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Explicit conditions for oscillation of neutral differential systems

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1. Introduction and preliminaries

In the past decade there has been much work on the oscillation of neutral differential equations. Necessary and sufficient conditions via the characteristic equations have been obtained for those with constant coefficients, and various explicit sufficient conditions have been obtained, see [2–6, 8, 9]. For neutral differential systems of first order, O. Arino and I. Györi also gave a necessary and sufficient condition via the characteristic equation, see [1]. Since this condition is not easy to verify, some explicit conditions are needed. But to the best of the authors' knowledge there are very few results so far; here we mention only the results by I. Györi and G. Ladas [7] for a very special system and a weaker definition of oscillation. In this paper we will give some explicit conditions for oscillation of neutral systems under a stronger definition. We will show that even for the scalar case our results for explicit conditions are still the best up to now.

Consider the neutral delay differential system in the form

$$\frac{d^N}{dt^N}[y(t) - Py(t-r)] + \sum_{j=1}^{m} Q_j y(t-\tau_j) = 0 \tag{1.1}$$

where P, Q_j $(j = 1,\dots,m)$ are given $n \times n$ matrices, r, τ_j $(j = 1,\dots,m)$ are nonnegative numbers, $v = \max\{r, \tau_1,\dots,\tau_m\}$ and N is a positive integer.

DEFINITION 1.1. By a solution of (1.1) on $[-v, \infty)$ we mean a function $y \in C([-v, \infty), \mathbf{R}^n)$ such that $y(t) - Py(t-r)$ is N-times continuously differentiable and satisfies (1.1) on $[0, \infty)$.

DEFINITION 1.2. A solution $y = (y_1,\dots,y_n)^T : [-v, \infty) \to \mathbf{R}^n$ of (1.1) is called nonoscillatory if there exists a $t_0 \geq 0$ and $i_0 \in \{1,\dots,n\}$ such that $|y_{i_0}(t)| > 0$, $t \geq t_0$. A solution y of (1.1) is called oscillatory if it is not nonoscillatory. Eq. (1.1) is called oscillatory if all of its solutions are oscillatory.

Note that the definition of oscillation here is much stronger than that in

[7] in the sense that oscillation according to our definition implies oscillation as in [7]. We will see later that even under the latter definition our results are still much better.

Our results are based on the following lemma which is an extension of Theorem 2.1 in [1].

LEMMA 1.1. *Eq.* (1.1) *is oscillatory if and only if its characteristic equation*

$$\det\,[\lambda^N(I - Pe^{-\lambda r}) + \sum_{j=1}^{m} Q_j e^{-\lambda \tau_j}] = 0 \tag{1.2}$$

has no real root.

For the criteria of oscillation we need the following notations and definitions.

For any $n \times n$ real matrix A we denote by $\lambda_i(A)$ $(i = 1,\dots,n)$ the eigenvalues of A satisfying

$$\operatorname{Re}\lambda_1(A) \geq \operatorname{Re}\lambda_2(A) \geq \cdots \geq \operatorname{Re}\lambda_n(A).$$

We define $\|A\|_i = \sup_{x\in\mathbf{R}^n, x\neq 0} \dfrac{\|Ax\|_i}{\|x\|_i}$, $i = 1, 2,\dots,\infty$, where $x = (x_1,\dots,x_n)$, $\|x\|_i = (\sum_{j=1}^{n} |x_j|^i)^{1/i}$, $i < \infty$ and $\|x\|_\infty = \max_{1\leq j\leq n}\{|x_j|\}$. For each $i = 1, 2,\dots,\infty$, the Lozenski measure $\mu_i(A)$ of A is defined as follows:

$$\mu_i(A) = \lim_{h\to 0^+} \frac{\|I + hA\|_i - 1}{h},$$

and $\nu_i(A) = -\mu_i(-A)$, $i = 1, 2,\dots,\infty$. In general, without specification, we denote by $\mu(A)$ and $\nu(A)$ any pair of $\mu_i(A)$ and $\nu_i(A)$, $i = 1, 2,\dots,\infty$. It has been shown that $\mu_i(A)$ and $\nu_i(A)$, $i = 1, 2,\dots,\infty$, exist for any $n \times n$ matrix A and can be explicitly calculated for $i = 1, 2$, and $i = \infty$:

$$\mu_1(A) = \sup_j \{a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}|\}, \qquad \nu_1(A) = \inf_j \{a_{jj} - \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}|\};$$

$$\mu_2(A) = \lambda_1\left(\frac{1}{2}(A + A^T)\right), \qquad \nu_2(A) = \lambda_n\left(\frac{1}{2}(A + A^T)\right);$$

$$\mu_\infty(A) = \sup_i \{a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\}, \qquad \nu_\infty(A) = \inf_i \{a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\}.$$

