# Conformal mapping of geodesically slit tori
# and an application to
# the evaluation of the hyperbolic span

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## Introduction

Throughout this paper an open Riemann surface of genus one is called an *open torus*; it will be called a *geodesically slit torus* if it arises from a *symmetric* torus by removing a *single* segment lying on the axis of symmetry. For the precise definitions, see Section 1.

Our aim is to find a conformal mapping of a geodesically slit torus onto another and to study the spans of such an open torus. More specifically, we consider two geodesically slit tori; one has a slit along a geodesic homotopic to a longitude and the other has a slit along a geodesic homotopic to a meridian. Our first problem is then to give criteria for such slit tori to be conformally equivalent. A conformal mapping between them, if any, will be constructed by means of Jacobi's elliptic functions. The method employed here is thus classical and the idea is also simple, but we would like to remind the reader of the fact that the corresponding classical problem of studying conformal mapping between a horizontal slit rectangle and a vertical slit rectangle is more difficult (cf. [5]). Slit tori can be sometimes dealt with more easily than slit rectangles. The reason is that in the case of tori we can normalize the slit so as to lie over the boundary of a fundamental region.

The second problem is to evaluate the hyperbolic span of a geodesically slit torus. We also find some estimates of the euclidean span of general open tori. These spans have been introduced in [8] and [9], as generalizations of the Schiffer span of plane domains.

We see that the hyperbolic span of a geodesically slit torus is exactly expressed by means of the complete elliptic integrals of the first kind. Some numerical examples will be also given. We finally observe that the cutting and pasting method yields all the tori from a single torus.

We remark that a theoretical treatment for plane rectangles was established by Jenkins ([3]), and its counterpart for open tori has been given in [8] and [9]. We also note that in [9] we define another span, the spherical span, of an open torus, and investigate the relation of these —euclidean, hyperbolic,

and spherical— spans to various extremal problems of the complementary area for general open tori.

## 1. Preliminaries

Let $R$ be an open Riemann surface of genus one with a fixed canonical homology basis $\chi = \{a, b\}$ modulo dividing cycles. We may and do consider $\chi$ a set of generators of the fundamental group of the Kerékjártó-Stoïlow compactification of $R$. For simplicity, we call $R$ an *open torus* and the pair $(R, \chi)$ a *marked open torus*. We keep on the other hand the routine terms a *torus* and a *marked torus* to mean a closed Riemann surface of genus one and such a surface with a marking—that is, with a fixed set of generators of the fundamental group, respectively. To avoid ambiguity we sometimes use the terms a *closed torus* and a *marked closed torus* for the same objects.

A *realization* of $(R, \chi)$ is, by definition, a triple $(R', \chi', I')$, where $R'$ is a closed torus, $\chi' = \{a', b'\}$ a set of generators of the fundamental group of $R'$, and $I'$ a conformal mapping of $R$ into $R'$ such that $I'(a)$ and $I'(b)$ are homotopic to $a'$ and $b'$ respectively. Two realizations $(R', \chi', I')$ and $(R'', \chi'', I'')$ of the marked open torus $(R, \chi)$ are said to be *equivalent*, if there is a conformal mapping $f$ of $R'$ onto $R''$ such that $f \circ I' = I''$. Equivalence classes are called *compact continuations* of $(R, \chi)$, and the compact continuation of $(R, \chi)$ determined by a realization $(R', \chi', I')$ is denoted by $[R', \chi', I']$. If $[R', \chi', I'] = [R'', \chi'', I'']$, two marked tori $(R', \chi')$ and $(R'', \chi'')$ define the same point in the Teichmüller space of genus one and have the same modulus, which we call the *modulus of the compact continuation* $[R', \chi', I']$. We know ([8], Theorem 5) that the set $M(R, \chi)$ of moduli of the compact continuations of $(R, \chi)$ is a closed disk in the upper half plane. The diameter of $M(R, \chi)$ gives an analogue of Schiffer's span of planar domains (cf. [8], Theorem 7 and the subsequent remark), which depends, however, not only on $R$ but also on $\chi$.

