)<0\quad\text{for some }\ \alpha\in\Gamma\},\end{aligned}$$

and

$$I^\pm(\Gamma)=I^+(\Gamma)\cap I^-(\Gamma).$$

Clearly, we have $I = I^0(\Gamma) \cup I^+(\Gamma) \cup I^-(\Gamma)$ and $I^0(\Gamma) \cap I^+(\Gamma) = I^0(\Gamma) \cap I^-(\Gamma) = \emptyset$. Next, we put $-\Gamma = \{-\alpha \mid \alpha \in \Gamma\}$, and using these notations, we define five subsets of Ω_X^m by

$$\Omega_{X,\mathrm{I}}^m = \{\Gamma \in \Omega_X^m \mid I^0(\Gamma) \neq \emptyset\},$$

$$\Omega_{X,\mathrm{II}}^m = \{\Gamma \in \Omega_X^m \mid \Gamma \text{ contains } \pm\alpha \text{ such that } \|\alpha\| = \sqrt{2}\},$$

$$\Omega_{X,\mathrm{III}}^m = \{\Gamma \in \Omega_X^m \mid \Gamma \text{ contains } \pm\alpha \text{ such that } \|\alpha\| = 1 \text{ or } 2\},$$

$$\Omega_{X,\mathrm{IV}}^m = \{\Gamma \in \Omega_X^m \mid I^\pm(\Gamma) \neq \emptyset \text{ and } \Gamma \cap (-\Gamma) = \emptyset\},$$

$$\Omega_{X,\mathrm{V}}^m = \{\Gamma \in \Omega_X^m \mid I^0(\Gamma) = I^\pm(\Gamma) = \emptyset\}.$$

Then, it is easy to see that

$$\Omega_X^m = \Omega_{X,\mathrm{I}}^m \cup \Omega_{X,\mathrm{II}}^m \cup \cdots \cup \Omega_{X,\mathrm{V}}^m,$$

and each subset $\Omega_{X,\mathrm{I}}^m \sim \Omega_{X,\mathrm{V}}^m$ is invariant with respect to the action of W_X^m because $w(\lambda_i) = \pm\lambda_j$ for $w \in W_X^m$. (Remark that the above union is not necessary disjoint.) Next, we put

$$q_{X,\mathrm{I}}^m = \max_{\Gamma \in \Omega_{X,\mathrm{I}}^m} a_m(\Gamma), \cdots, q_{X,\mathrm{V}}^m = \max_{\Gamma \in \Omega_{X,\mathrm{V}}^m} a_m(\Gamma).$$

(We consider $q_{X,*}^m = 0$ if $\Omega_{X,*}^m = \emptyset$.) Then, clearly we have $p_X^m = \max\{q_{X,\mathrm{I}}^m, \cdots, q_{X,\mathrm{V}}^m\}$. In the following, we evaluate the value $q_{X,\mathrm{I}}^m \sim q_{X,\mathrm{IV}}^m$ in terms of p_X^k $(k < m)$, and after calculating the exact value of $q_{X,\mathrm{V}}^m$, we determine the value p_X^m by induction on m.

4.4. First, we prove the following lemma.

LEMMA 4.5. $q_{X,\mathrm{I}}^m = p_X^{m-1} + 1$ $(m \geq 1)$.

PROOF. Let Γ be an element of $\Omega_{X,\mathrm{I}}^m$. Since $I^0(\Gamma) \neq \emptyset$, we may assume $m \in I^0(\Gamma)$ by considering $w(\Gamma)$ $(w \in W_X^m)$ instead of Γ if necessary. (cf. Lemma 4.2 (1) and Lemma 4.3 (2).) Then, we have $\Gamma \subset V^{m-1}$, and hence $\Gamma \in \Omega_X^{m-1}$. In particular, by Lemma 4.3 (1), we have the inequality $a_m(\Gamma) = a_{m-1}(\Gamma) + 1 \leq p_X^{m-1} + 1$. Next, we take $\Gamma \in \Omega_X^{m-1}$ such that $a_{m-1}(\Gamma) = p_X^{m-1}$. Then, Γ also belongs to Ω_X^m and $a_m(\Gamma) = a_{m-1}(\Gamma) + 1 = p_X^{m-1} + 1$. Combining these results, we have $q_{X,\mathrm{I}}^m = \max_{\Gamma \in \Omega_{X,\mathrm{I}}^m} a_m(\Gamma) = p_X^{m-1} + 1$. q.e.d.

LEMMA 4.6. *Assume $m \geq 3$ for $X = D$ and $m \geq 2$ otherwise. Then,*

$$q_{X,\mathrm{II}}^m = \begin{cases} p_X^{m-2} + 3, & X = A \text{ or } C, \\ p_X^{m-2} + 4, & X = B \text{ or } D. \end{cases}$$

PROOF. (i) The case $X = A$ or C. Assume $m \geq 2$ and $\Gamma \in \Omega_{X,\mathrm{II}}^m$. Then, by the definition, there exist $\pm\alpha\in\Gamma$ such that $\|\alpha\|=\sqrt{2}$. Since $m\geq 2$, we can apply Lemma 4.2 (3), and we may consider $\alpha=\lambda_{m-1}-\lambda_m$. We put $\Gamma'=\Gamma\setminus\{\pm\alpha\}$. Then, we have $\Gamma'\subset V^{m-2}$. In fact, an element $\beta\in\Gamma'$ such that $\beta\notin V^{m-2}$ must be of the form $\pm 2\lambda_{m-1}$, $\pm 2\lambda_m$, $\pm(\lambda_{m-1}+\lambda_m)$, $\pm\lambda_i\pm\lambda_{m-1}$, $\pm\lambda_i\pm\lambda_m$ $(1\leq i\leq m-2)$. But, using the facts $\pm(\lambda_{m-1}-\lambda_m)\in\Gamma$ and Γ is non-additive, we can easily see that these elements cannot belong to Γ, and hence $\Gamma'\subset V^{m-2}$. Since $\Gamma'\in\Omega_X^{m-2}$, $s={}^{\#}(\Gamma\setminus\Gamma')=2$ and $t=\dim(\boldsymbol{R}\Gamma/\boldsymbol{R}\Gamma')=1$, we have by Lemma 4.4 and Lemma 4.3 (1),

$$\begin{aligned}a_m(\Gamma)&=a_m(\Gamma')+s-t\\&=a_{m-2}(\Gamma')+2+s-t\\&=a_{m-2}(\Gamma')+3\\&\leq p_X^{m-2}+3,\end{aligned}$$

and hence, $q_{X,\mathrm{II}}^m\leq p_X^{m-2}+3$.

Conversely, we take $\Gamma'\in\Omega_X^{m-2}$ such that $a_{m-2}(\Gamma')=p_X^{m-2}$, and put $\Gamma=\Gamma'\cup\{\pm(\lambda_{m-1}-\lambda_m)\}$. Then, using the fact $\Gamma'\subset V^{m-2}$ and Γ' is non-additive, we can easily show that $\Gamma\in\Omega_{X,\mathrm{II}}^m$. In addition, in the same way as above, we have $a_m(\Gamma)=a_{m-2}(\Gamma')+3=p_X^{m-2}+3$, and hence, the equality $q_{X,\mathrm{II}}^m=p_X^{m-2}+3$ holds.

(ii) The case $X=B$ $(m\geq 2)$ or D $(m\geq 3)$. Let Γ be an element of $\Omega_{X,\mathrm{II}}^m$. As in the above case, we may assume $\pm(\lambda_{m-1}-\lambda_m)\in\Gamma$, and we put $\hat{\Gamma}=\Gamma\cup\{\pm(\lambda_{m-1}+\lambda_m)\}$. Then, we have $\hat{\Gamma}\in\Omega_X^m$. In fact, assume that $\beta\in\Gamma$ satisfies $\beta+(\lambda_{m-1}+\lambda_m)\in\Delta_X^m$. Such a β must be of the form $-\lambda_{m-1}$, $-\lambda_m$, $\pm\lambda_i-\lambda_{m-1}$, $\pm\lambda_i-\lambda_m$ $(1\leq i\leq m-2)$. But, since $\pm(\lambda_{m-1}-\lambda_m)\in\Gamma$ and Γ is non-additive, these elements cannot belong to Γ, and hence $\beta+(\lambda_{m-1}+\lambda_m)\notin\Delta_X^m$. In the same way, we can prove that $\beta-(\lambda_{m-1}+\lambda_m)\notin\Delta_X^m$ for $\beta\in\Gamma$, and therefore, we have $\hat{\Gamma}\in\Omega_X^m$. Now, we put $\hat{\Gamma}'=\Gamma\setminus\{\pm\lambda_{m-1}\pm\lambda_m\}$. Then, as in the case of (i), we can easily show that $\hat{\Gamma}'\in\Omega_X^m$ and $\hat{\Gamma}'\subset V^{m-2}$. Since $s={}^{\#}(\hat{\Gamma}\setminus\hat{\Gamma}')=4$ and $t=\dim(\boldsymbol{R}\hat{\Gamma}/\boldsymbol{R}\hat{\Gamma}')=2$, we have

$$\begin{aligned}a_m(\Gamma)&\leq a_m(\hat{\Gamma})\\&=a_m(\hat{\Gamma}')+s-t\\&=a_{m-2}(\hat{\Gamma}')+4\\&\leq p_X^{m-2}+4,\end{aligned}$$

and hence, $q_{X,\mathrm{II}}^m\leq p_X^{m-2}+4$.