For any $n \times n$ matrices A and B and any Lozenski measures we have

i) $\mu(A + B) \leq \mu(A) + \mu(B), \qquad \nu(A + B) \geq \nu(A) + \nu(B);$

ii) $\nu(-A) = -\mu(A), \qquad \mu(-A) = \nu(A);$

iii) $\mu(\alpha A) = \alpha\mu(A), \qquad \nu(\alpha A) = \alpha\nu(A),\ \alpha > 0;$

iv) $\mu(A) \geq \operatorname{Re}\lambda_1(A), \qquad \nu(A) \leq \operatorname{Re}\lambda_n(A).$ (1.3)

For more details concerning Lozenski measures, see [10]. In the sequel, we will obtain some criteria for oscillation by using Lozenski measures. Since the criteria are given by the general form of Lozenski measures μ and ν, we will actually have infinitely many different results corresponding to each criterion in the theorems. Moreover, three of them, which are given by μ_i and ν_i, $i = 1, 2, \infty$, can be expressed explicitly. But for the scalar case, where $\mu(A) = \nu(A) = A$, all of them coincide to give the same results.

2. Main results

To simplify the discussion and proofs we first consider a simpler equation

$$\frac{d^N}{dt^N}[y(t) - Py(t-r)] + Qy(t-\tau) = 0 \tag{2.1}$$

where P, Q are $n \times n$ matrices, $r, \tau \geq 0$. The following conditions will be used to determine the oscillation of eq. (2.1):

$$(\mathrm{A}_1) \qquad \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q)(kr+\tau)^N \geq \left(\frac{N}{e}\right)^N,$$

$$(\mathrm{A}_2) \qquad \sum_k{}^{*} [\mu(P)]^{-(k+1)} \nu(Q)[(k+1)r - \tau]^N \geq \left(\frac{N}{e}\right)^N,$$

$$(\mathrm{A}_3) \qquad \sum_{k*} [\mu(P)]^{-(k+1)} \nu(Q)[-(k+1)r + \tau]^N \geq \left(\frac{N}{e}\right)^N$$

where $\sum_k{}^{*}$ and $\sum_{k*}$ denote the sums over all the terms for $k \geq 0$ such that $(k+1)r - \tau > 0$ and $-(k+1)r + \tau > 0$, respectively.

THEOREM 2.1. *Assume N is odd and $\nu(Q) > 0$. Then each of the following is sufficient for* (2.1) *to be oscillatory*:

i) $\mu(P) = \nu(P) = 1,$

ii) $0 < \nu(P) \leq \mu(P) \leq 1,$ *and* (A_1) *holds,*

iii) $1 \leq \nu(P) \leq \mu(P),$ *and* (A_2) *holds,*

iv) $0 < \nu(P) < 1 < \mu(P),$ *and* $(A_1), (A_2)$ *hold.*

THEOREM 2.2. *Assume N is even and $\nu(Q) > 0$. Then each of the following is sufficient for* (2.1) *to be oscillatory*:

i) $0 < \mu(P) \leq 1$, *and* (A_3) *holds*,

ii) $\mu(P) > 1$, *and* $(A_2), (A_3)$ *hold.*

REMARK 2.1. The condition (A_3) in Theorem 2.2 is required in the sense that if the set $\{k \in \mathbf{Z}_+ : -(k+1)r + \tau > 0\}$ is empty and $\nu(P) > 0$, then eq. (2.1) must have a nonoscillatory solution. In fact, the above assumption implies that $r \geq \tau$. If (2.1) is oscillatory, then (3.1) has no real root. Let

$$F(\lambda) = \lambda^N (I - Pe^{-\lambda r}) + Qe^{-\lambda\tau}.$$

Then

$$\mu(F(0)) = \mu(Q) \geq \nu(Q) > 0$$

implies that $\mu(F(\lambda)) > 0$ for all $\lambda \in \mathbf{R}$. But

$$\mu(F(\lambda)) \leq \lambda^N (1 - \nu(P) e^{-\lambda r}) + \mu(Q) e^{-\lambda r} \longrightarrow -\infty$$

as $\lambda \to -\infty$. This contradiction shows that all solutions cannot be oscillatory.

