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A distance on conjugacy classes of HNN decompositions of a group

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Introduction

Let A be a group with a presentation $\langle X | R \rangle$, and let B be a subgroup of A and $\psi: B \to A$ a monomorphism. *The HNN extension of A relative to B and ψ with stable letter t is the group*

$$A *_{B,\psi,t} = \langle X, t \,|\, R, t^{-1}bt = \psi(b)(b \in B) \rangle,$$

which was introduced by G. Higman, B. H. Neumann and H. Neumann [4]. Note that $G = A *_{B,\psi,t}$ has a natural epimorphism $\lambda: G \to \mathbf{Z}$ obtained by killing A. Information about G can be deduced from information in A, B and ψ. However G may be expressed in a different way as an HNN extension with same natural epimorphism $\lambda: G \to \mathbf{Z}$ and same stable letter t. We shall study the problem to compare more than one different such expressions. Thus we consider the problem under the following situation:

Let G be a group, and suppose that an epimorphism $\lambda: G \to \mathbf{Z}$ and a section $\tau: \mathbf{Z} \to G$ are given. Put $G_\lambda = \operatorname{Ker} \lambda$, $t = \tau(1)$, and let φ denote an inner automorphism of G_λ given by $\varphi(g) = t^{-1}gt$:

$$(0.1) \qquad \begin{cases} 1 \longrightarrow G_\lambda \longrightarrow G \xrightarrow{\lambda} \mathbf{Z} \longrightarrow 0, \quad \varphi \circlearrowright G_\lambda, \quad \tau: \mathbf{Z} \to G \\ t = \tau(1) \quad \text{and} \quad \varphi(g) = t^{-1}gt. \end{cases}$$

Now, let $\mathscr{D}$ denote the set of HNN decompositions of G (with given λ and $t = \tau(1)$) relative to the condition (0.1). Also $\mathscr{D}_{fg}$ denotes the set of $\alpha = A *_B \in \mathscr{D}$ so that both A and B are finitely generated. In [6] we associated to each $\alpha \in \mathscr{D}$ a non-negative integer valued function $v_\alpha: G \to \mathbf{Z}_{\geq 0}$, and then, using these functions, we defined a distance on $\mathscr{D}_{fg}$ which reflects differences of combinatorial structures of the decompositions. For a decomposition $\alpha = A *_B \in \mathscr{D}$, $t^i \alpha t^{-i} (i \in \mathbf{Z})$ denote its *conjugate decompositions* $(t^i A t^{-i}) *_{(t^i B t^{-i})} \in \mathscr{D}$. We consider an equivalence relation '$\sim$' on $\mathscr{D}$: $\alpha \sim \beta$ if and only if $\beta = t^i \alpha t^{-i}$ for some $i \in \mathbf{Z}$. In many cases, it is more suitable to consider the set of equivalence classes $\mathscr{D}^* = \mathscr{D}/\sim$ and its subset $\mathscr{D}^*_{fg} = \mathscr{D}_{fg}/\sim$ rather than $\mathscr{D}$ and

$\mathscr{D}_{fg}$ themselves. In this paper we shall study several properties of $\mathscr{D}^*$ and $\mathscr{D}^*_{fg}$ by considering associated functions $v^*_\alpha: G \to \mathbf{Z}_{\geq 0}$ (cf. (0.4) below). Particularly, it is shown that a distance function d^* can be also defined on $\mathscr{D}^*_{fg}$ under the assumption that G is not of 'lobster pot type'.

We will state the results more precisely. For a given group G we assume the condition (0.1). Then *the set of HNN decompositions of G relative to* (0.1) is the set $\mathscr{D}$ of all subgroups $A \subset G_\lambda$ which satisfy the following condition (0.2) (see [6, §2]):

(0.2) The natural homomorphism $\iota: A*_{B,\varphi|B,s} \to G$ induced by the inclusion $A \subset G$ and satisfying $\iota(s) = t$ is an isomorphism, where $B = A \cap tAt^{-1}$.

Each element $\alpha \in \mathscr{D}$ will be written as $\alpha = A*_{B,\varphi,t}$ (or simply $A*_B$). We associate to each $\alpha \in \mathscr{D}$ a non-negative integer valued function $v_\alpha: G \to \mathbf{Z}_{\geq 0}$ as follows: Write each element $g \in G$ as a Britton's reduced word relative to α; $g = a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \cdots a_n t^{\varepsilon_n} a_{n+1}$ (see Lemma 1.1). Then we put

$$
\begin{aligned}
v^+_\alpha(g) &= \max\{0, \varepsilon_1, \varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\}, \\
v^-_\alpha(g) &= \min\{0, \varepsilon_1, \varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\} \text{ and} \\
v_\alpha(g) &= v^+_\alpha(g) - v^-_\alpha(g).
\end{aligned} \tag{0.3}
$$

These values are independent of the expression of g as a reduced word. Now, for each $[\alpha] \in \mathscr{D}^*$, where $[\alpha]$ denotes the equivalent class of $\alpha \in \mathscr{D}$, we define a function $v^*_{[\alpha]}$ (or v^*_α): $G \to \mathbf{Z}_{\geq 0}$ by

$$
v^*_\alpha(g) = \min_{i \in \mathbf{Z}} v_{t^{-i}\alpha t^i}(g) \left(= \min_{i \in \mathbf{Z}} v_\alpha(t^i g t^{-i})\right). \tag{0.4}
$$