Conversely, we take $\Gamma'\in\Omega_X^{m-2}$ such that $a_m(\Gamma')=p_X^{m-2}$, and put $\Gamma=\Gamma'\cup\{\pm\lambda_{m-1}\pm\lambda_m\}$. Then, as before, we have $\Gamma\in\Omega_{X,\mathrm{II}}^m$ and $a_m(\Gamma)=$

$a_{m-2}(\Gamma') + 4 = p_X^{m-2} + 4$, and therefore, we obtain the desired equality $q_{X,\mathrm{II}}^m = p_X^{m-2} + 4$. q.e.d.

LEMMA 4.7. *$q_{X,\mathrm{III}}^m = 0$ for $X = A$ or D,*

$$q_{B,\mathrm{III}}^m = p_D^{m-1} + 2, \quad q_{C,\mathrm{III}}^m = p_C^{m-1} + 2. \qquad (m \geq 1)$$

PROOF. Since Δ_A^m and Δ_D^m do not contain a root of length 1 or 2, we have $\Omega_{X,\mathrm{III}}^m = \emptyset$ for $X = A$ or D, and hence, $q_{A,\mathrm{III}}^m = q_{D,\mathrm{III}}^m = 0$. Next, we consider the case $X = B$. Assume $\Gamma \in \Omega_{B,\mathrm{III}}^m$. Then, Γ contains $\pm \alpha \in \Delta_B^m$ such that $\|\alpha\| = 1$. By considering the action of the Weyl group if necessary, we may assume that $\alpha = \lambda_m$. We put $\Gamma' = \Gamma \setminus \{\pm \lambda_m\}$. Then, by similar arguments in the proof of Lemma 4.6, we can show that $\Gamma' \subset V^{m-1}$. In addition, since Γ is non-additive, we have $\lambda_i, -\lambda_i \notin \Gamma'$ for $1 \leq i \leq m-1$, and hence, $\Gamma' \in \Omega_D^{m-1}$. By using the facts $s = {}^{\#}(\Gamma \setminus \Gamma') = 2$ and $t = \dim(\boldsymbol{R}\Gamma/\boldsymbol{R}\Gamma')$ $= 1$, we have

$$\begin{aligned} a_m(\Gamma) &= a_m(\Gamma') + s - t \\ &= a_{m-1}(\Gamma') + 1 + s - t \\ &= a_{m-1}(\Gamma') + 2 \\ &\leq p_D^{m-1} + 2, \end{aligned}$$

and hence $q_{B,\mathrm{III}}^m \leq p_D^{m-1} + 2$.

Conversely, we take $\Gamma' \in \Omega_D^{m-1}$ such that $a_{m-1}(\Gamma') = p_D^{m-1}$, and put $\Gamma = \Gamma' \cup \{\pm \lambda_m\}$. Then, we have easily $\Gamma \in \Omega_{B,\mathrm{III}}^m$ and $a_m(\Gamma) = a_{m-1}(\Gamma') + 2 = p_D^{m-1} + 2$, which implies $q_{B,\mathrm{III}}^m = p_D^{m-1} + 2$.

The proof of the equality $q_{C,\mathrm{III}}^m = p_C^{m-1} + 2$ can be done in completely the same way, and we omit it. q.e.d.

LEMMA 4.8. *$q_{X,\mathrm{IV}}^m = 0$ ($X = A$ or C), and $q_{X,\mathrm{IV}}^m \leq p_X^{m-1} + 2$ for $X = B$ ($m \geq 2$) and $X = D$ ($m \geq 3$).*

PROOF. Let Γ be an element of $\Omega_{X,\mathrm{IV}}^m$, i.e., $I^{\pm}(\Gamma) \neq \emptyset$ and $\Gamma \cap (-\Gamma) = \emptyset$. By considering the action of W_X^m if necessary, we may assume that $m \in I^{\pm}(\Gamma)$. We first show that $\lambda_m, -\lambda_m \notin \Gamma$ in the case $X = B$. Assume $\lambda_m \in \Gamma$. Then, since $\Gamma \cap (-\Gamma) = \emptyset$, we have $-\lambda_m \notin \Gamma$, and hence $\lambda_i - \lambda_m \in \Gamma$ or $-\lambda_i - \lambda_m \in \Gamma$ for some i ($1 \leq i \leq m-1$) because $m \in I^-(\Gamma)$. This contradicts the fact that Γ is non-additive since $\lambda_m + (\pm \lambda_i - \lambda_m) = \pm \lambda_i \in \Delta_B^m$. The property $-\lambda_m \notin \Gamma$ can be proved in the same way. Similarly, in the case $X = C$, we can show $2\lambda_m, -2\lambda_m \notin \Gamma$. In particular, for $X = A \sim D$, we have $\lambda_i + \lambda_m \in \Gamma$ or $-\lambda_i + \lambda_m \in \Gamma$ for some i ($1 \leq i \leq m-1$) because $m \in I^+(\Gamma)$, and hence, we have $\Omega_{X,\mathrm{IV}}^m = \emptyset$ if $m = 1$. In the following, we assume $m \geq 2$ for

$X = A$, B or C, and $m \geq 3$ for $X = D$. By considering the action of W_X^m, we may assume $\lambda_m - \lambda_{m-1} \in \Gamma$ (cf. Lemma 4.2 (1), (2). Note that in the case $X = D$ and $m = 2$, $w(\lambda_1 + \lambda_2) \neq \lambda_2 - \lambda_1$ for any $w \in W_D^2$). Next, since $m \in I^-(\Gamma)$ and $-\lambda_m$, $-2\lambda_m \notin \Gamma$, we have $\lambda_j - \lambda_m \in \Gamma$ or $-\lambda_j - \lambda_m \in \Gamma$ for some j $(1 \leq j \leq m-1)$. We assume that $\lambda_j - \lambda_m \in \Gamma$. Then, since Γ is non-additive, and $(\lambda_m - \lambda_{m-1}) + (\lambda_j - \lambda_m) = \lambda_j - \lambda_{m-1}$, we have $j = m-1$, i.e., $\pm(\lambda_m - \lambda_{m-1}) \in \Gamma$, which contradicts $\Gamma \cap (-\Gamma) = \emptyset$. Hence, we have $\lambda_j - \lambda_m \notin \Gamma$. In particular, we obtain the result $\Omega_{A,\mathrm{IV}}^m = \emptyset$, and for the remaining case $X = B$, C or D, we have $-\lambda_j - \lambda_m \in \Gamma$. Then, by the same argument, we have $-\lambda_{m-1} \pm \lambda_m \in \Gamma$. But, in the case $X = C$, $(-\lambda_{m-1} + \lambda_m) + (-\lambda_{m-1} - \lambda_m) = -2\lambda_{m-1} \in \Delta_C^m$, and hence we have $\Omega_{C,\mathrm{IV}}^m = \emptyset$. For the case $X = B$ or D, we put $\Gamma' = \Gamma \setminus \{-\lambda_{m-1} \pm \lambda_m\}$. Then, as in the proof of Lemma 4.6, we can easily show $\Gamma' \subset V^{m-1}$. Hence, $\Gamma' \in \Omega_X^{m-1}$, and by using the facts $s = {}^{\#}(\Gamma \setminus \Gamma') = 2$ and $t = \dim(\boldsymbol{R}\Gamma/\boldsymbol{R}\Gamma') \geq 1$, we have

$$a_m(\Gamma) = a_m(\Gamma') + s - t \leq a_{m-1}(\Gamma') + 2,$$

and therefore, $q_{X,\mathrm{IV}}^m \leq p_X^{m-1} + 2$ for $X = B$ or D. q.e.d.

REMARK. As is easy to see, the set $\Gamma = \Gamma' \cup \{-\lambda_{m-1} \pm \lambda_m\}$ is not necessary non-additive for $\Gamma' \in \Omega_X^{m-1}$ ($X = B$ or D), and the equality $q_{X,\mathrm{IV}}^m = p_X^{m-1} + 2$ does not hold in general.

4.5. Finally, for the type V, we have the following results.

LEMMA 4.9. *For* $m \geq 2$,

$$q_{A,\mathrm{V}}^m = [m^2/4] + 1,$$
$$q_{B,\mathrm{V}}^m = 1/2 \cdot m(m-1) + 1,$$
$$q_{C,\mathrm{V}}^m = 1/2 \cdot m(m+1),$$
$$q_{D,\mathrm{V}}^m = 1/2 \cdot m(m-1) + \delta_{m,2}.$$

PROOF. Since $I^0(\Gamma) = I^{\pm}(\Gamma) = \emptyset$, the set I is expressed as a disjoint union of $I^+(\Gamma)$ and $I^-(\Gamma)$, i.e., for each $i \in I$, (λ_i, α) is always positive or negative for all $\alpha \in \Gamma$.