The above oscillation criteria for eq. (2.1) can be easily extended to the eq. (1.1) by using the following conditions where $q = (\prod_{j=1}^{m} \nu(Q_j))^{1/m}$, $\tau = \frac{1}{m} \sum_{j=1}^{m} \tau_j$.

$$(\mathrm{B}_1) \qquad \sum_{j=1}^{m} \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q_j) (kr + \tau_j)^N \geq \left(\frac{N}{e}\right)^N, \text{ or} \tag{2.2}$$

$$mq \sum_{k=0}^{\infty} [\nu(P)]^k (kr + \tau)^N \geq \left(\frac{N}{e}\right), \tag{2.3}$$

$$(\mathrm{B}_2) \qquad \sum_{j,k}^{*} [\mu(P)]^{-(k+1)} \nu(Q_j) [(k+1)r - \tau_j]^N \geq \left(\frac{N}{e}\right)^N, \text{ or}$$

$$mq \sum_{k}^{*} [\mu(P)]^{-(k+1)} [(k+1)r - \tau]^N \geq \left(\frac{N}{e}\right)^N,$$

$$(\mathrm{B}_3) \qquad \sum_{*\, j,k} [\mu(P)]^{-(k+1)} \nu(Q_j) [-(k+1)r + \tau_j]^N \geq \left(\frac{N}{e}\right)^N, \text{ or}$$

$$mq \sum_{*\, k} [\mu(P)]^{-(k+1)} [-(k+1)r + \tau]^N \geq \left(\frac{N}{e}\right)^N$$

where $\sum_k^*$, $\sum_{*\,k}$ are defined the same as before; $\sum_{j,k}^*$, $\sum_{*\,j,k}$ denote the sums over all the terms for $1 \leq j \leq m$, $k \geq 0$ such that $(k+1)r - \tau_j > 0$ and $-(k+1)r +$

$\tau_j > 0$, respectively.

THEOREM 2.3. *Assume N is odd and $\nu(Q_j) \geq 0$ but not all zero, $j = 1, \ldots, m$. Then each one of the following is sufficient for (1.1) to be oscillatory:*

i) $\mu(P) = \nu(P) = 1$,

ii) $0 < \nu(P) \leq \mu(P) \leq 1$, *and* (B_1) *holds,*

iii) $1 \leq \nu(P) \leq \mu(P)$, *and* (B_2) *holds,*

iv) $0 < \nu(P) < 1 < \mu(P)$, *and* (B_1), (B_2) *hold.*

THEOREM 2.4. *Assume N is even and $\nu(Q_j) \geq 0$ but not all zero, $j = 1, \ldots, m$. Then each one of the following is sufficient for (1.1) to be oscillatory:*

i) $0 < \mu(P) \leq 1$, *and* (B_3) *holds,*

ii) $\mu(P) > 1$, *and* (B_2), (B_3) *hold.*

The idea in the proofs of the above theorems may also be applied to the case that $\nu(Q_j)$ are not all nonnegative. As an example we give a result for the equations of the form

$$\frac{d^N}{dt^N}[y(t) - Py(t-r)] + \sum_{j=1}^{m} [G_j y(t-\sigma_j) - H_j y(t-\tau_j)] = 0 \tag{2.4}$$

where P, G_j, H_j are $n \times n$ matrices, $\nu(P) > 0$, $\nu(G_j) > 0$, $\mu(H_j) > 0$, $\nu(G_j) - \mu(H_j) \geq 0$ and not all zero for $j = 1, \ldots, m$, r, σ_j, τ_j $(j = 1, \ldots, m) \geq 0$.

