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Ultimately positive (negative) solutions to a differential inclusion of order n

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1. The aim of this paper is to prove the existence of nonoscillatory solutions with the prescribed asymptotic behaviour of the differential inclusion

$$L_n x(t) \in F(t, x(\varphi(t))), \qquad n > 1, \tag{E}$$

where $L_n x(t)$ is the n-th quasiderivative of $x(t)$ with respect to the continuous functions $a_i(t)\colon J = [t_0, \infty) \to (0, \infty)$, $i = 0, 1, \cdots, n$, $L_0 x(t) = a_0(t)x(t)$, $L_i x(t) = a_i(t)(L_{i-1}x(t))'$, $i = 1, 2, \cdots, n$, $\int_{t_0}^{\infty} a_i^{-1}(t)\,dt = \infty$, $i = 0, 1, \cdots, n-1$, $F(t, x)\colon J \times \mathbf{R} \to \{$nonempty convex compact subsets of $\mathbf{R}\}$, $\mathbf{R} = (-\infty, \infty)$ and $\varphi(t)\colon J \to \mathbf{R}$ is a continuous function such that $\lim_{t\to\infty} \varphi(t) = \infty$.

We will use the following notation: $F(t, x)x > (<)0$ means that $yx > (<)0$ for each $y \in F(t, x)$; if $h\colon J \times \mathbf{R} \to \mathbf{R}$, then $F(t, x) \geqq (\leqq) h(t, x)$ means that $y \geqq (\leqq) h(t, x)$ for each $y \in F(t, x)$; if $B \subset \mathbf{R}$, then $|B| = \sup\{|x| : x \in B\}$, $\|B\| = \inf\{|x| : x \in B\}$. If C is a set, then $cf(C)$ is the set of all convex closed subsets of C.

The basic assumptions on $F(t, x)$ are as follows:

1° $F(t, x)$ is upper semicontinuous on $J \times \mathbf{R}$.

2° $F(t, 0) = \{0\}$ for each $t \in J$.

3° $F(t, x)x < 0$ for each $(t, x) \in J \times \mathbf{R}$, $x \neq 0$;

or

4° $F(t, x)x > 0$ for each $(t, x) \in J \times \mathbf{R}$, $x \neq 0$.

Let $t_0 \leqq b < t < \infty$. Then we denote

$$P_0(t, b) = 1,\ P_i(t, b) = \int_b^t a_1^{-1}(s_1) \int_b^{s_1} a_2^{-1}(s_2) \cdots \int_b^{s_{i-1}} a_i^{-1}(s_i)\,dw_i,$$

$dw_i = ds_i \cdots ds_1$, $i = 1, 2, \cdots, n-1$,

$$Q_n(t, b) = 1,\ Q_j(t, b) = \int_b^t a_{n-1}^{-1}(s_{n-1}) \int_b^{s_{n-1}} a_{n-2}^{-1}(s_{n-2}) \cdots \int_b^{s_{j+1}} a_j^{-1}(s_j)\,dz_j,$$

$dz_j = ds_j \cdots ds_{n-1}$, $j = 1, 2, \cdots, n-1$.

It is easy to see that

$$\lim_{t\to\infty} P_i(t, b) = \infty, \ \lim_{t\to\infty} Q_i(t, b) = \infty, \ i = 1, 2, \cdots, n-1,$$

$$\lim_{t\to\infty} P_i(t, b) P_j^{-1}(t, b) = 0, \ 0 \leqq i < j \leqq n-1,$$

$$\lim_{t\to\infty} Q_j(t, b) Q_i^{-1}(t, b) = 0, \ 0 < i < j \leqq n-1.$$

Moreover, let us denote

$$\gamma(t) = \sup\{s \geqq t_0 : \varphi(s) \leqq t\} \qquad \text{for all } t \geqq t_0.$$

In this paper we will state the conditions which guarantee the existence of nonoscillatory solutions of (E) which are asymptotic to the solutions of $L_n y(t) = 0$, more precisely, the existence of such solution $x(t)$ of (E) that

$$\lim_{t\to\infty} \frac{|L_0 x(t)|}{P_k(t, b)} = c_k > 0, \qquad k \in \{0, 1, \cdots, n-1\}. \tag{1}$$

On the other side we will state the conditions which guarantee the existence of nonoscillatory solution $x(t)$ of (E) which is asymptotic to none of the solutions of $L_n y(t) = 0$, more precisely, we will prove the existence of nonoscillatory solution $x(t)$ of (E) such that

$$\lim_{t\to\infty} \frac{L_0 x(t)}{P_k(t, b)} = 0, \ \lim_{t\to\infty} \frac{|L_0 x(t)|}{P_{k-1}(t, b)} = \infty, \qquad k \in \{1, 2, \cdots, n-1\}. \tag{2}$$

Such problems were discussed, in the case of a differential equation, by Hale and Onuchic [1], Kitamura [2], Kusano and Švec [3], Švec [4].

2. In this part we will prove the existence of the positive and also negative solution $x(t)$ of (E) which satisfies (1).

Taking into consideration the properties of $\varphi(t)$ we can find $T_0 \geqq \gamma(t_0)$ such that $\gamma(t) \geqq t_0$ for each $t \geqq T_0$.

THEOREM 1. *Let the assumptions* $1^o - 4^o$ *be sadisfied. Suppose that*:

(H_1) *To each measurable function* $z(t): J \to \mathbf{R}$ *there exists a measurable selector* $v(t): J \to \mathbf{R}$ *such that* $v(t) \in F(t, z(t))$ *a.e. on* J.
Denote $Mz(t) = \{$*the set of all measurable selectors belonging to* $z(t)\}$.

(H_2) *There exists a continuous function* $G(t, u): J \times [0, \infty) \to [0, \infty)$ *such that*:

a) $G(t, u)$ *is nondecreasing in* u *for each fixed* $t \in J$;

b) $|F(t, z)| \leqq G(t, |z|)$ *for each* $(t, z) \in J \times \mathbf{R}$;

c) $\int_{T_0}^{\infty} a_n^{-1}(s) Q_{k+1}(s, t_0) G(s, c a_0^{-1}(\varphi(s)) P_k(\varphi(s), t_0)) ds < \infty$

for some $c > 0$ *and* $k \in \{0, 1, \cdots, n-1\}$.