An HNN decomposition $\alpha = A*_{B,\varphi,t} \in \mathscr{D}$ is said to be *properly ascending*, if $A = B \neq \varphi(B)$ or $A = \varphi(B) \neq B$ holds. We will say that G is of *lobster pot type relative to* (0.1), if there is a properly ascending HNN decomposition $\alpha \in \mathscr{D}_{fg}$. It is shown that this condition is equivalent to the condition that all $\alpha \in \mathscr{D}_{fg}$ are properly ascending (Proposition 2.9). We also note that if G is of lobster pot type, then $v^*_\alpha = |\lambda|$ for all $[\alpha] \in \mathscr{D}^*_{fg}$ (Proposition 2.2). Now our main result is the following

THEOREM A. *Let G be a group, and suppose the condition* (0.1). *Suppose further that G is not of lobster pot type. Then, for every* $[\alpha], [\beta] \in \mathscr{D}^*_{fg}$,

$$
d^*([\alpha], [\beta]) = \sup_{g \in G} |v^*_\alpha(g) - v^*_\beta(g)| < \infty,
$$

and this function d^* *becomes a distance on* D^*_{fg}. *Moreover, if* $\alpha = A*_B$ *and* $\beta = C*_D$, *then*

$$d^*([\alpha], [\beta]) = \max\{\max v_\alpha^*(C),\ \max v_\beta^*(A)\}.$$

Theorem A is proved in §3, and in §4 we give examples of groups which have more than one conjugacy classes of HNN decompositions and calculate the distances between them.

The idea of the distance d^* has a motivation in our previous paper on incompressible spanning surfaces for a knot [5]. Each equivalence class of such surface determines a conjugacy class of HNN decomposition of the knot group. The relation between these two objects in terms of the distance d^* will be studied in [7].

1. Associated function $v_\alpha^*: G \to \mathbf{Z}_{\geq 0}$

Let G be a given group, and suppose the condition (0.1). In the introduction we have associated to each $[\alpha] \in \mathscr{D}^*$ a non-negative integer valued function $v_{[\alpha]}^*$ (or v_α^*): $G \to \mathbf{Z}_{\geq 0}$ by (0.4). Here $\mathscr{D}^* = \mathscr{D}/\sim$, and $\alpha \sim \beta$ if and only if $\beta = t^i \alpha t^{-i}$ for some $i \in \mathbf{Z}$. In this section we study its several properties. First we recall Britton's lemma.

LEMMA 1.1 (Reduced form theorem, Britton's lemma; [2], [3, Th. 32]). *Any $g \in A *_{B,\varphi,t}$ can be written as*

$$g = a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \cdots a_n t^{\varepsilon_n} a_{n+1}$$

where $n \geq 0$, $\varepsilon_i = \pm 1$, $a_i \in A$ and there is no consecutive subword $t^{-1} a_i t$ with $a_i \in B$ or $t a_i t^{-1}$ with $a_i \in \varphi(B)$; n and $\varepsilon_1, \ldots, \varepsilon_n$ are uniquely determined.

We call an expression as above a *reduced word relative to $A *_{B,\varphi,t}$*. The following proposition gives a characterization of $v_\alpha^*: G \to \mathbf{Z}_{\geq 0}$.

PROPOSITION 1.2. *Suppose $[\alpha] \in \mathscr{D}^*$ and $\alpha = A *_B$. Then, for each $g \in G_\lambda$,*

$$v_\alpha^*(g) = \min\{q - p \mid g \in A(p, q)\}.$$

Here, $A(p, q)$ denotes the minimal subgroup of G_λ containing $t^i A t^{-i}$ for $p \leq i \leq q$.

PROOF. For a given $g \in G_\lambda$, we suppose $g \in A(p, q)$. Then $t^{-p} g t^p \in A(0, p - q)$, and hence

$$t^{-p} g t^p = (t^{\delta_1} a_1 t^{-\delta_1}) \cdots (t^{\delta_n} a_n t^{-\delta_n}) = t^{\delta_1} a_1 t^{\delta_2 - \delta_1} a_2 \cdots t^{\delta_n - \delta_{n-1}} a_n t^{-\delta_n}$$

for some $a_j \in A$ and $0 \leq \delta_j \leq q - p$ $(1 \leq j \leq n)$. It follows that

$$v_\alpha^+(t^{-p} g t^p) = \max\{0, \delta_1, \delta_2, \ldots, \delta_n\} \quad \text{and} \quad v_\alpha^-(t^{-p} g t^p) = \min\{0, \delta_1, \delta_2, \ldots, \delta_n\}.$$

Since $0 \leq \delta_j \leq q - p$ for $1 \leq j \leq n$, we have

$$v_\alpha^*(g) \leq v_\alpha(t^{-p} g t^p) = v_\alpha^+(t^{-p} g t^p) - v_\alpha^-(t^{-p} g t^p) \leq q - p \quad \text{and}$$

$$v_\alpha^*(g) \leq \min\{q - p \mid g \in A(p, q)\}.$$

Conversely, for $g \in G_\lambda$, we can find an $r \in \mathbf{Z}$ such that

$$v_\alpha(t^r g t^{-r}) = v_\alpha^*(g) (= \min_{i \in \mathbf{Z}} v_\alpha(t^i g t^{-i})).$$

We express $t^r g t^{-r}$ as a reduced word relative to α; $t^r g t^{-r} = a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \cdots a_n t^{\varepsilon_n} a_{n+1}$. Note that $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = \lambda(t^r g t^{-r}) = \lambda(g) = 0$ since $g \in G_\lambda$. Putting $q^* = v_\alpha^+(t^r g t^{-r})$ and $p^* = v_\alpha^-(t^r g t^{-r})$, we have

$$\begin{aligned} t^r g t^{-r} &= a_1 (t^{\varepsilon_1} a_2 t^{-\varepsilon_1})(t^{\varepsilon_1+\varepsilon_2} a_2 t^{-\varepsilon_1-\varepsilon_2}) \cdots (t^{\varepsilon_1+\cdots+\varepsilon_{n-1}} a_n t^{-\varepsilon_1-\cdots-\varepsilon_{n-1}}) a_{n+1} \\ &\in A(p^*, q^*). \end{aligned}$$

