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Smooth linearization of vector fields near invariant manifolds

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1. Introduction

Linearization of vector fields and diffeomorphisms at a hyperbolic fixed point has been investigated by many authors. In this theory, one tries to locally reduce nonlinear vector fields and diffeomorphisms to linear ones. There are, roughly speaking, two groups of works according to the smoothness class to which the reduction belongs. One group tries to find a smooth (meaning C^r, $r \geq 1$) conjugacy under so-called non-resonance conditions [12] [13] [1] [11]. The second group seek a homeomorphism which conjugates a nonlinear vector field to a linear one [3] [2].

The idea of linearization around fixed points was extended by Pugh and Shub [6] to that around normally hyperbolic invariant manifolds. A similar result was later obtained by Osipenko [4] [5]. In both of the work by Pugh-Shub and that of Osipenko, the conjugacy between nonlinear vector fields (or diffeomorphisms) and linear ones is a homeomorphism. In this regard, their works fall into the second group in the above.

The purpose of this paper is to show that there are situations in which vector fields can be smoothly ($C^r, r > 0$) linearzed in a neighborhood of normally hyperbolic invariant manifolds. The conditions to be placed on the linear part of the vector fields in this paper are considered as a kind of non-resonance conditions. This work therefore falls into the first group in the above. Non-resonance conditions on the linear part of vector fields at fixed points are easy to state because they are algebraic relations between the eigenvalues of a matrix. When one deals with the linear part of vector fields near invariant manifolds, the eigenvalues of matrices are of little use except for special situations, e.g., singularly perturbed vector fields, see [8]. In this paper the non-resonance conditions will be given in terms of a certain relationship among growth and decay rates of solutions of linear differential systems. These can be regarded as gap conditions on the spectra of invariant manifolds in the sense of Sacker and Sell [7].

Although analyses are given to vector fields in this paper, the ideas and techniques employed can readily be modified to treat diffeomorphisms. It is also straight-forward to deal with non-autonomous vector fields based on the same ideas as the present work.

The organization of this paper is this: In section 2, main result is presented after the precise formulation of the probem. This is followed by several remarks. The proof of the main theorem is given in section 3. Section 4 is devoted to the proof of technical lemmas employed in section 3. To conclude this introduction, notations used in the sequel follow: $\dot{x}$ means the derivative of a function $x(t)$ with respect to time t. D_j means the derivative of a function with respect to j-th argument. For example, $D_2 g(x, y, z)$ stands for the Jacobian of g with respect to y. The expression $D_2 g(x, y, z)$ is sometimes denoted by $D_y g(x, y, z)$ if there is no chance of confusion. D without any subscript means the derivative of a function with respect to all the arguments. For example, $Dg(x, y, z)$ is the Jacobian of $g(x, y, z)$ with respect to (x, y, z). The symbol $gl(\mathbf{R}^n)$ stands for the set of n by n matrices with real entries, and $I_{n \times n}$ stands for the identity matrix.

2. Main Theorem

Let $\mathbf{M}$ be a compact manifold of class C^{r+1} and consider the following system of equations

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{u} = A(x)u + G(x, u) \end{cases} \tag{2.1}$$

where $x \in \mathbf{M}$, and $u \in \mathbf{R}^n$. We assume throughout the paper that f, A, G are C^r bounded functions with $r \geq 3$. Moreover, we assume $G(x, u) = O(|u|^2)$ as $|u| \to 0$ and we consider the system (2.1) as a perturbation of the following

$$\begin{cases} \dot{x} = f(x) \\ \dot{u} = A(x)u \end{cases} \quad \text{where } f(x) = f(x, 0). \tag{2.2}$$

Since we are interested in the flow structure of (2.1) near $u = 0$, we can, and will, assume the following:

For $\varepsilon > 0$ *small,*

$$\begin{cases} |G(x, u)| \leq M\varepsilon^2, \ |DG(x, u)| \leq M\varepsilon & \text{for } (x, u) \in \mathbf{M} \times \mathbf{R}^n \\ G(x, u) \equiv 0 & \text{for } (x, u) \notin \mathbf{M} \times \mathbf{D}(\varepsilon) \end{cases}$$

where $\mathbf{D}(\varepsilon) = \{u \in \mathbf{R}^n; |u| < \varepsilon\}$. Therefore, we are interested in the flow structure of (2.1) in the ε-neighborhood $\mathbf{M} \times \mathbf{D}(\varepsilon)$ of $\mathbf{M}$ in $\mathbf{M} \times \mathbf{R}^n$. We will also assume that $\mathbf{M}$ is embedded in $\mathbf{R}^m$ for some $m > 0$.

Let $\phi(t, x)$ stand for the flow on $\mathbf{M}$ generated by the first equation in

(2.2). We denote by $\Phi(t, s; x)$ the fundamental solution operator of the linear system

$$\dot{X} = Df(\phi(t, x))X. \tag{2.3}$$

Let $U(t, s; x)$ be the fundamental solution operator of

$$\dot{U} = A(\phi(t, x))U. \tag{2.4}$$

Our basic condition is that (2.4) has an exponential dichotomy on the whole line $\mathbf{R}$ uniformly in $x \in \mathbf{M}$ and that the dynamics of (2.3) is dominated by that of (2.4). More precisely, we assume:

(H1) There exist families of projections $P(x), Q(x): \mathbf{R}^n \to \mathbf{R}^n$ for $x \in \mathbf{M}$ such that $P(x) + Q(x) = I_{n \times n}$ $x \in \mathbf{M}$,

$$U(t, s; x)P(\phi(s, x)) = P(\phi(t, x))U(t, s; x)$$
$$U(t, s; x)Q(\phi(s, x)) = Q(\phi(t, x))U(t, s; x)$$

and *Rank* $P(x) = k$, *Rank* $Q(x) = l$, $x \in \mathbf{M}$, $k + l = n$. Moreover, there are constants $0 < \delta < \alpha < \beta$ and $K \geq 1$ such that $\alpha > r\delta$, (where r is the degree of smoothness of the functions f, A, G) and that

$$\begin{cases} K^{-1}e^{-\delta|t-s|} \leq |\Phi(t, s; x)| \leq Ke^{\delta|t-s|} & t, s \in \mathbf{R} \\ K^{-1}e^{-\beta(t-s)} \leq |U(t, s; x)P(\phi(s, x))| \leq Ke^{-\alpha(t-s)} & t \geq s \\ K^{-1}e^{\beta(t-s)} \leq |U(t, s; x)Q(\phi(s, x))| \leq Ke^{\alpha(t-s)} & t \leq s \end{cases} \tag{2.5}$$

In the language of the spectral theory of Sacker and Sell [7], the condition (2.5) is the same as saying that the tangential spectrum Σ_T and the normal spectrum Σ_N are sepatated as follows:

$$\Sigma_T \subset (-\delta, \delta),\ \Sigma_N \subset (-\beta, -\alpha) \cup (\alpha, \beta).$$

Under the condition (H1), let E^s and E^u be defined by:

$$E^s = \{(x, u) \in \mathbf{M} \times \mathbf{R}^n;\ \sup_{t \geq 0} |U(t, 0; x)u| e^{\alpha t} < \infty\}$$

$$E^u = \{(x, u) \in \mathbf{M} \times \mathbf{R}^n;\ \sup_{t \leq 0} |U(t, 0; x)u| e^{-\alpha t} < \infty\}.$$

These are, respectively, called the stable and the unstable bundles. It is known [8] [14] that under the condition (H1) E^s and E^u are C^r-vector bundles over the base $\mathbf{M}$. Without losing generality, we can assume that these bundles are trivial ones. The reason is this: If d_1 and d_2 are, respectively, the dimension of trivializing bundles of E^s and E^u, namely

$$E^s \oplus (\mathbf{M} \times \mathbf{R}^{d_1}) = \mathbf{M} \times \mathbf{R}^{d_1 + k} \quad \text{and} \quad E^u \oplus (\mathbf{M} \times \mathbf{R}^{d_2}) = \mathbf{M} \times \mathbf{R}^{d_2 + l},$$

then we can extend the system (2.1) by adding extra equations $\dot{v} = -\lambda v$ and $\dot{w} = \lambda w$ where $v \in \mathbf{R}^{d_1}$, $w \in \mathbf{R}^{d_2}$ and $\lambda = (\alpha + \beta)/2$. This modification does not affect the earlier conditions on (2.1) or the condition (H1). Therefore we assume the following condition.

(H2) The vector bundles E^s and E^u are trivial.