Then the differential inclusion (E) *has infinitely many solutions* $x(t)$ *satisfying* (1).

PROOF. Let $0 < |\alpha_k| < |\beta_k| \leqq c$, $\alpha_k \beta_k > 0$. Because of $(H_2 - c)$ we can choose $T \geqq T_0 \geqq \gamma(t_0)$ such that

$$\int_T^\infty a_n^{-1}(s) Q_{k+1}(s, t_0) G(s, c a_0^{-1}(\varphi(s)) P_k(\varphi(s), t_0))\, ds \leqq |\beta_k| - |\alpha_k|. \tag{3}$$

Let $C[t_0, \infty) = C(J)$ be the locally convex space of all continuous functions on J with the topology of uniform convergence on compact subintervals of J. We will seek the desired solution $x(t)$ of (E) in the set

$$Y = \{u(t) \in C(J) \colon |\alpha_k| P_k(t, t_0) \leqq a_0(t) |u(t)| \leqq |\beta_k| P_k(t, t_0)\}. \tag{4}$$

To prove our theorem we have to consider various situations.

α) Let 1^o, 2^o, 3^o be satisfied and let $n-k$ be even.

α_1) We will first seek a positive solution of (E) satisfying (1). Thus, let $0 < \alpha_k < \beta_k \leqq c$. In this case we have

$$Y = Y_1 = \{u(t) \in C[t_0, \infty) \colon \alpha_k P_k(t, t_0) \leqq a_0(t) u(t) \leqq \beta_k P_k(t, t_0)\}.$$

We will seek the desired solution of (E) in the set Y_1 as a fixed point of the multivalued operator A defined on Y_1 as follows: for $u(t) \in Y_1$

$$Au(t) = \Big\{ a_0^{-1}(t) \Big[\beta_k P_k(t, t_0) + \int_T^t a_1^{-1}(s_1) \int_T^{s_1} a_2^{-1}(s_2) \cdots \int_T^{s_{k-1}} a_k^{-1}(s_k) \cdot$$

$$\int_{s_k}^\infty a_n^{-1}(s) Q_{k+1}(s, s_k) v(s)\, ds\, dw_k \Big],\ v(t) \in Mu(\varphi(t)) \Big\},\ t \geqq T, \tag{5}$$

$$Au(t) = \beta_k a_0^{-1}(t) P_k(t, t_0),\ t_0 \leqq t \leqq T.$$

The operator A is well defined on Y_1. In fact, from (H_2) respecting the fact that Q_{k+1} and G are monotone we get

$$\left| \int_{s_k}^\infty a_n^{-1}(s) Q_{k+1}(s, s_k) v(s)\, ds \right| \leqq \int_{s_k}^\infty a_n^{-1}(s) Q_{k+1}(s, t_0) |F(s, u(\varphi(s)))|\, ds$$

$$\leqq \int_{s_k}^\infty a_n^{-1}(s) Q_{k+1}(s, t_0) G(s, |u(\varphi(s))|)\, ds$$

$$\leqq \int_{s_k}^\infty a_n^{-1}(s) Q_{k+1}(s, t_0) G(s, c a_0^{-1}(\varphi(s)) P_k(\varphi(s), t_0)))\, ds < \infty.$$

From the assumption 3^o we have $v(t) < 0$. Therefore, we get $Au(t) \leqq \beta_k a_0^{-1}(t)$ $P_k(t, t_0)$ for $t \geqq t_0$.

Furthermore, taking (3) into consideration, we get for $t \geqq T$

$$\beta_k P_k(t, t_0) - a_0(t)Au(t)$$
$$= -\int_T^t a_1^{-1}(s_1)\int_T^{s_1} a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds dw_k$$
$$\leqq \int_T^t a_1^{-1}(s_1)\int_T^{s_1} a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)dw_k\int_T^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)\cdot$$
$$\cdot G(s, ca_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds \leqq (\beta_k - \alpha_k)P_k(t, t_0)$$

and finally $\alpha_k P_k(t, t_0) \leqq a_0(t)Au(t)$. Thus, we have $Au(t) \subset Y_1$. It is easy to see that the set $Au(t)$ is nonempty and convex.

Now, we will prove that: $A\colon Y_1 \to cf(Y_1)$; A is upper semicontinuous on Y_1; $\overline{AY_1}$ is compact.

Let $\xi(t) \in Au(t)$, $u(t) \in Y_1$. Then for $t \geqq T$ we have

$$[a_0(t)\xi(t)]' \leqq \beta_k P_k'(t, t_0) + a_1^{-1}(t)\int_T^t a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)ds_k\cdots ds_2\cdot$$
$$\cdot\int_T^{\infty} a_T^{-1}(s)Q_{k+1}(s, t_0)G(s, ca_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds.$$

From this we conclude that $[a_0(t)\xi(t)]'$, $\xi(t) \in AY_1$ are uniformly bounded on each compact subinterval of J. Therefore, $a_0(t)\xi(t)$, $\xi(t) \in AY_1$ are equicontinuous on each compact subinterval of J. The uniform boundedness of the functions $a_0(t)\xi(t)$, $\xi(t) \in AY_1$, on each compact subinterval of J is clear. From all this we conclude that the sets $Au(t)$, $u(t) \in Y_1$ as well as the set AY_1, are relatively compact in the topology of $C[t_0, \infty)$.