We first treat the case $X = A$. For $\Gamma \in \Omega_{A,\mathrm{V}}^m$, we put ${}^{\#}I^+(\Gamma) = a$ and ${}^{\#}I^-(\Gamma) = b$. Then, clearly we have $a + b = m$ and $a, b \geq 1$ because $\lambda_i - \lambda_j \in \Gamma$ implies $i \in I^+(\Gamma)$ and $j \in I^-(\Gamma)$. Now, we define a subset $\Gamma_0 \subset \Delta$ by

$$\Gamma_0 = \{\lambda_i - \lambda_j \mid 1 \leq i \leq a,\ a+1 \leq j \leq m\}.$$

It is clear that Γ_0 is non-additive, and $I^0(\Gamma_0) = \emptyset$, $I^+(\Gamma_0) = \{1, \cdots, a\}$, $I^-(\Gamma_0) = \{a+1, \cdots, m\}$. In particular, $\Gamma_0 \in \Omega_{A,\mathrm{V}}^m$. For the set Γ, we can choose $w \in W_A^m$ such that $I^-(w(\Gamma)) = \{1, \cdots, a\}$ and $I^-(w(\Gamma)) = \{a+1, \cdots, m\}$.

Then, if $\lambda_i - \lambda_j \in w(\Gamma)$, we have $i \in \{1, \cdots, a\}$ and $j \in \{a+1, \cdots, m\}$, which implies $w(\Gamma) \subset \Gamma_0$. Since the independent roots $\lambda_1 - \lambda_i$ $(a+1 \le i \le m)$ and $\lambda_i - \lambda_{a+1}$ $(2 \le i \le a)$ span the space $\boldsymbol{R}\Gamma_0$, we have $\dim \boldsymbol{R}\Gamma_0 = a + b - 1 = m - 1$, and by Lemma 4.3 (2), Lemma 4.4,

$$\begin{aligned} a_m(\Gamma) &= a_m(w(\Gamma)) \le a_m(\Gamma_0) \\ &= {}^{\#}\Gamma_0 + m - \dim \boldsymbol{R}\Gamma_0 \\ &= ab + m - (m-1) \\ &= ab + 1 \\ &= a(m-a) + 1, \end{aligned}$$

and therefore $q^m_{A,\mathrm{V}} = \max_{1 \le a \le m-1} a(m-a) + 1 = [m^2/4] + 1$.

Next, we consider the case $X = B$, C or D. For $\Gamma \in \Omega^m_{X,\mathrm{V}}$, we first show that there exists $\Gamma' \in \Omega^m_{X,\mathrm{V}}$ such that $I^-(\Gamma') = \emptyset$ and $a_m(\Gamma') = a_m(\Gamma)$. For the case $X = B$, we put ${}^{\#}I^-(\Gamma) = a$. Then, by the action of W^m_X, we may assume $I^-(\Gamma) = \{1, \cdots, a\}$ and $I^+(\Gamma) = \{a+1, \cdots, m\}$, i.e.,

$$\begin{aligned} \Gamma \subset \{ &-\lambda_i\ (1 \le i \le a),\ \lambda_i\ (a+1 \le i \le m),\ \lambda_j - \lambda_i\ (1 \le i \le a < j \le m), \\ &\lambda_j + \lambda_i\ (a+1 \le i < j \le m),\ -\lambda_j - \lambda_i\ (1 \le i < j \le a)\}. \end{aligned}$$

By putting $w = S_{\lambda_1} \cdots S_{\lambda_a} \in W^m_B$, we have easily $w(\lambda_i) = -\lambda_i$ $(1 \le i \le a)$, and $w(\lambda_i) = \lambda_i$ $(a+1 \le i \le m)$. Then, we have $w(\Gamma) \subset \{\lambda_i\ (1 \le i \le m),\ \lambda_j + \lambda_i\ (1 \le i < j \le m)\}$, which implies $I^-(w(\Gamma)) = \emptyset$. The proof for the case $X = C$ is completely the same. For the case $X = D$, we may assume $I^-(\Gamma) = \{1, \cdots, a\}$ and $I^+(\Gamma) = \{a+1, \cdots, m\}$, as above. Then we have

$$\begin{aligned} \Gamma \subset \{\lambda_j - \lambda_i\ (1 \le i \le a < j \le m),\ \lambda_j + \lambda_i\ (a+1 \le i < j \le m), \\ -\lambda_j - \lambda_i\ (1 \le i < j \le a)\}. \end{aligned}$$

We put

$$\begin{aligned} &\beta_1 = \lambda_1 - \lambda_2,\quad \beta_2 = \lambda_3 - \lambda_4, \cdots,\quad \beta_{[a/2]} = \lambda_{2[a/2]-1} - \lambda_{2[a/2]}, \\ &\gamma_1 = \lambda_1 + \lambda_2,\quad \gamma_2 = \lambda_3 + \lambda_4, \cdots,\quad \gamma_{[a/2]} = \lambda_{2[a/2]-1} + \lambda_{2[a/2]}, \end{aligned}$$

and $\quad w = S_{\beta_1} S_{\gamma_1} \cdots S_{\beta_{[a/2]}} S_{\gamma_{[a/2]}}.$

Then we have $w(\lambda_i) = -\lambda_i$ $(1 \le i \le 2[a/2])$ and $w(\lambda_i) = \lambda_i$ $(2[a/2] + 1 \le i \le m)$. Hence, if a is even, we have $w(\Gamma) \subset \{\lambda_j + \lambda_i\ (1 \le i < j \le m)\}$, which implies $I^-(w(\Gamma)) = \emptyset$. In the case a is odd, we have $I^-(w(\Gamma)) = \{a\}$. In this case, we consider $w(\Gamma)$ as an element of Ω^{m+1}_D, and put $\beta = \lambda_a - \lambda_{a+1}$, $\gamma = \lambda_a + \lambda_{a+1}$, $w' = S_\beta S_\gamma w$. Then, since $S_\beta S_\gamma(\lambda_i) = -\lambda_i$ $(i = a, m+1)$ and $S_\beta S_\gamma(\lambda_i) = \lambda_i$ $(i \ne a, m+1)$; we have $w'(\Gamma) \in \Omega^m_D$. (Note that $w'(\Gamma) \subset V^m$.) Clearly, $w'(\Gamma) \subset$

$\{\lambda_j + \lambda_i\ (1 \le i < j \le m)\}$, and hence $I^-(w'(\Gamma)) = \emptyset$. In addition, by Lemma 4.3 (1),

$$a_m(w'(\Gamma)) = a_{m+1}(w'(\Gamma)) - 1 = a_{m+1}(\Gamma) - 1 = a_m(\Gamma),$$

which completes the proof. Thus, we may assume that $I^-(\Gamma) = \emptyset$.

Now, we put $\Gamma_0 = \{\lambda_i + \lambda_j\ (1 \le i < j \le m)\}$ for the case $X = B$, C or D. Then, we have $\Gamma_0 \in \Omega^m_{X,\mathrm{V}}$ because $I^0(\Gamma_0) = I^-(\Gamma_0) = \emptyset$. In addition, by using the facts that ${}^{\#}\Gamma_0 = 1/2 \cdot m(m-1)$ and $\dim \boldsymbol{R}\Gamma_0 = 1$ $(m = 2)$, $= m$ $(m \ge 3)$, it is easy to see that $a_m(\Gamma_0) = 1/2 \cdot m(m-1) + \delta_{m,2}$.

For the case $X = D$, by using $I^-(\Gamma) = \emptyset$, we have $\Gamma \subset \Gamma_0$, and hence $a_m(\Gamma) \le a_m(\Gamma_0)$. Since $\Gamma_0 \in \Omega^m_{D,\mathrm{V}}$, we obtain the equality $q^m_{D,\mathrm{V}} = a_m(\Gamma_0) = 1/2 \cdot m(m-1) + \delta_{m,2}$. Next, for the case $X = C$, by putting $\hat{\Gamma} = \Gamma_0 \cup \{2\lambda_1, \cdots, 2\lambda_m\}$, we have easily $\Gamma \subset \hat{\Gamma}$, and $\hat{\Gamma} \in \Omega^m_{C,\mathrm{V}}$. Hence, $a_m(\Gamma) \le a_m(\hat{\Gamma}) = {}^{\#}\hat{\Gamma} = 1/2 \cdot m(m+1)$, which implies $q^m_{C,\mathrm{V}} = 1/2 \cdot m(m+1)$. Finally, for the case $X = B$, assume $\lambda_i \notin \Gamma$. Then, we have $\Gamma \subset \Gamma_0$, and in particular, $a_m(\Gamma) \le a_m(\Gamma_0) \le 1/2 \cdot m(m-1) + 1$. If $\lambda_i \in \Gamma$ for some i, then other λ_j cannot belong to Γ because Γ is non-additive. Hence, by putting $\hat{\Gamma} = \Gamma_0 \cup \{\lambda_i\}$, we have $\Gamma \subset \hat{\Gamma}$. We can easily check that $\hat{\Gamma} \in \Omega^m_{B,\mathrm{V}}$ and $a_m(\Gamma) \le a_m(\hat{\Gamma}) = 1/2 \cdot m(m-1) + 1$, and therefore we have $q^m_{B,\mathrm{V}} = 1/2 \cdot m(m-1) + 1$. q.e.d.