Thus $g \in A(p^* - r, q^* - r)$, and

$$v_\alpha^*(g) = q^* - p^* = (q^* - r) - (p^* - r) \geq \min\{q - p \mid g \in A(p, q)\}. \qquad \square$$

PROPOSITION 1.3. *Suppose that* $[\alpha] \in \mathscr{D}^*$ *and* $\alpha = A*_B$. *Then, for any finitely generated subgroup* $C \subset G_\lambda$,

$$\max v_\alpha^*(C) (= \max_{g \in C} v_\alpha^*(g)) = \min\{q - p \mid C \subset A(p, q)\}.$$

To prove Proposition 1.3 we note the following easy fact

LEMMA 1.4. *Suppose that* $[\alpha] \in \mathscr{D}^*$ *and* $\alpha = A*_B$. *If for a subset* $X \subset G_\lambda$,

$$X \subset A(r, s) \cap A(u, v) \quad \text{and} \quad s - r > v - u,$$

then there are integers r', s' *so that*

$$X \subset A(r', s'), \ r \leq r' \leq s' \leq s \quad \text{and} \quad v - u \geq s' - r'.$$

PROOF OF PROPOSITION 1.3. Since C is finitely generated subgroup of G_λ, $C \subset A(p, q)$ for some $p \leq q$. By Proposition 1.2, we see that, for each $g \in C$,

$$v_\alpha^*(g) = \min\{s - r \mid g \in A(r, s)\} \leq q - p.$$

Hence $\max v_\alpha^*(C) \leq q - p$, and then

$$\max v_\alpha^*(C) \leq \min\{q - p \mid C \subset A(p, q)\}. \tag{1.5}$$

If the right hand of (1.5) is equal to 0, then so is the left one, and the equality holds. Thus we may assume that the right hand of (1.5) is positive:

$$C \subset A(p^*, q^*) \quad \text{and} \quad q^* - p^* = \min\{q - p \mid C \subset A(p, q)\} (> 0). \tag{1.6}$$

Then it follows that

(1.7) (a) there is $a \in C$ so that $a \notin A(p^* + 1, q^*)$, and
(b) there is $b \in C$ so that $b \notin A(p^*, q^*)$.

There are three cases.

Case 1: $a \notin A(p^*, q^* - 1)$. By Lemma 1.4, this assumption and (1.7) (a) imply that $v_\alpha^*(a) = q^* - p^*$. Hence, by (1.5) and (1.6), we have the desired equality.

Case 2: $b \notin A(p^* + 1, q^*)$. In this case the same argument as in Case 1 holds.

Case 3: $a \in A(p^*, q^* - 1)$ and $b \in A(p^* + 1, q^*)$. Consider $c = ab \in C$. Then, by the assumptions on a, b, we see that $c \notin A(p^* + 1, q^*)$ and $c \notin A(p^*, q^* - 1)$. By Lemma 1.4, this implies that $v_\alpha^*(c) = q^* - p^*$. From this together with (1.5) and (1.6), we have the desired equality. □

Now we define an order '$\preceq$' on $\mathscr{D}$ as follows: For $\alpha = A *_B$, $\beta = C *_D \in \mathscr{D}$,

$$\alpha \preceq \beta \quad \text{if and only if} \quad A \subset C$$

(note that $A \subset C$ implies $B \subset D$). Then we see

LEMMA 1.8 ([6, Lemma 3.4]).

$$\alpha \preceq \beta \quad \textit{if and only if} \quad v_\alpha(g) \geq v_\beta(g) (g \in G).$$

Using this order on $\mathscr{D}$, we can define a 'partial order' on $\mathscr{D}^*$ as follows: For $[\alpha], [\beta] \in \mathscr{D}^*$,

$$[\alpha] \preceq [\beta] \quad \text{if and only if} \quad \alpha \preceq t^{-i}\beta t^i \qquad \text{for some } i \in \mathbf{Z}.$$

Then we show the following

PROPOSITION 1.9. *For* $[\alpha], [\beta] \in \mathscr{D}^*$,

$$[\alpha] \preceq [\beta] \quad \textit{if and only if} \quad v_\alpha^*(g) \geq v_\beta^*(g) (g \in G).$$

PROOF. Suppose $[\alpha] \preceq [\beta]$. Then $\alpha \preceq t^{-k}\beta t^k$ for some $k \in \mathbf{Z}$. By Lemma 1.8,

$$v_\alpha(g) \geq v_{t^{-k}\beta t^k}(g) = v_\beta(t^k g t^{-k}) \text{ for any } g \in G.$$

Hence $v_\alpha(t^i g t^{-i}) \geq v_\beta(t^{i+k} g t^{-i-k})$ for every $i \in \mathbf{Z}$, and we have

$$v_\alpha^*(g) = \min_{i \in \mathbf{Z}} v_\alpha(t^i g t^{-i}) \geq \min_{i \in \mathbf{Z}} v_\beta(t^{i+k} g t^{-i-k}) = v_\beta^*(g).$$

Conversely we suppose that $v_\alpha^*(g) \geq v_\beta^*(g)$ for every $g \in G$. Since $v_\alpha^*(g) = 0$

for all $g \in A$, we have $v_\beta^*(A) = 0$. Hence, by Proposition 1.3, $A \subset t^i C t^{-i}$ for some $i \in \mathbf{Z}$. Thus we have $\alpha \preceq t^i \beta t^{-i}$ and $[\alpha] \preceq [\beta]$. □