THEOREM 2.5. *Assume N is odd. Then each one of the following is sufficient for (2.4) to be oscillatory.*

i) *$\nu(P) \geq 1$, $\sigma_j < \tau_j < r$, and (B_2) holds for the case where $\nu(Q_j)$ are replaced by $\nu(G_j) - \mu(H_j)$, $j = 1, \ldots, m$. Furthermore,*

$$-\left(\frac{N}{er}\right)^N + a^N \nu(P) + \sum_{j=1}^{m} [\nu(G_j)e^{-b_j(r-\sigma_j)} - \mu(H_j)e^{-b_j(r-\tau_j)}] > 0 \tag{2.5}$$

where

$$a = \min_{1 \leq j \leq m} \left\{ \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)}{\mu(H_j)} \right\}, \quad b_j = \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)(r-\sigma_j)}{\mu(H_j)(r-\tau_j)}, \quad j = 1, \ldots, m.$$

ii) *$\nu(P) > 0$, $\mu(P) \leq 1$, $\sigma_j > \tau_j > 0$, and (B_1) holds for the case where $\nu(Q_j)$ are replaced by $\nu(G_j) - \mu(H_j)$, $j = 1, \ldots, m$. Furthermore, $\nu(P^{-1}G_j) > 0$, $\mu(P^{-1}H_j) > 0$, $j = 1, \ldots, m$ and*

$$-\left(\frac{N}{er}\right)^N + a^{*N}\nu(P^{-1}) + \sum_{j=1}^{m} [\nu(P^{-1}G_j)e^{-b_j^*\sigma_j} - \mu(P^{-1}H_j)e^{-b_j^*\tau_j}] > 0,$$

where

$$a^* = \min_{1\le j\le m} \left\{\frac{1}{\sigma_j - \tau_j} \ln \frac{\nu(P^{-1}G_j)}{\mu(P^{-1}H_j)}\right\},\ b_j^* = \frac{1}{\sigma_j - \tau_j} \ln \frac{\nu(P^{-1}G_j)\sigma_j}{\mu(P^{-1}H_j)\tau_j},\ j = 1,\dots,m.$$

3. Proofs of the main results

The following lemma will be needed in the proofs of the results.

LEMMA 3.1. *Let A be an $n\times n$ real matrix. If either $\nu(A) > 0$ or $\mu(A) < 0$, then* $\det(A) \neq 0$.

PROOF. From (1.3), if $\nu(A) > 0$, then $\mathrm{Re}\,\lambda_n(A) > 0$. Hence $\mathrm{Re}\,\lambda_i(A) > 0$ for $i = 1, 2,\dots, n$. Thus

$$\det(A) = \lambda_1(A)\cdots\lambda_n(A) \neq 0.$$

The case that $\mu(A) < 0$ is similar. □

PROOF OF THEOREM 2.1. The characteristic equation of (2.1) is

$$\det(\lambda^N(I - Pe^{-\lambda r}) + Qe^{-\lambda\tau}) = 0. \tag{3.1}$$

i) Assume $\mu(P) = \nu(P) = 1$. Let

$$F(\lambda) = \lambda^N(I - Pe^{-\lambda r}) + Qe^{-\lambda r}.$$

Then $\nu(F(0)) = \nu(Q) > 0$, and

$$\nu(F(\lambda)) \ge \nu(\lambda^N(I - Pe^{-\lambda r})) + \nu(Q)e^{-\lambda r}.$$

For $\lambda > 0$

$$\begin{aligned}\nu(F(\lambda)) &\ge \lambda^N(1 - \mu(P)e^{-\lambda r}) + \nu(Q)e^{-\lambda\tau} \\ &= \lambda^N(1 - e^{-\lambda r}) + \nu(Q)e^{-\lambda\tau} > 0.\end{aligned} \tag{3.2}$$

For $\lambda < 0$

$$\begin{aligned}\nu(F(\lambda)) &\ge |\lambda|^N\nu(-I + Pe^{-\lambda r}) + \nu(Q)e^{-\lambda\tau} \\ &\ge |\lambda|^N(-1 + \nu(P)e^{-\lambda r}) + \nu(Q)e^{-\lambda\tau} > 0.\end{aligned}$$

Thus $\nu(F(\lambda)) > 0$ for all $\lambda\in\mathbf{R}$. By Lemma 3.1, $\det F(\lambda) \neq 0$ for $\lambda\in\mathbf{R}$, i.e., (3.1) has no real root.

ii) Assume $0 < \nu(P) \leq \mu(P) \leq 1$. Clearly $\lambda = 0$ is not a root of (3.1). For $\lambda > 0$, by (3.2)

$$\nu(F(\lambda)) \geq \lambda^N(1 - \mu(P)e^{-\lambda r}) + \nu(Q)e^{-\lambda\tau} > 0.$$