4.6. Now, under these preliminaries, we determine the value p^m_X for $X = A \sim D$. For this purpose, we prepare one more lemma.

LEMMA 4.10. *Assume $k \ge 3$ and $X = A$, C or D. If $p^m_X = q^m_{X,\mathrm{V}}$ for $m = k$ and $k+1$, then $p^m_X = q^m_{X,\mathrm{V}}$ for $m \ge k$.*

PROOF. We have only to show the equality in the case $m = k + 2$. First, for $m \ge 4$, by using Lemma 4.9, we have immediately,

$$\begin{aligned} q^{2s}_{A,\mathrm{V}} &= q^{2s-1}_{A,\mathrm{V}} + s, \\ q^{2s+1}_{A,\mathrm{V}} &= q^{2s}_{A,\mathrm{V}} + s, \end{aligned} \qquad (s \ge 2)$$

$$q^m_{C,\mathrm{V}} = q^{m-1}_{C,\mathrm{V}} + m,$$

$$q^m_{D,\mathrm{V}} = q^{m-1}_{D,\mathrm{V}} + m - 1,$$

and hence, $q^m_{X,\mathrm{V}} \ge q^{m-1}_{X,\mathrm{V}} + 1$ for $X = A$, C or D. Similarly, we can show that the inequality $q^m_{X,\mathrm{V}} \ge q^{m-2}_{X,\mathrm{V}} + 4$ holds for $m \ge 5$. Hence, we have by Lemma 4.5,

$$q^{k+2}_{X,\mathrm{I}} = p^{k+1}_X + 1 = q^{k+1}_{X,\mathrm{V}} + 1 \le q^{k+2}_{X,\mathrm{V}},$$

and by Lemma 4.6,

$$q^{k+2}_{X,\mathrm{II}} \le p^k_X + 4 = q^k_{X,\mathrm{V}} + 4 \le q^{k+2}_{X,\mathrm{V}}.$$

For the type III, we have by Lemma 4.7,

$$q_{A,\mathrm{III}}^{k+2} = q_{D,\mathrm{III}}^{k+2} = 0$$

and $$q_{C,\mathrm{III}}^{k+2} = p_C^{k+1} + 2 = q_{C,\mathrm{V}}^{k+1} + 2 \le q_{C,\mathrm{V}}^{k+1} + k + 2 = q_{C,\mathrm{V}}^{k+2}.$$

Similarly, by Lemma 4.8, we have

$$q_{A,\mathrm{IV}}^{k+2} = q_{C,\mathrm{IV}}^{k+2} = 0,$$

and $$q_{D,\mathrm{IV}}^{k+2} \le p_D^{k+1} + 2 = q_{D,\mathrm{V}}^{k+1} + 2 \le q_{D,\mathrm{V}}^{k+1} + k + 1 = q_{D,\mathrm{V}}^{k+2}.$$

Therefore, we have $p_X^{k+2} = \max\{q_{X,\mathrm{I}}^{k+2}, \cdots, q_{X,\mathrm{V}}^{k+2}\} = q_{X,\mathrm{V}}^{k+2}$. q.e.d.

PROOF OF THEOREM 3.1. We prove the theorem inductively by applying Lemma 4.10. First, we treat the case $X = A$. If $m = 1$, then we have $\Delta_A^1 = \emptyset$, and hence $p_A^1 = 1$. In the case $m = 2$, since the set of roots $\Delta_A^2 = \{\pm(\lambda_1 - \lambda_2)\}$ is itself non-additive, we have by Lemma 4.4, $p_A^2 = a_2(\Delta_A^2) = 2 + 2 - 1 = 3$. Then, using the equalities $q_{A,\mathrm{I}}^m = p_A^{m-1} + 1$, $q_{A,\mathrm{II}}^m = p_X^{m-2} + 3$, $q_{A,\mathrm{V}}^m = [m^2/4] + 1$ and $q_{A,\mathrm{III}}^m = q_{A,\mathrm{IV}}^m = 0$ (Lemmas 4.5 $\sim$ 4.9), we obtain the table

m	1	2	3	4	5	6
$q_{A,\mathrm{I}}^m$			4	5	7	8
$q_{A,\mathrm{II}}^m$			4	6	7	9
$q_{A,\mathrm{V}}^m$			3	5	7	10
p_A^m	1	3	4	6	7	10

Since $p_A^m = q_{A,\mathrm{V}}^m$ for $m = 5, 6$, we have by Lemma 4.10, $p_A^m = q_{A,\mathrm{V}}^m = [m^2/4] + 1$ for $m \ge 5$. Therefore, by using the equality $p_{U(m)}^c = p_{SU(m)}^c + 1$, we obtain the desired results for $G = SU(m)$.

Next, we consider the case $X = C$. For $m = 1$, the set of roots $\Delta_C^1 = \{\pm 2\lambda_1\}$ is non-additive, and we have by Lemma 4.4, $p_C^1 = a_1(\Delta_C^1) = 2 + 1 - 1 = 2$. As in the case of $X = A$, by using the equalities $q_{C,\mathrm{I}}^m = p_C^{m-1} + 1$, $q_{C,\mathrm{II}}^m = p_C^{m-2} + 3$, $q_{C,\mathrm{II}}^2 = 3$, $q_{C,\mathrm{III}}^m = p_C^{m-1} + 2$, $q_{C,\mathrm{V}}^m = 1/2 \cdot m(m+1)$ and $q_{C,\mathrm{IV}}^m = 0$, we have the following table

m	1	2	3	4
$q_{C,\mathrm{II}}^m$		3	5	7
$q_{C,\mathrm{III}}^m$		4	6	8
$q_{C,\mathrm{V}}^m$		3	6	10
p_C^m	2	4	6	10

(We may omit the value $q_{C,\mathrm{I}}^m$ because $q_{C,\mathrm{I}}^m < q_{C,\mathrm{III}}^m$ for $m \ge 2$.) Hence, as above, we have $p_C^m = q_{C,\mathrm{V}}^m = 1/2 \cdot m(m+1)$ for $m \ge 3$.

For the case $X = D$, since $\Delta_D^1 = \emptyset$, we have $p_D^1 = 1$. And in the case $m = 2$, the set $\Delta_D^2 = \{\pm\lambda_1 \pm \lambda_2\}$ is itself non-additive, which implies $p_D^2 = a_2(\Delta_D^2) = 4 + 2 - 2 = 4$. For $m = 3$, since the group $SO(6)$ is locally isomorphic to $SU(4)$, we have $p_D^3 = p_{SU(4)}^c = 5$, as we showed above. Then, by using the equalities in Lemmas 4.5 $\sim$ 4.9, we obtain the table

m	1	2	3	4	5	6
$q_{D,\mathrm{I}}^m$				6	9	11
$q_{D,\mathrm{II}}^m$				8	9	12
$q_{D,\mathrm{IV}}^m$				≤ 7	≤ 10	≤ 12
$q_{D,\mathrm{V}}^m$				6	10	15
p_D^m	1	4	5	8	10	15

Hence, as above, we have by Lemma 4.10, $p_D^m = q_{D,\mathrm{V}}^m = 1/2 \cdot m(m-1)$ for $m \geq 5$.

Finally, we determine the value p_X^m for $X = B$. Since the set $\Delta_B^1 = \{\pm\lambda_1\}$ is non-additive, we have $p_B^1 = a_1(\Delta_B^1) = 2 + 1 - 1 = 2$. Then, by using the results in Lemmas 4.5 $\sim$ 4.9 and the value p_D^m, we have the table

m	1	2	3	4	5	6
$q_{B,\mathrm{I}}^m$		3	5	7	9	12
$q_{B,\mathrm{II}}^m$		4	6	8	10	12
$q_{B,\mathrm{III}}^m$		3	6	7	10	12
$q_{B,\mathrm{IV}}^m$		≤ 4	≤ 6	≤ 8	≤ 10	≤ 13
$q_{B,\mathrm{V}}^m$		2	4	7	11	16
p_B^m	2	4	6	8	11	16

In particular, we have $p_B^m = q_{B,\mathrm{V}}^m$ for $m = 5, 6$. Then, in completely the same way as Lemma 4.10, we can prove that the equality $p_B^m = q_{B,\mathrm{V}}^m$ holds for $m \geq 5$. (We omit the details. Note that $q_{B,\mathrm{III}}^m = p_D^{m-1} + 2 = 1/2 \cdot (m-1)(m-2) + 2 < 1/2 \cdot m(m-1) + 1 = q_{B,\mathrm{V}}^m$ for $m \geq 6$.) q.e.d.