Recall that $\mathscr{D}_{fg}$ is the subset of $\mathscr{D}$ consisting of $\alpha = A *_B \in \mathscr{D}$ so that both A and B are finitely generated, and $\mathscr{D}_{fg}^* = \mathscr{D}_{fg}/\sim (\subset \mathscr{D}^*)$. We close this section by noting the following

THEOREM 1.10 (Bestvina and Feighn, Baumslag and Shalen, Miller; [1, Th. 1], [6, Th. 2.4]). *Let G be a finitely presented group, and suppose the condition* (0.1). *Then, for each $\alpha \in \mathscr{D}$, there is a $\gamma \in \mathscr{D}_{fg}$ so that $\gamma \preceq \alpha$.*

Since $G_\lambda *_{G_\lambda} \in \mathscr{D}$, as a direct consequence of this theorem, we have

COROLLARY 1.11. *Let G be a finitely presented group, and suppose the condition* (0.1). *Then $\mathscr{D}_{fg} \neq \emptyset$ and $\mathscr{D}_{fg}^* \neq \emptyset$.*

2. Ascending HNN decompositions

Let G be a given group, and suppose the condition (0.1). A decomposition $\alpha = A *_{B,\varphi,t} \in \mathscr{D}$ is said to be *ascending* (resp. *properly ascending*) if $B = A$ or $\varphi(B) = A$(resp. $B = A \neq \varphi(B)$ or $\varphi(B) = A \neq B$) holds. In this section we study properties of decompositions of these types. We first claim

LEMMA 2.1. *For $\alpha = A *_B \in \mathscr{D}$, the following three conditions are equivalent:*
(1) *α is ascending.*
(2) *$A \subset tAt^{-1}$ or $A \subset t^{-1}At$.*
(3) *$A \subset t^{-k}At^k$ for some $k \in \mathbf{Z} - \{0\}$.*

PROOF. The implications (1)⇔(2)⇒(3) are obvious. We will show that (3)⇒(2). Suppose that $A \subset t^{-k}At^k$ for some $k > 0$ (similar argument as below also holds in the case of $k < 0$). If $k = 1$, there is nothing to do. Thus we assume that $k \geq 2$. Consider a subgroup $A(0, k-1) *_{t^{1-k}Bt^{k-1}} (t^{-k}At^k) \subset G_\lambda$. By the assumption, A is contained in both $A(0, k-1)$ and $t^{-k}At^k$. This implies $A \subset t^{1-k}Bt^{k-1} \subset t^{1-k}At^{k-1}$. Thus, we get $A \subset t^{-1}At$ by induction. □

If $\alpha \in \mathscr{D}$ is ascending (resp. properly ascending), then so are $t^{-i}\alpha t^i$ for all $i \in \mathbf{Z}$. Thus we will say that $[\alpha] \in \mathscr{D}^*$ is *ascending* (resp. *properly ascending*) if so is α.

PROPOSITION 2.2. *For $[\alpha] \in \mathscr{D}_{fg}^*$,*

$$[\alpha] \text{ is ascending if and only if } v_\alpha^*(g) = |\lambda(g)| \ (g \in G).$$

PROOF. Suppose that $[\alpha] \in \mathscr{D}_{fg}^*$ is ascending and $\alpha = A *_B$. For any $g \in G$

we can write $g = g't^k$ for a $g' \in G_\lambda$. Then, $g' \in A(p, q)$ for some $p \leq q$. Note that

$$A(p, q) = \begin{cases} t^q A t^{-q} & (\text{if } A = B), \\ t^p A t^{-p} & (\text{if } A = \varphi(B)). \end{cases}$$

In the case of $g' \in t^q A t^{-q}$, we have $g' = t^q a t^{-q}$ for some $a \in A$. Hence $t^{-q} g t^q = a t^k$ and

$$|\lambda(g)| \leq v_\alpha^*(g) = \min_{i \in \mathbf{Z}} v_\alpha(t^{-i} g t^i) \leq v_\alpha(a t^k) \leq |k| = |\lambda(g)|.$$

The similar argument holds in the case of $g' \in t^p A t^{-p}$. Thus we have $v_\alpha^*(g) = |\lambda(g)|$.

Conversely, we suppose $v_\alpha^*(g) = |\lambda(g)|$ $(g \in G)$. Consider a subgroup $A(0, 1) = gp(A \cup tAt^{-1})$ of G_λ. Since $v_\alpha^* \mid G_\lambda = 0$, by Proposition 1.3,

$$\min \{q - p \mid gp(A \cup tAt^{-1}) \subset A(p, q)\} = v_\alpha^*(gp(A \cup tAt^{-1})) = 0.$$

This implies that $gp(A \cup tAt^{-1}) \subset t^i A t^{-i}$ for some $i \in \mathbf{Z}$. Moreover, by Lemma 1.4, we have $gp(A \cup tAt^{-1}) \subset A$ or $gp(A \cup tAt^{-1}) \subset tAt^{-1}$. The former implies $tAt^{-1} \subset A$, and the latter implies $A \subset tAt^{-1}$. Thus α is ascending. □

Now we show the following

PROPOSITION 2.3. *If there is an ascending decomposition* $\alpha \in \mathscr{D}_{fg}$, *then all* $\beta \in \mathscr{D}_{fg}$ *are ascending.*