The upper half plane carries the Poincaré metric, and the set $M(R, \chi)$ is again a closed disk with respect to this metric. Hence it makes sense to speak of the hyperbolic diameter of $M(R, \chi)$. It is, contrary to the case of euclidean metric, invariant under any change of canonical homology bases of $R$ modulo dividing cycles. Hence, we have the following definition (cf. [9]).

DEFINITION 1. The euclidean diameter of $M(R, \chi)$ is called the *euclidean span* of the marked open torus $(R, \chi)$ and the hyperbolic diameter of the disk $M(R, \chi)$ is called the *hyperbolic span* of the open torus $R$. They are denoted by $\sigma_E(R, \chi)$ and by $\sigma_H(R)$ respectively.

It is apparent that the hyperbolic span is more intrinsic and convenient than the euclidean. In the problem to evaluate the hyperbolic span, for example, we may change the marking so that the new marking enables us to solve the problem. In the present paper we show that the hyperbolic span is given in closed form for some special open tori. The class of open tori with which we are concerned in this paper will be described in the definitions below.

DEFINITION 2. A realization $(R', \chi', I')$ of $(R, \chi)$ is called a *strongly symmetric torus*, if $R'$ admits an anticonformal automorphism $j$ of order 2 with $j(\partial_{R'}R) = \partial_{R'}R$. Here, $\partial_R{}'R$ stands for the relative boundary of $I'(R)$ in $R'$.

Note that $j$ does not always fix $\partial_R{}'R$ pointwise. It is obvious that a strongly symmetric torus is symmetric in the ordinary sense; it admits an anticonformal automorphism of order two. The converse is not true, however. We recall (see, e.g. [2], Chapter 6) that a closed marked torus with modulus $\tau$ is symmetric if and only if $J(\tau)$ is real, where $J$ is the modular function. This condition is also equivalent to the existence of a unimodular matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\mathrm{Re}\left(\dfrac{\alpha\tau + \beta}{\gamma\tau + \delta}\right)$ is either 0 or $\dfrac{1}{2}$ (cf. [2] or [4], for instance).

We furthermore need the following definition.

DEFINITION 3. A realization $(R', \chi', I')$ of $(R, \chi)$ is called a *geodesically slit torus* if each component of the set $\Gamma := R' \setminus I'(R)$ is a geodesic Jordan arc on $R'$. The geodesically slit torus $(R', \chi', I')$ is called a *longitudinally* (resp. *meridionally*) *slit torus*, if the geodesics on which $\Gamma$ lie are homotopic to $a'$ (resp. $b'$) on $R'$, where $\{a', b'\} = \chi'$. An open torus $R$ is said to *admit a longitudinally* (resp. *meridionally*) *slit realization*, if there exists a canonical homology basis $\chi$ of $R$ modulo dividing cycles such that one of the realizations of $(R, \chi)$ is a longitudinally (resp. meridionally) slit torus.

As in the classical case of the Riemann mapping theorem for plane domains, we assume that $R$ has only one boundary component. For simplicity we introduce the following definition.

DEFINITION 4. An open torus is called a *once holed torus* if it has only one (Kerékjártó-Stoïlow) boundary component.

Our purpose is to study the hyperbolic span of an open torus $R$. Since there is no *a priori* definite measurement of a general open torus, we are forced to start with one of its realizations. To be more precise, suppose that a once holed torus $R$ admits a strongly symmetric longitudinally slit realization

$(R', \chi', I')$. Let $\chi$ be the corresponding marking of $R$. Then there are a complex number $\tau$ and a group $G$ of rank 2 generated by the translations

$$z \mapsto z + 1 \quad \text{and} \quad z \mapsto z + \tau$$

such that $R'$ is obtained as the quotient space $\mathbf{C}/G$. Because of the strong symmetry of $(R', \chi', I')$ we see that $\mathrm{Re}\left(\dfrac{\alpha\tau + \beta}{\gamma\tau + \delta}\right)$ is either 0 or $\dfrac{1}{2}$ for some unimodular matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

  *In this paper we are concerned with only the first case.* (The other case, the case of rhombi, would be dealt with similarly. Indeed, we can make use of the cn-function instead of the sn-function.) The relative boundary $\partial_{R'} R$ lies on the axis of symmetry, which is, by our assumption, homotopic to the loop $a'$. On the other hand, $a'$ can be taken as a geodesic. Consequently, via a parallel translation, we may assume that the slit lies on the loop $a'$. Changing the marking $\chi$ if necessary —which has no influence on the hyperbolic span of $R$—, we may also suppose that $\tau$ is purely imaginary. In other words, $(R', \chi', I')$ arises from a rectangle with a horizontal slit. In the next section we will find a conformal mapping of such a longitudinally slit torus onto a meridionally slit torus in closed form.