§5. Proof of Theorem 3.1. (The case of the compact exceptional Lie groups)

In this section, we determine the value p_G^c for the exceptional Lie groups $E_6 \sim E_8$, F_4 and G_2, by applying the results stated in Appendix (Theorem A1, A2). We first prepare the following lemma.

LEMMA 5.1. *Let Γ be a finite subset of the vector space V^m. If $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, then for positive integers k and l, the following equality holds*:

$$a_k(\Gamma) = a_l(\Gamma_1) + {}^{\#}\Gamma_2 + k - l - \dim(\boldsymbol{R}\Gamma_2/(\boldsymbol{R}\Gamma_1 \cap \boldsymbol{R}\Gamma_2)).$$

Using the definition and the fact $\boldsymbol{R}\Gamma = \boldsymbol{R}\Gamma_1 + \boldsymbol{R}\Gamma_2$, we can easily prove this lemma, and we omit the details. In this section, we denote by Δ_G the set of roots of G ($= E_6 \sim E_8$, F_4 or G_2), and by Ω_G the set of non-additive subsets Γ of Δ_G. As we showed in Corollary 3.5, we have the equality $p_G^c = \max_{\Gamma \in \Omega_G} a_m(\Gamma)$, where m is the rank of G, and we determine this integer for $G = E_6 \sim G_2$.

5.1. The case $G = E_6$, E_7 or E_8. It is well known that the set of roots $\Delta_{E_6} \sim \Delta_{E_8}$ can be expressed as

$$\Delta_{E_6} = \{\pm \lambda_i \pm \lambda_j \ (1 \le i < j \le 5),\ \pm 1/2 \cdot (\sum_{i=1}^{5} (-1)^{\varepsilon_i} \lambda_i - \lambda_6 - \lambda_7 + \lambda_8) \quad (\sum_{i=1}^{5} \varepsilon_i \text{ is even})\},$$

$$\Delta_{E_7} = \{\pm \lambda_i \pm \lambda_j \ (1 \le i < j \le 6),\ \pm(\lambda_7 - \lambda_8),\ \pm 1/2 \cdot (\sum_{i=1}^{6} (-1)^{\varepsilon_i} \lambda_i + \lambda_7 - \lambda_8) \quad (\sum_{i=1}^{6} \varepsilon_i \text{ is odd})\},$$

$$\Delta_{E_8} = \{\pm \lambda_i \pm \lambda_j \ (1 \le i < j \le 8),\ \pm 1/2 \cdot \sum_{i=1}^{8} (-1)^{\varepsilon_i} \lambda_i \quad (\sum_{i=1}^{8} \varepsilon_i \text{ is even})\},$$

where $\{\lambda_1, \cdots, \lambda_8\}$ is an orthonormal basis of V^8 (cf. [4]).

Now, assume that $\Gamma \in \Omega_{E_m}$ ($m = 6, 7$ or 8) satisfies $\Gamma \cap (-\Gamma) = \emptyset$, i.e., there does not exist a root $\alpha \in \Gamma$ satisfying $\pm\alpha \in \Gamma$. Then, it is easy to see that the space W^Γ which we defined in §3 is abelian. (Remind the proof of Proposition 3.4.) Hence, by combining Theorem A1 and A2 in Appendix, we have $a_m(\Gamma) = \dim_{\boldsymbol{C}} W^\Gamma \le 16, 27, 36$, according as $G = E_6, E_7, E_8$.

Next, we consider the case where $\Gamma \in \Omega_{E_m}$ contains roots $\pm\alpha$. We put $\Gamma' = \Gamma \setminus \{\pm\alpha\}$. Then, since Γ is non-additive, we have $\beta + \alpha, \beta - \alpha \notin \Gamma$ for $\beta \in \Gamma'$, which implies $(\alpha, \beta) = 0$. (Note that the length of α-series containing β is 1.) In particular, we have $\Gamma' \subset \langle\alpha\rangle^\perp$. If $G = E_6$, we may assume $\alpha = 1/2 \cdot (\lambda_1 + \cdots + \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8)$ by considering the action of the Weyl group of E_6. (Note that any two elements $\alpha, \beta \in \Delta_{E_m}$ ($m = 6, 7$ or 8) can be mapped to each other by the action of the Weyl group because all roots are of the same length and Δ_{E_m} is irreducible. cf. [4].) Then, it is easy to see that

$$\langle\alpha\rangle^{\perp}\cap\Delta_{E_6}=\{\pm(\lambda_i-\lambda_j)\quad(1\leq i<j\leq 5)\},$$

and hence, we have $\Gamma'\subset\Delta_A^5$. By putting $\Gamma_1=\Gamma'$ and $\Gamma_2=\{\pm\alpha\}$ in Lemma 5.1, it follows that

$$\begin{aligned} a_6(\Gamma) &= a_5(\Gamma')+2+1-\dim(\boldsymbol{R}\alpha/(\boldsymbol{R}\Gamma'\cap\boldsymbol{R}\alpha)) \\ &\leq a_5(\Gamma')+3 \\ &\leq 10<16. \end{aligned}$$

Therefore, combining with Theorem A1 and A2, we have $p_{E_6}^c=\max_{\Gamma\in\Omega_{E_6}} a_6(\Gamma)=16$. For the group E_7, we may assume $\alpha=\lambda_7-\lambda_8$ by the same reason as E_6. In this case, we have easily

$$\langle\alpha\rangle^{\perp}\cap\Delta_{E_7}=\{\pm\lambda_i\pm\lambda_j\quad(1\leq i<j\leq 6)\},$$

and by using the fact $p_D^6=15$, we have in the same way as above, $a_7(\Gamma)\leq 18<27$, which implies $p_{E_7}^c=27$. For the group E_8, we use the root $\alpha=\lambda_7+\lambda_8$, and carry out the same procedure. Since $\langle\alpha\rangle^{\perp}\cap\Delta_{E_8}=E_7$ and $p_{E_7}^c=27$, we have $a_8(\Gamma)\leq 30<36$, and therefore, $p_{E_8}^c=36$.

5.2. The case of $G=F_4$. In this case, it is known that the set of roots of F_4 is given by

$$\Delta_{F_4}=\{\pm\lambda_i\,(1\leq i\leq 4),\ \pm\lambda_i\pm\lambda_j\,(1\leq i<j\leq 4),\ 1/2\cdot(\pm\lambda_1\pm\lambda_2\pm\lambda_3\pm\lambda_4)\},$$

where $\{\lambda_1,\cdots,\lambda_4\}$ is an orthonormal basis of V^4. We apply the same method as E_m. First, if $\Gamma\in\Omega_{F_4}$ satisfies $\Gamma\cap(-\Gamma)=\emptyset$, we have $\max a_4(\Gamma)=9$ by Theorem A1 and A2. In the case where Γ contains roots $\pm\alpha$, we must divide the proof into two cases according as $\|\alpha\|=1$ or $\|\alpha\|=\sqrt{2}$.

(i) The case $\|\alpha\|=1$. In this case, we may assume $\alpha=\lambda_4$ by considering the action of the Weyl group of F_4. Then, we have

$$\langle\alpha\rangle^{\perp}\cap\Delta_{F_4}=\{\pm\lambda_i\ (1\leq i\leq 3),\ \pm\lambda_i\pm\lambda_j\ (1\leq i<j\leq 3)\}=\Delta_B^3,$$

and hence, by putting $\Gamma'=\Gamma\setminus\{\pm\alpha\}$, we have

$$\begin{aligned} a_4(\Gamma) &= a_3(\Gamma')+2+1-\dim(\boldsymbol{R}\alpha/(\boldsymbol{R}\Gamma'\cap\boldsymbol{R}\alpha)) \\ &\leq 9-1=8. \end{aligned}$$

(Note that $\dim(\boldsymbol{R}\alpha/(\boldsymbol{R}\Gamma'\cap\boldsymbol{R}\alpha))\geq 1$.)

(ii) The case $\|\alpha\|=\sqrt{2}$. In this case, we may assume $\alpha=\lambda_1-\lambda_2$. Then, we have

$$\langle\alpha\rangle^{\perp}\cap\Delta_{F_4}=\{\pm\lambda_3,\ \pm\lambda_4,\ \pm\lambda_3\pm\lambda_4,\ \pm(\lambda_1+\lambda_2),\ \pm 1/2\cdot(\lambda_1+\lambda_2\pm\lambda_3\pm\lambda_4)\}.$$

We define three vectors $\lambda_1', \lambda_2', \lambda_3'$ by

$$\lambda_1 + \lambda_2 = 2\lambda_1', \quad \lambda_3 + \lambda_4 = 2\lambda_2' \quad \text{and} \quad \lambda_3 - \lambda_4 = 2\lambda_3'.$$

Then, we have $(\lambda_i', \lambda_j') = 0$, $\|\lambda_i'\| = \|\lambda_j'\|$ for $i \neq j$, and

$$\langle\alpha\rangle^{\perp} \cap \Delta_{F_4} = \{\pm 2\lambda_i' \ (1 \leq i \leq 3), \ \pm \lambda_i' \pm \lambda_j' \ (1 \leq i < j \leq 3)\} \simeq \Delta_C^3.$$

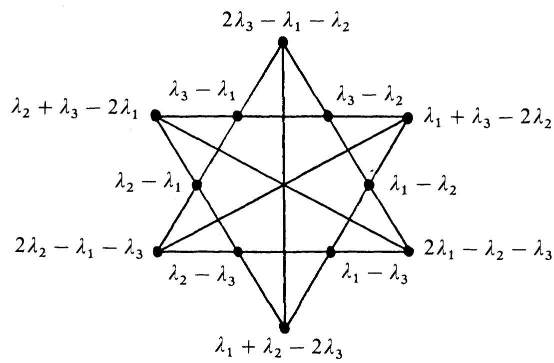
Therefore, as above, we have $a_4(\Gamma) \leq a_3(\Gamma') + 3 - 1 \leq 8$.

