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Fractional powers of operators with polynomially bounded resolvent and the semigroups generated by them

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0. Introduction

The first result on fractional powers of a closed operator was obtained by S. Bochner [4]. In 1949, he constructed fractional powers of $-\Delta$, using essentially that the Laplacian Δ generates a bounded C_0-semigroup. E. Hille and R. Phillips (see [8] and [21]) took up this idea and defined fractional powers of the negatives of arbitrary generators of bounded C_0-semigroups.

In 1960, A. V. Balakrishnan [2], giving a new definition, extended the theory of fractional powers to closed operators A for which the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ exists and satisfies

$$\|R(\lambda, A)\| \leq \frac{C}{|\lambda|}$$

on an open sector $\{\lambda \in \boldsymbol{C} : |\arg \lambda| < a\}$ for some $0 < a < \frac{\pi}{2}$. For such operators, other, by [14] equivalent definitions of fractional powers have been given, e.g. by T. Kato [10], H. Komatsu [12], H. W. Hövel and U. Westphal [9] and C. Martinez, M. Sanz and L. Marco [15].

Motivated by the examples below, we study in this paper fractional powers $(-A)^b$ $(b \in \boldsymbol{C})$ and the semigroups generated by their negatives (if any) in the case that the resolvent set $\rho(A)$ contains a closed sector $\Sigma(a) = \{\lambda \in \boldsymbol{C} : |\arg \lambda| \leq a\} \cup \{0\}$, and the resolvent satisfies

$$\|R(\lambda, A)\| \leq C(1 + |\lambda|)^n$$

for some $n \in \boldsymbol{N}_0$ and all $\lambda \in \Sigma(a)$.

EXAMPLES. (1) A closed, densely defined, linear operator A is the generator of an integrated semigroup (see for example [11] or [18]) if and only if the resolvent of A exists and is polynomially bounded on a right half plane $\{\lambda \in \boldsymbol{C} : \Re\lambda > \omega\}$. If in addition $[0, \infty) \subseteq \rho(A)$, then A belongs to the class of operators we will discuss in this paper. The domains of the fractional powers of such generators are important in the study of the associated abstract Cauchy problem

$$(ACP) \qquad u'(t) = Au(t) \ (t \geq 0), \qquad u(0) = x.$$

It is shown in [23], that the (ACP) has a classical solution for all $x \in D((-A)^{b+1})$, where $b \geq 0$ depends on the growth of the resolvent of A on the right half plane.

In defining fractional powers of the negatives of certain generators of integrated semigroups, approaches have been made by K. Yoshinaga [24], H. A. Emami-Rad [6] and M. Balabane [1] within the framework of distribution semigroups.

(2) In 1972, R. Beals [3] investigated the class of closed, densely defined, linear operators A such that $R(\lambda, A)$ exists and satisfies $\|R(\lambda, A)\| \leq C(1 + |\lambda|)^n$ for all $\lambda \in \boldsymbol{C}$ with $\Re\lambda \geq \psi(|\Im\lambda|)$ and $\Re\lambda > \omega$, where $\omega > 0$ and ψ is a continuous, nonnegative, concave function on $[0, \infty)$ such that $\psi(t) \to \infty$, $t^{-1}\psi(t) \to 0$ $(t \to \infty)$, and $\int_1^\infty t^{-2}\psi(t)dt < \infty$ (see also [19]). If in addition $[0, \infty) \subseteq \rho(A)$, then A satisfies the assumptions above.

The paper consists of two sections. In the first part, we construct fractional powers and study some of their properties. The definition is motivated by a functional calculus similar to the one used by H. O. Fattorini [7] and S. G. Krein [13] in the case $n = -1$.

One of the reasons for the construction of fractional powers is their application to incomplete abstract Cauchy problems

$$\left(\frac{d}{dt}\right)^m u(t) = (-1)^{m+1}Au(t) \ (t \geq 0), \qquad u(0) = x$$

for $m \geq 2$ (see example below). Here, it is of interest whether the fractional powers $-(-A)^{1/m}$ are infinitesimal generators. In Section 2, we show that for every exponent $0 < b \leq \frac{1}{2}$ the operator $-(-A)^b$ is the complete generator of an analytic semigroup of growth order α.

1. Construction of fractional powers

Let A be a closed, densely defined, linear operator on a complex Banach space X. We assume that there are constants $0 < a < \frac{\pi}{2}$, $C > 0$ and $n \in \boldsymbol{N}_0$ such that the closed sector

$$\Sigma(a) := \{\lambda \in \boldsymbol{C} : |\arg \lambda| \leq a\} \cup \{0\}$$

is contained in $\rho(A)$, and $\|R(\lambda, A)\| \leq C(1 + |\lambda|)^n$ for all $\lambda \in \Sigma(a)$.

Since $\rho(A)$ is open, the assumption $0 \in \rho(A)$ implies the existence of a constant $0 < d \leq 1$ such that the closed ball $B_d := \{\lambda \in \boldsymbol{C} : |\lambda| \leq d\}$ is contained in $\rho(A)$. The resolvent is analytic on $\rho(A)$ and therefore bounded on B_d.

Hence, we can assume that

$$\|R(\lambda, A)\| \leq C(1+|\lambda|)^n \qquad \text{for all } \lambda \in \Sigma(a) \cup B_d . \tag{1.1}$$

Throughout the paper, we use the following notations. For $0<\tilde{a} \leq a$ and $0<\tilde{d} \leq d$, the (upwards oriented) curve $\Gamma(\tilde{a}, \tilde{d})$ is given by $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 &:= \{\lambda \in \boldsymbol{C} : \arg \lambda = -\tilde{a}, |\lambda| \geq \tilde{d}\} = \{-te^{-i\tilde{a}} : -\infty < t \leq -\tilde{d}\} \\ \Gamma_2 &:= \{\lambda \in \boldsymbol{C} : |\arg \lambda| > \tilde{a}, |\lambda| = \tilde{d}\} = \{de^{i(2\pi - t)} : \tilde{a} < t < 2\pi - \tilde{a}\} \\ \Gamma_3 &:= \{\lambda \in \boldsymbol{C} : \arg \lambda = \tilde{a}, |\lambda| \geq \tilde{d}\} = \{te^{i\tilde{a}} : \tilde{d} \leq t < \infty\} . \end{aligned} \tag{1.2}$$

Note that the argument function used takes on values in $(-\pi, \pi]$.

For every $b \in \boldsymbol{C}$, the mapping $\lambda \mapsto (-\lambda)^b$ is given by $(-\lambda)^b := e^{b \log(-\lambda)}$, where we take the main branch of the complex logarithm. Hence, $\lambda \mapsto (-\lambda)^b$ is defined and analytic on $\boldsymbol{C} \backslash [0, \infty)$ and can be estimated by

$$|(-\lambda)^b| = e^{\Re b \log|\lambda| - \Im b \arg(-\lambda)} \leq |\lambda|^{\Re b} e^{\pi |\Im b|} . \tag{1.3}$$

Our definition of fractional powers $(-A)^b$ is based on a functional calculus, i.e. on improper integrals of the form

$$\frac{1}{2\pi i} \int_\Gamma (-\lambda)^b R(\lambda, A) x \, d\lambda . \tag{1.4}$$

Note that the curve Γ encloses the spectrum $\sigma(A)$ and is contained in the intersection of $\rho(A)$ with the domain of the analytic mapping $\lambda \mapsto (-\lambda)^b$.