Under the conditions (H1) and (H2), one can introduce a new coordinate system in terms of E^s and E^u. In this new coordinate system, the equations in(2.1) can be expressed as

$$\begin{cases} \dot{x} = f(x, y, z) \\ \dot{y} = B(x)y + g(x, y, z) \\ \dot{z} = C(x)z + h(x, y, z) \end{cases} \tag{2.6}$$

where $y \in \mathbf{R}^k$, $z \in \mathbf{R}^l$, $f(x, 0, 0) = f(x)$. The functions on the right hand side of (2.6) are C^r-bounded. When one transforms a system of differential equations by using a diffeomorphism, the vector field in the new system usually has less smoothness than the original. This loss of smoothness does not happen in the present case. The reason is the following: Although $DP(x)$ is only C^{r-1}-bounded, the product $[DP(x)]f(x)$ is C^r-bounded. This can be verified by differentiating the relation

$$U(t, 0; x)P(x) = P(\phi(t, x))U(t, 0; x)$$

at $t = 0$. The same remark holds for $Q(x)$. Since we use linearly independent column vectors of the projection matrices $P(x)$ and $Q(x)$ to define our coordinate transformation, the loss of smoothness does not happen. The functions g and h share the properties of G in (2.1). Namely, we have:

$$g \equiv 0, \ h \equiv 0 \text{ outside the set } \mathbf{M} \times \mathbf{D}(\varepsilon)$$

$$|g| \leq M\varepsilon^2, \ |h| \leq M\varepsilon^2, \ |Dg| \leq M\varepsilon, \ |Dh| \leq M\varepsilon.$$

The linear part of (2.6) has the same properties as that of (2.2). If we denote by $Y(t, s; x)$ and $Z(t, s; x)$ the fundamental solution operators of $\dot{y} = B(\phi(t, x))y$ and $\dot{z} = C(\phi(t, x))z$ then the following estimates are valid.

$$\begin{aligned} &K^{-1}e^{-\delta|t-s|} \leq |\Phi(t, s; x)| \leq Ke^{\delta|t-s|} && t, s \in \mathbf{R} \\ &K^{-1}e^{-\beta(t-s)} \leq |Y(t, s; x)| \leq Ke^{-\alpha(t-s)} && t \geq s \\ &K^{-1}e^{\beta(t-s)} \leq |Z(t, s; x)| \leq Ke^{\alpha(t-s)} && t \leq s \end{aligned} \tag{2.7}$$

We are now in a position to state the main result of this paper.

THEOREM. *Suppose the conditions* (H1) *and* (H2) *are satisfied. For* $\varepsilon > 0$ *sufficiently small the following are true.*

(A) *If the constants* α, β, *and* δ *in* (H1) *satisfy* $2\alpha - \beta > (r-1)\delta$, *then there is a* C^1*-bounded coordinate transformation in* $\mathbf{M} \times \mathbf{D}(\varepsilon)$ *which brings the equation* (2.6) *into*

$$\begin{cases} \dot{x} = f(x) + F(x, y, z) \\ \dot{y} = B(x)y \\ \dot{z} = C(x)z \end{cases} \tag{2.8}$$

where F *is a* C^1*-bounded function satisfying* $F(x, 0, z) = 0 = F(x, y, 0)$.

(B) *If* $2\alpha - \beta > (r-1)\delta$ *is satisfied and either* $Rank\, P = 0$ *or* $Rank\, Q = 0$ *holds, then there is a* C^{r-1}*-bounded transformation in* $\mathbf{M} \times \mathbf{D}(\varepsilon)$ *which brings the equation* (2.1) *into*

$$\begin{cases} \dot{x} = f(x) \\ \dot{y} = A(x)u. \end{cases} \tag{2.9}$$

The proof of Theorem (A)(B) will be given in the following two sections. The result in Theorem (A) calls for a special attention. The result by Pugh and Shub [6] and by Osipenko [4, 5] says that if $\alpha > \delta$ is satisfied there is a homeomorphism which conjugates the system (2.6) and the system (2.8) with $F(x, y, z) \equiv 0$. Let us say in general that the system (2.1) is C^r-synchronized with the flow on $\mathbf{M}$ if there is a C^r-diffeomorphism (or homeomorphism if $r = 0$) which transforms the system (2.1) into a form

$$\begin{cases} \dot{x} = f(x) \\ \dot{u} = H(x, u). \end{cases}$$

Theorem (A) indicates that in general the flow near a normally hyperbolic invariant manifold $\mathbf{M}$ is not C^1-synchronized with the flow on $\mathbf{M}$. The flow on the stable and unstable manifolds of $\mathbf{M}$, however, can be smoothly synchronized with the flow on $\mathbf{M}$. In fact, Theorem (B) and its proof show that the flow on $\mathbf{W}^s(\mathbf{M})$ and $\mathbf{W}^u(\mathbf{M})$ are C^{r-1}-synchronized with that on $\mathbf{M}$ if the original vector field is C^r, $r \geq 2$.

The extra condition $2\alpha - \beta > (r-1)\delta$ in Theorem (A)(B) can be considered as a non-resonance condition. An example by Hartman [3] says that such a condition cannot be dropped in general to have a C^1-linearization of flow near, even, hyperbolic fixed points. The condition $2\alpha - \beta > (r-1)\delta$, when specialized to the case where $\mathbf{M}$ is a point, is not satisfied by the example of Hartman [3]. It is natural to try to write down more general non-resonance conditions which ensure C^1-linearization. Such a generalization can be done by extending the techniques of Samovol [10] developed for linearization around fixed points.

One of the technical difficulties in the proof of Theorem (A)(B) is that the matrices $A(x)$, $B(x)$, $C(x)$ are not constant (in Samovol's work [10], these matrices are constant). This difficulty is overcome by Henry's perturbation theorem for exponential dichotomies and some of its consequences. An idea of Bo Deng is adapted to seek a special kind of coordinate changes leading to Lemmas 3.1 and 3.2.

3. Proof of Theorem (A)(B)

Throughout the rest of the article, we use the exponential estimates in (2.7) frequently. The outline of the proof is this: We first introduce a coordinate transformation which have the effect of straightening the stable and unstable fibers of **M**. We then linearize the flows on $\mathbf{W}^s(\mathbf{M})$ and $\mathbf{W}^s(\mathbf{M})$. Up to this point, the coordinate transformations are C^{r-1} (and hence the proof of (B) is complete). Finally, we linearize the flow in a full neighborhood of **M**. In the last step, the coordinate change is C^1.

3.1. Coordinate change by using the stable and the unstable manifolds

The system (2.6) has the stable manifold $\mathbf{W}^s(\mathbf{M})$ and the unstable manifold $\mathbf{W}^u(\mathbf{M})$. We refer the readers to [8] and [14] for the properties of these manifolds. Since the nonlinear terms in (2.6) has been modified appropriately outside a small neighborhood of **M**, local and global manifolds need not be distinguished. The unstable manifold $\mathbf{W}^u(\mathbf{M})$ is the union of unstable fibers $\mathbf{W}^u(\xi)$, $\xi \in \mathbf{M}$,

$$\mathbf{W}^u(\mathbf{M}) = \bigcup_{\xi \in \mathbf{M}} \mathbf{W}^u(\xi),$$

where $\mathbf{W}^u(\xi)$ is expressed as the graph of functions,

$$\mathbf{W}^u(\xi) = \{(x, y, z);\ x = q(\xi, z),\ y = p(\xi, z)\}.$$

Under the condition (H1) the functions q and p are C^r-bounded and satisfy $q(\xi, 0) = \xi$, $D_2 q(\xi, 0) = 0$, $p(\xi, 0) = 0$, $D_2 p(\xi, 0) = 0$. The fibe $\mathbf{W}^u(\xi)$ through a point $\xi \in \mathbf{M}$ has the following characterization. Let $(x, y, z) \cdot t = (x \cdot t, y \cdot t, z \cdot t)$ represent the flow generated by (2.6). Then for each $\gamma \in (\delta, \alpha)$,

$$\mathbf{W}^u(\xi) = \{(x, y, z);\ \sup_{t \le 0} |x \cdot t - \phi(t, x)| e^{-\gamma t} < \infty,$$
$$\sup_{t \le 0} |y \cdot t| e^{-\gamma t} < \infty,\ \sup_{t \le 0} |z \cdot t| e^{-\gamma t} < \infty\},$$

where $\phi(t, x)$ represents the flow on **M**. As we have assumed that the manifold **M** is embedded in $\mathbf{R}^m$ for some $m > 0$, we used the notation $|x \cdot t - \phi(t, x)|$ instead of the distance between two points $x \cdot t$, $\phi(t, x)$ on **M**. Moreover,

$\mathbf{W}^u(\phi(t, x)) = \mathbf{W}^u(\xi) \cdot t$ holds true. In terms of the functions p and q, the last fact translates into the following. If $x_0 = q(\xi, z_0)$, $y_0 = p(\xi, z_0)$ for $\xi \in \mathbf{M}$ then $(x_0 \cdot t, y_0 \cdot t, z_0 \cdot t)$ satisfies $x_0 \cdot t = q(\phi(t, x), z_0 \cdot t)$, $y_0 \cdot t = p(\phi(t, x), z_0 \cdot t)$. Therefore, by differentiating with respect to t, one finds that $p(\xi, z)$ and $q(\xi, z)$ satisfy the functional equations

$$\begin{aligned} &D_1 q(\xi, z) f(\xi) + D_2 q(\xi, z)[C(q(\xi, z))z + h(q(\xi, z), p(\xi, z), z)] \\ &= f(q(\xi, z), p(\xi, z), z), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &D_1 p(\xi, z) f(\xi) + D_2 p(\xi, z)[C(q(\xi, z))z + h(q(\xi, z), p(\xi, z), z)] \\ &= B(q(\xi, z))p(\xi, z) + g(q(\xi, z), p(\xi, z), z). \end{aligned} \tag{3.2}$$

Change the variables in (2.6) through $(x, y, z) \to (\bar{x}, \bar{y}, \bar{z})$ defined by $x = q(\bar{x}, \bar{z})$, $y = \bar{y} + p(\bar{x}, \bar{z})$, $z = \bar{z}$. By using the functional equations (3.1) and (3.2) one finds the equations for $(\bar{x}, \bar{y}, \bar{z})$ as follows. The bars are dropped from $(\bar{x}, \bar{y}, \bar{z})$.