Hence $\det(F(\lambda)) \neq 0$ for $\lambda > 0$ by Lemma 3.1. Let $\lambda = -s$ and denote

$$G(s) = -s^N(I - Pe^{sr}) + Qe^{s\tau}. \tag{3.3}$$

Then $\lambda < 0$ is a root of (3.1) if and only if $s > 0$ is a root of $\det(G(s)) = 0$. By Lemma 3.1, if $\det(G(s)) = 0$ has a root $s > 0$, then $\nu(G(s)) \leq 0$. Since N is odd,

$$0 \geq \nu(G(s)) \geq s^N(-1 + \nu(P)e^{sr}) + \nu(Q)e^{s\tau}$$

which implies that $0 < \nu(P)e^{sr} < 1$. As a result

$$\begin{aligned} s^N &\geq \nu(Q)e^{s\tau}(1 - \nu(P)e^{sr})^{-1} \\ &= \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q) e^{s(kr+\tau)} \\ &> \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q) \left[\frac{s(kr+\tau)e}{N}\right]^N. \end{aligned} \tag{3.4}$$

The equality can not hold since $e^{s(kr+\tau)}$ attains its minimal value at different point s for different k. Thus

$$\sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q)(kr+\tau)^N < \left(\frac{N}{e}\right)^N,$$

contradicting (A_1). Therefore (3.1) has no real root.

iii) Assume $1 \leq \nu(P) \leq \mu(P)$. Similar to i) we see that $\lambda \leq 0$ is not a root of (3.1). Assume $\lambda > 0$ is a root of (3.1). By Lemma 3.1 we have

$$0 \geq \nu(F(\lambda)) \geq \lambda^N(1 - \mu(P)e^{-\lambda r}) + \nu(Q)e^{-\lambda\tau} \tag{3.5}$$

which implies that $\mu(P)e^{-\lambda r} > 1$. So

$$\begin{aligned} \lambda^N &\geq [\mu(P)]^{-1}\nu(Q)e^{\lambda(r-\tau)}(1 - [\mu(P)]^{-1}e^{\lambda r})^{-1} \\ &= \sum_{k=0}^{\infty} [\mu(P)]^{-(k+1)}\nu(Q)e^{\lambda[(k+1)r-\tau]} \\ &> \sum_k{}^* [\mu(P)]^{-(k+1)}\nu(Q)\left[\frac{\lambda((k+1)r-\tau)e}{N}\right]^N, \end{aligned}$$

that is,

$$\sum_k^* [\mu(P)]^{-(k+1)} \nu(Q) [(k+1)r - \tau]^N < \left(\frac{N}{e}\right)^N,$$

contradicting (A_2).

iv) Clearly, $\lambda = 0$ is not a root of (3.1). From the proof of ii) and iii) we see that (A_1) and (A_2) imply that any $\lambda > 0$ and $\lambda < 0$ can not be a root of (3.1). □

PROOF OF THEOREM 2.2.

i) Assume $\mu(P) \leq 1$. Similar to the proof of Theorem 2.1 ii), we see that any $\lambda \geq 0$ is not a root of (3.1). Assume $\lambda = -s < 0$ is a root of (3.1). Then by Lemma 3.1 and from (3.1)

$$\nu(F(-s)) = \nu(s^N(1 - Pe^{sr}) + Qe^{s\tau}) \leq 0.$$

Since N is even,

$$0 \geq \nu(F(-s)) \geq s^N(1 - \mu(P)e^{sr}) + \nu(Q)e^{s\tau}$$

which implies that $\mu(P)e^{sr} > 1$. As a result

$$\begin{aligned} s^N &\geq [\mu(P)]^{-1} \nu(Q) e^{s(-r+r)} (1 - [\mu(P)]^{-1} e^{-sr})^{-1} \\ &= \sum_{k=0}^{\infty} [\mu(P)]^{-(k+1)} \nu(Q) e^{s[-(k+1)r+\tau]} \\ &> \sum_{k*} [\mu(P)]^{-(k+1)} \nu(Q) \left[\frac{s[-(k+1)r+\tau]e}{N}\right]^N, \end{aligned}$$

that is,

$$\sum_{k*} [\mu(P)]^{-(k+1)} \nu(Q) [-(k+1)r + \tau]^N < \left(\frac{N}{e}\right)^N.$$

ii) Assume $\mu(P) > 1$. If λ is a real root of (3.1), then $\lambda \neq 0$ and $\lambda^N > 0$. By the proof of i), (A_3) implies that $\lambda < 0$ cannot be a root of (3.1). By the proof of Theorem 2.1 iii), (A_2) implies that $\lambda > 0$ cannot be a root of (3.1). □