Lemma 1.1. *Let $\Re b < -(n+1)$. Then, the improper integral in* (1.4) *exists for all $x \in X$ and is independent of the particular choice of $\Gamma = \Gamma(\tilde{a}, \tilde{d})$. Further,*

$$\left\| \frac{1}{2\pi i} \int_\Gamma (-\lambda)^b R(\lambda, A) x \, d\lambda \right\| \leq \frac{\Re b + n}{\Re b + n + 1} e^{\pi |\Im b|} C 2^n \tilde{d}^{\Re b + 1} \|x\| .$$

Proof. Using the estimates (1.1) and (1.3), one can show that

$$\begin{aligned} \frac{1}{2\pi} \int_{\tilde{d}}^\infty \|(-te^{i\tilde{a}})^b e^{i\tilde{a}} R(te^{i\tilde{a}}, A) x\| \, dt &\leq \frac{1}{2\pi} \int_{\tilde{d}}^\infty t^{\Re b} e^{\pi |\Im b|} C(1+t)^n \|x\| \, dt \\ &\leq \frac{-C}{2(\Re b + n + 1)} e^{\pi |\Im b|} (\tilde{d}+1)^n \tilde{d}^{\Re b + 1} \|x\| . \end{aligned}$$

Considering the parametrizations given in (1.2), this yields the existence of the curve integral over Γ_3. Similarly, with the same final estimate, one obtains the existence of the integral over Γ_1. The integral over Γ_2 exists because

Γ_2 is a finite path. Further, since $\|(-\lambda)^b R(\lambda, A)\| \leq \tilde{d}^{\Re b} e^{\pi|\Im b|} C(1+\tilde{d})^n$ for all $\lambda \in \Gamma_2$,

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_2} (-\lambda)^b R(\lambda, A) x d\lambda \right\| \leq 2\pi \tilde{d} \frac{e^{\pi|\Im b|} C(\tilde{d}+1)^n \tilde{d}^{\Re b}}{2\pi} \|x\| .$$

To see the independence of the choice of $\Gamma = \Gamma(\tilde{a}, \tilde{d})$, it is sufficient to show that all integrals coincide with the one over $\Gamma(a, d)$. By Cauchy's Theorem, we have

$$\frac{1}{2\pi i} \left\{ \int_{\Gamma(a,d)} - \int_{\Gamma(\tilde{a},\tilde{d})} \right\} (-\lambda)^b R(\lambda, A) x d\lambda = \lim_{m \to \infty} \frac{1}{2\pi i} \int_{\Gamma_m} (-\lambda)^b R(\lambda, A) x d\lambda ,$$

where the curve Γ_m $(m > d)$ is given by $\Gamma_m := \{me^{it} : -a < t < -\tilde{a}\} \cup \{me^{it} : \tilde{a} < t < a\}$. Along Γ_m, the integrand $(-\lambda)^b R(\lambda, A)x$ is bounded by $m^{\Re b} e^{\pi|\Im b|} C(1+m)^n \|x\|$. Hence,

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_m} (-\lambda)^b R(\lambda, A) x d\lambda \right\| \leq 2\pi m \frac{e^{\pi|\Im b|} m^{\Re b} C(1+m)^n}{2\pi} \|x\| \to 0 \qquad (m \to \infty) .$$

■

Similar to the case $n = -1$ (see for example [2] or [7]), we introduce first linear operators J^b $(b \in \boldsymbol{C})$ which will later form the basis of our definition of fractional powers of A. Here, $[b]$ $(b \in \boldsymbol{C})$ denotes the largest integer smaller than or equal to $\Re b$. Note that $-(n+2) \leq \Re(b - [b] - n - 2) < -(n+1)$ for all $b \in \boldsymbol{C}$.

DEFINITION 1.2. *Let $b \in \boldsymbol{C}$. On $D(A^{[b]+n+2})$, the operator J^b is defined by*

$$J^b x := \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma} (-\lambda)^b R(\lambda, A) x d\lambda & \text{if } -(n+2) \leq \Re b < -(n+1) \\ J^{b-[b]-n-2}(-A)^{[b]+n+2} x & \text{otherwise} . \end{cases}$$

REMARKS. (1) The operators J^b are well-defined since, by Lemma 1.1, our definition gives a bounded linear operator for $-(n+2) \leq \Re b < -(n+1)$. Note that we use 'bounded operator' in the meaning of 'bounded operator on X'.

(2) With the generalized resolvent equation

$$(1.5) \qquad (-\lambda)^{-(m+1)} R(\lambda, A)(-A)^{m+1} x = R(\lambda, A) x + \sum_{j=0}^{m} (-\lambda)^{-(j+1)} (-A)^j x$$

for all $\lambda \in \rho(A)$, $\lambda \neq 0$, $m \in \boldsymbol{N}_0$ and $x \in D(A^{m+1})$, and the equality $\frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^b d\lambda = 0$ if $\Re b < -1$, we obtain the following representation of the operators J^b ($b \in \boldsymbol{C}$). For every $x \in D(A^{[b]+n+2})$,

$$(1.6) \qquad J^b x = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (-\lambda)^b R(\lambda, A) x d\lambda & \text{if } \Re b < 0 \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (-\lambda)^{b-[b]-1} R(\lambda, A)(-A)^{[b]+1} x d\lambda & \text{if } \Re b \geq 0 . \end{cases}$$

(3) The operator J^b is bounded if $\Re b < -(n+1)$.

PROPOSITION 1.3. *If $\Re b \geq -(n+1)$, then the operator J^b is closable.*

PROOF. It follows from the closedness of $(-A)^{[b]+n+2}$ that the operators $J^{b-[b]-n-2}$ and $(-A)^{[b]+n+2}$ commute on $D(A^{[b]+n+2})$. Therefore,

$$J^b = J^{b-[b]-n-2}(-A)^{[b]+n+2} \subseteq (-A)^{[b]+n+2} J^{b-[b]-n-2} .$$

With its maximal domain, the operator to the right is closed. Thus, J^b is closable. ■

Next, we show that the operators J^b ($b \in \boldsymbol{C}$) satisfy the properties expected of fractional powers. We begin with the semigroup property. Here, $\langle b \rangle$ denotes $\max\{0, [b]+n+2\}$ for any $b \in \boldsymbol{C}$.

LEMMA 1.4. *Let $b, c \in \boldsymbol{C}$. Then, $J^b J^c x = J^{b+c} x$ for every $x \in D(A^{\langle b \rangle + \langle c \rangle})$.*

PROOF. The claim is proved in two steps. First, assume that $\Re b, \Re c < -(n+1)$ which implies $\langle b \rangle = \langle c \rangle = 0$. By (1.6) and Lemma 1.1, we obtain for every $x \in X$

$$J^b J^c x = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_{\tilde{\Gamma}} (-\lambda_1)^b (-\lambda_2)^c R(\lambda_1, A) R(\lambda_2, A) x d\lambda_2 d\lambda_1 ,$$

where $\Gamma = \Gamma(a, d)$ and $\tilde{\Gamma} = \Gamma(\tilde{a}, \tilde{d})$ for $0 < \tilde{a} < a$, $0 < \tilde{d} < d$. Since $\Gamma \cap \tilde{\Gamma} = \varnothing$, the resolvent equation $(\lambda - \mu) R(\lambda, A) R(\mu, A) = (R(\mu, A) - R(\lambda, A))$ and Fubini's Theorem yield

$$\begin{aligned} J^b J^c x = & \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \left\{ \frac{1}{2\pi i} \int_\Gamma \frac{(-\lambda_1)^b}{\lambda_1 - \lambda_2} d\lambda_1 \right\} (-\lambda_2)^c R(\lambda_2, A) x d\lambda_2 \\ & + \frac{1}{2\pi i} \int_\Gamma (-\lambda_1)^b \left\{ \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{(-\lambda_2)^c}{\lambda_2 - \lambda_1} d\lambda_2 \right\} R(\lambda_1, A) x d\lambda_1 . \end{aligned}$$

It follows from Cauchy's Integral Formula that

$$\frac{1}{2\pi i} \int_\Gamma \frac{(-\lambda_1)^b}{\lambda_1 - \lambda_2} d\lambda_1 = 0 \quad (\lambda_2 \in \tilde{\Gamma}) \quad \text{and} \quad \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{(-\lambda_2)^c}{\lambda_2 - \lambda_1} d\lambda_2 = (-\lambda_1)^c \quad (\lambda_1 \in \Gamma) ,$$

since $\tilde{\Gamma}$ runs to the right of Γ. Therefore, $J^b J^c x = J^{b+c} x$.