$$\begin{cases} \dot{x} = f(x) + f_0(x, y, z) = f_1(x, y, z) \\ \dot{y} = B(x)y + g_0(x, y, z)y \\ \dot{z} = C(x)z + \bar{h}(x, y, z) \end{cases} \tag{3.3}$$

where f_0, g_0, and $\bar{h}$ are given by

$$\begin{aligned} f_0(x, y, z) &= [D_1 q]^{-1}[f(q, y + p, z) - f(q, p, z)] \\ &\quad - [D_1 q]^{-1} D_2 q[h(q, y + p, z) - h(q, p, z)], \\ g_0(x, y, z) &= B(q) - B(x) + \int_0^1 D_2 g(q, sy + p, z)ds - D_2 p \int_0^1 D_2 h(q, sy + p, z)\,ds \\ &\quad - D_1 p[D_1 q]^{-1}\left[\int_0^1 D_2 f(q, sy + p, z)\,ds - D_2 q \int_0^1 D_2 h(q, sy + p, z)ds\right], \\ \bar{h}(x, y, z) &= h(q, y + p, z) + [C(q) - C(x)]x. \end{aligned}$$

Here the functions p, q, $D_j p$, $D_j q$ are evaluated at (x, z). One should notice that f_0, g_0 and $D_2 f_0$ are C^{r-1}-bounded and $\bar{h}$ is C^r-bounded. Moreover, $f_0(x, 0, z) \equiv 0$. This and the fact that the second equation in (3.3) vanishes for $y = 0$ mean that the unstable manifold $\mathbf{W}^u(\mathbf{M})$ for (3.3) is given by $\mathbf{M} \times \{0\} \times \mathbf{R}^l$ and that the flow on $\mathbf{W}^u(\mathbf{M})$ is synchronized with that on $\mathbf{M}$.

The stable fibers of $\mathbf{M}$ for the equation (3.3) (not for (2.6)) are now used to further simplify (3.3). Let $\mathbf{W}^s(\mathbf{M})$ be the stable manifold of $\mathbf{M}$ for the system (3.3). Let $\mathbf{W}^s(\mathbf{M}) = \bigcup_{\xi \in \mathbf{M}} \mathbf{W}^s(\xi)$ be the decomposition of $\mathbf{W}^s(\mathbf{M})$ into the stable fibers. Each fiber has a dynamical characterization as before, and can be expressed as a graph of functions r and s:

$$\mathbf{W}^s(\xi) = \{(x, y, z) \colon x = r(\xi, y),\ z = s(\xi, y)\}.$$

Under the condition (H1), r and s are C^{r-1}-bounded and satisty

$$r(\xi, 0) = \xi,\ D_2 r(\xi, 0) = 0,\ s(\xi, 0) = 0,\ D_2 s(\xi, 0) = 0.$$

These functions satisfy functional equations similar to those in (3.1) and (3.2). In terms of these functions, a new coordinate system $(\bar{x}, \bar{y}, \bar{z})$ is introduced via $x = r(\bar{x}, \bar{y})$, $y = \bar{y}$, $z = \bar{z} + s(\bar{x}, \bar{y})$. With the help of the functinal equations mentioned above, the equations for the new variables become, after dropping the bars, as follows.

$$\begin{cases} \dot{x} = f(x) + f_2(x, y, z) \\ \dot{y} = B(x)y + g_1(x, y, z)y \\ \dot{z} = C(x)z + h_1(x, y, z)z \end{cases} \tag{3.4}$$

where

$$\begin{aligned} f_2(x, y, z) &= [D_1 f]^{-1}[f_1(r, y, z+s) - f_1(r, y, s)] \\ &\quad - [D_1 r]^{-1} D_2 r[g_0(r, y, z+s)y - g_0(r, y, s)y], \\ g_1(x, y, z) &= [B(r) - B(x)] + g_0(r, y, z+s), \\ h_1(x, y, z) &= C(r) - C(x) + \int_0^1 D_3 \bar{h}(r, y, \tau z + s)d\tau - D_2 s \int_0^1 D_3 g_0(r, y, \tau z + s)y d\tau \\ &\quad - D_1 s[D_1 r]^{-1}\left[\int_0^1 D_3 f_1\,(r, y, \tau z + s)d\tau \right. \\ &\quad \left. - D_2 r \int_0^1 D_3 g_0(r, y, \tau z + s)y d\tau \right]. \end{aligned}$$

The functions r, s, $D_j r$, $D_j s$ are evaluated at (x, y). It is easy to verify that $f_2(x, 0, z) = 0 = f(x, y, 0)$, that g_1 is C^{r-1}-bounded, and that f_2, h_1 are C^{r-2}-bounded. The stable manifold $\mathbf{W}^s(\mathbf{M}) = \mathbf{M} \times \mathbf{R}^k \times \{0\}$ and the unstable manifold $\mathbf{W}^u(\mathbf{M}) = \mathbf{M} \times \{0\} \times \mathbf{R}^l$ for the system (3.4) are straightened.

3.2. Linearization on $\mathbf{W}^s(\mathbf{M})$ and on $\mathbf{W}^u(\mathbf{M})$

The equations for the flows on $\mathbf{W}^s(\mathbf{M})$ and $\mathbf{W}^u(\mathbf{M})$ for (3.4) are given, respectively, by

$$\dot{x} = f(x),\ \dot{y} = B(x)y + g_1(x, y, 0)y, \tag{3.5}$$

$$\dot{x} = f(x),\ \dot{z} = C(x)z + h_1(x, 0, z)z. \tag{3.6}$$

Here one should observe that $g_1(x, y, 0)$ and $h_1(x, 0, z)$ are C^{r-1}-bounded functions. To see this for $h_1(x, 0, z)$, recall that $D_2 s(x, 0) = 0$, $s(x, 0) = 0$. Therefore,

$$h_1(x, 0, z) = \int_0^1 D_3\bar{h}(x, 0, \tau z)d\tau,$$

and $\bar{h}$ is C^r-bounded, and hence $h_1(x, 0, z)$ is C^{r-1}-bounded.

We will show that there is a coordinate transformation which linearizes the system (3.5) and (3.6) simultaneously. We look for the change of coordinates in the form $\bar{y} = y + P(x, y)y$, $\bar{z} = z + Q(x, z)z$, where P and Q are matrix-valued functions (different from those appearing in section 2) satisfying $P(x, 0) = 0$, $Q(x, 0) = 0$. They are also required to be C^{r-1}-bounded. We will show the existence of such P and Q. Since the argument is the same for both P and Q we only give a treatment for P in detail.

The equation for $\bar{y}$ is explicitly given by

$$\dot{\bar{y}} = \ = B(x)\bar{y} + [\dot{P} - B(x)P + PB(x) + g_1(x, y, 0) + Pg_1(x, y, 0)]y,$$

in which $\dot{P}$ stands for the derivative of $P(x, y)$ along the solution of (3.5). If the function P can be chosen so that the quantity inside the square bracket in the equation for $\bar{y}$ vanishes, then the equation for $\bar{y}$ becomes linear in $\bar{y}$ as desired. In order to show the existence of such a function, we consider the following differential equation,

$$\begin{aligned}\dot{p} = B(\phi(t, x))p - p[B(\phi(t, x)) + g_1(\phi(t, x), \psi(t, x, y), 0)] \\ - g_1(\phi(t, x), \psi(t, x, y), 0)\end{aligned} \tag{3.7}$$

where p is in $gl(\mathbf{R}^k)$ and $(\phi(t, x), \psi(t, x, y))$ is the unique solution of (3.5) with the initial value (x, y). The equation (3.7) is linear inhomogeneous. The fundamental solution operator of the principal part of this equation is given by

$$\mathbf{L}(t, s; x, y)(p) = Y(t, s; x)pY(s, t; x, y),$$

where $Y(t, s; x)$ is the fundamental solution operator of $\dot{y} = B(\phi(t, x))y$ and $Y(s, t; x, y)$ is that of $\dot{y} = [B(\phi(t, x)) + g_1(\phi(t, x), \psi(t, x, y), 0)]y$. Therefore, by applying the variation of constants formula to (3.7), we have

$$\begin{aligned}p(t) = {} & Y(t, s; x)p(s)Y(s, t; x, y) \\ & - \int_s^t Y(t, \tau; x)g_1(\phi(\tau, x), \psi(\tau, x, y), 0)Y(\tau, t; x, y)d\tau.\end{aligned}$$

If we seek a solution of (3.7) which satisfies $|Y(t, s; x)p(s)Y(s, t; x, y)| \to 0$ as $s \to \infty$, then, by using the estimates in (2.7), we have

$$p(t) = - \int_\infty^t Y(t, \tau; x)g_1(\phi(\tau, x), \psi(\tau, x, y), 0)Y(\tau, t; x, y)d\tau.$$

We now define our function P as the initial value of this special solution.

$$P(x, y) = p(0) = -\int_{\infty}^{t} Y(0, \tau; x) g_1(\phi(\tau, x), \psi(\tau, x, y), 0) Y(\tau, 0; x, y) d\tau. \tag{3.8}$$

The following lemma completes the proof of Theorem (B).

LEMMA 3.1. *Under the conditions* (H1)(H2) *and* $2\alpha - \beta > (r-1)\delta$, *the formula* (3.8) *defines a* C^{r-1}*-bounded function* $P(x, y)$ *satisfying* $P(x, 0) = 0$ *and*

$$\dot{P} = B(x)P - P[B(x) + g_1(x, y, 0)] - g_1(x, y, 0),$$

where $\dot{P}$ *is the derivative of* $P(x, y)$ *along the solution of* (3.5).

The proof of this lemma will be given in section 4. By using this function, we achieve the linearization of the equation (3.5) on $\mathbf{W}^s(\mathbf{M})$. The argument for $Q(x, z)$ is similar.