PROOF OF THEOREM 2.3 AND 2.4. Similar to those of Theorem 2.1 and 2.2. To show the difference we only give an outline of the proof of Theorem 2.3 ii) as an example.

Corresponding to (3.4) we now have

$$s^N \geq \sum_{j=1}^{m} \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q_j) e^{s(kr+\tau_j)} \tag{3.6}$$

$$> \sum_{j=1}^{m} \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q_j) \left[\frac{s(kr + \tau_j)e}{N} \right]^N,$$

that is,

$$\sum_{j=1}^{m} \sum_{k=0}^{\infty} [\nu(P)]^k \nu(Q_j) [kr + \tau_j]^N < \left(\frac{N}{e} \right)^N,$$

contradicting (2.2). From (3.6) we also have

$$\begin{aligned} s^N &\geq \left(\sum_{j=1}^{m} \nu(Q_j) e^{s\tau_j} \right) \left(\sum_{k=0}^{\infty} [\nu(P)]^k e^{skr} \right) \\ &\geq m \left(\prod_{j=1}^{m} \nu(Q_j) e^{s\tau_j} \right)^{1/m} \left(\sum_{k=0}^{\infty} [\nu(P)]^k e^{skr} \right) \\ &= mq \sum_{k=0}^{\infty} [\nu(P)]^k e^{s(kr+\tau)} \\ &> mq \sum_{k=0}^{\infty} [\nu(P)]^k \left[\frac{s(kr + \tau)e}{N} \right]^N, \end{aligned}$$

that is,

$$mq \sum_{k=0}^{\infty} [\nu(P)]^k (kr + \tau)^N < \left(\frac{N}{e} \right)^N,$$

contradicting (2.3). □

Proof of Theorem 2.5. The characteristic equation of eq. (2.4) is

$$\det \left(\lambda^N (I - Pe^{-\lambda r}) + \sum_{j=1}^{m} (G_j e^{-\lambda \sigma_j} - H_j e^{-\lambda \tau_j}) \right) = 0. \tag{3.7}$$

Let

$$F(\lambda) = \lambda^N (I - Pe^{-\lambda r}) + \sum_{j=1}^{m} (G_j e^{-\lambda \sigma_j} - H_j e^{-\lambda \tau_j}).$$

Then $\lambda = 0$ is not a root of (3.7) since the assumption before Theorem 2.5 implies that

$$\nu(F(0)) = \nu\left(\sum_{j=1}^{m} (G_j - H_j) \right) \geq \sum_{j=1}^{m} (\nu(G_j) - \mu(H_j)) > 0.$$

Hence $\det F(0) \neq 0$.

i) Assume $\nu(P) \geq 1$ and $\sigma_j < \tau_j < r$. If $\lambda > 0$ is a root of (3.7), then by Lemma 3.1

$$0 \geq \nu(F(\lambda)) \geq \lambda^N(1 - \mu(P)e^{-\lambda r}) + \sum_{j=1}^{m} (\nu(G_j)e^{-\lambda\sigma_j} - \mu(H_j)e^{-\lambda\tau_j})$$

$$\geq \lambda^N(1 - \mu(P)e^{-\lambda r}) + \sum_{j=1}^{m} (\nu(G_j) - \mu(H_j))e^{-\lambda\tau_j}.$$

This is a similar inequality to (3.5) for eq. (2.1). By a similar discussion we can get a contradiction to (B_2) where $\nu(Q_j)$ are replaced by $\nu(G_j) - \mu(H_j), j = 1,\dots,m$. If $\lambda < 0$ is a root of (3.7), let $\lambda = -s$, and denote

$$\phi(s) = -s^N(Ie^{-sr} - P) + \sum_{j=1}^{m} (G_j e^{-s(r-\sigma_j)} - H_j e^{-s(r-\tau_j)}), \tag{3.8}$$

$$\alpha(s) = -s^N(Ie^{-sr} - P)$$

$$\beta_j(s) = G_j e^{-s(r-\sigma_j)} - H_j e^{-s(r-\tau_j)}, \qquad j = 1,\dots,m.$$