Now, let b, c be arbitrary complex numbers and $x \in D(A^{\langle b\rangle+\langle c\rangle})$. Then, $x \in D(J^c)$ and, since the operators $(-A)^{-1}$ and J^c commute on $D(J^c)$, $J^c x \in D(A^{\langle b\rangle}) = D(J^b)$. Hence, $x \in D(J^b J^c)$. The inequality $\langle b\rangle + \langle c\rangle \leq [b+c] + n + 2$ yields that $x \in D(J^{b+c})$ as well. By Definition 1.2, we have

$$J^b J^c x = J^{b-[b]-n-2}(-A)^{[b]+n+2} J^{c-[c]-n-2}(-A)^{[c]+n+2} x\,.$$

The operators $J^{c-[c]-n-2}$ and $-A$ commute on $D(A)$. Thus, the first part of the proof yields

$$J^b J^c x = J^{b+c-[b]-[c]-2n-4}(-A)^{[b]+[c]+2n+4} x\,.$$

The statement $J^b J^c x = J^{b+c} x$ follows from

$$J^b = J^{b+1}(-A)^{-1} \qquad \text{for all } b \in \boldsymbol{C}\,. \tag{1.7}$$

■

LEMMA 1.5. *$J^b = (-A)^b|_{D(A^{b+n+2})}$ for all integers b.*

PROOF. First, let $b = 1$ and $x \in D(A^{n+3})$. By (1.6), we have to show that

$$\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-1} R(\lambda, A)(-A)^2 x d\lambda = -Ax\,.$$

Applying Cauchy's Integral Formula, we obtain

$$\lim_{m\to\infty} \frac{1}{2\pi i}\int_{\Gamma_m} \frac{1}{\lambda} R(\lambda, A)(-A)^2 x d\lambda = R(0, A)(-A)^2 x + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda} R(\lambda, A)(-A)^2 x d\lambda\,,$$

where the curve Γ_m is given by $\Gamma_m := \{me^{it} : -a < t < a\}$. The resolvent equation (1.5) and the resolvent estimate (1.1) yield the existence of a constant $M > 0$ such that

$$\|R(\lambda, A)x\| \leq \frac{M}{|\lambda|}(\|x\| + \|Ax\| + \cdots + \|A^{n+1}x\|)$$

for all $x \in D(A^{n+1})$ and $\lambda \in \Sigma(a) \cup \Gamma$ with $|\lambda| \geq d$. Hence,

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_m} \frac{1}{\lambda} R(\lambda, A)(-A)^2 x d\lambda\right\|$$

$$\leq 2\pi m \frac{M}{2\pi m^2}(\|A^2 x\| + \|A^3 x\| + \cdots + \|A^{n+3}x\|) \to 0 \qquad (m \to \infty)$$

and $\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda} R(\lambda, A)(-A)^2 x d\lambda = Ax$. The assertion follows with (1.7). ■

PROPOSITION 1.6. *$\lim_{b\to 1} J^b x = -Ax$ for every $x \in D(A^{n+3})$.*

PROOF. Let $S := \{b \in \boldsymbol{C} : 0 < \Re b < 2\}$. By (1.6) and (1.7), we have

$$J^b x = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{b-n-3}R(\lambda, A)(-A)^{n+3}x\,d\lambda$$

for all $b \in S$. Now, the mapping $(b, \lambda) \mapsto (-\lambda)^{b-n-3}R(\lambda, A)(-A)^{n+3}x$ is continuous on $S \times \Gamma$. By Lemma 1.1, the integral above converges uniformly on every compact subset of S. Therefore, $b \mapsto J^b x$ is continuous on S. In particular, $J^b x \to J^1 x$ $(b \to 1)$. But, by Lemma 1.5, $J^1 x = -Ax$. ∎

The remaining part of this section is devoted to the closures $\overline{J^b}$ of the operators J^b $(b \in \boldsymbol{C})$ (see Proposition 1.3). In the proof of Proposition 1.3, we have shown that, for every $b \in \boldsymbol{C}$, J^b is a restriction of the closed operator

$$C^b := (-A)^{[b]+n+2}J^{b-[b]-n-2}$$

with maximal domain. By Lemma 1.1, $\overline{J^b} = J^b \in \mathscr{B}(X)$ if $\Re b < -(n+1)$.

PROPOSITION 1.7. *(a) For every integer b,* $\overline{J^b} = (-A)^b$.
(b) If $\Re b > n+1$, *then* $\overline{J^b} = C^b$.

PROOF. (a) By Lemma 1.6, $J^b = (-A)^b|_{D(A^{b+n+2})}$. Therefore, it is sufficient to show that $D(A^{b+n+2})$ is a core for $(-A)^b$.

If $b \le 0$, this follows immediately since $(-A)^b \in \mathscr{B}(X)$ and $D(A^{b+n+2})$ is dense in X. Thus, assume $b > 0$ and $x \in D(A^b)$. Since $D(A^{n+2})$ is dense in X, there is a sequence $(y_m)_{m\in \boldsymbol{N}}$ in $D(A^{n+2})$ such that $y_m \to (-A)^b x$ as $m \to \infty$. Define $x_m := (-A)^{-b}y_m \in D(A^{b+n+2})$. Then, $x_m = (-A)^{-b}y_m \to (-A)^{-b}(-A)^b x = x$ and $(-A)^b x_m = y_m \to (-A)^b x$.

(b) As mentioned above, J^b is the restriction of the closed operator C^b to $D(A^{[b]+n+2})$. Similar to part (a), we will prove that $D(J^b) = D(A^{[b]+n+2})$ is a core for C^b.

Let $x \in D(C^b)$, i.e. $J^{b-[b]-n-2}x \in D(A^{[b]+n+2})$. Since $D(A^{[b]+n+2})$ is dense in X, there is a sequence $(y_m)_{m\in \boldsymbol{N}}$ in $D(A^{[b]+n+2})$ such that $y_m \to C^b x$ $(m \to \infty)$. We take $x_m := J^{-b}y_m$. Note that $J^{-b} \in \mathscr{B}(X)$. Since the operators J^{-b} and $(-A)^k$ $(k \in \boldsymbol{Z})$ commute on $D(A^k)$, the set $D(A^k)$ $(k \ge 0)$ is invariant under J^{-b}. Hence, $x_m \in D(A^{[b]+n+2})$. Moreover, using Lemma 1.4 and part (a),

$$J^{-b}C^b x = J^{-b}(-A)^{[b]+n+2}J^{b-[b]-n-2}x = (-A)^{[b]+n+2}J^{-[b]-n-2}x = x\,.$$

Analogously, one can show that $J^b x_m = (-A)^{[b]+n+2}J^{b-[b]-n-2}J^{-b}y_m = y_m$ for all $m \in \boldsymbol{N}$. Thus, $x_m = J^{-b}y_m \to J^{-b}C^b x = x$ and $J^b x_m = y_m \to C^b x$ as $m \to \infty$. ∎

Unfortunately, we do not know whether the above result holds for exponents $-(n+1) < \Re b < n+1$, $b \notin \boldsymbol{Z}$, i.e. it is not clear at present whether $\overline{J^b} = C^b$ for every $b \in \boldsymbol{C}$ and all operators A. An example in which the equality holds is given next.