3.3. Proof of Theorem (A)

If we change variables in (3.4) in terms of $\bar{y} = y + P(x, y)y$, $\bar{z} = z + Q(x, z)z$ with the C^{r-1}-bounded functions P and Q obtained in the previous subsection, we find the equations for the new variables $(x, \bar{y}, \bar{z})$. By using the functional equations that P and Q satisfy, the equations for $(x, \bar{y}, \bar{z})$ are given as follows. Here, we again drop the bars from $\bar{y}$ and $\bar{z}$.

$$\begin{cases} \dot{x} = f(x) + f_3(x, y, z) \\ \dot{y} = B(x)y + g_2(x, y, z)z \\ \dot{z} = C(x)z + h_2(x, y, z)y. \end{cases} \tag{3.9}$$

The formulae for g_2 and h_2 are a little more complicated than those for g_1 and h_1. Let us define $\bar{g}_1$ and $\bar{h}_1$ by

$$\bar{g}_1(x, y, z) = [I_{k\times k} + P(x, y)][g_1(x, y, z) - g_1(x, y, 0)]y,$$
$$\bar{h}_1(x, y, z) = [I_{l\times l} + Q(x, z)][h_1(x, y, z) - h_1(x, 0, z)]z.$$

Let $(x, \bar{y}, \bar{z}) \to (x, \bar{P}(x, \bar{y}), \bar{Q}(x, \bar{z}))$ be the inverse map of

$$(x, y, z) \longrightarrow (x, \bar{y}, \bar{z}) = (x, y + P(x, y)y, z + Q(x, z)z).$$

The inverse exists in $\mathbf{M} \times \mathbf{D}(\varepsilon)$ for small $\varepsilon > 0$, since $P(x, 0) = 0$ and $Q(x, 0) = 0$ hold. The functions $\bar{P}$ and $\bar{Q}$ are also C^{r-1}-bounded. Now we define

$$\tilde{g}_1(x, y, z) = \bar{g}_1(x, \bar{P}(x, y), \bar{Q}(x, z)), \quad \tilde{h}_1(x, y, z) = \bar{h}_1(x, \bar{P}(x, y), \bar{Q}(x, z)).$$

In terms of these, g_2 and h_2 are defined by

$$g_2(x, y, z) = \int_0^1 D_3\tilde{g}_1(x, y, \tau z)d\tau, \quad h_2(x, y, z) = \int_0^1 D_2\tilde{h}_1(x, \tau y, z)d\tau,$$

and f_3 is given by

$$f_3(x, y, z) = f_2(x, \bar{P}(x, y), \bar{Q}(x, z)).$$

These functions have the following properties.

f_3 is C^{r-1}-bounded,
g_2 and h_2 are C^{r-2}-bounded,
$f_3(x, 0, z) = 0 = f_3(x, y, 0)$, $g_2(x, 0, z) = 0$, $h_2(x, y, 0) = 0$.

So far, we have transformed the equation (2.6) into (3.9) in terms of a C^{r-1} diffeomorphism. We shall transform (3.9) into (2.8). We can, however, do this final step with a C^1-diffeomorphism. We seek our change of variables in the form: $\bar{y} = y + R(x, y, z)z$, $\bar{z} = z + S(x, y, z)y$, where R and S are C^1-functions (matrix-valued) to be determined, satistying $R(x, 0, 0) = 0$, $S(x, 0, 0) = 0$. The equations for $\bar{y}$ and $\bar{z}$ are given by

$$\dot{\bar{y}} = B(x)\bar{y} + [\dot{R} - B(x)R + RC(x) + g_2(x, y, z) + R\bar{h}_2(x, y, z)]z,$$
$$\dot{\bar{z}} = C(x)\bar{z} + [\dot{S} - C(x)S + SB(x) + h_2(x, y, z) + S\bar{g}_2(x, y, z)]y,$$

where $\dot{R}$ and $\dot{S}$ stand for the derivative of R and S along the solution of (3.9) and $\bar{g}_2$ and $\bar{h}_2$ are defined by

$$\bar{g}_2(x, y, z) = \int_0^1 D_2\tilde{g}_1(x, \tau y, z)d\tau, \quad \bar{h}_2(x, y, z) = \int_0^1 D_3\tilde{h}_1(x, y, \tau z)d\tau.$$

It is easy to verify:

$$g_2(x, y, z)z = \bar{g}_2(x, y, z)y, \quad h_2(x, y, z)y = \bar{h}_2(x, y, z)z,$$

and $\bar{g}_2$, $\bar{h}_2$ are C^{r-1}-bounded. Moreover, we have

$$g_2(x, 0, z) = 0, \ \bar{g}_2(x, y, 0) = 0, \ h_2(x, y, 0) = 0, \ \bar{h}_2(x, 0, z) = 0.$$

We will show the existence of functions R and S satisfying

$$\dot{R} - B(x)R + RC(x) + g_2(x, y, z) + R\bar{h}_2(x, y, z) = 0, \tag{3.10}$$

$$\dot{S} - C(x)S + SB(x) + h_2(x, y, z) + S\bar{g}_2(x, y, z) = 0. \tag{3.11}$$

The idea of proof is the same as that for P and Q used in the previous subsection. Namely, we define our functions R and S as the initial value of a special solution to (3.10) and (3.11) respectively. Let $w(t) = (\xi(t), \eta(t), \zeta(t))$ denote the solution of (3.9) with the initial condition $w(0) = (x, y, z) = u \in$

$\mathbf{M} \times \mathbf{D}(\varepsilon)$. The matrices

$$Y_0(t, s; u),\ Z_0(t, s; u),\ Y_1(t, s; u),\ Z_1(t, s; u)$$

respectively stand for the fundamental solution operators of

$$\dot{y} = B(\xi(t))y,\ \dot{z} = C(\xi(t))z,\ \dot{y} = [B(\xi(t)) + \bar{g}_2(w(t))]y,\ \dot{z} = [C(\xi(t)) + \bar{h}_2(w(t))]z.$$

With these notations in place, $R(u) = R(x, y, z)$ and $S(u) = S(x, y, z)$ are defined by

$$R(u) = -\int_{-\infty}^{0} Y_0(0, t; u) g_2(w(t)) Z_1(t, 0; u) dt, \tag{3.12}$$

$$S(u) = -\int_{\infty}^{0} Z_0(0, t; u) h_2(w(t)) Y_1(t, 0; u) dt, \tag{3.13}$$

where $u = (x, y, z) \in \mathbf{M} \times \mathbf{D}(\varepsilon)$. The following lemma will be proven in section 4.

LEMMA 3.2. *If the conditions* (H1)(H2) *and* $2\alpha - \beta > (r - 1)\delta$ *are satisfied, then* (3.12)(3.13) *define* C^1*-functions satisfying* $R(x, 0, z) = 0$, $S(x, y, 0) = 0$ *and the equations* (3.10)(3.11).

Let $(x, \bar{y}, \bar{z}) \to (x, \bar{R}(x, \bar{y}, \bar{z}), \bar{S}(x, \bar{y}, \bar{z}))$ be the inverse of the map

$$(x, y, z) \longrightarrow (x, y + R(x, y, z)z, z + S(x, y, z)y).$$

The inverse exists in $\mathbf{M} \times \mathbf{D}(\varepsilon)$ and is C^1. Since $R(x, 0, z) = 0$ and $S(x, y, 0) = 0$ are valid, one can easily verify that $\bar{R}(x, 0, \bar{z}) = 0$ and $\bar{S}(x, \bar{y}, 0) = 0$. We have therefore transformed the system (3.9) into (2.8) where

$$F(x, y, z) = f_3(x, \bar{R}(x, y, z), \bar{S}(x, y, z)).$$

Since $f_3(x, y, 0) = 0 = f_3(x, 0, z)$, $\bar{S}(x, y, 0) = 0$, and $\bar{R}(x, 0, z) = 0$, the function F has the desired property $F(x, 0, z) = 0 = F(x, y, 0)$. This complete the proof of Theorem (A).

4. Proof of technical lemmas

The proof of Lemma 3.1 and that of Lemma 3.2 are given in this section. The ideas and techniques used are the same for both cases. The exponential estimates in (2.7) are used frequently in the below.

4.1. Proof of Lemma 3.1

Recall the equation (3.5);

$$\dot{x} = f(x), \quad \dot{y} = B(x)y + g_1(x, y, 0)y.$$

Note that the following estimates are valid for $i \geq 1,\ j \geq 1,\ i + j \leq r - 1$:

$$|g_1(x, y, 0)| \leq M|y|, \quad |D_1^j g_1(x, y, 0)| \leq M|y|,$$
$$|D_2 g_1(x, y, 0)| \leq M, \quad |D_1^i D_2^j g_1(x, y, 0)| \leq M.$$

We also have

$$|g_1(x, y, 0)| \leq M\varepsilon, \quad |D_1^j g_1(x, y, 0)| \leq M\varepsilon, \quad j = 1,\dots,r-1.$$

PROPOSITION 4.1. *Let $(\phi(t, x), \psi(t, x, y))$ be the solution of* (3.5) *with the initial value (x, y) at $t = 0$. There exist constants $M_j > 0$, $j = 1,\dots,r-1$, which depend only on j, k, α, β, δ, and ε such that the following statements are true.*