Then $\det(\phi(s)) = 0$, and since N is odd,

$$\nu(\alpha(s)) \geq s^N(-e^{-sr} + \nu(P)) > 0,$$

$$\nu(\beta_j(s)) \geq \nu(G_j)e^{-s(r-\sigma_j)} - \mu(H_j)e^{-s(r-\tau_j)} \triangleq \ell_j(s), \qquad j = 1,\dots,m.$$

We have $\ell_j(s) \geq 0$ if and only if

$$s \leq \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)}{\mu(H_j)} \triangleq a_j, \qquad j = 1,\dots,m.$$

Set $a = \min_{1 \leq j \leq m} \{a_j\}$. We have $\nu(\phi(s)) > 0$ for $0 < s \leq a$. Consider the case that $s > a$. Then

$$\nu(\alpha(s)) \geq -\left(\frac{N}{er}\right)^N + a^N \nu(P),$$

and since

$$\ell_j'(s) = -\nu(G_j)(r - \sigma_j)e^{-s(r-\sigma_j)} + \mu(H_j)(r - \tau_j)e^{-s(r-\tau_j)}$$

we see the minimum of ℓ_j are attained at

$$s_j = \frac{1}{\tau_j - \sigma_j} \ln \frac{\nu(G_j)(r - \sigma_j)}{\mu(H_j)(r - \tau_j)} = b_j,$$

and thus

$$\nu(\beta_j(s)) \geq \ell_j(s) \geq \ell_j(s_j) = \nu(G_j)e^{-b_j(r-\sigma_j)} - \mu(H_j)e^{-b_j(r-\tau_j)}, \qquad j = 1,\dots,m.$$

Therefore by (3.8) and (2.5)

$$\nu(\phi(s)) \geq \nu(\alpha(s)) + \sum_{j=1}^{m} \nu(\beta_j(s))$$

$$\geq -\left(\frac{N}{er}\right)^N + a^N \nu(P) + \sum_{j=1}^{m} [\nu(G_j)e^{-b_j(r-\sigma_j)} - \mu(H_j)e^{-b_j(r-\tau_j)}] > 0$$

contradicting that $\det(\phi(s)) = 0$.

ii) By Lemma 3.1, $\nu(P) > 0$ implies that P^{-1} exists. Then (3.7) is equivalent to

$$\det(-\lambda^N(I - P^{-1}e^{\lambda r}) + \sum_{j=1}^{m} [P^{-1}G_j e^{\lambda(r-\sigma_j)} - P^{-1}H_j e^{\lambda(r-\tau_j)}]) = 0.$$

With $\lambda = -s$, we have

$$\det(s^N(I - P^{-1}e^{-sr}) + \sum_{j=1}^{m} [P^{-1}G_j e^{-s(r-\sigma_j)} - P^{-1}H_j e^{-s(r-\tau_j)}]) = 0. \tag{3.9}$$

Then we have a duality between (3.7) and (3.9) as follows:

$$(P, G_j, H_j, \sigma_j, \tau_j) \longleftrightarrow (P^{-1}, P^{-1}G_j, P^{-1}H_j, r - \sigma_j, r - \tau_j).$$

Using this duality and part i) we obtain the desired result. □

4. Discussion for special cases

Here we mention the work by Győri and Ladas in [7] where eq. (1.1) is considered for the case that $N = 1$, P is a diagonal matrix with diagonal entries $p^1, \dots, p_n$ such that $0 \leq p_i \leq 1$, $i = 1, \dots, n$. $\nu_1(Q_j) \geq 0$ where $\nu_1(Q_j)$ denotes the Lozenski measure defined in Section 1. Some comparison results with delay equations are obtained. An explicit condition for oscillation (actually componentwise oscillation) is given there:

If $P = pI$ for $p \in [0, 1]$ (hence $\nu_1(P) = p$), and

$$\sum_{j=1}^{m} \sum_{k=0}^{\infty} [\nu_1(P)]^k \nu_1(Q_j)(kr + \tau_j) > \frac{1}{e},$$

then eq. (1.1) is (componentwise) oscillatory.

