HIROSHIMA MATH. J.
24 (1994), 607–612

On the upper bounds of Green potentials

Dedicated to Professor M. Nakai on the occasion of his 60th birthday

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(Received June 14, 1993)

1. Introduction

Let D be a domain in $\boldsymbol{R}^n$ ($n \geq 2$) with the Green function $G(x, y)$ for the Laplace equation. By $|D|$ we denote the volume of D. In [5] Cranston and McConnell proved the following result.

THEOREM A. *Let $n = 2$ and let D be a domain of finite area. Then there exists an absolute constant c such that for any function $h \geq 0$ harmonic on D,*

$$\int_D G(x, y)h(y)dy \leq c|D|h(x).$$

Their methods are highly probabilistic; they use the life time of conditioned Brownian motion. Chung [4] gave a simplified proof of Theorem A. His proof is based on the up-crossing and the down-crossing inequalities in the martingale theory. Bañuelos [2] extended Theorem A to general elliptic equations and $n \geq 3$. (For the higher dimensional case we need to assume some boundary regularity.) His proof is also probabilistic.

The purpose of this note is to give an elementary analytic proof of Theorem A. Throughout this note we let h be a positive harmonic function on D. We say that u is an h-Green potential of density f if

$$u(x) = \frac{1}{h(x)} \int_D G(x, y)f(y)dy.$$

In other words, u is the h-Green potential of density f if hu is the Green potential of density f. In this terminology, the conclusion of Theorem A reads as follows: *the upper bound of the h-Green potential of density h is dominated by $c|D|$.*

Let us consider the upper bound of h-Green potentials. In the simplest

Part of this work was made while the author was visiting the Department of Mathematics, University of Uppsala. He wishes to thank the Royal Swedish Academy of Sciences and the Japan Society of Promotion of Science.

case when h is a constant, the h-Green potential reduces to a usual Green potential, whose upper bound is estimated as follows.

LEMMA 1. *Let D be a domain of finite volume. Let $R > 0$ be such that $|B(0, R)| = |D|$ and let G^* be the Green function for $B(0, R)$. Then*

$$\sup_{x \in D} \int_D G(x, y)dy \le \int_{B(0,R)} G^*(0, y)dy = c_0 |D|^{2/n} .$$

where c_0 is a positive constant depending only on n.

We observe that the right estimate $c_0|D|$ appears in case $n = 2$. This lemma can be proved by the symmetrization [1, Theorem 2.8] or by the Faber-Krahn inequality type argument [2, Lemma 1].

The main trick of Theorem A is how to reduce a general case to Lemma 1. It may be natural to consider subdomains in each of which h is nearly constant. For example let $D_j = \{x \in D : 2^{j-1} < h(x) < 2^{j+1}\}$ and let G_j be the Green function for D_j. Then, in each D_j, the h-Green potential

$$u_j(x) = \frac{1}{h(x)} \int_{D_j} G_j(x, y)h(y)dy$$

is comparable to the usual Green potential $\int_{D_j} G_j(x, y)dy$, whose upper bound is estimated by Lemma 1. Thus, if the upper bound over D of the h-Green potential

$$u(x) = \frac{1}{h(x)} \int_D G(x, y)h(y)dy$$

is estimated by the sum of the upper bounds over D_j of u_j, then Theorem A follows. This is the most difficult step in the proof of Theorem A. Roughly speaking, a difficulty arises from the fact that u_j "vanishes" on ∂D_j, while u is positive on $D \cap \partial D_j$. Also the estimate heavily depends on the way of decomposition. It seems that only probabilistic proofs have been known. We shall generalize this step by considering essential conditions for the decomposition. As a result we shall arrive at a purely analytic proof of Theorem A.

Suppose D is decomposed into at most countably many open subsets $D_j \neq \emptyset$, $N_1 < j < N_2$ where $-\infty \le N_1 < N_2 \le \infty$. We assume that open subsets D_j satisfy

(i) $D = \bigcup_j D_j$,

(ii) $\overline{D_j} \cap D_k = \emptyset$ if $|j - k| \ge 2$.

Condition (ii) implies that D is covered by D_j at most twice.