  NOTATION. A marked closed torus with modulus $\tau$ is denoted by $(T(\tau), \{a, b\})$. It is also denoted simply by $T(\tau)$, since a marking is tacitly considered whenever we speak of the modulus. A (once holed) strongly symmetric longitudinally (resp. meridionally) slit torus is denoted by $T_L(\tau, \ell)$ (resp. $T_M(\tau, \ell)$), if the modulus is $\tau$ and the length of the longitudinal (resp. meridional) slit $\Gamma$ is $\ell$. In particular, $T_L(\tau, 0) = T_M(\tau, 0)$, and they are obtained from $T(\tau)$ by removing a point. We often use the same notation $T_L(\tau, \ell)$ or $T_M(\tau, \ell)$ to denote a realization of an open torus $R$, if there is no fear of confusion.

## 2.  Conformal mapping of a longitudinally slit torus onto a meridionally slit torus

  Let $T_L(\tau_0, \ell_0) = T(\tau_0) \setminus \Gamma_0$ be a (strongly symmetric, once holed) longitudinally slit torus with $0 \le \ell_0 < 1$. In this section we try to find a meridionally slit torus $T_M(\tau_1, \ell_1) = T(\tau_1) \setminus \Gamma_1$ which is another realization of $T_L(\tau_0, \ell_0)$ and to determine the modulus $\tau_1$ and the slit length $\ell_1$ explicitly. The geodesically slit tori $T_L(\tau_0, \ell_0)$ and $T_M(\tau_1, \ell_1)$ are, as (abstract) Riemann surfaces, completely the same, which we denote by $R$. The set $\chi$ of generators of the fundamental group of $T(\tau_0)$ induces a canonical homology basis of $R$ [modulo dividing cycles], so that the pair $(R, \chi)$ is a marked open torus. The tori

$T(\tau_0)$ and $T(\tau_1)$ determine compact continuations of $(R, \chi)$, and $\operatorname{Im}\tau_0$ and $\operatorname{Im}\tau_1$ are respectively the minimum and the maximum in the set $\{t\in\mathbf{R}\,|\,t=\operatorname{Im}\tau$ for some $\tau\in M(R, \chi)\}$. Hence, by the corollary to Proposition 1 in [8], $\tau_1$ is also purely imaginary, so that the spans can be easily found. In fact the euclidean span $\sigma_E(R, \chi)$ is given by $|\tau_1-\tau_0|=-i(\tau_1-\tau_0)$ and the hyperbolic span $\sigma_H(R)$ is given by $\log(\tau_1/\tau_0)$.

Let $G$ be the group described in Section 1. The open torus $T_L(\tau_0, \ell_0)$ is represented as the quotient space $(\mathbf{C}\setminus\bigcup_{m, n\in\mathbf{Z}}[m+n\tau_0, m+n\tau_0+\ell_0])/G$, where $[t_1, t_2]$ stands for the closed segment joining $t_1$ and $t_2$. Let $P_0$ be the fundamental region for the group $G$ defined by

$$\{z\in\mathbf{C}\,|\,0\le\operatorname{Re}z<1,\ 0\le\operatorname{Im}z<\operatorname{Im}\tau_0\}.$$

## 2.1. We start with

$$(1) \qquad\qquad q_0=\exp(2\pi i\tau_0),$$

and consider the Jacobi theta functions $\theta_2(v, q_0)$ and $\theta_3(v, q_0)$ with the parameter $q_0$ (cf. [1], pp. 223–280, for example):

$$\theta_2(v, q_0)=q_0^{1/4}(e^{\pi iv}+e^{-\pi iv})\prod_{n=1}^{\infty}(1-q_0^{2n})(1+q_0^{2n}e^{2\pi iv})(1+q_0^{2n}e^{-2\pi iv})$$

$$\theta_3(v, q_0)=\prod_{n=1}^{\infty}(1-q_0^{2n})(1+q_0^{2n-1}e^{2\pi iv})(1+q_0^{2n-1}e^{-2\pi iv}).$$