EXAMPLE. On $X = C_0(\boldsymbol{R}, \boldsymbol{C}^2)$, consider the matrix multiplication operator

$$\mathscr{A}_q : f \mapsto qf,\ f \in D(\mathscr{A}_q) := \{f \in X : qf \in X\}\,, \tag{1.8}$$

where $q : \boldsymbol{R} \to M_2(\boldsymbol{C})$ is defined by

$$q(x) := \begin{pmatrix} x + i(x+1) & x^2 \\ 0 & x - i(x+1) \end{pmatrix} \qquad (x \in \boldsymbol{R})\,.$$

The spectrum of $\mathscr{A}_q$ is given by $\sigma(\mathscr{A}_q) = \{x + i(x+1), x - i(x+1) : x \in \boldsymbol{R}\}$, and $R(\lambda, \mathscr{A}_q)f = (\lambda - q(\cdot))^{-1}f (f \in X)$, where

$$(\lambda - q(x))^{-1} = \begin{bmatrix} \dfrac{1}{\lambda - x - i(x+1)} & \dfrac{x^2}{(\lambda - x - i(x+1))(\lambda - x + i(x+1))} \\ 0 & \dfrac{1}{\lambda - x + i(x+1)} \end{bmatrix} \qquad (x \in \boldsymbol{R})\,.$$

Hence, the set $S := \Sigma\left(\frac{\pi}{8}\right) \cup B_{\sqrt{2}/4}$ is contained in $\rho(\mathscr{A}_q)$, and $\|R(\lambda, \mathscr{A}_q)\| \leq M$ for all $\lambda \in S$. Thus, $\mathscr{A}_q$ satisfies the assumptions of this paper with $a = \frac{\pi}{8}$ and $n = 0$.

By Cauchy's Integral Formula and induction, it follows that the operators J^b $(b \in \boldsymbol{C})$ with domain $D(\mathscr{A}_q^{[b]+2})$ concur with multiplication by

$$j_b(x) = \begin{bmatrix} (-x - i(x+1))^b & \dfrac{x^2}{2i(x+1)}[(-x - i(x+1))^b - (-x + i(x+1))^b] \\ 0 & (-x + i(x+1))^b \end{bmatrix}. \tag{1.9}$$

Obviously, J^b is bounded if $\Re b < -1$ (compare to Remark 3). For $\Re b \geq -1$, the operator J^b is a restriction of the closed matrix multiplication operator $M_{j_b} : f \mapsto j_b f$ (with maximal domain). Moreover, since $D(J^b) = D(\mathscr{A}_q^{[b]+2})$ contains $C_{00}(\boldsymbol{R}, \boldsymbol{C}^2)$ which is a core for M_{j_b}, it follows that $\overline{J^b} = M_{j_b}$. Calculating the pointwise product of j_b and $-q$ shows that

$$j_b(x)(-q(x)) = (-q(x))j_b(x) = j_{b+1}(x) \qquad \text{for all } x \in \boldsymbol{R}\,. \tag{1.10}$$

Consequently, the operators $\overline{J^b}$ and C^b coincide on the core $C_{00}(\boldsymbol{R}, \boldsymbol{C}^2)$. This yields $\overline{J^b} = C^b$ for all $b \in \boldsymbol{C}$.

Note that $\overline{J^b}$ is unbounded if $-1 < \Re b < 0$. ■

By definition, the space $D(A^{[b]+n+2}) = D(J^b)$ is a core for $\overline{J^b}$ $(b \in \boldsymbol{C})$. In order to extend the semigroup property of the operators J^b $(b \in \boldsymbol{C})$ to their closures, we need to know that the spaces $D(A^{[b]+n+2+k})$ $(k \geq 1)$ have the same property. For $\Re b < -(n+1)$, this follows instantly since the operator J^b is bounded and the spaces $D(A^k)$ $(k \in N)$ are dense in X.

LEMMA 1.8. *Let* $\Re b \geq -(n+1)$. *Then,* $D(A^{[b]+n+2+k})$ $(k \geq 1)$ *is a core for* $\overline{J^b}$.

PROOF. Let $k \geq 1$ and $x \in D(A^{[b]+n+2})$. Since $D(A^k)$ is dense in X, there exists a sequence $(y_m)_{m \in N}$ in $D(A^k)$ such that $y_m \to (-A)^{[b]+n+2}x$ as $m \to \infty$. We define $x_m := (-A)^{-([b]+n+2)}y_m \in D(A^{[b]+n+2+k})$. Now,

$$x_m = (-A)^{-([b]+n+2)}y_m \to (-A)^{-([b]+n+2)}(-A)^{[b]+n+2}x = x\,,$$

$$(-A)^{[b]+n+2}x_m = y_m \to (-A)^{[b]+n+2}x\,,$$

and

$$J^b x_m = J^{b-[b]-n-2}(-A)^{[b]+n+2}x_m \to J^{b-[b]-n-2}(-A)^{[b]+n+2}x = J^b x$$

as $m \to \infty$. Note that $J^{b-[b]-n-2} \in \mathscr{B}(X)$. ∎

We can now prove a weakened semigroup property for the operators $\overline{J^b}$ $(b \in C)$.

PROPOSITION 1.9. *For all $b, c \in C$, $\overline{J^{b+c}} \subseteq \overline{\overline{J^b}\,\overline{J^c}}$ with equality if $\overline{J^{b+c}} = C^{b+c}$.*

PROOF. By Lemma 1.4, $J^{b+c}|_{D(A^{\langle b\rangle+\langle c\rangle})} = J^b J^c|_{D(A^{\langle b\rangle+\langle c\rangle})} \subseteq \overline{J^b}\,\overline{J^c}$. Taking closures on both sides, the first part of the assertion follows by Lemma 1.8.

Now, assume that $\overline{J^{b+c}} = C^{b+c}$. Take any $x \in D(C^b C^c)$, i.e. $J^{c-[c]-n-2} \in D(A^{[c]+n+2})$ and $J^{b-[b]-n-2}(-A)^{[c]+n+2}J^{c-[c]-n-2}x \in D(A^{[b]+n+2})$. Since the operators $J^{b-[b]-n-2}$ and $(-A)^{[c]+n+2}$ commute on $D(A^{[c]+n+2})$, it follows by Lemma 1.4 and equality (1.7) that $J^{b+c-[b+c]-n-2}x \in D(A^{[b+c]+n+2})$. Hence, $x \in D(C^{b+c})$ and $C^{b+c}x = C^b C^c x$. Therefore, $C^b C^c \subseteq C^{b+c}$ and, since C^{b+c} is a closed operator, also $\overline{C^b C^c} \subseteq C^{b+c}$. With the first part of our proof, it follows that

$$\overline{J^{b+c}} \subseteq \overline{\overline{J^b}\,\overline{J^c}} \subseteq \overline{C^b C^c} \subseteq C^{b+c}\,.$$

But, by assumption, we have $\overline{J^{b+c}} = C^{b+c}$ which gives the assertion. ∎

Next, we consider the special case $c = -b$. By Proposition 1.7, we have $\overline{J^0} = I_X$. Therefore, Proposition 1.9 yields

$$\overline{\overline{J^{-b}}\,\overline{J^b}} = I_X$$

for all $b \in C$. However, one can even get a more precise result.

LEMMA 1.10. $\overline{J^{-b}}\,\overline{J^b} = I_{D(\overline{J^b})}$ *for every* $b \in C$.

PROOF. Let $x \in D(\overline{J^b})$. By Lemma 1.8, there is a sequence $(x_m)_{m \in N}$ in $D(A^{\langle b\rangle+\langle -b\rangle})$ such that $x_m \to x$ and $J^b x_m \to \overline{J^b}x$ as $m \to \infty$. Taking $y_m :=$

$J^b x_m \in D(A^{[-b]+n+2})$ yields $y_m = J^b x_m \to \overline{J^b}x$ and, by Lemma 1.5, $J^{-b}y_m = J^{-b}J^b x_m = x_m \to x$ $(m \to \infty)$. But this means $\overline{J^b}x \in D(\overline{J^{-b}})$ and $\overline{J^{-b}}\,\overline{J^b}x = x$. ∎

Hence, as expected of fractional powers, the operator $\overline{J^{-b}}$ is the possibly unbounded inverse of $\overline{J^b}$. In particular, the operators $\overline{J^b}$ $(b \in \boldsymbol{C})$ are injective, and for $\Re b > n + 1$, we have $0 \in \rho(\overline{J^b})$. However, a spectral mapping theorem

$$\sigma(\overline{J^b}) = (-\sigma(A))^b$$

can in general not be expected, as the following example shows.