PROOF. Suppose that $\alpha = A*_B \in \mathscr{D}_{fg}$ is ascending. We only consider the case of $A \subset tAt^{-1}$; in the case of $A \subset t^{-1}At$, similar argument as below remains valid. Take any $\beta = C*_D \in \mathscr{D}_{fg}$. By Proposition 2.2, $v_\alpha^*(C) = 0$. Hence, by Proposition 1.3, $C \subset t^k A t^{-k}$ for some $k \in \mathbf{Z}$. While, from $A \subset tAt^{-1}$,

(2.4) $$A \subset t^i A t^{-i} \quad \text{for every } i \geq 0.$$

Thus, we have

(2.5) $$C \subset t^m A t^{-m} \quad \text{for some } m > 0.$$

On the other hand, since A is finitely generated,

(2.6) $$A \subset C(p, p + r) \quad \text{for some } p \in \mathbf{Z} \text{ and } r \geq 0.$$

Here $C(p, q)$ denotes the minimal subgroup of G_λ containing $t^i C t^{-i}$ for $p \leq i \leq q$. In the case of $p < 0$, we see, by (2.4) that

$$A \subset t^{-p} A t^p \subset C(0, r).$$

This shows that, in (2.6), we can always take p to be non-negative; we assume now $p \geq 0$. We consider a subgroup

$$A(p, p+r)*_{(t^{p+r}Dt^{-p-r})}A(p+r+1, p+2r+1) \subset G_\lambda.$$

Since $r+1 \geq 0$, by (2.4) and (2.6) we have

$$t^{r+1}At^{-r-1} \subset A(p+r+1, p+2r+1) \quad \text{and} \quad A \subset t^{r+1}At^{-r-1}.$$

Then, these and (2.6) imply that

$$A \subset t^{p+r}Dt^{-p-r} \subset t^{p+r}Ct^{-p-r}. \tag{2.7}$$

By (2.5), (2.6) and (2.7) together with the fact $m > 0$, $r \geq 0$, $p \geq 0$, we have $C \subset t^{m+p+r}Ct^{-m-p-r}$ and $m+p+r > 0$. Thus β is ascending by Lemma 2.1. □

As a corollary of the proof of Proposition 2.3, we have

PROPOSITION 2.8. *If there is* $\alpha = A*_B \in \mathscr{D}_{fg}$ *so that* $A = B = \varphi(B)$, *then* $\alpha = G_\lambda *_{G_\lambda}$ *and* $\mathscr{D}_{fg} = \{\alpha\}$.

Propositions 2.3 and 2.8 imply the following

PROPOSITION 2.9. *If there is a properly ascending decomposition* $\alpha \in \mathscr{D}_{fg}$, *then all decompositions in* $\mathscr{D}_{fg}$ *are properly ascending.*

We will say that G is of *lobster pot type relative to* (0.1), if G has a properly ascending decomposition $\alpha \in \mathscr{D}_{fg}$.

PROPOSITION 2.10. *Suppose that G is not of lobster pot type relative to* (0.1). *Then, for every* $[\alpha], [\beta] \in \mathscr{D}^*_{fg}$, $[\alpha] = [\beta]$ *is equivalent to* $v^*_\alpha = v^*_\beta$.

PROOF. We may assume that $\mathscr{D}_{fg} \neq \{G_\lambda *_{G_\lambda}\}$. Suppose that $v^*_\alpha = v^*_\beta$ for $\alpha = A*_B$ and $\beta = C*_D$. Then $\max v^*_\alpha(C) = \max v^*_\beta(C) = 0$, and hence $C \subset t^iAt^{-i}$ for some $i \in \mathbf{Z}$ by Proposition 2.3. Similarly we see that $A \subset t^jCt^{-j}$ for some $j \in \mathbf{Z}$. These imply that

$$A \subset t^jCt^{-j} \subset t^{i+j}At^{-i-j}.$$

If $i+j \neq 0$, α is ascending by Lemma 2.1. Hence $i+j = 0$ and $A = t^jCt^{-j}$. This means that $[\alpha] = [\beta]$. □

By Propositions 1.9 and 2.10, we have

COROLLARY 2.11. *Suppose that G is not of lobster pot type relative to* (0.1). *Then the partial order* '$\preceq$' *on* $\mathscr{D}^*_{fg}$ *becomes an order.*

3. Distance on $\mathscr{D}_{fg}^*$ –Proof of Theorem A–

Let G be a given group, and suppose the condition (0.1). In this section we give the proof of Theorem A. We first show the following

THEOREM 3.1. *For* $[\alpha]\in\mathscr{D}_{fg}^*$, *we have*

$$v_\alpha^*(gt^k)=\max\{|k|,\ v_\alpha^*(g)\}\qquad \textit{for every}\ g\in G_\lambda\ \textit{and}\ k\in\mathbf{Z}.$$

PROOF. Put $\alpha=A*_B$ and fix $g\in G_\lambda$ and $k\in\mathbf{Z}$. Then

$$(3.2)\qquad v_\alpha^*(g)=v_\alpha(t^{-m}gt^m)\qquad \text{for some}\ m\in\mathbf{Z}.$$

We put $g'=t^{-m}gt^m$ and write it as a reduced word relative to α;

$$g'=a_1t^{\varepsilon_1}a_2t^{\varepsilon_2}\cdots a_nt^{\varepsilon_n}a_{n+1}.$$

Consider elements $g_i=t^{-i}(g't^k)t^i=t^{-i}g't^{k+i}$ $(i\in\mathbf{Z})$. Since $v_\alpha^*(gt^k)=v_\alpha^*(g't^k)=\min v_\alpha(g_i)$, we will calculate the values of $v_\alpha^+(g_i)$, $v_\alpha^-(g_i)$ and $v_\alpha(g_i)$.