The other two, $\theta_1(v, q_0)$ and $\theta_4(v, q_0)$, will not be explicitly used in the sequel. It should be noted that $q_0$ is not defined to be $\exp(\pi i\tau_0)$. We finally set

$$(2) \qquad\qquad k_0=\left(\frac{\theta_2(0, q_0)}{\theta_3(0, q_0)}\right)^2,\qquad k_0'=\sqrt{1-k_0^2}$$

and

$$(3) \qquad\qquad K_0=\frac{\pi}{2}\theta_3^2(0, q_0),\qquad K_0'=-2iK_0\tau_0.$$

Then, as is well known, $0<k_0, k_0'<1$ and obviously $iK_0'/K_0=2\tau_0$.

The function $\zeta=4K_0z$ maps the rectangle $P_0$ onto

$$\Pi_0\colon 0\le\operatorname{Re}\zeta<4K_0,\quad 0\le\operatorname{Im}\zeta<2K_0'.$$

We may now assume that $\Gamma$ is realized on $\Pi_0$ as the horizontal segment

$$S_0\colon K_0-2K_0\ell_0\le\operatorname{Re}\zeta\le K_0+2K_0\ell_0,\quad\operatorname{Im}\zeta=0.$$

If this is the case, $\partial_{T(\tau_0)}R$ consists of two edges of the segment $\Gamma$. We apply the Jacobi sn-function $x=\operatorname{sn}(\zeta, k_0)$ to map $\Pi_0$ onto the two-sheeted covering surface

$$F_0 \colon X^2 = (x^2 - 1)\left(x^2 - \frac{1}{k_0^2}\right)$$

over the $x$-plane. Then $\partial_{T(\tau_0)} R$ is realized as the doubly traced slit on $F_0$ over the horizontal segment

$$\Sigma_0 \colon \lambda_0 \leq \operatorname{Re} x \leq 1, \quad \operatorname{Im} x = 0$$

with $-1 < \lambda_0 \leq 1$, where $\lambda_0$ is determined by

(4) $$\lambda_0 = \operatorname{sn}(K_0 \pm 2K_0 \ell_0 \,; k_0).$$

**2.2.** There are a unique Möbius transformation $y = S(x)$ and a real number $k_1$, $0 < k_1 < 1$, such that $S$ maps the four points $x = -\dfrac{1}{k_0}, -1, \lambda_0, \dfrac{1}{k_0}$ to the four points $X = -\dfrac{1}{k_1}, -1, 1, \dfrac{1}{k_1}$ in this order. If we set $\lambda_1 = S(1)$, then it is obvious that $\lambda_1$ is a real number with $1 \leq \lambda_1 < \dfrac{1}{k_1}$. After simple computations we have

(5) $$k_1 = k_0 \frac{\sqrt{\dfrac{1 - k_0^2 \lambda_0^2}{1 - k_0^2}} + \lambda_0}{\sqrt{\dfrac{1 - k_0^2 \lambda_0^2}{1 - k_0^2}} + 1}$$

and

(6) $$\lambda_1 = \frac{\sqrt{\dfrac{1 - k_0^2 \lambda_0^2}{1 - k_0^2}} + \dfrac{1 - \lambda_0}{2}}{\sqrt{\dfrac{1 - k_0^2 \lambda_0^2}{1 - k_0^2}} - \dfrac{1 - \lambda_0}{2}}.$$

**2.3.** Now we can use an argument similar to the above (but in the opposite direction) to obtain a new two-sheeted covering surface

$$F_1 \colon Y^2 = (y^2 - 1)\left(y^2 - \frac{1}{k_1^2}\right)$$

with the doubly traced slit over the horizontal segment

$$\Sigma_1 \colon 1 \leq \operatorname{Re} y \leq \lambda_1, \quad \operatorname{Im} y = 0.$$