Suppose $0 < \inf_{D_j} h \le \sup_{D_j} h < \infty$ for each j; and if $N_2 = \infty$, then

$$\lim_{j \to \infty} \left(\inf_{D_j} h \right) = \infty . \tag{1}$$

Let $\omega_j = \omega_j(x, E)$ be the h-harmonic measure of D_j. This means that $\omega_j(x, E)$ is the PWB^h solution of the characteristic function χ_E (see [3, Chapter XVI] and [6, Chapter VIII]). We assume that there is a constant λ, $0 < \lambda < \frac{1}{2}$, for which the following inequalities hold uniformly for j

$$
\begin{aligned}
&\omega_j(\cdot, \partial D_j \cap D_{j-1}) \le \lambda \qquad \text{on } K_j, \\
&\omega_j(\cdot, \partial D_j \cap D_{j+1}) \le 1 - \lambda \qquad \text{on } K_j,
\end{aligned} \tag{2}
$$

where $K_j = D_j \backslash (D_{j-1} \cup D_{j+1})$. We observe from (ii) that K_j is covered only by D_j; K_j includes $D_j \cap \partial D_{j-1}$ and $D_j \cap \partial D_{j+1}$. We understand the inequalities in (2) always hold if $K_j = \emptyset$. Our main result is as follows.

THEOREM 1. *Let D, D_j, h and λ be as above. Let u be an h-Green potential on D with density $f \ge 0$ and let u_j be the h-Green potential on D_j with density $f|_{D_j}$. Then*

$$\sup_D u \le \frac{2 - 2\lambda}{1 - 2\lambda} \sum_j \sup_{D_j} u_j .$$

REMARK. If $D_j = \{x \in D : 2^{j-1} < h(x) < 2^{j+1}\}$, then all the assumptions for Theorem 1 are satisfied with $\lambda = \frac{1}{3}$. In fact, $K_j = \{x \in D : h(x) = 2^j\}$ and

$$
\begin{aligned}
\omega_j(x, \partial D_j \cap D_{j-1}) &\le \left(\frac{1}{h(x)} - \frac{1}{2^{j+1}}\right) \Big/ \left(\frac{1}{2^{j-1}} - \frac{1}{2^{j+1}}\right), \\
\omega_j(x, \partial D_j \cap D_{j+1}) &\le \left(\frac{1}{2^{j-1}} - \frac{1}{h(x)}\right) \Big/ \left(\frac{1}{2^{j-1}} - \frac{1}{2^{j+1}}\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\omega_j(\cdot, \partial D_j \cap D_{j-1}) \le \frac{1}{3} \qquad \text{on } K_j, \\
&\omega_j(\cdot, \partial D_j \cap D_{j+1}) \le \frac{2}{3} \qquad \text{on } K_j.
\end{aligned} \tag{2'}
$$

This is the case of [4] and [5]. In Section 3 we shall give a proof of Theorem A by using Theorem 1.

I am grateful to Minoru Murata for introducing the work of Cranston and McConnell. I would like to thank Matts Essén for drawing my attention to the book of Bandle [1] and F.-Y. Maeda for helpful comments.

2. Proof of Theorem 1

PROOF OF THEOREM 1. In view of the monotone convergence theorem, we may assume that $f \not\equiv 0$ a.e. is bounded and has compact support F. It follows that the Green potential $hu = \int G(\cdot, y) f(y) dy$ satisfies

$$0 < hu \le A < \infty \qquad \text{on } D\,,$$

(3)

$$\lim_{x \to \xi, x \in D} h(x)u(x) = 0 \qquad \text{for q.e. } \xi \in \partial D\,,$$

where q.e. (quasi everywhere) means that the equality holds outside a polar set (see [7, Lemma 6.24 and Theorem 8.34]). We see that

$$\frac{hu}{\sup_F (hu)} \le 1 \le \frac{h}{\inf_F h} \qquad \text{on } F\,.$$

Since hu is harmonic in $D \backslash F$, it follows from (3) and the maximum principle that

$$\frac{hu}{\sup_F (hu)} \le \frac{h}{\inf_F h} \qquad \text{on } D\,,$$

whence

$$u \le \frac{\sup_F (hu)}{\inf_F h} < \infty \qquad \text{on } D\,.$$

Thus u is bounded on D. Let $M_j = \sup_{K_j} u$. If $K_j = \emptyset$, then we let $M_j = 0$. Observe that M_j is bounded. Moreover, by (1) and (3), if $N_2 = \infty$, then

(4) $$M_j \to 0 \text{ as } j \to \infty\,.$$

Hereafter we assume that $N_1 = -\infty$ and $N_2 = \infty$. If N_1 or N_2 is finite, we can argue in the same fashion by letting $M_j = 0$ and $K_j = \emptyset$ for $j \le N_1$ or $j \ge N_2$.