EXAMPLE. Consider again the matrix multiplication operator $\mathscr{A}_q$ on $X = C_0(\boldsymbol{R}, \boldsymbol{C}^2)$ given in (1.8). As shown above, the operators $\overline{J^b}$ $(b \in \boldsymbol{C})$ correspond to matrix multiplication by j_b (see (1.9)). Hence, if the inverse of $\lambda - \overline{J^b}$ $(\lambda \in \boldsymbol{C})$ exists, it must concur with multiplication by

$$(\lambda - j_b(x))^{-1} = \begin{pmatrix} \dfrac{1}{\lambda-(-x-i(x+1))^b} & \dfrac{x^2}{2i(x+1)}\left[\dfrac{1}{\lambda-(-x-i(x+1))^b} - \dfrac{1}{\lambda-(-x+i(x+1))^b}\right] \\ 0 & \dfrac{1}{\lambda-(-x+i(x+1))^b} \end{pmatrix}.$$

Take any $0 < \Re b < 1$. The term in the upper right of $(\lambda - j_b)^{-1}$ is unbounded for every $\lambda \in \boldsymbol{C}$. Thus, $\sigma(\overline{J^b}) = \boldsymbol{C}$. ∎

Coming to the end of the first section, and justified by the properties shown above, we finally define fractional powers of $-A$.

DEFINITION 1.11. *For every $b \in \boldsymbol{C}$, we define the fractional power $(-A)^b$ of the operator $-A$ by*

$$(-A)^b := \overline{J^b}\,.$$

We summarize the results of this section.

THEOREM 1.12. *The fractional powers $(-A)^b$ $(b \in \boldsymbol{C})$ satisfy the following.*

(*i*) *For every $b \in \boldsymbol{C}$, the operator $(-A)^b$ is closed. If $\Re b < -(n + 1)$, then $(-A)^b \in \mathscr{B}(X)$.*

(*ii*) *If $b \in \boldsymbol{N}$, then $(-A)^b = \underbrace{(-A)\ldots(-A)}_{b\ \text{times}}$ and $(-A)^{-b} = ((-A)^b)^{-1}$. Also, $(-A)^0 = I_X$.*

(*iii*) *The operators $(-A)^b$ $(b \in \boldsymbol{C})$ are injective, and $(-A)^{-b}(-A)^b = I_{D((-A)^b)}$ for every $b \in \boldsymbol{C}$.*

(*iv*) *For all b, $c \in \boldsymbol{C}$, $(-A)^{b+c} \subseteq \overline{(-A)^b(-A)^c}$ with equality if $(-A)^{b+c} = C^{b+c}$.*

(*v*) *$\lim_{b\to 1}(-A)^b x = -Ax$ for every $x \in D(A^{n+3})$.*

REMARK. In the case that A is not densely defined, the results for the operators J^b ($b \in C$), that is Lemma 1.4, Lemma 1.5 and Proposition 1.6, still hold.

2. Fractional powers as generators

As in the previous section, the linear operator A is closed and densely defined. Its resolvent exists and satisfies $\|R(\lambda, A)\| \leq C(1 + |\lambda|)^n$ on the set $\Sigma(a) \cup B_d$.

Regarding applications to higher order abstract differential equations, it is of interest whether the negative fractional powers $-(-A)^b$ as defined above are infinitesimal generators. In the following, we will show that for exponents $0 < b < \dfrac{\pi}{2(\pi - a)}$, thus in particular for $0 < b \leq \frac{1}{2}$, the negative fractional power $-(-A)^b$ generates an analytic semigroup $(T_b(t))_{t>0}$ of growth order $\dfrac{n+1}{b}$ given by

$$T_b(t) := \frac{1}{2\pi i} \int_\Gamma e^{-t(-\lambda)^b} R(\lambda, A) d\lambda \qquad (t > 0). \tag{2.1}$$

Semigroups of growth order α were introduced by G. Da Prato [5] in 1966 (see also I. Miyadera, S. Oharu and N. Okazawa [16], N. Okazawa [20], P. E. Sobolevskii [22] and A. V. Zafievskii [25]). We recall the definition.

DEFINITION 2.1. *Let $\alpha > 0$. A family $(T(t))_{t>0}$ of bounded, linear operators on X is called a semigroup of growth order α if it satisfies the following conditions.*

(*i*) $T(t + s) = T(t)T(s)$ *for all* $t, s > 0$.
(*ii*) *The mapping* $t \mapsto T(t)$ $(t > 0)$ *is strongly continuous.*
(*iii*) *If* $T(t)x = 0$ *for all* $t > 0$, *then* $x = 0$.
(*iv*) $\|t^\alpha T(t)\|$ *is bounded as* t *tends to zero.*
(*v*) $X_0 := \bigcup_{t>0} T(t)X$ *is dense in* X.

As a first step, we prove the existence of the integral in (2.1).

PROPOSITION 2.2. *Let Y be a complex Banach space. Assume that $F : \rho(A) \to Y$ is analytic, and that there are constants $K > 0$ and $k \in N_0$ such that $\|F(\lambda)\| \leq K(1 + |\lambda|)^k$ for all $\lambda \in \Sigma(a) \cup B_d$. If $0 < b < \dfrac{\pi}{2(\pi - a)}$ and $\Gamma = \Gamma(\tilde{a}, \tilde{d})$ for any $\max\left\{\dfrac{2b-1}{2b}\pi, 0\right\} < \tilde{a} \leq a$ and $0 < \tilde{d} \leq d$, then the improper* *integral*

$$U_\Gamma(t) := \frac{1}{2\pi i}\int_\Gamma e^{-t(-\lambda)^b} F(\lambda)d\lambda$$

converges for all $t \in W(b, \tilde{a}) := \{t \in \boldsymbol{C}: |\arg t| < \arctan(\cos(b(\pi - \tilde{a})))\}$ *uniformly on compact subsets of* $W(b, \tilde{a})$ *and coincides with the curve integral over* $\Gamma(\hat{a}, \hat{d})$ *for every* $\tilde{a} \le \hat{a} \le a$ *and* $0 < \hat{d} \le d$. *Moreover, there exists a constant* $M > 0$ *depending on* Γ, b, K, k *such that* $\|U_\Gamma(t)\| \le M(ct_1 - |t_2|)^{-(k+1/b)}$, *where* $c := \cos(b(\pi - \tilde{a}))$, *for all* $t = t_1 + it_2 \in W(b, \tilde{a})$.