Obviously this result is included in Theorem 2.3 ii), and the equality in (B_1) is not valid there.

A special case for eq. (1.1) is that P and $Q_j, j = 1, \dots, m$, are symmetric matrices. Since for any symmetric matrix A, $\mu_2(A) = \lambda_1(A)$, $\nu_2(A) = \lambda_n(A)$ and $\mu(A) \geq \lambda_1(A)$, $\nu(A) \leq \lambda_n(A)$ for any Lozenski measures, then if we use μ_2 and ν_2 in the previous theorems, they will give the best results among those using all the Lozenski measures.

Another special case for eq. (1.1) is the scalar case, i.e., P and Q_j, $j = 1,\dots,m$ are constants, hence

$$\mu(P) = \nu(P) = P, \quad \mu(Q_j) = \nu(Q_j) = Q_j, \qquad j = 1,\dots,m.$$

Therefore, if we substitute P and Q_j into $\mu(P)$ or $\nu(P)$, $\mu(Q_j)$ or $\nu(Q_j)$, $j = 1,\dots,m$, respectively, the previous theorems will give criteria for oscillations of scalar equations. It is easy to see they include and improve the following known sufficient conditions for oscillation, see [2–6, 8, 9]:

A. $N = 1$, $0 < p < 1$, $\sum_{j=1}^{m} q_j\tau_j > \frac{1-p}{e}$ or $(\prod_{j=1}^{m} q_j)^{1/m}(\sum_{j=1}^{m} \tau_j) > \frac{1-p}{e}$, or

$p \geq 1$, $r > \tau_j$ $(j = 1,\dots,m)$ and $\sum_{j=1}^{m} q_j(r - \tau_j) > \frac{p-1}{e}$.

B. $N = 2$, $m = 1$, $0 < p < 1$, $p^{-1}q(\tau - r)^2 > \left(\frac{2}{e}\right)^2$.

C. N is odd, $m = 1$, $0 < p < 1$, and either one of the following:

i) $q\tau^N > \left(\frac{N}{e}\right)^N$,

ii) $\frac{q}{1-p}(\tau - r)^N > \left(\frac{N}{e}\right)^N$,

iii) $q[\tau^N + p(\tau + r)^N] > \left(\frac{N}{e}\right)^N$,

iv) $q\frac{p^{n+1}-1}{p-1}\tau^N > \left(\frac{N}{e}\right)^N$, or

v) $\max_i (ir + \tau)^N qp^i > \left(\frac{N}{e}\right)^N$.

D. N is even, $m = 1$, $0 < p \leq 1$ and $p^{-1}q(\tau - r)^N > \left(\frac{N}{e}\right)^N$.

References

[1] O. Arino and I. Győri, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays,* J. Differential Equations, **81** (1989), 98–105.

[2] Y. Chen, *Oscillation of a class of neutral functional differential equations,* Chin. Ann. Math., **10A** (5) (1989), 545–553.

[3] K. Gopalsamy, *Oscillation and nonoscillation in neutral differential equations with variable parameters,* Jour. Math. Phy. Sci., **21** (1987), 593–611.

[4] K. Gopalsamy and B. G. Zhang, *Oscillation and nonoscillation in first order neutral differential equations*, J. Math. Anal. Appl., **151** (1990), 42–57.

[5] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, *Oscillation and asymptotic behavior of second order neutral differential equations*, Ann. Mat. Pura Appl., (1987), 29–40.

[6] E. A. Grove, G. Ladas and J. Schinas, *Sufficient conditions for the oscillation of delay and neutral delay equations*, Canad. Math. Bull., **31 (4)** (1988), 459–466.

[7] I. Györi and G. Ladas, *Oscillations of systems of neutral differential equations*, Differential and Integral Equations, **1** (1988), 281–286.

[8] G. Ladas and Y. G. Sficas, *Oscillations of higher-order neutral equations*, J. Austral. Math. Soc., Ser. **B 27** (1986), 502–511.

[9] I. P. Stavroulakis, *Oscillations of mixed neutral equations*, Hiroshima Math. J., **19** (1989), 441–456.

[10] M. Vidyasagar, "Nonlinear Systems Analysis," Prentice-Hall, Inc., Englewood Cliffs, 1978.

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