Let $u_i(t) \in Y_1$, $i = 1, 2, \cdots$, and let the sequence $\{u_i(t)\}$ converge to $u(t)$ in $C[t_0, \infty)$. Furthermore, let $z_i(t) \in Au_i(t)$, $i = 1, 2, \cdots$. The set AY_1 being relatively compact, there exists a subsequence $\{z_{i_j}(t)\}$ of $\{z_i(t)\}$ which converges to a function $z(t) \in \overline{AY_1} \subset Y_1$ in the topology of $C[t_0, \infty)$. We have

$$z_i(t) = a_0^{-1}(t)\Big\{\beta_k P_k(t, t_0)$$
$$+ \int_T^t a_1^{-1}(s_1)\int_T^{s_1} a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)Q_{k+1}(s, s_k)v_i(s)ds dw\Big\},\ t \geqq T,$$
$$z_i(t) = a_0^{-1}(t)\beta_k P_k(t, t_0),\ t_0 \leqq t \leqq T,$$

where $v_i(t) \in Mu_i(\varphi(t))$. From (H_2) and (3) we get

$$\int_T^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)|v_i(s)|ds$$

$$\leqq \int_T^\infty a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ca_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds \leqq \beta_k - \alpha_k.$$

Let $L_1(T, \infty)$ denote the set of all measurable functions f on $[T, \infty)$ such that

$$\|f(t)\|_1 = \int_T^\infty a_n^{-1}(s)Q_{k+1}(s, t_0)|f(s)|ds < \infty.$$

Thus, we see that the sequence $\{v_i(t)\}$ is bounded in the space $L_1(T, \infty)$. Furthermore, if $\{E_m\}$, $E_m \subset [T, \infty)$, is a decreasing sequence of sets such that $\bigcap_{m=1}^\infty E_m = \emptyset$, then

$$\lim_{m\to\infty}\left|\int_{E_m} a_n^{-1}(s)Q_{k+1}(s, t_0)v_i(s)ds\right| \leqq \lim_{m\to\infty}\int_{E_m} a_n^{-1}(s)Q_{k+1}(s, t_0)|v_i(s)|ds$$

$$\leqq \lim_{m\to\infty}\int_{E_m} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ca_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds = 0.$$

Then (see [2, Th. IV. 8.9]) it is possible to choose a subsequence $\{v_{i_j}(t)\}$ of $\{v_i(t)\}$ which weakly converges to some $v(t) \in L_1(T, \infty)$.

Because $\{u_{i_j}(t)\}$ converges to $u(t)$ in $C[t_0, \infty)$ and $v_{i_j}(t) \in F(t, u_{i_j}(\varphi(t)))$, $j = 1, 2, \cdots$, using the assumption 1^o, to given $\varepsilon > 0$ and $t \in J$ there exists $N = N(t, \varepsilon)$ such that for any $i_j \geqq N$ we have $F(t, u_{i_j}(\varphi(t))) \subset O_\varepsilon(F(t, u(\varphi(t)))$, where $O_\varepsilon(F(t, u(\varphi(t))))$ is the ε-neighbourhood of the set $F(t, u(\varphi(t)))$.

Consider the sequence $\{v_{i_j}(t)\}$, $i_j \geqq N$. Then (see [2, Corollary V. 3.14]) it is possible to construct such convex combinations from $v_{i_j}(t)$, $i_j \geqq N$, denoted by $g_m(t)$; $m = 1, 2, \cdots$, that the sequence $\{g_m(t)\}$ converges to $v(t)$ in $L_1(T, \infty)$. Then by the Riesz theorem there exists a subsequence $\{g_{m_i}(t)\}$ of $\{g_m(t)\}$ which converges to $v(t)$ a.e. on $[T, \infty)$. From the convexity of $O_\varepsilon(F(t, u(\varphi(t))))$ and from the fact that $v_{i_j}(t) \in O_\varepsilon(F(t, u(\varphi(t))))$ it follows that $g_{m_i}(t) \in O_\varepsilon(F(t, u(\varphi(t))))$, $i = 1, 2, \cdots$ and, therefore, $v(t) \in \bar{O}_\varepsilon(F(t, u(\varphi))))$. In the limit as $\varepsilon \to 0$ we see that $v(t) \in F(t, u(\varphi(t)))$. We note that in our considerations t was a fixed point and that $F(t, u(\varphi(t)))$ is a compact convex subset of **R**.

Thus, the function

$$z(t) = a_0^{-1}(t)\Big\{\beta_k P_k(t, t_0)$$

$$+ \int_T^t a_1^{-1}(s_1)\int_T^{s_1} a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^\infty a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds dw\Big\}$$

for $t \geqq T$,

$$z(t) = a_0^{-1}(t)\beta_k P_k(t, t_0) \qquad \text{for } t_0 \leqq t \leqq T$$

is well defined and $z(t) \in Au(t)$ for $t \in J$.

Now, it follows from the weak convergence of $\{v_{i_j}(t)\}$ to $v(t)$ in $L_1(T, \infty)$ that the subsequence $\{z_{i_j}(t)\}$ of $\{z_i(t)\}$ converges to $z(t)$ a.e. on J. However, the functions $z_{i_j}(t)$ belong to the compact set $\overline{AY_1}$. Therefore, there exists a subsequence of the sequence $\{z_{i_j}(t)\}$ which converges to a function $\bar{z}(t)$ in the topology of $C[t_0, \infty)$. This means that $\bar{z}(t) = z(t) \in Au(t)$ a.e. on J. With this the upper semicountinuity of the operator A on Y_1 is proved.

The similar considerations as in the proof of upper semicontinuity of A on Y_1, made for case that $z_i(t) \in Au(t)$ and $\{z_i(t)\}$ converges to $z(t)$ in $C[t_0, \infty)$ give us that $z(t) \in Au(t)$. This means that the set $Au(t)$ is closed. Thus, we have proved that $Au(t)$ is compact and A maps Y_1 into $cf(Y_1)$.

From all this we conclude by Ky Fan's theorem that the operator A has a fixed point in Y_1, i.e. there exists $u(t) \in Y_1$ such that $u(t) \in Au(t)$.

It is easy to see that $u(t)$ is the desired positive solution of (E) satisfying (1). In fact, from the positiveness of $u(t)$ on $[t_0, \infty)$ and from the assumption 3^o it follows that $L_n u(t)$ has a constant sign on some interval $[T_u, \infty)$ and all quasiderivatives $L_i u(t)$, $i = 0, 1, \cdots, n-1$ are monotone on some ray $[T_1, \infty)$, $T_1 \geqq T_u$. By l'Hospital's rule we get

$$0 < \alpha_k \leqq \lim_{t \to \infty} \frac{L_0 u(t)}{P_k(t, t_0)} = \lim_{t \to \infty} L_k u(t) = c_k \leqq \beta_k.$$

From the construction of the operator A it is evident that there exist infinitely many solutions of (E) satisfying (1).