This covering surface produces a meridionally slit torus. It is conformally equivalent to the longitudinally slit torus with which we have started. To know the modulus $\tau_1$ and the slit lingth $\ell_1$ of the meridionally slit torus, we

first set

(7) $$k_1' = \sqrt{1 - k_1^2}$$

as usual and let $K_1 = K(k_1^2)$ and $K_1' = K'(k_1'^2)$ be the complete elliptic integrals of the first kind with the parameter $k_1^2$ and $k_1'^2$, respectively (cf. [1]):

(8) $$K_1 = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k_1^2 t^2)}}, \qquad K_1' = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k_1'^2 t^2)}}.$$

(Note that the notation $K(k^2)$ is different from the one employed in [6], where $K(k^2)$ is denoted by $K(k)$.) Then we can cut the surface $F_1$ and choose a single-valued branch of the elliptic integral of the first kind

$$\xi = F(y) := \int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - k_1^2 t^2)}}$$

on the cut surface so that it yields a conformal mapping of the resulting slit surface onto the rectangle

$$\Pi_1 : 0 \le \operatorname{Re} \xi \le 4K_1, \quad 0 \le \operatorname{Im} \xi \le 2K_1'$$

with the vertical slit

$$S_1 : \operatorname{Re} \xi = K_1, \quad -2K_1 \ell_1 \le \operatorname{Im} \xi \le 2K_1 \ell_1,$$

where $\ell_1$ is determined by

(9)
$$\ell_1 = \frac{i}{2k_1}\left( K_1 - \int_0^{\lambda_1} \frac{dt}{\sqrt{(1 - t^2)(1 - k_1^2 t^2)}} \right)$$
$$= \frac{1}{2K_1} \int_1^{\lambda_1} \frac{dt}{\sqrt{(t^2 - 1)(1 - k_1^2 t^2)}}.$$

It is easily seen that $0 < \ell_1 < \dfrac{K_1'}{2K_1}$. The mapping $\xi \mapsto w = \dfrac{1}{4K_1} \xi$ maps the rectangle $\Pi_1$ onto the rectangle

$$P_1 : 0 \le \operatorname{Re} w < 1, \quad 0 \le \operatorname{Im} w < \operatorname{Im} \tau_1$$

with

(10) $$\tau_1 := i \frac{K_1'}{2K_1}.$$

The slit $S_1$ corresponds to a vertical slit on $P_1$ of length $\ell_1$.

Summing up, we have

THEOREM 1.  *The longitudinally slit torus $T_L(\tau_0, \ell_0)$ is mapped conformally onto a meridionally slit torus $T_M(\tau_1, \ell_1)$, where $\tau_1$ and $\ell_1$ are uniquely determined by $\tau_0$ and $\ell_0$ through (1)–(10).*

## 3. New parameters

Now we introduce another parameter

$$(11) \qquad \mu_0 := \sqrt{\frac{1 - k_0^2 \lambda_0^2}{1 - k_0^2}}.$$

We then have

$$(12) \qquad k_1 = k_0 \frac{\mu_0 + \lambda_0}{\mu_0 + 1}$$

and

$$(13) \qquad \lambda_1 = \frac{\mu_0 + \dfrac{1 - \lambda_0}{2}}{\mu_0 - \dfrac{1 - \lambda_0}{2}}.$$

It is apparent that $0 < k_1 \le k_0$.  The transformation $S$ is given by

$$y = S(x) = \frac{(\mu_0 + 1)x + (\mu_0 - \lambda_0)}{(\mu_0 - 1)x + (\mu_0 + \lambda_0)}.$$

Setting

$$(11') \qquad \mu_1 := \sqrt{\frac{1 - k_1^2 \lambda_1^2}{1 - k_1^2}},$$

we see from (12) and (13)

$$(14) \qquad \mu_1 = \frac{\dfrac{1 + \lambda_0}{2}}{\mu_0 - \dfrac{1 - \lambda_0}{2}}.$$