We claim

(5) $$u \le M_{j-1}\omega_j(\cdot, \partial D_j \cap D_{j-1}) + M_{j+1}\omega_j(\cdot, \partial D_j \cap D_{j+1}) + u_j \text{ on } D_j\,.$$

To this end we rewrite (5) as

(6) $$h(u - u_j) \le M_{j-1} h\omega_j(\cdot, \partial D_j \cap D_{j-1}) + M_{j+1} h\omega_j(\cdot, \partial D_j \cap D_{j+1}) \text{ on } D_j\,.$$

Note that both sides of (6) are bounded harmonic functions on D_j. We shall apply the maximum principle on D_j. It follows from (3) that the left hand side has nonpositive boundary values q.e. on $\partial D_j \cap \partial D$. Assumption (ii) yields

$$\partial D_j \cap D \subset (\partial D_j \cap D_{j-1}) \cup (\partial D_j \cap D_{j+1}) \subset K_{j-1} \cup K_{j+1}\,.$$

By the definitions of M_j and ω_j, the boundary values on $\partial D_j \cap D$ of the right hand side of (6) are greater than or equal to those of the left hand side q.e. Hence the maximum principle yields (6). Thus (5) follows.

For simplicity we let $U_j = \sup_{D_j} u_j$. Then we have from (2) and (5)

(7) $$\sup_{D_j} u \le \max\{M_{j-1}, M_{j+1}\} + U_j\,,$$

$$M_j \le \lambda M_{j-1} + (1-\lambda)M_{j+1} + U_j . \tag{8}$$

We shall show from (8) that

$$M_j \le \frac{1}{1-2\lambda}\sum_{k=-\infty}^{\infty} U_k \qquad \text{for all } j . \tag{9}$$

Then the conclusion readily follows from (7) and (9).

Let us now prove (9). We write (8) as

$$M_j - \alpha M_{j-1} \le M_{j+1} - \alpha M_j + \beta U_j ,$$

where $0 < \alpha = \lambda/(1-\lambda) < 1$ and $\beta = 1/(1-\lambda) > 0$. Apply the above inequality for $j, \dots, J$ to obtain

$$M_j - \alpha M_{j-1} \le M_{J+1} - \alpha M_J + \beta \sum_{k=j}^{J} U_k .$$

Letting $J \to \infty$, we find from (4) that

$$M_j \le \alpha M_{j-1} + \beta \sum_{k=j}^{\infty} U_k \le \alpha M_{j-1} + \beta \sum_{k=-\infty}^{\infty} U_k .$$

Apply this inequality for $j-1$ to obtain

$$M_{j-1} \le \alpha M_{j-2} + \beta \sum_{k=-\infty}^{\infty} U_k ,$$

whence

$$M_j \le \alpha^2 M_{j-2} + \beta(1+\alpha) \sum_{k=-\infty}^{\infty} U_k .$$

Repeating this procedure, we get

$$M_j \le \frac{\beta}{1-\alpha}\sum_{k=-\infty}^{\infty} U_k = \frac{1}{1-2\lambda}\sum_{k=-\infty}^{\infty} U_k ,$$

since M_j is bounded and so $\alpha^k M_{j-k} \to 0$ as $k \to \infty$. Thus (9) follows. The proof is complete.

3. Proof of Theorem A

PROOF OF THEOREM A. Let

$$u(x) = \frac{1}{h(x)}\int_D G(x, y)h(y)dy .$$

Let $D_j = \{x \in D : 2^{j-1} < h(x) < 2^{j+1}\}$ and let

$$u_j(x) = \frac{1}{h(x)} \int_D G_j(x, y) h(y) dy ,$$

where G_j is the Green function for D_j. By definition

$$u_j(x) \le \frac{1}{2^{j-1}} \int_{D_j} G_j(x, y) 2^{j+1} dy \le 4 \int_{D_j} G_j(x, y) dy .$$

Applying Lemma 1 to D_j, we obtain

$$\sup_{D_j} u_j \le 4c_0 |D_j|^{2/n} .$$

Hence, Theorem 1 and (2′) yield

$$\sup_D u \le \frac{2 - 2/3}{1 - 2/3} \sum_{j=-\infty}^{\infty} 4c_0 |D_j|^{2/n} \le 16c_0 \sum_{j=-\infty}^{\infty} |D_j|^{2/n} .$$

For $n = 2$ this implies the required inequality. The theorem is proved.

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