Since $g_i=t^{-i}a_1t^{\varepsilon_1}a_2t^{\varepsilon_2}\cdots a_nt^{\varepsilon_n}a_{n+1}t^{k+i}$ is a reduced word relative to α, we have

$$v_\alpha^+(g_i)=\max\{0,\ v_\alpha^+(g')-i,\ k\}\quad\text{and}\quad v_\alpha^-(g_i)=\min\{0,\ v_\alpha^-(g')-i,\ k\}.$$

Plotting the values of $v_\alpha^+(g_i)$ and $v_\alpha^-(g_i)$ in the (i,k)-plane, we have the value of $v_\alpha^+(g_i)$ as in Figure 1.

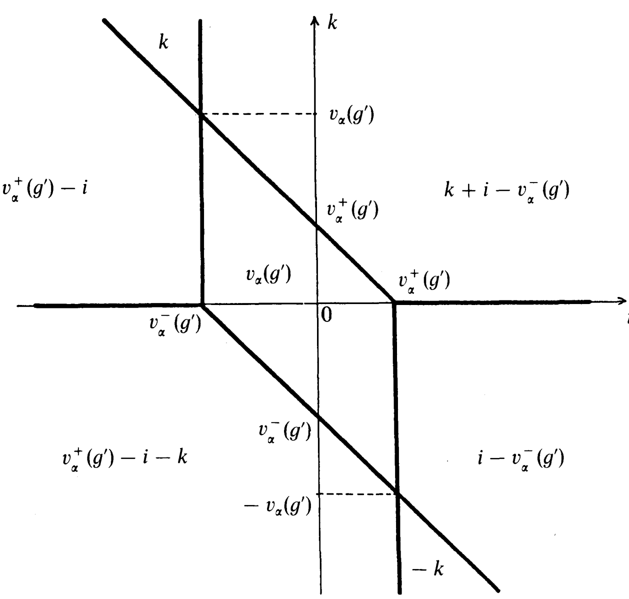

Figure 1. $v_\alpha(g_i)$

From Figure 1, we see that

$$v_\alpha^*(gt^k) = v_\alpha^*(g't^k) = \min_{i\in\mathbf{Z}} v_\alpha(g_i) = \begin{cases} k & (k \geq v_\alpha(g')) \\ v_\alpha(g') & (-v_\alpha(g') \leq k \leq v_\alpha(g')) \\ -k & (k \leq -v_\alpha(g')) \end{cases}$$

Since $v_\alpha(g') = v_\alpha^*(g)$ by (3.2), we have $v_\alpha^*(gt^k) = \max\{|k|, v_\alpha^*(g)\}$ as desired. □

We now suppose that G is not of lobster pot type. For each $[\alpha], [\beta] \in \mathscr{D}_{fg}^*$, we set

$$(3.3) \qquad d^*([\alpha], [\beta]) = \sup_{g\in G} |v_\alpha^*(g) - v_\beta^*(g)|.$$

The boundedness $d^*([\alpha], [\beta]) < \infty$ follows from Proposition 3.4 below. In fact, since A and C are finitely generated, $\max v_\alpha^*(C) < \infty$ and $\max v_\beta^*(A) < \infty$ by Proposition 1.3.

PROPOSITION 3.4. *Suppose that* $[\alpha], [\beta] \in \mathscr{D}_{fg}^*$ *and* $\alpha = A*_B$, $\beta = C*_D$. *Then*

$$d^*([\alpha], [\beta]) = \max\{\max v_\alpha^*(C), \max v_\beta^*(A)\}.$$

Moreover, the following conditions are immediate from the definition of d^* and Proposition 2.10:

(3.5) For each $[\alpha], [\beta], [\gamma] \in \mathscr{D}_{fg}^*$,

(1) $d^*([\alpha], [\beta]) = 0$ if and only if $[\alpha] = [\beta]$,
(2) $d^*([\alpha], [\beta]) = d^*([\beta], [\alpha])$ and
(3) $d^*([\alpha], [\gamma]) \leq d^*([\alpha], [\beta]) + d^*([\beta], [\gamma])$.

To prove Proposition 3.4 we prepare

LEMMA 3.6. *For each* $[\alpha], [\beta] \in \mathscr{D}_{fg}^*$,

$$\sup_{x\in G} |v_\alpha^*(x) - v_\beta^*(x)| = \sup_{g\in G_\lambda} |v_\alpha^*(g) - v_\beta^*(g)|.$$

PROOF. Take $x \in G$ and write $x = gt^k$ where $g \in G_\lambda$, $k \in \mathbf{Z}$. By Proposition 3.1, we have

$$v_\alpha^*(x) = \max\{|k|, v_\alpha^*(g)\}, \quad v_\beta^*(x) = \max\{|k|, v_\beta^*(g)\}.$$

This implies that

$$|v_\alpha^*(x) - v_\beta^*(x)| \leq |v_\alpha^*(g) - v_\beta^*(g)|.$$

Thus $\sup_{x\in G}|v_\alpha^*(x) - v_\beta^*(x)| = \sup_{g\in G_\lambda}|v_\alpha^*(g) - v_\beta^*(g)|$ follows. □

PROOF OF PROPOSITION 3.4. Take any $g \in G_\lambda$. Then, by Proposition 1.2,

$$(3.7) \qquad g \in C(r, s) \quad \text{and} \quad s - r = v_\beta^*(g) \qquad \text{for some } r \leq s.$$

By Proposition 1.3,

$$(3.8) \qquad C \subset A(p, q) \quad \text{and} \quad q - p = \max v_\alpha^*(C) \qquad \text{for some } p \leq q.$$