α_2) Now, we will seek a negative solution $x(t)$ of (E) satisfying (1). In this case we put $\beta_k < \alpha_k < 0$, $0 < |\alpha_k| < |\beta_k| \leqq c$ and

$$Y_2 = \{u(t) \in C[t_0, \infty) \colon \beta_k P_k(t, t_0) \leqq a_0(t) u(t) \leqq \alpha_k P_k(t, t_0)\}.$$

The desired solution will be obtained a fixed point of the operator A in Y_2.

If $u(t) \in Y_2$ and $v(t) \in Mu(\varphi(t))$, then from assumption 3^o we see that $v(t) > 0$ and from (5) we have $a_0(t) Au(t) \geqq \beta_k P_k(t, t_0)$ for $t \geqq t_0$ and

$$\begin{aligned} &a_0(t) Au(t) - \beta_k P_k(t, t_0) \\ &\leqq \int_T^t a_1^{-1}(s_1) \int_T^{s_1} a_2^{-1}(s_2) \cdots \int_T^{s_{k-1}} a_k^{-1}(s_k) dw_k \cdot \int_T^\infty a_n^{-1}(s) Q_{k+1}(s, t_0) \cdot \\ &\quad \cdot G(s, c a_0^{-1}(\varphi(s)) P_k(\varphi(s), t_0)) ds \leqq (-\beta_k + \alpha_k) P_k(t, t_0), \end{aligned}$$

where we have used the fact that Q_{k+1} and G are monotone and the fact that $v(s) \leqq G(s, |u(\varphi(s))|) \leqq G(s, c a_0^{-1}(\varphi(s)) P_k(\varphi(s), t_0))$. From this we have $a_0(t) Au(t) \leqq \alpha_k P_k(t, t_0) < 0$. Thus, we have $Au \subset Y_2$.

The proof that: $A \colon Y_2 \to cf(Y_2)$, A is upper semicontinuous on Y_2, $\overline{AY_2}$

is compact can be made in the same way as it was done for Y_1. The end of the proof is similar to that in case α_1).

β) Let 1^o, 2^o, 3^o be satisfied and let $n-k$ be odd.

β_1) We shall investigate the existence of a positive solution $x(t)$ of (E) satisfying (1). We seek this solution in the set Y_1 as a fixed point of the operator B: $u(t)\in Y_1$,

$$Bu(t)=\Bigg\{a_0^{-1}(t)\Bigg[\alpha_k P_k(t,t_0) - \int_T^t a_1^{-1}(s_1)\int_T^{s_1} a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s,s_k)v(s)\,ds\,dw_k\Bigg], \quad v(s)\in Mu(\varphi(s))\Bigg\}, \qquad t\geqq T,$$

$$Bu(t)=\alpha_k a_0^{-1}(t)P_k(t,t_0), \qquad t_0\leqq t\leqq T.$$

Applying similar arguments as in the preceeding cases, we obtain a fixed point $u(t)$ of B in the set Y_1 which is the desired positive solution of (E) with the asymptotic behavior (1).

β_2) We get the existence of negative solution of (E) satisfying (1) as a fixed point of the operator B in the set Y_2 using similar procedure as in the previous cases.

γ) Let 1^o, 2^o, 4^o be satisfied let $n-k$ be even.

γ_1) We seek a positive solution $x(t)$ of (E) satisfying (1). In this case we use the set Y_1 and the operator C: $u(t)\in Y_1$,

$$Cu(t)=\Bigg\{a_0^{-1}(t)\Bigg[\alpha_k P_k(t,t_0) + \int_T^t a_1^{-1}(s_1)\int_T^{s_1} a_2^{-1}(s_2)\cdots\int_T^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s,s_k)v(s)\,ds\,dw_k\Bigg], \quad v(s)\in Mu(\varphi(s))\Bigg\}, \qquad t\geqq T,$$

$$Cu(t)=\alpha_k a_0^{-1}(t)P_k(t,t_0), \qquad t_0\leqq t\leqq T.$$

As in the previous cases it is easy to prove that this operator C is well defined on Y_1 and maps Y_1 into Y_1. The rest of the proof can be made in the same way as in the previous cases.

γ_2) We seek a negative solution of (E) satisfying (1) as a fixed point of the operator C defined on Y_2. The considerations are similar as in the previous cases.

δ) Let 1^o, 2^o, 4^o be satisfied and let $n-k$ be odd.

δ_1) A positive solution of (E) satisfying (1) can be found as a fixed point of the operator D defined on Y_1 as follows: $u(t) \in Y_1$,

$$Du(t) = \Big\{ a_0^{-1}(t) \Big[\alpha_k P_k(t, t_0) - \int_T^t a_1^{-1}(s_1) \int_T^{s_1} a_2^{-1}(s_2) \cdots \int_T^{s_{k-1}} a_n^{-1}(s_k) Q_{k+1}(s, s_k) v(s) ds dw_k \Big],$$
$$v(s) \in Mu(\varphi(s)) \Big\}, \qquad t \geqq T,$$
$$Du(t) = \alpha_k a_0^{-1}(t) P_k(t, t_0), \qquad t_0 \leqq t \leqq T.$$

The proof is similar to those in the previous cases.

δ_2) The desired negative solution of (E) satisfying (1) can be found as a fixed point of the operator D defined on Y_2 using similar arguments as in the previous cases.

3. In this part we will deal with the existence of positive (negative) solutions of (E) which have the asymptotic behaviour (2).