(a) *The functions $\phi(t, x)$, $\psi(t, x, y)$ are C^{r-1} in $(x, y) \in \mathbf{M} \times \mathbf{R}^k$ and satisfy*

$$|\psi(t, x, y)| \leq K|y|e^{-(\alpha - \varepsilon MK)t}, \qquad t \geq 0,$$
$$|D_x^j \phi(t, x)| \leq M_j e^{j\delta t}, \qquad t \geq 0,\ j = 1,\dots,r-1,$$
$$|D_x^i D_y^j \psi(t, x, y)| \leq M_{i+j} e^{-(\alpha - \varepsilon KM)t} e^{i\delta t}, \qquad t \geq 0,\ i + j = 1,\dots,r-1.$$

(b) *Let $Y(t, s; x)$ and $Y(t, s; x, y)$ be as defined before* Lemma 3.1. *For $t, s \geq 0$, these functions are C^{r-1} in (x, y) and satisfy*

$$|D_x^j Y(t, s; x)| \leq M_j e^{-\alpha(t-s) + j\delta t}, \qquad j = 0, 1,\dots,r-1, \quad t \geq s,$$
$$|D_x^j Y(t, s; x)| \leq M_j e^{\beta(s-t) + j\delta s}, \qquad j = 0, 1,\dots,r-1, \quad t \leq s,$$
$$|D_x^i D_y^j Y(t, s; x, y)| \leq M_{i+j} e^{-(\alpha - 2\varepsilon KM)(t-s)} e^{(i+j)\delta t}, \qquad i + j = 0, 1,\dots,r-1, \ t \geq s,$$
$$|D_x^i D_y^j Y(t, s; x, y)| \leq M_{i+j} e^{(\beta + 2\varepsilon KM)(s-t)} e^{(i+j)\delta s}, \qquad i + j = 0, 1,\dots,r-1, \ t \leq s,$$

The proof of this proposition will be given after we complete the proof of Lemma 3.1. Now choose $\varepsilon > 0$ so small that $2\alpha - \beta - (r-1)\delta > 3\varepsilon MK$. Recall the definition of $P(x, y)$.

$$P(x, y) = -\int_\infty^0 Y(0, \tau; x) g_1(\phi(\tau, x), \psi(\tau, x, y), 0) Y(\tau, 0; x, y) d\tau.$$

We immediately obtain the following by using Proposition 4.1.

$$|P(x, y)| \leq \int_0^\infty K e^{\beta t} M |\psi(t, x, y)| K e^{-\alpha t} dt \leq |y| K^3 M / (2\alpha - \beta - \varepsilon MK).$$

We will show that formal derivatives of $P(x, y)$ are well-defined. The verification that these formal derivatives are actually the derivatives of $P(x, y)$ is omitted, although it is not so complicated to do so in the present situation. We will go through the verification in the proof of Lemma 3.2. The procedure is the same and easier for the present case. The (formal) derivative of $P(x, y)$ with respect to x is

$$D_xP(x, y) = -I_1 - I_2 - I_3$$

where I_1, I_2, I_3 are given by ($g_1(x, y) := g_1(x, y, 0)$ here)

$$I_1 = \int_\infty^0 D_x Y(0, t; x) g_1(\phi(t), \psi(t)) Y(t, 0; x, y) dt,$$

$$I_2 = \int_\infty^0 Y(0, t; x) [D_1 g_1(\phi(t), \psi(t))(D_x\phi(t)) + D_2 g_1(\phi(t), \psi(t))(D_x\psi(t))] Y(t, 0; x, y) dt,$$

$$I_3 = \int_\infty^0 Y(0, t; x) g_1(\phi(t), \psi(t)) D_x Y(t, 0; x, y) dt.$$

The estimates in Proposition 4.1 allow us to give bounds on I_j, $j = 1, 2, 3$.

$$|I_1| \le \int_0^\infty M_1 e^{(\beta+\delta)t} K|y| e^{-(\alpha-\varepsilon MK)t} M_0 e^{-(\alpha-\varepsilon MK)t} dt = (M_0 M_1 K|y|)/(2\alpha - \beta - \delta - 2\varepsilon MK).$$

This computation is valid since $2\alpha - \beta - (r-1)\delta > 3\varepsilon MK$, $r \ge 3$. We also have

$$|I_2| \le K^2 \int_0^\infty e^{\beta t}[MK|y|e^{-(\alpha-\varepsilon MK)t} M_1 e^{\delta t} + MM_1 e^{-(\alpha-2\varepsilon MK)t} e^{\delta t}] e^{-(\alpha-\varepsilon MK)t} dt$$
$$\le K^2 M(KM_1|y| + MM_1)/2\alpha - \beta - \delta - 3\varepsilon MK)$$
$$|I_3| \le K^2 M|y| \int_0^\infty e^{\beta t} e^{-(\alpha-\varepsilon MK)t} M_1 e^{-(\alpha-2\varepsilon MK)t+\delta t} dt$$
$$\le K^2 MM_1|y|/(2\alpha - \beta - \delta - 2\varepsilon MK).$$

By arguing inductively, it is not difficult to show that

$$|D_x^i D_y^j P(x, y)| \le C_{ij}(M_0, \dots, M_{i+j}, \alpha, \beta, \delta, \varepsilon), \qquad i + j \le r - 1,$$

where C_{ij} is a positive polynomial in $M_0, \dots, M_{i+j}$, and $(2\alpha - \beta - m\delta - 2\varepsilon MK)^{-1}$, $m \le i + j$. To conclude the proof of Lemma 3.1, let us show that the function $P(x, y)$ satisfies the desired functional equation in Lemma 3.1. Recall that $P(x, y)$ is defined as the initial value of the special solution to (3.7). By definition

$$P(\phi(t, x)), \psi(t, x, y))$$
$$= -\int_{\infty}^{0} Y(0, \tau; \phi(t, x))g_1(\phi(t+\tau, x), \psi(t+\tau, x, y), 0)\, Y(\tau, 0; \phi(t, x), \psi(t, x, y))d\tau.$$

Making use of the identities $Y(0, \tau; \phi(t, x)) = Y(t, \tau + t; x)$ and $Y(\tau, 0; \phi(t, x), \psi(t, x, y)) = Y(t + \tau, t; x, y)$, the last integral is expressed as

$$P(\phi(t, x)), \psi(t, x, y))$$
$$= -\int_{\infty}^{t} Y(t, \tau; x)g_1(\phi(\tau, x), \psi(\tau, x, y), 0)\, Y(\tau, t; x, y)d\tau.$$

By differntiating with respect to t and setting $t = 0$, one can verify that P satisfies the functional equation in Lemma 3.1. The property $P(x, 0) = 0$ follows from $\psi(t, x, 0) \equiv 0$.

Proof of Proposition 4.1. (a) The estimate $|D_x\phi(t, x)| \leq Ke^{\delta t}$, $t \geq 0$ is valid from (2.7). The function $a(t) := D_x^2\phi(t, x)$ satisfies

$$\dot{a}(t) = Df(\phi(t))a(t) + D^2f(\phi(t))(D_x\phi(t), D_x\phi(t)).$$

The variation of constants formula implies

$$|a(t)| = \left|\int_0^t \Phi(t, s; x)D^2f(\phi(s))(D_x\phi(s), D_x\phi(s))ds\right|$$
$$\leq \int_0^t Ke^{\delta(t-s)} \sup_{\xi\in\mathbf{M}} |D^2f(\xi)|\,|D_x\phi(s)|^2 ds$$
$$\leq K^3(\sup|D^2f|)\int_0^t e^{\delta(t-s)}e^{2\delta s}ds \leq K^3(\sup|D^2f|)e^{2\delta t}/\delta.$$

By induction, one obtains $|D_x^j\phi(t)| \leq M_j e^{j\delta t}$. The function $\psi(t, x, y)$ satisfies

$$\psi(t) = Y(t, 0; x)y + \int_0^t Y(t, s; x)g_1(\phi(s), \psi(s), 0)\psi(s)ds.$$

Bounds $|g_1| \leq M\varepsilon$ and $|Y(t, s; x)| \leq Ke^{-\alpha(t-s)}$ are available for $t \geq s$. Gronwall's inequality applies and

$$|\psi(t)| \leq Ke^{-(\alpha - \varepsilon MK)t}, \qquad t \geq 0$$

follows. Let $b(t) = D_x\psi(t, x, y)$. It satisfies $b(0) = 0$ and

$$\dot{b}(t) = B(\phi(t))b(t) + D_2g_1(\phi(t), \psi(t), 0)b(t)$$
$$+ [DB(\phi(t))(D_x\phi(t)) + D_1g_1(\phi(t), \psi(t), 0)(D_x\phi(t))]\psi(t).$$

By using the variation of constants formula and the estimates

$$|D_x\phi(t)| \leq M_1 e^{\delta t}, \ (|DB(\phi)| + |D_1 g_1|)|\psi| \leq (M + \varepsilon M)\varepsilon \leq 2\varepsilon M, \ t \geq 0$$

together with Gronwall's inequality, one obains

$$|D_x\psi(t, x, y)| \leq 2\varepsilon M M_1 K^2 |y| e^{-(\alpha-\delta-2\varepsilon MK)t}, \ t \geq 0.$$

Arguing by induction, one can obtain the remaining estimates in (a).

(b) $Y(t, s; x)$ is $(r-1)$ times differentiable in x for t, s fixed. The derivative satisfies (see the appendix of [8])

$$D_x Y(t, s; x) = \int_s^t Y(t, \tau; x)[DB(\phi(\tau))D_x\phi(\tau)]Y(\tau, s; x)d\tau.$$

By using (2.7) and $|D_x\phi(t)| \leq Ke^{\delta|t|}$, one immediately obtains

$$|D_x Y(t, s; x)| \leq \begin{cases} (K^3 M/\delta)e^{-\alpha(t-s)+\delta t}, & t \geq s \geq 0 \\ (K^3 M/\delta)e^{\beta(s-t)+\delta s}, & s \geq t \geq 0. \end{cases}$$

The remaining cases are handled similarly.