Combining these results, we obtain the desired result $p_{F_4}^c = \max_{\Gamma \in \Omega_{F_4}} a_4(\Gamma) = 9$.

5.3. The case of $G = G_2$. In this case, we prove the equality $p_{G_2}^c = 4$. It is known that the set of roots of G_2 is expressed as

$$\{\pm(\lambda_i - \lambda_j) \quad (1 \leq i < j \leq 3), \ \pm(2\lambda_1 - \lambda_2 - \lambda_3),$$
$$\pm(2\lambda_2 - \lambda_1 - \lambda_3), \ \pm(2\lambda_3 - \lambda_1 - \lambda_2)\},$$

by using an orthonormal basis $\{\lambda_i\}$ of V^3. Since the rank of G_2 is two, we can express this set in the plane as follows.



We first show that ${}^{\#}\Gamma \leq 4$ for $\Gamma \in \Omega_{G_2}$. The roots with length $\sqrt{2}$ constitute a small regular hexagon, and it is easy to see that among them, $\alpha + \beta \notin \Delta_{G_2}$ if and only if $\alpha + \beta = 0$. Hence the number of roots of Γ with length $\sqrt{2}$ is at most two. Similarly, in a large regular hexagon, $\alpha + \beta \notin \Delta_{G_2}$ if and only if either "$\alpha + \beta = 0$" or "α and β are adjacent", which implies that the number of roots of Γ with length $\sqrt{6}$ is also at most two. Therefore, we have ${}^{\#}\Gamma \leq 4$ for $\Gamma \in \Omega_{G_2}$. Now, we take $\Gamma \in \Omega_{G_2}$ such that $\Gamma \neq \emptyset$. Then, since $\dim \boldsymbol{R}\Gamma \geq 1$, we have

$$\begin{aligned} a_2(\Gamma) &= {}^{\#}\Gamma + 2 - \dim \boldsymbol{R}\Gamma \\ &\leq 6 - \dim \boldsymbol{R}\Gamma \\ &\leq 5. \end{aligned}$$

If $a_2(\Gamma) = 5$, we have $^{\#}\Gamma = 4$ and $\dim \boldsymbol{R}\Gamma = 1$. But, in this case Γ is contained in a line, which contradicts $^{\#}\Gamma = 4$. Hence, we have $a_2(\Gamma) \leq 4$. On the other hand, it is easy to see that the set $\Gamma = \{\pm(\lambda_1 - \lambda_2),\ \pm(2\lambda_3 - \lambda_1 - \lambda_2)\}$ is non-additive and $a_2(\Gamma) = 4$. Combining these results, we obtain the equality $p^c_{G_2} = 4$.

REMARK. For the groups $G = E_6, E_7, E_8$ and F_4, the non-additive set Γ with maximum $a_m(\Gamma)$ satisfies $\Gamma \cap (-\Gamma) = \emptyset$, while the group G_2 possesses the non-additive set Γ satisfying $a_2(\Gamma) = 4$ and $\Gamma = -\Gamma$.

§6. Some facts on the values p_G

In this final section, we determine the value p_G for compact Lie groups G with small rank. The results are stated as follows.

PROPOSITION 6.1. *For the groups* $G = U(m)$ $(1 \leq m \leq 5)$, $SU(m)$ $(2 \leq m \leq 5)$, $SO(2m+1)$ $(1 \leq m \leq 4)$, $Sp(m)$ $(1 \leq m \leq 3)$, $SO(2m)$ $(1 \leq m \leq 4)$ *and* G_2, *the value* p_G *is equal to* p^c_G.

To prove this proposition, we first prepare the following lemma.

LEMMA 6.2. *Let G be a compact Lie group. If there exists $\Gamma \in \Omega$ such that $\Gamma = -\Gamma$, then the inequality $p_G \geq a(\Gamma)$ holds. In addition, if $\Gamma \in \Omega$ satisfies $\Gamma = -\Gamma$ and $a(\Gamma) = p^c_G$, then we have $p_G = p^c_G$.*

PROOF. Let τ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. Then, by the definition of roots, we have $\tau\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ for each $\alpha \in \Delta$. Now, assume $\Gamma \in \Omega$ satisfies $\Gamma = -\Gamma$. For $\alpha \in \Gamma \cap \Delta^+$, we put

$$U_\alpha = 1/\sqrt{2}\cdot(Z_\alpha + \tau(Z_\alpha)) \quad \text{and} \quad V_\alpha = \sqrt{-1}/\sqrt{2}\cdot(Z_\alpha - \tau(Z_\alpha)),$$

where Z_α is the basis of $\mathfrak{g}_\alpha$ which we defined in §3. Then, U_α and V_α are real vectors, i.e., $U_\alpha, V_\alpha \in \mathfrak{g}$. Now, using the set Γ, we define a subspace $W_0 \subset \mathfrak{g}$ by

$$W_0 = \sum_{\alpha \in \Gamma \cap \Delta^+} (\boldsymbol{R}U_\alpha + \boldsymbol{R}V_\alpha) + (\boldsymbol{R}\Gamma)^\perp,$$

where $(\boldsymbol{R}\Gamma)^\perp$ implies the orthogonal complement of $\boldsymbol{R}\Gamma$ in $\mathfrak{t}$. Then, we have $[W_0, W_0] \subset \mathfrak{t}$. In fact, since Γ is non-additive, we have $[Z_\alpha, Z_\beta] = [Z_{-\alpha}, Z_{-\beta}] = 0$ for $\alpha, \beta \in \Gamma \cap \Delta^+$, and in addition $[Z_\alpha, Z_{-\beta}] = 0$ $(\alpha \neq \beta)$ because $-\beta \in -\Gamma = \Gamma$. Hence, we have $[U_\alpha, U_\beta] = [U_\alpha, V_\beta] = [V_\alpha, V_\beta] = 0$ for $\alpha, \beta \in \Gamma \cap \Delta^+$ $(\alpha \neq \beta)$, and $[U_\alpha, V_\alpha] = 2/(\alpha, \alpha)\cdot\alpha \in \mathfrak{t}$. Therefore, combining with the equalities $[H, Z_\alpha] = [H, Z_{-\alpha}] = 0$ for $H \in (\boldsymbol{R}\Gamma)^\perp$, $\alpha \in \Gamma \cap \Delta^+$, we have $[W_0, W_0] \subset \mathfrak{t}$. Next, since $\Gamma = -\Gamma$, the complexification of W_0 is equal to

$$\begin{aligned}
&\sum_{\alpha\in\Gamma\cap\Delta^+} \mathfrak{g}_\alpha + \sum_{\alpha\in\Gamma\cap\Delta^+} \mathfrak{g}_{-\alpha} + (\boldsymbol{R}\Gamma)^{\perp c} \\
&= \sum_{\alpha\in\Gamma\cap\Delta^+} \mathfrak{g}_\alpha + \sum_{\alpha\in\Gamma\cap\Delta^-} \mathfrak{g}_\alpha + (\boldsymbol{R}\Gamma)^{\perp c} \\
&= \sum_{\alpha\in\Gamma} \mathfrak{g}_\alpha + (\boldsymbol{R}\Gamma)^{\perp c} \\
&= W^\Gamma.
\end{aligned}$$

Therefore, we have

$$p_G = \max_{W\in\mathcal{N}_G} \dim_{\boldsymbol{R}} W \geq \dim_{\boldsymbol{R}} W_0 = \dim_{\boldsymbol{C}} W^\Gamma = a(\Gamma).$$

The second statement follows immediately from the fact $p_G^c \geq p_G \geq a(\Gamma)$.

q.e.d.