We easily see

$$(15) \qquad 4\mu_0\mu_1 = (1 + \lambda_0)(1 + \lambda_1)$$

and

(16)
$$\frac{1 - \lambda_1}{1 - \mu_1} = \frac{1 - \lambda_0}{1 - \mu_0}.$$

**3.1** The procedure (7)–(10) is, roughly speaking, the reverse of (1)–(4). If we rewrite (7)–(10) in a way similar to (1)–(4), we shall easily be aware of the fact that the only distinguishable difference occurs between equation (4) and its counterpart. Namely, we had only to replace (4) with

(4')
$$\lambda_1 = \mathrm{sn}\,(K_1 \pm 2iK_1\ell_1\,;\,k_1).$$

The other equations (1)–(3) remain valid if we replace the suffix 0 with 1. Hence, we obtain the parameters $(\lambda_0, \mu_0)$ and $(\lambda_1, \mu_1)$ for (strongly symmetric, once holed) longitudinally and meridionally slit tori respectively.

To deal with $T_L(\tau_0, \ell_0)$ and $T_M(\tau_1, \ell_1)$ simultaneously, we denote by $T_X$ one of them; if $X = L$, it is a longitudinally slit torus, and if $X = M$, it is a meridionally slit torus. We set

(17)
$$\lambda(T_X) := \begin{cases} \mathrm{sn}\,(K_0 \pm 2K_0\ell_0\,;\,k_0) & \text{if } X = L \\ \mathrm{sn}\,(K_1 \pm 2iK_1\ell_1\,;\,k_1) & \text{if } X = M \end{cases}$$

and

(18)
$$\mu(T_X) := \sqrt{\frac{1 - k(T_X)^2 \lambda(T_X)^2}{1 - k(T_X)^2}},$$

where $k(T_X)$ is equal to $k_0$ if $X = L$ and $k_1$ if $X = M$. We know that

$$-1 < \lambda(T_X) < \infty, \qquad 0 < \mu(T_X) < \infty$$

and the slit reduces to a point if and only if $\lambda(T_X) = 1$, or equivalently, $\mu(T_X) = 1$.

It is apparent that $(\tau, \ell)$ determines $(\lambda, \mu)$ uniquely and vice versa. Indeed, equations (7)–(10) show how to determine $(\tau, \ell)$ from $(\lambda, \mu)$. It should be noted here that $\ell_0$ and $\ell_1$ are determined by different formulae:

(19)
$$\begin{cases} \ell_0 = \dfrac{1}{2K_0} \displaystyle\int_{\lambda_0}^{1} \frac{dt}{\sqrt{(1 - t^2)(1 - k_0^2 t^2)}} \\[3mm] \ell_1 = \dfrac{1}{2K_1} \displaystyle\int_{1}^{\lambda_1} \frac{dt}{\sqrt{(1 - t^2)(1 - k_1^2 t^2)}} \end{cases}.$$

Hence, we can use $(\lambda, \mu)$ instead of $(\tau, \ell)$. It is more convenient to introduce the parameter

$$\nu := \frac{1 + \lambda}{2}.$$

Then the parameter domain is precisely $\mathscr{L} \cup \mathscr{M} \cup \mathscr{N}$, where $\mathscr{L} := \{0 < v < 1,$ $\mu > 1\}$, $\mathscr{M} := \{v > 1, 0 < \mu < 1\}$ and $\mathscr{N} := \{v = \mu = 1\}$, respectively. The longitudinally slit tori are parametrized by $(v, \mu) \in \mathscr{L}$ and the meridionally slit tori are parametrized by $(v, \mu) \in \mathscr{M}$. The set $\mathscr{N}$ corresponds to the degenerate cases. (Note that $\mathscr{N}$ represents the family of once punctured tori — not a single open torus! Cf. the remark below.) Now the definition of $T[v, \mu]$ will be self-explanatory; for example, if $(v, \mu) \in \mathscr{L}$, it is the longitudinally slit torus with parameter value $(v, \mu)$. This is one of the advantages of using the new parameters $(v, \mu)$.