4.2. Proof of Lemma 3.2

In this subsection, we write the system (3.9) as

$$\begin{cases} \dot{x} = f(x) + f_3(x, y, z) \\ \dot{y} = B(x)y + g_3(x, y, z) \\ \dot{z} = C(x)z + h_3(x, y, z) \end{cases} \tag{3.9}$$

where

$$g_3(x, y, z) = g_2(x, y, z)z = \bar{g}_2(x, y, z)y, \quad h_3(x, y, z) = h_2(x, y, z)y = \bar{h}_2(x, y, z)z.$$

The functions f, g_3 and h_3 vanish identically outside the set $\mathbf{M} \times \mathbf{D}(\varepsilon)$. The estimates $|f_3| \leq M\varepsilon^2$, $|Df_3| \leq M\varepsilon$ and similar estimates for g_3 and h_3 are available. We also assume that M is chosen large enough to have $\max\{|B|, |DB|, |C|, |DC|\} \leq M$. To avoid writing unessential constants repeatedly, we also assume that M is large enough to bound such expressions as $\sup\{|B| + |DB| + |Dg_3|\} \leq M$, and so on.

The function $R(x, y, z)$ and $S(x, y, z)$ as defined in (3.12) and (3.13) do not depend on the value of solution $w(t)$ which are outside of $\mathbf{M} \times \mathbf{D}(\varepsilon)$. This is an important observation to obtain the estimates in the sequel. Therefore, we need to estimate the solution $w(t)$ of (3.9) only so long as it remains in $\mathbf{M} \times \mathbf{D}(\varepsilon)$. Let $\Phi(t, s; u)$ denot the fundamental solution operator of $\dot{x} = [Df(\xi(t)) + D_x f_3(w(t))]x$.

PROPOSITION 4.2. *Let* $w(t)$, $w_0(t)$, *and* $w_1(t)$ *be solutions of* (3.9) *with* $w(0) = u$, $w_0(0) = u_0$, $w_1(0) = u_1$ *where* $u, u_0, u_1 \in \mathbf{M} \times \mathbf{D}(\varepsilon)$. *The estimates in the sequel are valid so long as* $w(t), w_0(t), w_1(t) \in \mathbf{M} \times \mathbf{D}(\varepsilon)$.

(i) *For* $(\alpha_1, \beta_1, \delta_1)$ *such that* $\alpha_1 < \alpha$, $\beta_1 > \beta$, $\delta_1 > \delta$, *there exists a constant* $K_1 \geq 1$ *which depends only on* $(K, \alpha, \beta, \delta, \alpha_1, \beta_1, \delta_1)$ *such that for* $\varepsilon > 0$ *small the following estimates are valid.*

$$|\Phi(t, s; u)| \leq K_1 e^{\delta_1 |t-s|},$$

$$|Y_0(t, s; u)| \leq K_1 e^{-\alpha_1 (t-s)} \quad t \geq s, \qquad |Y_0(t, s; u)| \leq K_1 e^{-\beta_1 (t-s)} \quad t \leq s,$$

$$|Z_0(t, s; u)| \leq K_1 e^{\alpha_1 (t-s)} \quad t \leq s, \qquad |Z_0(t, s; u)| \leq K_1 e^{\beta_1 (t-s)} \quad t \geq s.$$

With $\alpha_2 = \alpha_1 - \varepsilon M K_1$, $\beta_2 = \beta_1 + \varepsilon M K_1$, *the following also hold true.*

$$|Y_1(t, s; u)| \leq K_1 e^{-\alpha_2 (t-s)} \quad t \geq s, \qquad |Y_1(t, s; u)| \leq K_1 e^{-\beta_2 (t-s)} \quad t \leq s,$$

$$|Z_1(t, s; u)| \leq K_1 e^{\alpha_2 (t-s)} \quad t \leq s, \qquad |Z_1(t, s; u)| \leq K_1 e^{\beta_2 ((t-s)} \quad t \geq s.$$

(ii)

$$|w_1(t) - w_0(t)| \leq K_1 |u_1 - u_0| e^{\beta_2 |t|},$$

$$|D_u w_0(t) \hat{u}| \leq K_1 |\hat{u}| e^{\beta_2 |t|},$$

$$|w_1(t) - w_0(t) - D_u w_0(t)(u_1 - u_0)| = o(|u_1 - u_0|) e^{(\beta_2 + \delta)|t|}.$$

Here and below, $o(|u_1 - u_0|)$ *is a quantity such that* $o(|u_1 - u_0|)/|u_1 - u_0| \to 0$ *as* $|u_1 - u_0| \to 0$.

(iii) *Let* $L = K_1^3 M / \beta_2$.

$$|Y_0(0, t; u_1) - Y(0, t; u_0)| \leq L |u_1 - u_0| e^{(\alpha_1 - \beta_2)t}, \qquad t \leq 0,$$

$$|Z_0(0, t; u_1) - Z_0(0, t; u_0)| \leq L |u_1 - u_0| e^{-(\alpha_1 - \beta_2)t}, \qquad t \geq 0,$$

$$|Y_1(t, 0; u_1) - Y_1(t, 0; u_0)| \leq L |u_1 - u_0| e^{-(\alpha_2 - \beta_2)t}, \qquad t \geq 0,$$

$$|Z_1(t, 0; u_1) - Z_1(t, 0; u_0)| \leq L |u_1 - u_0| e^{(\alpha_1 - \beta_2)t}, \qquad t \leq 0.$$

(iv)

$$|D_u Y_0(0, t; u)| \leq L e^{(\alpha_1 - \beta_2)t}, \qquad t \leq 0,$$

$$|D_u Z_0(0, t; u)| \leq L e^{-(\alpha_1 - \beta_2)t}, \qquad t \geq 0,$$

$$|D_u Y_1(t, 0; u)| \leq L e^{-(\alpha_2 - \beta_2)t}, \qquad t \geq 0,$$

$$|D_u Z_1(t, 0; u)| \leq L e^{(\alpha_2 - \beta_2)t}, \qquad t \leq 0.$$

(v) *For* $t \leq 0$, *we have the following*

$$|Y_0(0, t; u_1) - Y_0(0, t; u_0) - D_u Y_0(0, t; u_0)(u_1 - u_0)| \le o(|u_1 - u_0|)e^{(\alpha_1 - \beta_2 - \delta)t},$$
$$|Z_1(t, 0; u_1) - Z_1(t, 0; u_0) - D_u Z_1(t, 0; u_0)(u_1 - u_0)| \le o(|u_1 - u_0|)e^{(\alpha_2 - \beta_2 - \delta)t}.$$

Similar estimates are valid for $Z_0(0, t)$ *and* $Y_1(t, 0)$ *when* $t \ge 0$.

(iv)

$$|D_u w_1(t) - D_u w_0(t)| = o(1)e^{(\beta_2 + \delta)|t|}, \qquad t \in \mathbf{R},$$
$$|Y_0(0, t; u_1) - Y_0(0, t; u_0)| = o(1)e^{(\alpha_1 - \delta)t}, \qquad t \le 0,$$
$$|Z_1(t, 0; u_1) - Z_1(t, 0; u_0)| = o(1)e^{(\alpha_2 - \delta)t}, \qquad t \le 0.$$

Similar estimates for $Z_0(0, t)$ *and* $Y_1(t, 0)$ *are valid when* $t > 0$. *If* $H(w)$ *is a continuous bounded function of* w, *then* $|H(w_1(t)) - H(w_0(t))| = o(1)e^{\delta|t|}$. *Here* $o(1)$ *is such that* $o(1) \to 0$ *as* $|u_1 - u_0| \to 0$.

PROOF. (1) The first three extimates follow from Henry's perturbation theorem (see [9]). The constant K_1 is given by

$$K_1 = 2K^2 e^{(\beta + 1)l}$$

where

$$l = (\log 2K) \max\{1/(\alpha - \alpha_1), 1/(\beta_1 - \beta), 1/(\delta_1 - \delta)\}.$$

By the variation of constants formula applied to $\dot{y} = B(\xi(t))y + \bar{g}_2(w(t))y$, one has

$$Y_1(t, s; u) = Y_0(t, s; u) + \int_s^t Y_0(t, \tau; u)\bar{g}_2(w(t))Y_1(\tau, s; u)d\tau.$$

The Gronwall's inequality establishes the estimate on Y_1. The argument for Z_1 is the same.