THEOREM 2. *Let all assumptions of the Theorem* 1 *be satisfied. Moreover, let the following assumption be satisfied*:

(H_3) *There exists a continuous function* $G_1(t, u): J \times [0, \infty) \to [0, \infty)$ *nondecreasing in u for each fixed* $t \in J$ *such that*

$$G_1(t, |x|) \leqq \| F(t, x) \|, \qquad x \in R \tag{6}$$

and

$$\int_{T_0}^{\infty} a_n^{-1}(s) \int_{T_0}^{s} Q_{k+1}(s, z) a_k^{-1}(z) dz \, G_1\Big(s, \frac{a}{2} a_0^{-1}(\varphi(s)) P_{k-1}(\varphi(s), t_0)\Big) ds = \infty \tag{7}$$

where $k \in \{1, 2, \cdots, n-1\}$, $0 < 2a < c$ *where* c *is from* (H_2) − (c).

i) *If the assumption* 3° *is satisfied and if* $n - k$ *is odd, then the inclusion* (E) *has infinitely many positive as well as negative solutions satisfying* (2).

ii) *If the assumption* 4° *is satisfied and if* $n - k$ *is even, then the inclusion* (E) *has infinitely many positive as well as negative solutions satisfying* (2).

iii) *If the assumption* 3° *is satisfied and if* $n - k$ *is even or if the assumption* 4° *is satisfied and if* $n - k$ *is odd, then there is no positive* (*negative*) *solution of* (E) *satisfying* (2).

PROOF. i) Let the assumption 3° be satisfied and let $n - k$ be odd. First, we will prove the existence of a positive solution $x(t)$ of (E) satisfying (2). Let

$$Y_3 = \left\{u(t) \in C(J): \frac{1}{2} a P_{k-1}(t, t_0) \leqq a_0(t)u(t) \leqq a[P_{k-1}(t, t_0) + P_k(t, t_0)]\right\}.$$

We define the operator B_1 on Y_3 as follows: $u(t) \in Y_3$,

$$B_1 u(t) = \left\{a_0^{-1}(t)\left[aP_{k-1}(t, t_0) - \int_{T'}^{t} a_1^{-1}(s_1) \int_{T'}^{s_1} a_2^{-1}(s_2) \cdots \int_{T'}^{s_{k-1}} a_k^{-1}(s_k) \cdot \right.\right.$$

$$\left.\left. \cdot \int_{s_k}^{\infty} a_n^{-1}(s)Q_{k-1}(s, s_k)v(s)ds dw_k\right], \; v(s) \in Mu(\varphi(s))\right\}, \qquad t \geqq T'.$$

$$B_1 u(t) = aa_0^{-1}(t)P_{k-1}(t, t_0), \qquad t_0 \leqq t \leqq T',$$

where $a > 0$ and $T' \geqq T_0$ are such that $P_{k-1}(\varphi(s), t_0) + P_k(\varphi(s), t_0) \leqq 2P_k(\varphi(s), t_0)$ for $s \geqq T'$ and

$$\int_{T'}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, 2aa_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds \leqq a. \tag{8}$$

The existence of such T' follows from $(H_2)-(c)$. As in the proof of Theorem 1 it is easy to prove that B_1 is well defined on Y_3. It follows from the assumption 3° that $u(t) \in Y_3$ implies $v(t) < 0$ for $t \geqq T'$. Therefore, we have $B_1 u(t) \geqq aa_0^{-1}(t)P_{k-1}(t, t_0) \geqq \frac{1}{2} aa_0^{-1}(t)P_{k-1}(t, t_0)$ for $t \geqq t_0$. On the other side, respecting (H_2) and (8), we obtain

$$\begin{aligned} B_1 u(t) \leqq a_0^{-1}(t)\Big\{ & aP_{k-1}(t, t_0) \\ & + \int_{T'}^{t} a_1^{-1}(s_1) \int_{T'}^{s_1} a_2^{-1}(s_2) \cdots \int_{T'}^{s_{k-1}} a_k^{-1}(s_k)ds_k \cdot \\ & \cdot \int_{T'}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, 2aa_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds\Big\} \\ \leqq a_0^{-1}(t) & \{aP_{k-1}(t, t_0) + aP_k(t, t_0)\}, \qquad t \geqq T'. \end{aligned}$$

Thus, we get $B_1 Y_3 \subset Y_3$. It is easy to see that $B_1 u(t)$ is nonempty and convex.

The proof that $B_1: Y_3 \to cf(Y_3)$, B_1 is upper semicontinuous on Y_3, $\overline{B_1 Y_3}$ is compact can be made in the same way as it was done for A_1 in the proof of Theorem 1. Therefore, Ky Fan's theorem can be applied. It gives the existence of a fixed point of B_1 in Y_3. Denote it by $x(t)$. To finish the proof we have to prove that $x(t)$ satisfies (2). We have

$$x(t) = a_0^{-1}(t)\Big\{aP_{k-1}(t, t_0)$$

$$-\int_{T'}^{t} a_1^{-1}(s_1)\int_{T'}^{s_1} a_2^{-1}(s_2)\cdots\int_{T'}^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds\,dw_k\Bigg\}$$

for $t \geqq T'$, where $v(s)$ is an appropriate element from $Mx(\varphi(s))$. Then

$$L_{k-1}x(t) = a - \int_{T'}^{t} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds\,ds_k,$$

$$L_k x(t) = -\int_{t}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t)v(s)ds, \qquad t \geqq T'$$

and

$$0 \leqq L_k x(t) \leqq \int_{t}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, a_0^{-1}(\varphi(s))2aP_k(\varphi(s), t_0))ds,$$

respecting (H_2) and (8). From this we obtain $\lim_{t\to\infty} L_k x(t) = 0$.

For $L_{k-1}x(t)$, using Fubini's theorem, we get

$$L_{k-1}x(t) = a - \int_{T'}^{t} a_n^{-1}(s)v(s)\int_{T'}^{s} a_k^{-1}(z)Q_{k+1}(s, z)dz\,ds$$
$$- \int_{T'}^{t} a_k^{-1}(s)\int_{t}^{\infty} a_n^{-1}(z)Q_{k+1}(z, s)v(z)dz\,ds.$$

From this, respecting (6) and (7), we obtain

$$L_{k-1}x(t) \geqq a$$
$$+ \int_{T'}^{t} a_n^{-1}(s)\int_{T'}^{s} a_k^{-1}(z)Q_{k+1}(s, z)dz\,G_1\left(s, \frac{1}{2}aa_0^{-1}(\varphi(s))P_{k-1}(\varphi(s), t_0)\right)ds$$

where the function on the right hand side tends to infinity for $t \to \infty$.