We now consider the two functions

$$\Phi(v, \mu) := \frac{\mu}{v} \quad \text{and} \quad \Psi(v, \mu) := \frac{1 - v}{1 - \mu}$$

of two real variables $v$ and $\mu$ with $v, \mu > 0$. Then by (15) and (16) we have

THEOREM 1'. *Let $T[v', \mu']$ and $T[v'', \mu'']$ be two geodesically slit tori with nondegenerate slits. Then they are conformally equivalent if and only if*

$$\Phi(v', \mu') \cdot \Phi(v'', \mu'') = 1$$

*and*

$$\Psi(v', \mu') = \Psi(v'', \mu'')$$

*hold.*

REMARK. As we have noticed earlier (see the end of Section 1), the excluded case with degenerate slits can easily be dealt with. Indeed, if this is the case, $T[v', \mu']$ and $T[v'', \mu'']$ are the same torus. Note that $v' = v'' = \mu' = \mu'' = 1$ and $k_0 = k_1$.

## 4. The hyperbolic span of a geodesically slit torus

We are now ready to give the hyperbolic span of the longitudinally slit torus $T[v, \mu]$ in closed form.

Recall that $K(m)$ denotes the complete elliptic integral of the first kind with parameter

$$m := k^2$$

(see [1]). From equations (11), (12) and (7) we have

$$(20) \qquad k_0^2 = \frac{1 - \mu_0^2}{\lambda_0^2 - \mu_0^2} = \frac{1 - \mu_0}{\lambda_0 - \mu_0} \cdot \frac{1 + \mu_0}{\lambda_0 + \mu_0}.$$

Hence we have

$$(21) \qquad k_0'^2 = 1 - k_0^2 = \frac{1 - \lambda_0}{\mu_0 - \lambda_0} \cdot \frac{1 + \lambda_0}{\mu_0 + \lambda_0},$$

$$(22) \qquad k_1^2 = k_0^2 \cdot \left( \frac{\mu_0 + \lambda_0}{\mu_0 + 1} \right)^2 = \frac{1 - \mu_0}{1 + \mu_0} \cdot \frac{\lambda_0 + \mu_0}{\lambda_0 - \mu_0},$$

and

$$(23) \qquad k_1'^2 = 1 - k_1^2 = \frac{2\mu_0}{1 + \mu_0} \cdot \frac{\lambda_0 - 1}{\lambda_0 - \mu_0}.$$

Introducing auxiliary functions

$$\xi(x, y) := \frac{1 - y}{x - y}$$

and

$$\Xi(x, y) := \frac{K\left( \dfrac{\xi(y, x)}{\xi(y, -y)} \right) \cdot K(\xi(x, y)\xi(x, -y))}{K\left( \dfrac{\xi(x, y)}{\xi(x, -y)} \right) \cdot K(\xi(y, x)\xi(y, -x))},$$

we have the following theorem.

THEOREM 2. *If a once holed torus R admits a strongly symmetric longitudinally (resp. meridionally) slit realization $T_L(\lambda_0, \mu_0)$ (resp. $T_M(\lambda_1, \mu_1)$), then the hyperbolic span $\sigma_H(R)$ is equal to*

$$\log \frac{\tau_1}{\tau_0} = \log \Xi(\lambda_0, \mu_0) = - \log \Xi(\lambda_1, \mu_1).$$

## 5. Examples

We could study asymptotic behavior of the hyperbolic and euclidean spans. Here, we will give some numerical examples, which show the approximate size of the moduli disk.

We can choose, for example, $k_0^2 = 0.5$. Then the modulus of the torus is exactly $0.5i$; the period parallelogram is a double unit square. We take a longitudinal slit of length $\ell_0 = 0.5$ (half a length of the geodesic loop $a$). Then, we have by simple calculations

$$\sigma_H(R) \approx 0.3, \qquad \sigma_E(R, \chi) \approx 0.08.$$

Similarly, if $k_0 = 0.9$ ($\tau_0 = 0.363i$) and $\ell_0 = 0.5$, we have

$$\sigma_H(R) \approx 0.6, \qquad \sigma_E(R, \chi) \approx 0.3.$$

Since the inclusion relation $R_1 \subset R_2$ between two Riemann surfaces implies the inequality $\sigma_H(R_1) \geq \sigma_H(R_2)$, we can give an estimate of the hyperbolic span of an open tori which arises from a rectangle. The same holds for the euclidean span.