(ii) The first two estimates follow from the variation of constants formula and the Gronwall's inequality together with the estimates in part (i). It is, however, crucial to keep in mind that we are concerned only with $w_1(t)$ and $w_0(t)$ in $\mathbf{M} \times \mathbf{D}(\varepsilon)$. Therefore one can bound $|\eta(t)|$ and $|\zeta(t)|$, whenever necessary, from above by ε. In order to prove the third inequality, let

$$\bar{\xi} = \xi_1 - \xi_0 - D_u\xi_0(u_1 - u_0),$$
$$\bar{\eta} = \eta_1 - \eta_0 - D_u\eta_0(u_1 - u_0),$$
$$\bar{\zeta} = \zeta_1 - \zeta_0 - D_u\zeta_0(u_1 - u_0),$$

and $\bar{w} = (\bar{\xi}, \bar{\eta}, \bar{\zeta})$. By using the mean value theorem, one finds that they satisfy

$$\dot{\bar{\xi}} = Df(\xi_0)\bar{\xi} + Df_3(w_0)\bar{w} + a(t),$$

$$\dot{\bar{\eta}} = B(\xi_0)\bar{\eta} + DB(\xi_0)\bar{\xi}\eta_0 + Dg_3(w_0)\bar{w} + b(t),$$
$$\dot{\bar{\zeta}} = C(\xi_0)\bar{\zeta} + DC(\xi_0)\bar{\xi}\zeta_0 + Dh_3(w_0)\bar{w} + c(t),$$

where $a(t)$, $b(t)$, and $c(t)$ are given by

$$\begin{aligned} a(t) = &\int_0^1 [Df(\xi_0 + l(\xi_1 - \xi_0)) - Df(\xi_0)]dl[\xi_1 - \xi_0] \\ &+ \int_0^1 [Df_3(w_0 + l(w_1 - w_0)) - Df_3(w_0)]dl[w_1 - w_0], \\ b(t) = &\int_0^1 [DB(\xi_0 + l(\xi_1 - \xi_0)) - DB(\xi_0)]dl[\xi_1 - \xi_0] \\ &+ [B(\xi_1) - B(\xi_0)][\eta_1 - \eta_0] \\ &+ \int_0^1 [Dg_3(w_0 + l(w_1 - w_0)) - Dg_3(w_0)]dl[w_1 - w_0], \end{aligned}$$

and $c(t)$ is the same as $b(t)$ with C, ζ_1, ζ_0, and h_3 replacing, respectively, B, η_1, η_0, and g_3. Let us take the first term in $a(t)$. The coefficient of $\xi_1 - \xi_0$ in this term can be bounded as

$$\begin{aligned} &\left|\int_0^1 [Df(\xi_0(t) + l(\xi_1(t) - \xi_0(t))) - Df(\xi_0(t))]dl\right| \\ &\quad \le e^{\delta|t|}\left|\int_0^1 [Df(\xi_0 + l(\xi_0 + l(\xi_1 - \xi_0)) - Df(\xi_0)]dl\right|_\delta \end{aligned}$$

where $|p(\cdot)|_\delta = \sup_{t\in\mathbf{R}} |p(t)|e^{-\delta|t|}$ is an weighted norm of a function $p(t)$. Since Df is continuous and bounded, and $\xi_1(t) - \xi_0(t) \to 0$ as $|u_1 - u_0| \to 0$ uniformlu on every compact time interval,

$$\left|\int_0^1 [Df(\xi_0 + l(\xi_1 - \xi_0)) - Df(\xi_0)]dl\right|_\delta \longrightarrow 0$$

as $|u_1 - u_0| \to 0$. Other coefficients in a, b, and c are estimated in the same manner. Therefore, using the estimate on $|w_1 - w_0|$, one obtains

$$\max\{|a|, |b|, |c|\} = o(|u_1 - u_0|)e^{(\delta+\beta_2)|t|}.$$

Using the variation of constants formula together with the estimates in part (i), $\bar{w}(0) = 0$, and

$$|DB(\xi_0)\eta_0| + |Dg_3| \le M\varepsilon, \quad |DC(\xi_0)\zeta_0| + |Dh_3| \le M\varepsilon,$$

one obtains, for $t \ge 0$,

$$|\bar{w}(t)| \le \int_0^t \varepsilon KMe^{\beta_1(t-s)}|\bar{w}(s)|ds + o(|u_1 - u_0|)\int_0^t e^{\beta_1(t-s)}e^{(\delta+\beta_2)s}ds.$$

The Gronwall's inequality applied to $|\bar{w}(t)|e^{-\beta_1 t}$ yields $|\bar{w}(t)| = o(|u_1 - u_0|)e^{(\beta_2+\delta)t}$ for $t \ge 0$. Here we used $\beta_2 - \beta_1 = \varepsilon MK_1$. The argument for $t \le 0$ is the same.

(iii) The variation of constants formula yield, for $t \le 0$,

$$\begin{aligned} |Y_0(0, t; u_1) - Y_0(0, t; u_0)| &= \left|\int_t^0 Y_0(0, s; u_0)[B(\xi_1(s)) - B(\xi_0(s))]Y_0(s, t; u_1)ds\right| \\ &\le K_1^2 \int_t^0 e^{\alpha_1 s}|B(\xi_1(s)) - B(\xi_0(s))|e^{-\alpha_1(s-t)}ds \\ &\le K_1^2 e^{\alpha_1 t}\int_t^0 MK_1|u_1 - u_0|e^{-\beta_2 s}ds \\ &\le (K_1^3 M/\beta_2)e^{(\alpha_1-\beta_2)t}|u_1 - u_0|. \end{aligned}$$

The remaining estimates follow in the same way.

(iv) The derivative $D_u Y_0$ is given by

$$D_u Y_0(0, t; u_0)(\hat{u}) = \int_t^0 Y_0(0, s; u_0)[DB(\xi_0(s))D_u\xi_0(s)\hat{u}]Y_0(s, t; u_0)ds.$$

Using this and the estimates in parts (i) and (iii), one can verify (see the proof of (vi) below):

$$\begin{aligned} &\left|\int_t^0 Y_0(0, t, ; u_1) - Y_0(0, t; u_0) - D_u Y_0(0, t; u_0)(u_1 - u_0)\right| \\ &= o(|u_1 - u_0|)e^{(\alpha_1-\beta_2-\delta)t}, \qquad t \le 0. \end{aligned}$$

Therefore $D_u Y_0$ defined in the above is the actual derivative of Y_0. The estimate on $|D_u Y_0|$ is obtained by the same computation as in the proof of part (iii). The remaining cases are similar. The argument here also proves part (v).

(vi) We have for $t \le s \le 0$,

$$\begin{aligned} &|Y_0(s, t; u_1) - Y_0(s, t; u_0)| \\ &= \left|\int_t^s Y_0(s, \tau; u_0)[B(\xi_1(\tau)) - B(\xi_0(\tau))]Y_0(\tau, t; u_1)d\tau\right| \\ &\le K_1^2|B(\xi_1) - B(\xi_0)|_\delta \int_t^s e^{-\alpha_1(s-\tau)}e^{-\delta\tau}e^{-\alpha_1(\tau-t)}d\tau \end{aligned}$$

$$= o(1)e^{\alpha_1(t-s)-\delta t}.$$

The second estimate in (vi) is estabilshed by setting $s = 0$. The estimate in the above is used in the proof of part (iii). The argument for Z_1 is the same. The estimate on $|D_u w_1(t) - D_u w_0(t)|$ follows from the same line of argument as the proof of part (ii). The last statement in part (vi) follows since $|w_1(t) - w_0(t)| \to 0$ as $|u_1 - u_0| \to 0$ uniformly on every compact set.

We are now in a position to give the proof of Lemma 3.2.

Since $2\alpha - \beta > (r-1)\delta$, $r \geq 3$, one can choose $\alpha_1 < \alpha$, $\beta_1 > \beta$ such that $2\alpha_1 - \beta_1 > \delta$. Then choose $\varepsilon > 0$ so small that $2\alpha_1 - \beta_1 - \delta > 2\varepsilon MK_1$. As before, we use a short hand notation: $\alpha_2 = \alpha_1 - \varepsilon MK_1$, $\beta_2 = \beta_2 + \varepsilon MK_1$. In the sequel, we will give the proof for $R(u) = R(x, y, z)$. The proof for $S(u)$ is almost identical.

Now, Proposition 4.2 (i) and $|g_2| \leq \varepsilon M$ give rise to

$$|R(u)| \leq \int_{-\infty}^{0} K_1 e^{\alpha_1 t} \varepsilon M e^{\alpha_2 t} dt = \varepsilon M K_1^2/(\alpha_2 + \alpha_2) < \infty.$$

For $u_1, u_0 \in \mathbf{M} \times \mathbf{D}(\varepsilon)$, consider the difference $R(u_1) - R(u_0) = -J_1 - J_2 - J_3$ where

$$J_1 = \int_{-\infty}^{0} [Y_0(0, t; u_1) - Y_0(0, t; u_0)] g_2(w_1(t)) Z_1(t, 0, ; u_1) dt,$$

$$J_2 = \int_{-\infty}^{0} Y_0(0, t; u_0)[g_2(w_1(t)) - g_2(w_0(t))] Z_1(t, 0, ; u_1) dt,$$

$$J_3 = \int_{-\infty}^{0} Y_0(0, t; u_0) g_2(w_0(t))[Z_1(t, 0, ; u_1) - Z_1(t, 0; u_0)] dt.$$

By using Proposition 4.2 (i)(iii),

$$\begin{aligned}|J_1| &\leq (K_1^3 M |u_1 - u_0|/\beta_2) \int_{-\infty}^{0} e^{(\alpha_1 - \beta_2)t} \varepsilon M K_1 e^{\alpha_2 t} dt \\ &= (\varepsilon K_1^4 M^2/\beta(\alpha_1 + \alpha_2 - \beta_2))|u_1 - u_0|. \qquad (4.1)\end{aligned}$$

Note that $\alpha_1 + \alpha_2 - \beta_2 = 2\alpha_1 - \beta_1 - 2\varepsilon MK_1 > \delta$. The estimates $|J_2| = O(|u_1 - u_0|)$ and $|J_3| = O(|u_1 - u_0|)$ follow in the same way. Therefore $R(u)$ is Lipschitz continuous with a Lipschitz constant

$$\operatorname{Lip} R = (2\varepsilon K_1^4 M^2)/\beta_2(\alpha_1 + \alpha_2 - \beta_2) + (\varepsilon K_1^3 M)/(\alpha_1 + \alpha_2 - \beta_2).$$