PROOF. The assumption $\tilde{a} > \max\left\{\frac{2b-1}{2b}\pi, 0\right\}$ yields $b < \frac{\pi}{2(\pi - \tilde{a})}$. Thus, for all $\lambda \in \Gamma$, $0 < b|\arg(-\lambda)| \le b(\pi - \tilde{a}) < \frac{\pi}{2}$ and therefore $\cos(b \arg(-\lambda)) \ge \cos(b(\pi - \tilde{a})) = c > 0$. Now, take $t = t_1 + it_2 \in W(b, \tilde{a})$. Since $|t_2| < ct_1$, which follows from the assumption $|\arg t| = \arctan\frac{|t_2|}{t_1} < \arctan c$, we obtain

$$|e^{-t(-\lambda)^b}| = e^{-t_1|\lambda|^b \cos(b\arg(-\lambda)) + t_2|\lambda|^b \sin(b\arg(-\lambda))} \le e^{-(ct_1 - |t_2|)|\lambda|^b} \tag{2.2}$$

for all $\lambda \in \Gamma$. By this, it follows that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\tilde{d}}^\infty \|e^{-t(-re^{i\tilde{a}})^b} e^{i\tilde{a}} F(re^{i\tilde{a}})\| dr \\
&\le \frac{1}{2}\int_{\tilde{d}}^1 e^{-(ct_1 - |t_2|)r^b} K(1 + r)^k dr + \frac{1}{2}\int_1^\infty e^{-(ct_1 - |t_2|)r^b} K(2r)^k dr \\
&\le (1 - \tilde{d})e^{-(ct_1 - |t_2|)\tilde{d}^b} K2^{k-1} + K2^{k-1}\int_0^\infty e^{-(ct_1 - |t_2|)r^b} r^k dr \\
&= K2^{k-1}\left[(1 - \tilde{d})e^{-(ct_1 - |t_2|)\tilde{d}^b} + \frac{1}{b}\Gamma\left(\frac{k+1}{b}\right)(ct_1 - |t_2|)^{-(k+1)/b}\right],
\end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function. Considering the parametrizations given in (1.2), this yields the existence of the curve integral over Γ_3. Analogously, with the same final estimate, one obtains the convergence of the integral over Γ_1. The integral over the finite path Γ_2 obviously exists, and, since $\|e^{-t(-\lambda)^b} F(\lambda)\| \le e^{-(ct_1 - |t_2|)\tilde{d}^b} K2^k$ for all $\lambda \in \Gamma_2$, it can be estimated by

$$\|U_{\Gamma_2}(t)\| \le K2^k e^{-(ct_1 - |t_2|)\tilde{d}^b}.$$

The coincidence of the integrals follows with Cauchy's Theorem. In fact, if $\tilde{\Gamma} = \Gamma(\hat{a}, \hat{d})$ for any $\tilde{a} \le \hat{a} \le a$ and $0 < \hat{d} < d$, then $U_\Gamma(t) - U_{\tilde{\Gamma}}(t) = \lim_{r\to\infty} U_{\Gamma_r}(t)$, where $\Gamma_r = \{re^{i\varphi}: -\hat{a} < \varphi < -\tilde{a}\} \cup \{re^{i\varphi}: \tilde{a} < \varphi < \hat{a}\}$. Along Γ_r $(r > 1)$, the integrand is bounded by $e^{-(ct_1-|t_2|)r^b}K(1+r)^k$. Thus,

$$\|U_{\Gamma_r}(+)\| \leq 2\pi r \frac{e^{-(ct_1-|t_2|)r^b}K(1+r)^k}{2\pi} \to 0 \qquad (r \to \infty). \qquad \blacksquare$$

PROPOSITION 2.3. *Let F, b and Γ be as in Proposition 2.2. Further, let $\lambda_0 \in \rho(A)\backslash(\Sigma(a) \cup B_d)$. If $m \in \{0, 1\}$, then the improper integral*

$$\int_\Gamma \frac{-(-\lambda)^{mb}e^{-t(-\lambda)^b}}{(\lambda-\lambda_0)^{k+2}}F(\lambda)d\lambda$$

converges uniformly for $t \in \overline{W(b, \tilde{a})}$.

PROOF. Let $m \in \{0, 1\}$ and $t \in \overline{W(b, \tilde{a})}$. Note that $\lambda_0 \notin \Gamma$. By (2.2), we have

$$|e^{-t(-\lambda)^b}| \leq e^{-(ct_1-|t_2|)|\lambda|^b} \leq 1$$

for all $\lambda \in \Gamma$. Hence,

$$\int_{\tilde{a}}^\infty \left\| \frac{-(-re^{i\tilde{a}})^{mb}e^{-t(-re^{i\tilde{a}})^b}e^{i\tilde{a}}}{(re^{i\tilde{a}}-\lambda_0)^{k+2}}F(re^{i\tilde{a}}) \right\| dr \leq \int_{\tilde{a}}^\infty \frac{r^{mb}K(1+r)^k}{|re^{i\tilde{a}}-\lambda_0|^{k+2}}dr.$$

This estimate yields the uniform convergence of the curve integrals over Γ_3. Similarly, one shows that the curve integrals over Γ_1 converge uniformly. $\blacksquare$

It follows from our assumptions on $R(\lambda, A)$ and Proposition 2.2 that the following definition is justified.

DEFINITION 2.4. *Let $0 < b < \dfrac{\pi}{2(\pi - a)}$ and $\Gamma = \Gamma(a, d)$. For $t > 0$, the operator $T_b(t) \in \mathscr{B}(X)$ is defined by*

$$T_b(t) := \frac{1}{2\pi i}\int_\Gamma e^{-t(-\lambda)^b}R(\lambda, A)d\lambda.$$

We will show that the operator family $(T_b(t))_{t>0}$ satisfies Definition 2.1.

It follows immediately from Proposition 2.2 that there is a constant $N_b > 0$ such that for all $t > 0$

$$\|T_b(t)\| \leq N_b t^{-(n+1)/b}. \tag{2.3}$$

PROPOSITION 2.5. *The mapping $t \mapsto T_b(t)$ has an analytic extension into the open sector $W(b, a)$. For all $m \in N$,*

$$\left(\frac{d}{dt}\right)^m T_b(t) = \frac{(-1)^m}{2\pi i}\int_\Gamma (-\lambda)^{mb}e^{-t(-\lambda)^b}R(\lambda, A)d\lambda \qquad (t \in W(b, a)).$$

PROOF. For every $t \in \boldsymbol{C}$, $\lambda \mapsto e^{-t(-\lambda)^b}$ is analytic on $\boldsymbol{C}\backslash[0, \infty)$. Hence, for $m = 0, 1, 2, \dots$, the mapping $(\lambda, t) \mapsto \left(\frac{\partial}{\partial t}\right)^m e^{-t(-\lambda)^b} R(\lambda, A) = (-1)^m(-\lambda)^{mb} e^{-t(-\lambda)^b} R(\lambda, A)$ is analytic in t and continuous on $\Gamma \times W(b, a)$. Further, we have for all $m \geq 0$ and $\lambda \in \Sigma(a) \cup B_d$

$$\|(-\lambda)^{mb} R(\lambda, A)\| \leq |\lambda|^{mb} C(1 + |\lambda|)^n \leq C(1 + |\lambda|)^{n+[mb]+1} .$$

By Proposition 2.2, the improper integrals

$$U_m(t) := \frac{(-1)^m}{2\pi i} \int_\Gamma (-\lambda)^{mb} e^{-t(-\lambda)^b} R(\lambda, A) d\lambda$$

converge uniformly on compact subsets of $W(b, a)$. Hence, for every $m \geq 0$, the mapping $t \mapsto U_m(t)$ is analytic on $W(b, a)$, and its derivative is given by $\frac{d}{dt} U_m(t) = U_{m+1}(t)$. ■

The following part treats the semigroup property of the operator family $(T_b(t))_{t>0}$.

PROPOSITION 2.6. *$T_b(t) T_b(s) = T_b(t + s)$ for all $t, s \in W(b, a)$.*

PROOF. If $t, s > 0$ then, by Proposition 2.2, we can write $T_b(s)$ as a curve integral over $\Gamma(\tilde{a}, \tilde{d})$, where $\max\left\{\frac{2b-1}{2b}\pi, 0\right\} < \tilde{a} < a$, $0 < \tilde{d} < d$. Similar to the proof of Lemma 1.4, one can show that the semigroup property holds. Since $t \mapsto T_b(t)$ is analytic on $W(b, a)$ (see Proposition 2.5), the assertion follows by the uniqueness theorem for analytic functions. ■

The set $X_b := \bigcup_{t>0} T_b(t) X$ is a dense subset of the continuity set

$$\Omega_b := \{x \in X : T_b(t)x \to x \ (t \to 0)\}$$

of $(T_b(t))_{t>0}$. In order to show that statement (v) of Definition 2.1 is fulfilled, it is therefore sufficient to prove that Ω_b is dense in X.