From (3.7) and (3.8), we have

$$g \in A(p + r, q + s),$$

and then

$$v_\alpha^*(g) \leq (q + s) - (p + r) = \max v_\alpha^*(C) + v_\beta^*(g).$$

Hence $v_\alpha^*(g) - v_\beta^*(g) \leq \max v_\alpha^*(C)$. Similarly we have $v_\beta^*(g) - v_\alpha^*(g) \leq \max v_\beta^*(A)$. These imply $|v_\alpha^*(g) - v_\beta^*(g)| \leq \max\{\max v_\alpha^*(C), \max v_\beta^*(A)\}$. Thus, by Lemma 3.6,

$$d^*([\alpha], [\beta]) \leq \max\{\max v_\alpha^*(C), \max v_\beta^*(A)\}.$$

On the other hand, choosing an element $c \in C$ with $v_\alpha^*(c) = \max v_\alpha^*(C)$, we see that

$$d^*([\alpha], [\beta]) \geq |v_\alpha^*(c) - v_\beta^*(c)| = \max v_\alpha^*(C)$$

since $v_\beta^*(c) = 0$. Similarly, by taking an element $a \in A$ with $v_\beta^*(a) = \max v_\beta^*(A)$, we see that $d^*([\alpha], [\beta]) \geq \max v_\beta^*(A)$. Thus the desired equality follows. □

The proof of Theorem A is now completed.

4. Examples

In this section we give two typical examples of groups which have more that one conjugacy classes of HNN decompositions.

EXAMPLE 4.1. Let G be a graph product of the following square diagram of groups where $\varphi: B \to C$ and $\psi: D \to A$ are monomorphisms:

$$\begin{array}{ccc} A & \xleftarrow{\supset} & B \\ \psi \uparrow & & \downarrow \varphi \\ D & \xrightarrow{\subset} & C. \end{array}$$

Then G has an HNN decomposition

$$\alpha = (A *_{B,\varphi} C) *_{D,\psi,t}$$

Let $\mathscr{D}$ be the set of HNN decompositions associated with natural epimorphism $\lambda: G \to \mathbf{Z}$ and natural section τ with $\tau(1) = t$. Putting $C' = t^{-1}Ct$, $D' = t^{-1}Dt$, $\varphi'(b) = t^{-1}\varphi(b)t(b \in B)$ and $\psi'(d') = \psi(td't^{-1})(d' \in D')$, we can also express G as graph product of the following diagram of groups;

$$\begin{array}{ccc} A & \overset{\supset}{\longleftarrow} & B \\ {\scriptstyle\psi'}\uparrow & & \downarrow{\scriptstyle\varphi'} \\ D' & \overset{\subset}{\longrightarrow} & C'. \end{array}$$

In fact we have

$$G = \langle A, C, t \mid relA,\ relC,\ b = \varphi(b)(b \in B),\ t^{-1}dt = \psi(d)(d \in D)\rangle$$
$$= \langle A, C', t \mid relA,\ relC',\ d' = \psi'(d')(d' \in D'),\ t^{-1}bt = \varphi'(b)(b \in B)\rangle,$$

and hence G has a decomposition

$$\beta = (A *_{D',\psi'} C') *_{B,\varphi',t} \in \mathscr{D}.$$

If A, B, C and D are finitely generated, then $\alpha, \beta \in \mathscr{D}_{fg}$.

PROPOSITION 4.2. *Under the above situation, suppose further that A, B, C and D are finitely generated and that $A \neq B$ and $C \neq \varphi(B)$. Then G is not of lobster pot type, and $d^*([\alpha], [\beta]) = 1$.*

PROOF. By the assumption that $A \neq B$ and $C \neq \varphi(B)$, we see that $D \neq A *_B C$ and $\varphi(D) \neq A *_B C$, and hence α is not ascending. This implies that G is not of lobster pot type by Proposition 2.3. We next note that

$$gp(t^{-1}(A *_B C)t \cup A *_B C) = gp(t^{-1}At *_{t^{-1}Bt} t^{-1}Ct \cup A *_B C)$$
$$\supset A *_{D'} C', \quad \text{and}$$
$$gp(A *_{D'} C' \cup t(A *_{D'} C')t^{-1}) \supset A *_B C.$$

Hence we have $d^*([\alpha], [\beta]) \leq 1$. To see $d^*([\alpha], [\beta]) = 1$, it suffices to find an element $g \in G$ satisfing the condition $|v_\alpha^*(g) - v_\beta^*(g)| = 1$. By the assumption that $A \neq B$ and $C \neq \varphi(B)$, we take $a \in A - B$ and $c \in C - \varphi(B)$. Putting $g = ac$, we have $v_\alpha^*(g) = 0$. On the other hand $g = atc't^{-1}$ $(c' = t^{-1}at \in C')$ is an expression of g relative to β. Consider the elements $t^i g t^{-i} = t^i atc't^{-1-i}$ $(i \in \mathbf{Z})$. We show that the right hand is a reduced word relative to β. We see that $a \in A *_{D'} C' - B$ by $a \in A - B$, and $t^i atc't^{-1-i} = t^{1+i}(t^{-1}at)t^{-1-i}$ is reduced relative to β for $i \leq -1$. Also $c' \in A *_{D'} C' - \varphi'(B)$ by $c \in C - \varphi(B)$, and hence $t^i atc't^{-1-i} = t^i a(tc't^{-1})t^{-i}$ is reduced relative to β for $i \geq 0$. It follows that

$$v_\beta(t^i g t^{-i}) = \begin{cases} i+1 & (i \geq 0) \\ -i & (i \leq -1), \end{cases}$$

and then we have $v_\beta^*(g) = 1$. Thus the assertion $d^*([\alpha], [\beta]) = 1$ is proved. □

Next we give an example of group which has infinitely many HNN decompositions.