Thus, the solution $x(t)$ satisfies $\lim_{t\to\infty} L_k x(t) = 0$, $\lim_{t\to\infty} L_{k-1}x(t) = \infty$ which is equivalent to (2) by l' Hospital's rule.

From the fact that $G_1(t, u)$ is nondecreasing in u it follows that if (7) is satisfied for some a, $0 < 2a < c$, then (7) will be satisfied also if instead of a we put arbitrary a', $2a < 2a' < c$. From this we conclude that there exist infinitely many positive solutions $x(t)$ of (E) satisfying (2).

Now, we will prove the existence of a negative solution of (E) satisfying (2) assuming that 3^o is satisfied and $n - k$ is odd. Let

$$Y_4 = \left\{u(t) \in C(J)\colon -a[P_{k-1}(t, t_0) + P_k(t, t_0)] \leqq a_0(t)u(t) \leqq -\frac{1}{2}aP_{k-1}(t, t_0)\right\}.$$

We use the operator B_2 defined on Y_4 as follows: $u(t) \in Y_4$,

$$B_2u(t) = \left\{a_0^{-1}(t)\left[-aP_{k-1}(t, t_0)\right.\right.$$

$$\left.-\int_{T'}^{t} a_1^{-1}(s_1)\int_{T'}^{s_1} a_2^{-1}(s_2)\cdots\int_{T'}^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds dw_k\right],$$

$$\left. v(s)\in Mu(\varphi(s))\right\}, \qquad t \geqq T'.$$

$$B_2u(t) = -aa_0^{-1}(t)P_{k-1}(t, t_0), \qquad t_0 \leqq t \leqq T',$$

where $a > 0$ and T' are such that (8) is satisfied.

It is easy to see that B_2 is well defined on Y_4. If $u(t)\in Y_4$, then from 3° we see that $v(t) > 0$, $v(t)\in Mu(\varphi(t))$. Therefore, $B_2u(t) \leqq -aa_0^{-1}(t)P_{k-1}(t, t_0) \leqq -\frac{1}{2}aa_0^{-1}(t)P_{k-1}(t, t_0)$, $t \geqq t_0$. Respecting (H_2) and (8), we obtain $B_2u(t) \geqq -a_0^{-1}(t)a[P_{k-1}(t, t_0) + P_k(t, t_0)]$. Thus, $B_2Y_4 \subset Y_4$.

Then, applying the same arguments as in the proof of Theorem 1, we see that $B_2\colon Y_4 \to cf(Y_4)$, B_2 is upper semicontinuous on Y_4, $\overline{B_2Y_4}$ is compact. Therefore, application of Ky Fan's theorem gives the existence of a fixed point of B_2 in Y_4. Denote it by $y(t)$. Then

$$y(t) = a_0^{-1}(t)\left\{-aP_{k-1}(t, t_0)\right.$$

$$\left.-\int_{T'}^{t} a_1^{-1}(s_1)\int_{T'}^{s_1} a_2^{-1}(s_2)\cdots\int_{T'}^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds dw_k\right\}$$

for $t \geqq T'$, where $v(t)$ is an appropriate element from $My(\varphi(t))$ and

$$L_{k-1}y(t) = -a - \int_{T'}^{t} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds ds_k,$$

$$L_k y(t) = -\int_{t}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t)v(s)ds, \quad t \geqq T'.$$

Respecting the assumption (H_2), we get $\lim_{t\to\infty} L_k y(t) = 0$. Use of Fubini's theorem and of (6) gives

$$L_{k-1}y(t) \leqq -a$$

$$-\int_{T'}^{t} a_n^{-1}(s)\int_{T'}^{s} a_k^{-1}(z)Q_{k+1}(s, z)dz\, G_1\left(s, \frac{1}{2}aa_0^{-1}(\varphi(s))P_{k-1}(\varphi(s), t_0)\right)ds,$$

which combined with (7) implies that $L_{k-1}y(t) \to -\infty$ as $t \to \infty$. The same arguments as in the case of positive solutions of (E) satisfying (2) give us the existence of infinitely many negative solutions of (E) satisfying (2).

ii) Let 4^o be satisfied and let k, $1 \leqq k \leqq n-1$, be an integer such that $n-k$ is even.

First, we will prove the existence of a positive solution $x(t)$ of (E) satisfying (2). In this case we take the set Y_3 and we define the operator B_3 on Y_3 as follows: $u(t) \in Y_3$,

$$B_3 u(t) = \Big\{ a_0^{-1}(t) \Big[aP_{k-1}(t, t_0) + \int_{T'}^{t} a_1^{-1}(s_1) \int_{T'}^{s_1} a_2^{-1}(s_2) \cdots \int_{T'}^{s_{k-1}} a_k^{-1}(s_k) \int_{s_k}^{\infty} a_n^{-1}(s) Q_{k+1}(s, s_k) v(s) ds dw_k \Big], \quad v(s) \in Mu(\varphi(s)) \Big\}, \qquad t \geqq T',$$

$$B_3 u(t) = a_0^{-1}(t) aP_{k-1}(t, t_0), \qquad t_0 \leqq t \leqq T'.$$

The assumption 4^o implies that for $u(t) \in Y_3$ we have $v(t) > 0$ for $t \geqq T'$. Therefore, $a_0(t) B_3 u(t) \geqq aP_{k-1}(t, t_0) \geqq \frac{1}{2} aP_{k-1}(t, t_0)$, $t \geqq t_0$, and, respecting (H_2) and (8), we get $a_0(t) B_3 u(t) \leqq a[P_{k-1}(t, t_0) + P_k(t, t_0)]$ for $t \geqq t_0$. Thus, $B_3 Y_3 \subset Y_3$. Applying the similar arguments as in the proof of Theorem 1, we obtain a fixed point $x(t)$ of B_3 in Y_3 which gives rise to a positive solution $x(t)$ of (E) existing in $[T', \infty)$. Then