LEMMA 2.7. *If $x \in D(A^{n+2})$, then*

$$\frac{1}{t}(T_b(t)x - x) \to -\frac{1}{2\pi i} \int_\Gamma (-\lambda)^{b-1} R(\lambda, A)(-A) x d\lambda = -J^b x$$

as $t \to 0$, $t \in W(b, a)$. In particular, $D(A^{n+2}) \subseteq \Omega_b$.

PROOF. Let $x \in D(A^{n+2})$. Fix $\lambda_0 \in \rho(A)\backslash(\Sigma(a) \cup B_d)$. For every $\lambda \in \rho(A)$, $\lambda \neq \lambda_0$, (2.4) $$R(\lambda, A)x = \sum_{j=0}^{n+1} \frac{(-1)^j}{(\lambda - \lambda_0)^{j+1}}(\lambda_0 - A)^j x + \frac{(-1)^n}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x .$$

We insert this in the definition of $T_b(t)$. By Proposition 2.2, we can interchange integration and summation. Cauchy's Integral Formula for the derivatives yields

$$\begin{aligned} T_b(t)x &= \frac{1}{2\pi i}\int_\Gamma e^{-t(-\lambda)^b}\left\{\sum_{j=0}^{n+1} \frac{(-1)^j}{(\lambda - \lambda_0)^{j+1}}(\lambda_0 - A)^j x \right. \\ &\qquad \left. + \frac{(-1)^n}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x\right\} d\lambda \\ &= \sum_{j=0}^{n+1} \frac{(-1)^j}{j!}\left(\frac{d}{d\lambda}\right)^j e^{-t(-\lambda)^b}\bigg|_{\lambda=\lambda_0} (\lambda_0 - A)^j x \\ &\qquad + \frac{(-1)^n}{2\pi i}\int_\Gamma \frac{e^{-t(-\lambda)^b}}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x d\lambda \end{aligned}$$

for all $t \in W(b, a)$. Therefore,

$$\begin{aligned} \frac{1}{t}[T_b(t)x - x] &= \frac{1}{t}[e^{-t(-\lambda)^b} - 1]x + \sum_{j=1}^{n+1} \frac{(-1)^j}{j!}\frac{1}{t}\left(\frac{d}{d\lambda}\right)^j e^{-t(-\lambda)^b}\bigg|_{\lambda=\lambda_0} (\lambda_0 - A)^j x \\ &\qquad + \frac{1}{t}\frac{(-1)^n}{2\pi i}\int_\Gamma \frac{e^{-t(-\lambda)^b}}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x d\lambda \end{aligned}$$

for all $t \in W(b, a)$. Now, $\frac{1}{t}[e^{-t(-\lambda)^b} - 1] \to -(-\lambda)^b$ for every $\lambda \in C\backslash[0, \infty)$ and

$$\frac{1}{t}\left(\frac{d}{d\lambda}\right)^j e^{-t(-\lambda)^b}\bigg|_{\lambda=\lambda_0} \to (-1)^{j-1}\left[\prod_{i=0}^{j-1}(b - i)\right](-\lambda_0)^{b-j} \qquad (1 \leq j \leq n + 1)$$

as $t \to 0$. Further,

$$\begin{aligned} &\frac{1}{t}\frac{1}{2\pi i}\int_\Gamma \frac{e^{-t(-\lambda)^b}}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x d\lambda \\ &\quad = \frac{1}{t}\frac{1}{2\pi i}\int_\Gamma \frac{e^{-t(-\lambda)^b} - 1}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x d\lambda \\ &\quad \to \frac{1}{2\pi i}\int_\Gamma \frac{-(-\lambda)^b}{(\lambda - \lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x d\lambda \end{aligned}$$

since, by means of Proposition 2.3, we can interchange differentiation and integration. Hence $\frac{1}{t}[T_b(t)x - x]$ converges as $t \to 0$, $t \in W(b, a)$, and

$$\begin{aligned}
&\lim_{\substack{t\to 0\\ t\in W(b,a)}} \frac{1}{t}[T_b(t)x - x] \\
&\quad = -(-\lambda_0)^b x + \sum_{j=1}^{n+1} \frac{(-1)^j}{j!}(-1)^{j-1}\left[\prod_{i=0}^{j-1}(b-i)\right](-\lambda_0)^{b-j}(\lambda_0 - A)^j x \\
&\quad\quad + \frac{(-1)^n}{2\pi i}\int_\Gamma \frac{-(-\lambda)^b}{(\lambda-\lambda_0)^{n+2}} R(\lambda, A)(\lambda_0 - A)^{n+2} x\, d\lambda\,.
\end{aligned}$$

Using Cauchy's Integral Formula for the derivatives and (2.4), one obtains

$$\lim_{\substack{t\to 0\\ t\in W(b,a)}} \frac{1}{t}[T_b(t)x - x] = -(-\lambda_0)^b x + \frac{1}{2\pi i}\int_\Gamma -(-\lambda)^b\left[R(\lambda, A) - \frac{1}{\lambda - \lambda_0}\right] x\, d\lambda\,.$$

Note that Cauchy's Integral Formula can not be applied to the term $-(-\lambda_0)^b$ since the integral $\frac{1}{2\pi i}\int_\Gamma \frac{-(-\lambda)^b}{\lambda-\lambda_0} d\lambda$ does not exist. But, with the identity $R(\lambda, A)(\lambda - A)x = x$ for all $\lambda \in \rho(A)$ and Cauchy's Integral Formula, it follows that

$$\begin{aligned}
&-(-\lambda_0)^b x + \frac{1}{2\pi i}\int_\Gamma -(-\lambda)^b\left[R(\lambda, A) - \frac{1}{\lambda-\lambda_0}\right] x\, d\lambda \\
&\quad - \frac{1}{2\pi i}\int_\Gamma -(-\lambda)^{b-1} R(\lambda, A)(-A) x\, d\lambda \\
&= -(-\lambda_0)^b x + \left\{\frac{-\lambda_0}{2\pi i}\int_\Gamma \frac{(-\lambda)^{b-1}}{\lambda - \lambda_0} d\lambda\right\} x \\
&= 0\,.
\end{aligned}$$

By (1.6), this gives the assertion. ∎

PROPOSITION 2.8. *The operators $T_b(t)$ $(t \in W(b, a))$ are injective.*

PROOF. Let $t_0 \in W(b, a)$. Assume $T_b(t_0)x = 0$ for some $x \in X$. Proposition 2.6 yields $T_b(t_0 + t)x = T_b(t)T_b(t_0)x = 0$ for all $t \in W(b, a)$. Since, by Proposition 2.5, $t \mapsto T_b(t)x$ is analytic, it follows by the uniqueness theorem for analytic functions that $T_b(t)x = 0$ for all $t \in W(b, a)$. Using that the operators $(-A)^{-1}$ and $T_b(t)$ commute, we obtain $T_b(t)(-A)^{-(n+2)}x = (-A)^{-(n+2)}T_b(t)x = 0$. But $(-A)^{-(n+2)}x \in D(A^{n+2})$. Therefore, Lemma 2.7 yields $(-A)^{-(n+2)}x = \lim_{t\to 0} T_b(t)(-A)^{-(n+2)}x = 0$ which implies $x = 0$. ∎

From the results above, it follows that the operator family $(T_b(t))_{t>0}$ forms an analytic semigroup of growth order $\frac{n+1}{b}$. The generator A_b of this semigroup is defined by

$$A_b x := \lim_{t\to 0} \frac{1}{t}[T_b(t)x - x] \quad \text{on} \quad D(A_b) := \left\{x \in X : \lim_{t\to 0} \frac{1}{t}[T_b(t)x - x] \text{ exists}\right\}.$$

In general, the generator of a semigroup of growth order α is not closed. But, by [20], Lemma 3.1, it is always closable. Its closure is called the complete generator of the semigroup. In the following, we will show that the complete generator $\overline{A_b}$ of $(T_b(t))_{t>0}$ coincides with the negative fractional power $-(-A)^b$ given in Definition 1.11.