EXAMPLE 4.3 (compare with [6, Example 6.4]). Let G_i $(i = 1, 2)$ be groups which have HNN decompositions $\alpha_i = A_i *_{B_i, \varphi_i, t}$ with common stable letter t. Let G be the amalgamation of G_1 and G_2 by the common infinite cyclic subgroup $\langle t \rangle$; $G = G_1 *_{\langle t \rangle} G_2$. Then G has an HNN decomposition

$$\beta = \alpha_1 * \alpha_2 = (A_1 * A_2) *_{(B_1 * B_2), \varphi, t}$$

where $\varphi | B_i = \varphi_i$ $(i = 1, 2)$. Let $\mathscr{D}$ be the set of HNN decompositions associated with natural epimorphism $G \to \mathbf{Z}$ and the stable letter t. Using conjugate decompositions $t^k \alpha_1 t^{-k}$ $(k \in \mathbf{Z})$ of G_1, we have a family of decompositions of G;

$$\beta_k = (t^k \alpha_1 t^{-k}) * \alpha_2 = ((t^k A_1 t^{-k}) * A_2) *_{(t^k B_1 t^{-k} * B_2), \varphi, t} \in \mathscr{D}.$$

If A_i and B_i $(i = 1, 2)$ are finitely generated, then $\beta_k \in \mathscr{D}_{fg}$ for every $k \in \mathbf{Z}$.

PROPOSITION 4.4. *Under the above situation, suppose further that $\beta \in \mathscr{D}_{fg}$, neither α_1 nor α_2 are ascending. Then G is not of lobster pot type, and $d^*([\beta_k], [\beta_j]) = k - j$ for $k \geq j$.*

PROOF. By the assumption, $A_i \neq B_i$ and $A_i \neq \varphi_i(B_i)$ $(i = 1, 2)$. Hence we have $A_1 * A_2 \neq B_1 * B_2$ and $A_1 * A_2 \neq \varphi_1(B_1) * \varphi_2(B_2) = \varphi(B_1 * B_2)$, and then β is not ascending. Thus G is not of lobster pot type by Proposition 2.3. To prove the latter half of the proposition, it suffices to show the following assertion:

$$d^*([\beta_k], [\beta]) = k \qquad \text{for every } k \geq 1. \tag{4.5}$$

Now, by the definition of β_j, we see that

$$gp((t^{j-1} A_1 t^{1-j} * A_2) \cup t(t^{j-1} A_1 t^{1-j} * A_2) t^{-1}) \supset t^j A_1 t^{-j} * A_2 \quad \text{and}$$

$$gp(t^{-1}(t^j A_1 t^{-j} * A_2) t \cup (t^j A_1 t^{-j} * A_2)) \supset t^{j-1} A_1 t^{1-j} * A_2.$$

Hence $d^*([\beta_j], [\beta_{j-1}]) \leq 1$ for every j. Thus $d^*([\beta_k], [\beta]) \leq k$ for every $k \geq 1$. To see the desired equality, it suffices to find an element $g \in G$ so that $|v_\beta^*(g) - v_{\beta_k}^*(g)| = k$. We choose elements $x \in A_1 - B_1$ and $y \in A_2 - \varphi_2(B_2)$, and put $g=xy$. Then $v_\beta^*(g)=0$ since $g \in A_1 * A_2$. We will show that $v_{\beta_k}^*(g)=k$. We express g as a word relative to β_k; $g=t^{-k}x_k t^k y$ where $x_k=t^k x t^{-k} \in t^k A_1 t^{-k}$. Consider elements

$$t^i g t^{-i} = t^{i-k} x_k t^k y t^{-i} \quad (i \in \mathbf{Z}). \tag{4.6}$$

If the right hand of (4.6) is not reduced relative to β_k, then one of the following conditions hold since $k \geq 1$:

(1) $i \leq k-1$ and $x_k \in t^k B_1 t^{-k} * B_2$.

(2) $i \geq 1$ and $y \in t^{-1}(t^k B_1 t^{-k} * B_2)t$.

However, by the assumption that $x \in A_1 - B_1$ and $y \in A_2 - \varphi_2(B_2)$, neither (1) nor (2) are satisfied. Hence the right hand of (4.6) is a reduced word relative to β_k. It follows from this that

$$v_{\beta_k}(t^i g t^{-i}) = \begin{cases} i & (i \geq k) \\ k & (1 \leq i \leq k-1) \\ k-i & (i \leq 0). \end{cases}$$

Hence we have $v_{\beta_k}(g) = \min_{i \in \mathbf{Z}} v_{\beta_k}(t^i g t^{-i}) = k$. Thus (4.5) is proved. □

References

[1] G. Baumslag and P. B. Shalen, Amalgamated products and finitely presented groups, Coment. Math. Helv. **65** (1990), 243–254.

[2] J. L. Britton, The word problem, Ann. of Math. **77** (1963), 16–32.

[3] D. E. Cohen, "Combinatorial Group Theory: a topological approach," Cambridge Univ. Press, Camblidge, 1989.

[4] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, J. London Math. Soc. **26** (1949), 247–254.

[5] O. Kakimizu, Finding disjoint incompressible spanning surfaces for a link, Hiroshima Math. J. **22** (1992), 225–236.

[6] O. Kakimizu, Combinatorial distance between HNN decompositions of a group, J. Pure and Appl. Algebra **82** (1992), 273–288.

[7] O. Kakimizu, Minimal HNN decompositions of a knot group and incompressible spanning surfaces, preprint.

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