Let us now prove that $R(u)$ is differentiable in u with the derivative given by:

$$\begin{aligned} D_uR(u)(\hat{u}) = & -\int_{-\infty}^{0} [D_u Y_0(0, t; u)(\hat{u})]g_2(w(t))Z_1(t, 0, ; u)dt \\ & -\int_{-\infty}^{0} D_u Y_0(0, t; u)[Dg_2(wt))D_u w(t)(\hat{u})]Z_1(t, 0, ; u)dt \\ & -\int_{-\infty}^{0} Y_0(0, t; u)g_2(w(t))[D_u Z_1(t, 0, ; u)(\hat{u})]dt. \end{aligned}$$

The function $D_uR(u)$ is well defined and Proposition 4.2 (ii)(iv) give

$$|D_uR(u)| \leq \text{Lip } R.$$

To show that $D_uR(u)$ is the actual derivative of $R(u)$, consider the difference

$$R(u_1) - R(u_0) - D_uR(u)(u_1 - u_0) = -\sum_{i=4}^{9} J_i$$

where

$$\begin{aligned} J_4 = & \int_{-\infty}^{0} [Y_0(0, t; u_1) - Y_0(0, t; u_0) - D_u Y_0(0, t; u_0)(u_1 - u_0)] \\ & \times g_2(w_1(t))Z_1(t, 0; u_1)dt, \\ J_5 = & \int_{-\infty}^{0} D_u Y_0(0, t; u_0)(u_1 - u_0)[g_2(w_1(t)) - g_2(w_0(t))]Z_1(t, 0; u_1)dt, \\ J_6 = & \int_{-\infty}^{0} D_u Y_0(0, t; u_0)(u_1 - u_0)g_2(w_0(t))[Z_1(t, 0; u_1) - Z_1(t, 0; u_0)]dt, \\ J_7 = & \int_{-\infty}^{0} Y_0(0, t; u_0)[g_2(w_1(t)) - g_2(w_0(t)) - Dg_2(w_0(t))D_u w_0(t)(u_1 - u_0)] \\ & \times Z_1(t, 0; u_1)dt, \\ J_8 = & \int_{-\infty}^{0} Y_0(0, t; u_0)Dg_2(w_0(t))D_u w_0(t)(u_1 - u_0)[Z_1(t, 0; u_1) - Z_1(t, 0; u_0)]dt, \\ J_9 = & \int_{-\infty}^{0} Y_0(0, t; u_0)g_2(w_0(t)) \\ & \times [Z_1(t, 0; u_1) - Z_1(t, 0; u_0) - D_u Z_1(t, 0; u_0)(u_1 - u_0)]dt. \end{aligned}$$

We will now show that $|J_i| = o(|u_1 - u_0|)$ as $|u_1 - u_0| \to 0$, $i = 4,\ldots,9$. By Proposition 4.2 (v), $|J_4|$ is bounded by

$$|J_4| = o(|u_1 - u_0|)\int_{-\infty}^{0} e^{(\alpha_1 - \beta_2 - \delta)t}e^{\beta_2 t}dt = o(|u_1 - u_0|)/(\alpha_1 - \delta).$$

Applying the fourth estimate in Proposition 4.2 (vi), one has, for $t \le 0$,

$$|g_2(w_1(t)) - g_2(w_0(t))| = o(1)e^{-\delta t}.$$

This, together with Proposition 4.2 (iv), gives

$$|J_5| = o(|u_1 - u_0|).$$

By Proposition 4.2 (iv)(vi), one also obtains $|J_6| = o(|u_1 - u_0|)$. The estimates on $|J_7|$, $|J_8|$, and $|J_9|$ are similar. Thus we have proven that $D_u R(u)$ is the true derivative of $R(u)$.

We will now show that $D_u R(u)$ is continuous in u with respect to the operator norm. To show this, consider the difference

$$D_u R(u_1)(\hat{u}) - D_u R(u_0)(\hat{u}) = -\sum_{j=1}^{9} I_j,$$

where I_j, $j = 1,\dots,9$, are given by

$$I_1 = \int_{-\infty}^{0} [D_u Y_0(0, t; u_1)(\hat{u}) - D_u Y_0(0, t; u_0)(\hat{u})] g_2(w_1(t)) Z_1(t, 0; u_1) dt,$$

$$I_2 = \int_{-\infty}^{0} D_u Y_0(0, t; u_0)(\hat{u}) [g_2(w_1(t)) - g_2(w_0(t))] Z_1(t, 0; u_1) dt,$$

$$I_3 = \int_{-\infty}^{0} D_u Y_0(0, t; u_0)(\hat{u}) g_2(w_0(t)) [Z_1(t, 0; u_1) - Z_1(t, 0; u_0)] dt,$$

$$I_4 = \int_{-\infty}^{0} [Y_0(0, t; u_1) - Y_0(0, t; u_0)] [Dg_2(w_1(t)) D_u w_1(t)(\hat{u})] Z_1(t, 0; u_1) dt,$$

$$I_5 = \int_{-\infty}^{0} Y_0(0, t; u_0) [Dg_2(w_1(t)) D_u w_1(t)(\hat{u}) - Dg_2(w_0(t)) D_u w_0(t)(\hat{u})] \times [Z_1(t, 0; u_1) - Z_1(t, 0; u_0)] dt,$$

$$I_6 = \int_{-\infty}^{0} Y_0(0, t; u_0) [Dg_2(w_0(t)) D_u w_0(t)(\hat{u})] [Z_1(t, 0; u_1) - Z_1(t, 0; u_0)] dt,$$

$$I_7 = \int_{-\infty}^{0} [Y_0(0, t; u_1) - Y_0(0, t; u_0)] g_2(w_1(t)) [D_u Z_1(t, 0; u_1)(\hat{u})] dt,$$

$$I_8 = \int_{-\infty}^{0} Y_0(0, t; u_0) [g_2(w_1(t)) - g_2(w_0(t))] [D_u Z_1(t, 0; u_1)(\hat{u})] dt,$$

$$I_9 = \int_{-\infty}^{0} Y_0(0, t; u_0) g_2(w_0(t)) [D_u Z_1(t, 0; u_1)(\hat{u}) - D_u Z_1(t, 0; u_0)(\hat{u})] dt.$$

We will show how to estimate $|I_1| = o(1)|\hat{u}|$. The computation for the remaining I_j's is similar. Now we consider the difference

$$
\begin{aligned}
&|D_u Y_0(0, t; u_1)(\hat{u}) - D_u Y_0(0, t; u_0)(\hat{u})| \\
&= \left| \int_t^0 Y_0(0, s; u_1)[DB(\xi_1(s))D_u\xi_1(s)(\hat{u})] Y_0(s, t; u_1)ds \right. \\
&\quad \left. - \int_t^0 Y_0(0, s; u_0)[DB(\xi_0(s))D_u\xi_0(s)(\hat{u})] Y_0(s, t; u_0)ds \right| \\
&\leq \sum_{j=1}^{3} |I_{1j}|
\end{aligned}
$$

where

$$I_{11} = \int_t^0 [Y_0(0, s; u_1) - Y_0(0, s; u_0)][DB(\xi_1(s))D_u\xi_1(s)(\hat{u})] Y_0(s, t; u_1)ds,$$

$$I_{12} = \int_t^0 Y_0(0, s; u_0)[DB(\xi_1(s))D_u\xi_1(s)(\hat{u}) - DB(\xi_0(s))D_u\xi_0(s)(\hat{u})] Y_0(s, t; u_1)ds,$$

$$I_{13} = \int_t^0 Y_0(0, s; u_0)[DB(\xi_0(s))D_u\xi_0(s)(\hat{u})][Y_0(s, t; u_1) - Y_0(s, t; u_0)]ds.$$

Applying Proposition 4.2, we obtain the following estimates:

$$
\begin{aligned}
|I_{11}| &\leq (|\hat{u}|MK_1^3/\delta(\beta_2 + \delta))(|B(\xi_1) - B(\xi_0)|_\delta)e^{(\alpha_1 - \beta_2 - \delta)t} \\
&= o(1)|\hat{u}|e^{(\alpha_1 - \beta_2 - \delta)t}, \\
|I_{12}| &\leq K_1^2 \int_t^0 e^{\alpha_1 t}|DB(\xi_1)D_u\xi_1 - DB(\xi_0)D_u\xi_0||\hat{u}|ds \\
&= o(1)|\hat{u}|e^{(\alpha_1 - \beta_2 - \delta)t}, \\
|I_{13}| &= o(1)|\hat{u}|e^{(\alpha_1 - \beta_2 - \delta)t}.
\end{aligned}
$$

Therefore, we have

$$|I_1| \leq o(1)|\hat{u}| \int_{-\infty}^0 e^{(\alpha_1 - \beta_2 - \delta)t} e^{\alpha_2 t} dt = o(1)|\hat{u}|/(\alpha_1 + \alpha_2 - \beta_2 - \delta).$$

The computation is the same for I_j, $j = 2,\dots,9$. We have thus proven that $R(x, y, z)$ is a C^1-function of $(x, y, z) \in \mathbf{M} \times \mathbf{D}(\varepsilon)$, for $\varepsilon > 0$ small. To show that $R(x, 0, z) = 0$, one should observe that $w(t) = (\xi(t), 0, \zeta(t))$ when $w(0) = (x, 0, z)$ and that $g_2(x, 0, z) = 0$. Therefore $R(x, 0, z) \equiv 0$ follows.

It is also easy to show that $R(x, y, z)$ satisfies the functional equation

(3.10) by the same type of arguments as in the proof of Lemma 3.1. This completes the proof of Lemma 3.2.

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