Proof of Proposition 6.1. By Lemma 6.2, we have only to find $\Gamma\in\Omega$ satisfying $\Gamma = -\Gamma$ and $a(\Gamma) = p_G^c$ for each G. First, for the group $U(m)$, we put

$$\begin{aligned}
&\Gamma = \emptyset, && m = 1, \\
&\Gamma = \{\pm(\lambda_1 - \lambda_2)\}, && m = 2, 3, \\
&\Gamma = \{\pm(\lambda_1 - \lambda_2), \pm(\lambda_3 - \lambda_4)\}, && m = 4, 5.
\end{aligned}$$

Then, it is easy to see that the above Γ satisfy the desired conditions. For the group $SU(m)$, the results follow immediately from the equalities $p_{U(m)} = p_{SU(m)} + 1$ and $p_{U(m)}^c = p_{SU(m)}^c + 1$ (cf. Proposition 1.2). The remaining case can be checked in completely the same way, and in the following, we only list up such Γ for each group.

$$\begin{aligned}
&SO(3): \Gamma = \{\pm\lambda_1\} \\
&SO(5): \Gamma = \{\pm\lambda_1 \pm \lambda_2\} \\
&SO(7): \Gamma = \{\pm\lambda_1 \pm \lambda_2, \pm\lambda_3\} \\
&SO(9): \Gamma = \{\pm\lambda_1 \pm \lambda_2, \pm\lambda_3 \pm \lambda_4\} \\
&Sp(1): \Gamma = \{\pm 2\lambda_1\} \\
&Sp(2): \Gamma = \{\pm 2\lambda_1, \pm 2\lambda_2\} \\
&Sp(3): \Gamma = \{\pm 2\lambda_1, \pm 2\lambda_2, \pm 2\lambda_3\} \\
&SO(2): \Gamma = \emptyset \\
&SO(4), SO(6): \Gamma = \{\pm\lambda_1 \pm \lambda_2\}
\end{aligned}$$

$$SO(8): \Gamma = \{\pm\lambda_1 \pm \lambda_2, \pm\lambda_3 \pm \lambda_4\}$$
$$G_2 \quad : \Gamma = \{\pm(\lambda_1 - \lambda_2), \pm(2\lambda_3 - \lambda_1 - \lambda_2)\}. \qquad \text{q.e.d.}$$

REMARK. For the compact classical Lie groups, we showed in §4 that the non-additive set with maximum $a(\Gamma)$ is of type V if the rank is sufficiently large. Then, since $I^{\pm}(\Gamma) = \emptyset$, we have $\Gamma \neq -\Gamma$, and hence, we cannot calculate the value p_G for these groups by only using Lemma 6.2. The same phenomena occur for the groups E_6, E_7, E_8 and F_4. (See Remark at the end of §5.)

Finally, we give some estimate on the value p_G for general compact simple Lie group G. For this purpose, we define integers $s_0(\mathfrak{g})$ for compact simple Lie algebras $\mathfrak{g}$ by

$$s_0(\mathfrak{g}) = \begin{cases} \operatorname{rank}\mathfrak{g} & \mathfrak{g} \neq \mathfrak{su}(m),\ \mathfrak{o}(2m) \text{ or } E_6 \\ [m/2] & \mathfrak{g} = \mathfrak{su}(m) \\ 2[m/2] & \mathfrak{g} = \mathfrak{o}(2m) \\ 4 & \mathfrak{g} = E_6. \end{cases}$$

Then, we have

PROPOSITION 6.3. *Let G be a compact simple Lie group with the Lie algebra $\mathfrak{g}$. Then, we have* $p_G \geq \operatorname{rank}\mathfrak{g} + s_0(\mathfrak{g})$.

PROOF. In Appendix of the paper [3], we constructed a subset $\Gamma_0 = \{\beta_1, \cdots, \beta_{s_0}\} \subset \Delta^+$ satisfying $\beta_i \pm \beta_j \notin \Delta \cup \{0\}$ $(i \neq j)$. Using this set Γ_0, we put $\Gamma = \Gamma_0 \cup (-\Gamma_0) = \{\pm\beta_1, \cdots, \pm\beta_{s_0}\}$. Then, we have clearly $\Gamma \in \Omega$ and $\Gamma = -\Gamma$. In addition, since $(\beta_i, \beta_j) = 0$ $(i \neq j)$, we have $\dim \boldsymbol{R}\Gamma = s_0$. Therefore, $a(\Gamma) = {}^{\#}\Gamma + \operatorname{rank}\mathfrak{g} - \dim \boldsymbol{R}\Gamma = \operatorname{rank}\mathfrak{g} + s_0$, which implies $p_G \geq \operatorname{rank}\mathfrak{g} + s_0$. q.e.d.

For the simple Lie groups listed up in Proposition 6.1 (except for $U(m)$, $SO(2)$, $SO(4)$), the equality $p_G = \operatorname{rank}\mathfrak{g} + s_0(\mathfrak{g})$ actually holds. But at present, we do not know whether the above equality holds for all compact simple Lie groups.

Appendix. Maximum dimensions of abelian subalgebras of complex simple Lie algebras

In this appendix, we refer to the relation between our results (Theorem 3.1 for classical Lie algebras) and the maximum dimensions of abelian subalgebras of complex simple Lie algebras. Our purpose is to determine such dimensions by using the results in §4 and the theorem of Malcev

[10]. We need this result in order to complete the proof of Theorem 3.1 for exceptional Lie algebras (see § 5).

Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$. We denote by $\mathscr{A}(\mathfrak{g}^c)$ the family of abelian subalgebras of $\mathfrak{g}^c$ and by $\mathscr{A}_{nil}(\mathfrak{g}^c)$ the subfamily of $\mathscr{A}(\mathfrak{g}^c)$ consisting of abelian subalgebras all whose elements are nilpotent in $\mathfrak{g}^c$. (For a complex Lie algebra $\mathfrak{l}$, an element $X \in \mathfrak{l}$ is called *nilpotent* (resp. *semi-simple*) if $\mathrm{ad}(X)$ is a nilpotent (resp. semi-simple) endomorphism of $\mathfrak{l}$.) By the very definition, we have

$$\mathscr{A}_{nil}(\mathfrak{g}^c) \subset \mathscr{A}(\mathfrak{g}^c) \subset \mathscr{N}_G^c,$$

where G denotes the adjoint group of $\mathfrak{g}$. Then, by putting

$$a(\mathfrak{g}^c) = \max_{A \in \mathscr{A}(\mathfrak{g}^c)} \dim_C A, \quad a_{nil}(\mathfrak{g}^c) = \max_{A \in \mathscr{A}_{nil}(\mathfrak{g}^c)} \dim_C A,$$

we have clearly

$$(*) \qquad a_{nil}(\mathfrak{g}^c) \leq a(\mathfrak{g}^c) \leq p_G^c.$$

Our purpose in this appendix is to determine the value $a(\mathfrak{g}^c)$ for all compact simple Lie algebras.

Concerning the value $a_{nil}(\mathfrak{g}^c)$, Malcev [10] obtained the following result.

THEOREM A1 (cf. [10]). *Let $\mathfrak{g}$ be a compact simple Lie algebra. Then the integer $a_{nil}(\mathfrak{g}^c)$ is given by*

$\mathfrak{g}$	$a_{nil}(\mathfrak{g}^c)$
$A_m (m \geq 1)$	$[(m+1)^2/4]$
$B_m (m \geq 4)$	$1/2 \cdot m(m-1)+1$
B_3	5
$C_m (m \geq 2)$	$1/2 \cdot m(m+1)$
$D_m (m \geq 4)$	$1/2 \cdot m(m-1)$

$\mathfrak{g}$	$a_{nil}(\mathfrak{g}^c)$
E_6	16
E_7	27
E_8	36
F_4	9
G_2	3

Malcev [10] stated a plan to obtain the integer $a(\mathfrak{g}^c)$ on the basis of the above theorem. However, details were not shown there.

In the following, we prove the following theorem.

THEOREM A2. *Let $\mathfrak{g}$ be a compact simple Lie algebra. Then the equality $a(\mathfrak{g}^c) = a_{nil}(\mathfrak{g}^c)$ holds.*

First, we note that Theorem A2 holds for the following classical compact simple Lie algebras of large rank:

$$A_m (m \geq 4), \ B_m (m \geq 5), \ C_m (m \geq 3), \ D_m (m \geq 5).$$

In fact, comparing the results in Theorem 3.1 and Theorem A1, we can observe that the equality $p_G^c = a_{nil}(\mathfrak{g}^c)$ holds for each $\mathfrak{g}$ stated above. Therefore, on account of the inequlity $(*)$, we have $a(\mathfrak{g}^c) = a_{nil}(\mathfrak{g}^c)$.

To complete the proof of Theorem A2, we prepare the following two lemmas.

LEMMA A3. *Let $\mathfrak{l}$ be a complex semi-simple Lie algebra and H a non-zero semi-simple element of $\mathfrak{l}$. Let $\mathfrak{l}'$ denote the centralizer of H in $\mathfrak{l}$, i.e., $\mathfrak{l}' = \{X \in \mathfrak{l} \mid [H, X] = 0\}$. Then:*

(1) *$\mathfrak{l}'$ is a reductive Lie algebra, i.e., the radical of $\mathfrak{l}'$ is congruent with the center $\mathfrak{c}$ of $\mathfrak{l}'$. Consequently, the derived ideal $\mathfrak{l}'' = [\mathfrak{l}', \mathfrak{l}']$ is a complex semi-simple Lie algebra and $\mathfrak{l}'$ can be expressed as $\mathfrak{l}' = \mathfrak{c} + \mathfrak{l}''$ (direct sum).*

(2) $\operatorname{rank} \mathfrak{l} = \operatorname{rank} \mathfrak{l}' = \dim_{\boldsymbol{C}} \mathfrak{c} + \operatorname{rank} \mathfrak{l}''$.