In [10] Strebel showed that any open torus of finite connectivity is realized on a closed torus so that the complement consists of disks (and / or points). It is also interesting to give the hyperbolic span of such an open torus.

## 6. Deformation of tori by cutting and pasting along a geodesic arc

If an open torus *varies continuously*, so do the hyperbolic and euclidean spans. This is a consequence of a more general theorem, which will be shown in a forthcoming paper. For the present case, however, the continuity immediately follows from Theorem 1. Namely, we have

THEOREM 3. *The hyperbolic span* $\sigma_H(T_X(\tau, \ell))$ *of a geodesically slit torus is a continuous function of the modulus* $\tau$ *and the slit length* $\ell$ *of* $T_X(\tau, \ell)$, $X = L$ *or* $M$.

From this theorem we also have

THEOREM 4. *Let there be given two closed tori* $T'$ *and* $T''$ *and assume that either*

$$\text{Im } \tau(T') \neq \text{Im } \tau(T'')$$

*or*

$$\text{Im } \tau(T') - \text{Im } \tau(T'') = \text{Re } \tau(T') + \text{Re } \tau(T'') = 0$$

*holds. Then we can cut* $T'$ *and* $T''$ *along geodesic arcs* $\Gamma'$ *and* $\Gamma''$ *respectively, so that* $T' \setminus \Gamma'$ *and* $T'' \setminus \Gamma''$ *are conformally equivalent.*

PROOF. By an elementary geometric observation, we can find a point $Q$ on the imaginary axis such that a circle $C$ centered at $Q$ passes through the points $\tau(T')$ and $\tau(T'')$. The circle intersects with the imaginary axis at two points $\tau_0$ and $\tau_1$. We assume $\text{Im } \tau_0 < \text{Im } \tau_1$. Let $T_0$ be the closed torus which is obtained from the lattice $\{m + n\tau_0 \,|\, m, n \in \mathbf{Z}\}$. Now, we remove a horizontal segment of length $\ell_0$ to obtain an open torus $T_L(\tau_0, \ell_0)$. By Theorem 3 there exists a number $\ell_0$ such that $C = \partial M(T_L(\tau_0, \ell_0))$. This completes the proof.

THEOREM 4'. *Let there be given two closed tori $T'$ and $T''$ and assume that*

$$\text{Im } \tau(T') = \text{Im } \tau(T'') \quad and \quad |\text{Re } \tau(T')| \neq |\text{Re } \tau(T'')|.$$

*Then we can find geodesic arcs $\Gamma'$ and $\Gamma''$ on $T'$ and $T''$, respectively, and an analytic arc $\gamma'$ emanating from a point on $\Gamma'$ so that $T' \setminus (\Gamma' \cup \gamma')$ and $T'' \setminus \Gamma''$ are conformally equivalent.*

For the proof we have only to take an auxiliary point $\tau$ such that $\text{Im } \tau$ is equal to neither $\text{Im } \tau(T')$ nor $\text{Im } \tau(T')$, and repeat the argument in Proof of Theorem 4 for the pair $\tau$ and $\tau(T')$ as well as for the pair $\tau$ and $\tau(T'')$. Note that the geodesic arc in Theorem 4' need not lie on a closed geodesic.

These theorems show that all the tori can be obtained from a single torus through a discontinuous deformation of a set of measure zero. More precisely, we have the following theorem immediately from Theorems 4 and 4'.

THEOREM 5. *Let $T^*$ be a fixed marked torus. Then, for any marked torus $T$, there exist a geodesic arc $\Gamma^*$ on $T^*$ and an analytic arc $\gamma^*$ emanating from a point of $\Gamma^*$ such that the slit torus $T^* \setminus (\Gamma^* \cup \gamma^*)$ can be conformally sewn to produce the torus $T$.*

Note that $\gamma^*$ does not meet $\Gamma^*$ except for the initial point of $\gamma^*$. Note also that $\gamma^*$ may reduce to a point.

Roughly speaking, slit deformations of a single torus — cutting it along a possibly branched arc and pasting the resulting boundary curve in another way — yield the whole Teichmüller space of genus one.

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