$$x(t) = a_0^{-1}(t) \Big\{ aP_{k-1}(t, t_0) + \int_{T'}^{t} a_1^{-1}(s_1) \int_{T'}^{s_1} a_2^{-1}(s_2) \cdots \int_{T'}^{s_{k-1}} a_k^{-1} \int_{s_k}^{\infty} a_n^{-1}(s) Q_{k+1}(s, s_k) v(s) ds dw_k \Big\}$$

for $t \geqq T'$, where $v(s)$ is an appropriate element from $Mx(\varphi(s))$ and

$$0 \leqq L_{k-1} x(t) = a + \int_{T'}^{t} a_k^{-1}(s_k) \int_{s_k}^{\infty} a_n^{-1}(s) Q_{k+1}(s, s_k) v(s) ds ds_k,$$

$$0 \leqq L_k x(t) = \int_{t}^{\infty} a_n^{-1}(s) Q_{k+1}(s, t) v(s) ds, \qquad t \geqq T'.$$

Use of (H_2) leads to the conclusion that $\lim_{t \to \infty} L_k x(t) = 0$ and use of Fubini's theorem, (6) and (7) give $\lim_{t \to \infty} L_{k-1} x(t) = \infty$. This is equivalent to (2) by l'Hospital's rule. Similar arguments as in the preceeding case i) give the existence of infinitely many positive solutions of (E) existing on $[T', \infty)$ and satisfying (2).

We obtain the existence of a negative solution $y(t)$ of (E) satisfying (2) as a fixed point of the operator

$$B_4u(t) = \Big\{a_0^{-1}(t)\Big[-aP_{k-1}(t, t_0)$$

$$-\int_{T'}^{t} a_1^{-1}(s_1)\int_{T'}^{s_1} a_2^{-1}(s_2)\cdots\int_{T'}^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds\,dw_k\Big],$$

$$v(s)\in Mu(\varphi(s))\Big\}, \qquad t \geqq T',$$

$$B_4u(t) = -aa_0^{-1}(t)P_{k-1}(t, t_0), \qquad t_0 \leqq t \leqq T'$$

in the set Y_4 by use of the similar procedurre as in the preceding cases. The same arguments as before give also the existence of infinitely many negative solutions of (E) existing on $[T', \infty)$ and satisfying (2).

iii) Let 3° be satisfied and let $n-k$ be even. Let $x(t)$ be a solution of (E) such that $|x(t)|>0$ on some interval $[T_x, \infty)$ and satisfies (2). Then the assumption 3° implies that $x(t)L_nx(t)<0$ for all $t \geqq T_x$ and this implies (see [5, Lemma 4]) that $|L_kx(t)|$ is increasing on some interval $[T_1, \infty)$, $T_1 \geqq T_x$, which leads to the contradiction with the assumption $\lim_{t\to\infty} L_kx(t)=0$.

Let 4° be satisfied and let $n-k$ be odd. Let $y(t)$ be a solution of (E) such that $|y(t)|>0$ on some interval $[T_y, \infty)$ and satisfies (2). Then the assumption 4° implies that $y(t)L_ny(t)>0$ on $[T_y, \infty)$. From this we see (see [5, Lemma 6]) that $|L_ky(t)|$ is increaisng on some interval $[T_2, \infty)$, $T_2 \geqq T_y$. This leads to the contradiction with the assumption $\lim_{t\to\infty} L_ky(t)=0$.

4. The existence of the desired solutions in Theorem 1 was proved on the interval $[T, \infty)$, $T \geqq T_0 \geqq t_0$ and in Theorem 2 on the interval $[T', \infty)$, $T' \geqq T_0 \geqq t_0$. The definition of T is given by the condition (3) and the definition of T' by the condition (8). We will show that, under some hypotheses concerning the sublinearity of $G(t, u)$, we will be able to prove the existence of the desired solutions on the interval $[T_0, \infty)$.

THEOREM 3. *Let all assumptions of Theorem* 1 *be satisfied. Moreover, suppose that*:

$u^{-1}G(t, u)$ *is nonincreasing in* u for $u \geqq 0$ *and each fixed* $t\in J$, (9)

$\lim_{u\to\infty} u^{-1}G(t, u)=0$ *for each fixed* $t\in J$. (10)

Then the inclusion (E) *has infinitely many positive as well as negative solutions* $x(t)$ *existing on* $[T_0, \infty)$ *and satisfying* (1).

PROOF. We sketch the proof for the case α_1) from Theorem 1. Similar procedure can be used in the remaining cases. Thus, let the condition 3^o be satisfied and let $n-k$ be even. From the assumption $(H_2)-(c)$ we conclude that the function

$$a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ca_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))$$

is integrable on $[T_0, \infty)$. Then, respecting (9), we get for $b > c$

$$\begin{aligned} & a_n^{-1}(s)Q_{k+1}(s, t_0)b^{-1}G(s, ba_0^{-1}(\varphi(s))P_k(\varphi(s), t_0)) \\ & \quad \leqq a_n^{-1}(s)Q_{k+1}(s, t_0)c^{-1}G(s, ca_0^{-1}(\varphi(s))P_k(\varphi(s), t_0)), \qquad s \geqq T_0 \end{aligned} \tag{11}$$

and by (10)

$$\lim_{b\to\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)b^{-1}G(s, ba_0^{-1}(\varphi(s))P_k(\varphi(s), t_0)) = 0 \tag{12}$$

pointwise on $[T_0, \infty)$. Use of the Lebesgue dominated convergence theorem gives

$$\lim_{b\to\infty} b^{-1}\int_{T_0}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ba_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds = 0. \tag{13}$$