(3) *$\mathfrak{l}''$ is a regular semi-simple subalgebra of $\mathfrak{l}$.*

For the definition of "regular subalgebra", see Dynkin [5], where all the regular semi-simple Lie subalgebras were completely determined. The proof of Lemma A3 is easy, and hence it is left to the readers.

LEMMA A4. *Let $\mathfrak{l}$ be a complex semi-simple Lie algebra with* $\operatorname{rank} \mathfrak{l} = n$. *Let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{l}$. Then it holds*

$$(**) \qquad \dim_{\boldsymbol{C}} \mathfrak{a} \leq 1/2 \cdot n(n+1).$$

In addition, if $\mathfrak{a}$ contains a non-nilpotent element, it holds

$$(***) \qquad \dim_{\boldsymbol{C}} \mathfrak{a} \leq 1/2 \cdot n(n-1) + 1.$$

PROOF. We prove the lemma by induction on n. In the case $n = 1$, $\mathfrak{l}$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$. As is easily seen, the dimension of any abelian subalgebra of $\mathfrak{sl}(2, \boldsymbol{C})$ is at most 1. Hence, the lemma holds in the case $n = 1$.

Now, we assume that the lemma holds in case $\operatorname{rank} \mathfrak{l} < n$ $(n \geq 2)$. We first consider the case where $\mathfrak{l}$ is expressed as a direct sum of two proper semi-simple ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$. Then, there are abelian subalgebras $\mathfrak{a}_1 \subset \mathfrak{l}_1$ and $\mathfrak{a}_2 \subset \mathfrak{l}_2$ such that $\mathfrak{a} \subset \mathfrak{a}_1 + \mathfrak{a}_2$. We put $n_i = \operatorname{rank} \mathfrak{l}_i$ $(i = 1, 2)$. Then, we have $n = n_1 + n_2$ and $n_i < n$. Hence, by the induction hypothesis, we have

$$\begin{aligned} \dim_{\boldsymbol{C}} \mathfrak{a} &\leq \dim_{\boldsymbol{C}} \mathfrak{a}_1 + \dim_{\boldsymbol{C}} \mathfrak{a}_2 \\ &\leq 1/2 \cdot n_1(n_1+1) + 1/2 \cdot n_2(n_2+1) \\ &< 1/2 \cdot n(n+1). \end{aligned}$$

Moreover, in case $\mathfrak{a}$ contains a non-nilpotent element, either $\mathfrak{a}_1$ or $\mathfrak{a}_2$ also contains a non-nilpotent element. Assume that $\mathfrak{a}_1$ contains a non-nilpotent element of $\mathfrak{l}_1$. Then, by the induction hypothesis, we have

$$\begin{aligned}\dim_{\boldsymbol{C}} \mathfrak{a} &\leq \dim_{\boldsymbol{C}} \mathfrak{a}_1 + \dim_{\boldsymbol{C}} \mathfrak{a}_2 \\ &\leq 1/2 \cdot n_1(n_1 - 1) + 1 + 1/2 \cdot n_2(n_2 + 1) \\ &\leq 1/2 \cdot n(n - 1) + 1.\end{aligned}$$

We next consider the case where $\mathfrak{l}$ is simple. Then, by Malcev [10], we have the following two possibilities:

(i) All element of $\mathfrak{a}$ are nilpotent;
(ii) $\mathfrak{a}$ contains a non-zero semi-simple element.

In view of Theorem A1, we can easily observe that the inequality $a_{nil}(\mathfrak{g}^c) \leq 1/2 \cdot n(n+1)$ holds for each compact simple Lie algebra $\mathfrak{g}$ with $\operatorname{rank} \mathfrak{g} = n$. Therefore, in the case (i), we have $\dim_{\boldsymbol{C}} \mathfrak{a} \leq 1/2 \cdot n(n+1)$.

Now, we consider the case (ii). Let $H \in \mathfrak{a}$ be a non-zero semi-simple element of $\mathfrak{l}$. We denote by $\mathfrak{l}'$ the centralizer of H in $\mathfrak{l}$. Let $\mathfrak{c}$ (resp. $\mathfrak{l}''$) be the center of $\mathfrak{l}'$ (resp. the derived ideal of $\mathfrak{l}'$). Then, by Lemma A3, it follows that $\mathfrak{l}''$ is semi-simple; $\mathfrak{l}' = \mathfrak{c} + \mathfrak{l}''$ (direct sum); and $n = \operatorname{rank} \mathfrak{l} = \dim_{\boldsymbol{C}} \mathfrak{c} + \operatorname{rank} \mathfrak{l}''$. Since $\mathfrak{a} \subset \mathfrak{l}'$, there is an abelian subalgebra $\mathfrak{a}''$ in $\mathfrak{l}''$ such that $\mathfrak{a} \subset \mathfrak{c} + \mathfrak{a}''$. Put $k = \operatorname{rank} \mathfrak{l}''$. Then, since $H \in \mathfrak{c}$, we have $k = n - \dim_{\boldsymbol{C}} \mathfrak{c} \leq n - 1$. Therefore, by the induction hypothesis, we have $\dim_{\boldsymbol{C}} \mathfrak{a}'' \leq 1/2 \cdot k(k+1)$. Consequently, we have

$$\begin{aligned}\dim_{\boldsymbol{C}} \mathfrak{a} &\leq \dim_{\boldsymbol{C}} \mathfrak{c} + \dim_{\boldsymbol{C}} \mathfrak{a}'' \\ &\leq n - k + 1/2 \cdot k(k+1) \\ &= 1/2 \cdot k(k-1) + n.\end{aligned}$$

Since the last expression takes its maximum value in the case $k = n - 1$, we have $\dim_{\boldsymbol{C}} \mathfrak{a} \leq 1/2 \cdot (n-1)(n-2) + n = 1/2 \cdot n(n-1) + 1$. This completes the proof of the lemma. q.e.d.

REMARK. Viewing the proof of Lemma A4, we can easily verify that if the equality holds in (**), then $\mathfrak{l}$ is a complex simple Lie algebra.

Now, using Lemma A4, we prove Theorem A2 for the remaining simple Lie algebras of small rank:

$$A_m (m = 1, 2, 3),\ B_m (m = 2, 3, 4),\ D_4,\ E_m (m = 6, 7, 8),\ F_4,\ G_2.$$

Let $\mathfrak{g}$ be one of the compact simple Lie algebras listed above. Put $m = \operatorname{rank} \mathfrak{g}$. Then, if $\mathfrak{g}$ is not of type D_4, we can easily check that the inequality $a_{nil}(\mathfrak{g}^c) \geq 1/2 \cdot m(m-1) + 1$ holds. Therefore, we have $\dim_{\boldsymbol{C}} \mathfrak{a} \leq a_{nil}(\mathfrak{g}^c)$ for all $\mathfrak{a} \in \mathscr{A}(\mathfrak{g}^c)$ (see Lemma A4), which implies that $a(\mathfrak{g}^c) = a_{nil}(\mathfrak{g}^c)$.

Finally, we assume that $\mathfrak{g}$ is of type D_4. Then, by Theorem A1, we have

$a_{nil}(\mathfrak{g}^c) = 6$. We now suppose that there exists an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}^c$ with $\dim_{\boldsymbol{C}} \mathfrak{a} > 6$. By the assumption, we may assume that $\mathfrak{a}$ contains a non-zero semi-simple element H of $\mathfrak{g}^c$. We denote by $\mathfrak{l}'$ the centralizer of H in $\mathfrak{g}^c$ and by $\mathfrak{l}''$ the derived ideal of $\mathfrak{l}'$. Applying the inequality $(***)$ in Lemma A4, we have $\dim_{\boldsymbol{C}} \mathfrak{a} \leq 7$ and hence $\dim_{\boldsymbol{C}} \mathfrak{a} = 7$. This implies that the equality holds in $(***)$. Thus, in view of the proof of Lemma A4, we can verify that $\operatorname{rank} \mathfrak{l}'' = 3$ and that $\mathfrak{l}''$ contains an abelian subalgebra $\mathfrak{a}''$ with $\dim_{\boldsymbol{C}} \mathfrak{a}'' = 6$. Taking account of Remark after Lemma A4, we can conclude that $\mathfrak{l}''$ is a regular simple subalgebra of $\mathfrak{g}^c$ with $\operatorname{rank} \mathfrak{l}'' = 3$. By the result of Dynkin [5], it follows that $\mathfrak{l}''$ is of type A_3. On the other hand, as we have proved in the above discussion, the complex simple Lie algebra of type A_3 does not contain any abelian subalgebra whose dimension is greater than 4. This is a contradiction. Thus we have $a(\mathfrak{g}^c) = 6$, which completes the proof of Theorem A2.

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