LEMMA 2.9. $-(-A)^b \subseteq \overline{A_b}$.

PROOF. By Lemma 2.7, $-J^b = A_b|_{D(A^{n+2})}$. Thus $-J^b \subseteq A_b$. Taking closures on both sides, the assertion follows by Definition 1.11. ■

To prove Lemma 2.9, we used that the operators $-(-A)^b$ and $\overline{A_b}$ coincide on the core $D(A^{n+2})$ for $-(-A)^b$. The opposite inclusion can be shown in a similar way. Consider the space

$$X_b := \bigcup_{t>0} T_b(t)X .$$

LEMMA 2.10. $X_b \subseteq D(A^\infty)$.

PROOF. Fix $t > 0$ and $x \in X$. For $m \in N_0$, the function $g_m(\lambda) := e^{-t(-\lambda)^b}\lambda^m R(\lambda, A)x$ is continuous on Γ, $g_m(\lambda) \in D(A)$ and $Ag_m(\lambda) = g_{m+1}(\lambda) - e^{-t(-\lambda)^b}\lambda^m x$ for all $\lambda \in \Gamma$. By Proposition 2.2, the integrals $\frac{1}{2\pi i}\int_\Gamma g_k(\lambda)d\lambda$ $(k \in N_0)$ exist. Cauchy's Theorem yields $\frac{1}{2\pi i}\int_\Gamma e^{-t(-\lambda)^b}\lambda^k d\lambda = 0$ for all $k \in N_0$. Since A is closed, we can conclude that

$$\frac{1}{2\pi i}\int_\Gamma g_m(\lambda)d\lambda \in D(A) \quad \text{and} \quad A\left(\frac{1}{2\pi i}\int_\Gamma g_m(\lambda)d\lambda\right) = \frac{1}{2\pi i}\int_\Gamma g_{m+1}(\lambda)d\lambda .$$

By induction, it follows that for all $m \in N$

$$T_b(t)x \in D(A^m) \quad \text{and} \quad A^m T_b(t)x = \frac{1}{2\pi i}\int_\Gamma e^{-t(-\lambda)^b}\lambda^m R(\lambda, A)x d\lambda . \qquad ■$$

In particular, X_b is contained in $D(A^{n+2})$ and therefore in the domains of $-(-A)^b$ and $\overline{A_b}$. To see that X_b is a core for $\overline{A_b}$ we need the following considerations.

It $x \in \Omega_b$, then, by (2.3), the mapping $t \mapsto \|T_b(r)x\|$ $(t > 0)$ is bounded, i.e.

$$N_b(x) := \sup_{t>0} \|T_b(t)x\|$$

exists. Further, $N_b(x) \geq \|x\|$ for every $x \in \Omega_b$. Obviously, $N_b(\cdot)$ defines a norm on Ω_b, and the space Ω_b provided with the norm $N_b(\cdot)$ is a Banach space.

Since $T_b(t)X \subseteq \Omega_b$ for all $t > 0$, we can restrict the semigroup $(T_b(t))_{t>0}$ to the space Ω_b, i.e. we study the operator family $(U_b(t))_{t\geq 0}$ given by

$$U_b(t) := T_b(t)|_{\Omega_b} \qquad (t > 0)$$
$$U_b(0) := I_{\Omega_b} .$$

Further, let E_b be the restriction of the generator A_b of $(T_b(t))_{t>0}$ to the space Ω_b, i.e.

$$E_b x = A_b x \qquad \text{for all } x \in D(E_b) := \{x \in D(A_b) : A_b x \in \Omega_b\} .$$

Being a restriction of the closed operator $\overline{A_b}$, E_b is closable in X. By [16], Lemma 3.2, we even have that $D(E_b)$ is a core for $\overline{A_b}$ or, equivalently,

$$\overline{E_b} = \overline{A_b} . \tag{2.5}$$

Moreover, Theorem 2.2 in [16] yields that the operator family $(U_b(t))_{t\geq 0}$ forms a strongly continuous semigroup of contractions with generator E_b on $(\Omega_b, N_b(\cdot))$.

PROPOSITION 2.11. *The space X_b is a core for the operator $\overline{A_b}$.*

PROOF. We will first show that X_b is a core for E_b in the Banach space $(\Omega_b, N_b(\cdot))$.

Since the operators A_b and J^b coincide on $D(A^{n+2})$, and J^b commutes with $(-A)^{-1}$ on $D(J^b) = D(A^{n+2})$, it follows that $A_b D(A^{2n+4}) \subseteq D(A^{n+2}) \subseteq D(A_b)$. By definition of E_b, we therefore have $D(A^{2n+4}) \subseteq D(E_b)$, and Lemma 2.10 yields $X_b \subseteq D(E_b)$.

Let $t > 0$ and $x \in X$. By definition of the semigroup $(U_b(t))_{t\geq 0}$, we obtain $U_b(s)T_b(t)x = T_b(t + s)x \in X_b$ for all $s \geq 0$. Hence, X_b is $(U_b(\cdot))$-invariant.

If $x \in \Omega_b$, then $N_b(U_b(t)x - x) \to 0$ $(t \to 0)$ since $(U_b(t))_{t\geq 0}$ is strongly continuous on $(\Omega_b, N_b(\cdot))$. But, for $t > 0$, $U_b(t)x = T_b(t)x \in X_b$. Thus, X_b is $N_b(\cdot)$-dense in Ω_b.

By [17], A-I, Proposition 1.9, it follows that $\overline{E_b|_{X_b}}^{N_b(\cdot)} = E_b$.

On the other hand, being a restriction of the closed operator $\overline{A_b}$, $E_b|_{X_b}$ is also closable in X, i.e. with respect to the norm $\|\cdot\|$. Since $N_b(x) \geq \|x\|$ for all $x \in \Omega_b$, we obtain

$$E_b = \overline{E_b|_{X_b}}^{N_b(\cdot)} \subseteq \overline{E_b|_{X_b}}^{\|\cdot\|} \subseteq \overline{A_b} .$$

Moreover, since $\overline{E_b|_{X_b}}^{\|\cdot\|}$ is closed in X, it follows that $\overline{E_b} \subseteq \overline{E_b|_{X_b}}^{\|\cdot\|} \subseteq \overline{A_b}$. By (2.5), this gives the assertion. ■

Lemma 2.9 and Proposition 2.11 allow the following conclusion.

PROPOSITION 2.12. $\overline{A_b} = -(-A)^b$.

PROOF. By Lemma 2.10, we have $X_b \subseteq D(A^\infty) \subseteq D(A^{n+2})$. Hence, Lemma 2.7 yields $A_b|_{X_b} \subseteq -J^b$. Taking closures on both sides, we obtain

$$\overline{A_b} \subseteq -(-A)^b$$

by means of Proposition 2.11 and Definition 1.11. Now, Lemma 2.9 gives the assertion. ■

EXAMPLE. If A is the generator of a n-times integrated semigroup such that $R(\lambda, A)$ exists on $\{\lambda \in \boldsymbol{C}: \Re\lambda > 0\} \cup \{0\}$, and $D(A)$ is dense in X, then, for every $0 < b < 1$, the negative fractional power $-(-A)^b$ is the generator of a semigroup $(T_b(t))_{t>0}$ of growth order $\dfrac{n+1}{b}$. In particular, the incomplete abstract Cauchy problem

$$\left(\frac{d}{dt}\right)^m u(t) = (-1)^{m+1} Au(t) \qquad (t \geq 0), \qquad u(0) = x,$$

where $m \geq 2$, has a bounded solution given by $T_{1/m}(\cdot)x$ for every $x \in D(A^{m(n+2)})$. ■

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