Therefore, for $p > 0$ there exists $b_0(p) > 0$ such that

$$\int_{T_0}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ba_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds \leqq pb \tag{14}$$

for each $b > b_0 > c$. Let $p = 1/2$, $b_k > b_0(1/2)$, $0 < a_k < b_k/2$. We define

$$Y_1 = \{u(t) \in C(J) \colon a_kP_k(t, t_0) \leqq a_0(t)u(t) \leqq b_kP_k(t, t_0)\}$$

and the operator A on Y_1 as follows: $u(t) \in Y_1$,

$$\begin{aligned} Au(t) = \Bigg\{ & a_0^{-1}(t)\Bigg[b_kP_k(t, t_0) \\ & + \int_{T_0}^{t} a_1^{-1}(s_1)\int_{T_0}^{s_1} a_2^{-1}(s_2)\cdots\int_{T_0}^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} a_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds\,dw_k\Bigg], \\ & v(s) \in Mu(\varphi(s))\Bigg\}, \qquad t \geqq T_0, \end{aligned}$$

$$Au(t) = a_0^{-1}(t)b_kP_k(t, t_0), \qquad t_0 \leqq t \leqq T_0.$$

By the assumption 3^o, $u(t) \in Y_1$ implies $v(t) < 0$. Therefore, $a_0(t)Au(t) \leqq b_kP_k(t, t_0)$ for $t \geqq t_0$. Furthermore, taking (14) into consideration, we get for $t \geqq T_0$

$$b_k P_k(t, t_0) - a_0(t)Au(t)$$
$$\leqq \int_{T_0}^{t} a_1^{-1}(s_1)\int_{T_0}^{s_1} a_2^{-1}(s_2)\cdots\int_{T_0}^{s_{k-1}} a_k^{-1}(s_k)dw_k \int_{T_0}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)\cdot$$
$$\cdot\, G(s, b_k a_0^{-1}(\varphi(s))P_k(\varphi(s), t_0))ds \leqq \frac{1}{2} b_k P_k(t, t_0))$$

and finally $0 < a_k P_k(t, t_0) < \frac{1}{2} b_k P_k(t, t_0) \leqq a_0(t)Au(t)$. This all proves that $AY_1 \subset Y_1$. Proceeding as in the proof of Theorem 1 (for the case α_1), we can show that A has a fixed element $x(t)$ in Y_1 and this element $x(t)$ is a positive solution of (E) which exists on $[T_0, \infty)$ and satisfies (1). Since any number b_k greater than $b_0(1/2)$ can be taken in defining Y_1 and A, it is clear that there exist infinitely many such solutions of (E).

THEOREM 4. *Let all assumptions of Theorem* 2 *be satisfied. Moreover, let* (9) *and* (10) *be satisfied. Then all statements of Theorem* 2 *hold and the solutions in* i) *as well as in* ii) *exist on* $[T_0, \infty)$ *and satisfy* (2).

PROOF. We only sketch the proof of i), since the proof of ii) is similar and the proof of iii) is the same as in Theorem 2. From the assumption $(H_2)-(c)$ we conclude that

$$a_n^{-1}(s)Q_{k+1}(s, t_0)G\left(s, \frac{c}{2} a_0^{-1}(\varphi(s))[P_{k-1}(\varphi(s), t_0) + P_k(\varphi(s), t_0)]\right)$$

is integrable on $[T_0, \infty)$. Let $2b > c$. Then, in view of (9), we obtain

$$a_n^{-1}(s)Q_{k+1}(s, t_0)b^{-1}G(s, ba_0^{-1}(\varphi(s))[P_{k-1}(\varphi(s), t_0) + P_k(\varphi(s), t_0)])$$
$$\leqq a_n^{-1}(s)Q_{k+1}(s, t_0)2c^{-1}G\left(s, \frac{c}{2} a_0^{-1}(\varphi(s))[P_{k-1}(\varphi(s), t_0) + P_k(\varphi(s), t_0)]\right)$$

for $s \geqq T_0$ and, respecting (10),

$$\lim_{b\to\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)b^{-1}G(s, ba_0^{-1}(\varphi(s))[P_{k-1}(\varphi(s), t_0) + P_k(\varphi(s), t_0)]) = 0$$

pointwise on $[T_0, \infty)$. Use of the Lebesgue dominated convergence theorem gives

$$\lim_{b\to\infty} b^{-1}\int_{T_0}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ba_0^{-1}(\varphi(s))[P_{k-1}(\varphi(s), t_0)+P_k(\varphi(s), t_0)])ds = 0.$$

It means that for $p > 0$ there exists $b_0(p) > 0$ such that

$$\int_{T_0}^{\infty} a_n^{-1}(s)Q_{k+1}(s, t_0)G(s, ba_0^{-1}(\varphi(s))[P_{k-1}(\varphi(s), t_0) + P_k(\varphi(s), t_0)])ds \leqq pb$$

for each $b \geqq b_0(p)$. Put $p = 1$ and let $a = b > b_0(1)$ be fixed. Consider the set

$$Y_3 = \left\{u(t) \in C(J): \frac{1}{2} aP_{k-1}(t, t_0) \leqq a_0(t)u(t) \leqq a[P_{k-1}(t, t_0) + P_k(t, t_0)]\right\}$$

and the operator B_1 defined on Y_3 as follows: $u(t) \in Y_3$,

$$B_1 u(t) = \left\{a_0^{-1}(t)\left[aP_{k-1}(t, t_0) \right.\right.$$
$$\left. - \int_{T_0}^{t} a_1^{-1}(s_1)\int_{T_0}^{s_1} a_k^{-1}(s_k)\cdots\int_{T_0}^{s_{k-1}} a_k^{-1}(s_k)\int_{s_k}^{\infty} s_n^{-1}(s)Q_{k+1}(s, s_k)v(s)ds\,dw_k\right],$$
$$\left. v(s) \in Mu(\varphi(s))\right\}, \qquad t \geqq T_0,$$
$$B_1 u(t) = a_0^{-1}(t)aP_{k-1}(t, t_0), \qquad t_0 \leqq t \leqq T_0.$$

It is easy to prove that $B_1 Y_3 C Y_3$. If we proceed as in the proof of i) of Theorem 2, then we can prove that B_1 has a fixed element $y(t) \in Y_3$ and that this element is a solution of (E) existing on $[T_0, \infty)$ and satisfying (2). Since any number a greater than $b_0(1)$ can be taken in defining Y_3 and B_1, there exist infinitely many such solutions of (E).

The existence of infinitely many negative solutions of (E) existing on $[T_0, \infty)$ and satisfying (2) can be proved in the similar way.

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