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On the primary decomposition of differential ideals of strongly Laskerian rings

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Introduction

Let R be a ring and let I be an ideal of R. Let $D = (D_0, D_1, D_2, \dots)$ be a higher derivation of rank m ($1 \leqq m \leqq \infty$) of R such that I is D-differential. On the primary decomposition of such an ideal I, the following is known:

If R is Noetherian and $m = \infty$, then the associated prime ideals of I are D-differential and I can be written as an irredundant intersection $Q_1 \cap \cdots \cap Q_n$ of primary ideals which are D-differential (cf. [4, Theorem 1]).

This result was extended by S. Sato [10] to the case of a set of higher derivations of finite rank. The works of Sato [10] yield the following: Let R be a Noetherian ring and let $\underline{H}$ be a set of higher derivations of rank m ($1 \leqq m \leqq \infty$) of R. Then every $\underline{H}$-differential ideal can be represented as an intersection of a finite number of $\underline{H}$-differential primary ideals of R.

In [10], S. Sato used essentially the assumption that the ring R is Noetherian.

In this paper, we treat the same problem for differential ideals of rings which may be non-Noetherian. The following is obtained: Let R be a ring and let I be an ideal of R. Let $\underline{H}$ be a set of higher derivations of rank m such that I is $\underline{H}$-differential. Then:

(1) If I is a decomposable ideal, $1 \leqq m < \infty$ and $char(R) \neq 0$, then I can be represented as an irredundant intersection of a finite number of $\underline{H}$-differential primary ideals of R.

(2) If R is strongly Laskerian and $1 \leqq m \leqq \infty$, then I can be represented as an irredundant intersection of a finite number of $\underline{H}$-differential primary ideals of R.

1. Preliminaries

All rings in this paper are assumed to be commutative with a unit element. Let R be a ring. A *derivation* of R is an additive endomorphism $d: R \to R$ such that $d(ab) = d(a)b + ad(b)$ for every $a, b \in R$. The set of all derivations of R is denoted by $Der(R)$. Let m be a positive integer. A *higher* *derivation* of rank m of R is a sequence $D=(D_0, D_1, \ldots, D_m)$ of additive endomorphisms $D_n: R \to R$ such that:

(1) D_0 is the identity map.

(2) $D_n(ab) = \Sigma\{D_i(a)D_j(b) \mid i+j=n\}$ for all $a, b \in R$ and $n \geqq 1$.

The set of all higher derivations of rank m of R is denoted by $HDer^m(R)$. A *higher derivation* of rank ∞ of R is an infinite sequence $D=(D_0, D_1, D_2, \ldots)$ such that $(D_0, D_1, \ldots, D_m) \in HDer^m(R)$ for every $m \geqq 1$. The set of all higher derivations of rank ∞ of R is denoted by $HDer^\infty(R)$. Let I be an ideal of R and let $\underline{D}$, $\underline{H}$ be subsets of $Der(R)$, $HDer^m(R)$ $(1 \leqq m \leqq \infty)$ respectively. We shall say that I is *$\underline{D}$-differential* if $d(I) \subset I$ for all $d \in \underline{D}$. Similarly we shall say that I is *$\underline{H}$-differential* if for all $D=(D_0, D_1, \ldots) \in \underline{H}$, $D_n(I) \subset I$ for all $n \geqq 0$.

Let I be an ideal of R. If I can be expressed as an intersection of finite number of primary ideals, we shall say that I is *decomposable*. The (uniquely determined) associated prime ideals of the primary ideals occurring in an irredundant primary representation of a decomposable ideal I are called the *associated prime ideals* of I. A ring R is called *Laskerian* if every ideal of R is decomposable. A Laskerian ring R is called *strongly Laskerian* if each primary ideal of R contains a power of its radical (cf. [2, Chap. 4, Ex. 23] or [7]). It is well known that

$$\text{Noetherian} \to \text{strongly Laskerian} \to \text{Laskerian}.$$

The following results will be needed repeatedly in this paper.

THEOREM A [6, p. 190–191]. *Let R be a ring containing a copy of the rational numbers. Given a sequence $\{d_i \mid d_i \in Der(R), i=1, 2, \ldots, m\}$ $(1 \leqq m \leqq \infty)$, the sequence $D=(D_0, D_1, \ldots, D_m)$ is in $HDer^m(R)$, where*

$$D_n = \Sigma\{d_{i_1} \ldots d_{i_r}/r! \mid i_1 + \cdots + i_r = n\}, \quad 1 \leqq n \leqq m.$$

Moreover, the correspondence $\{d_i\}_1^m \to D$ is a bijection between the set of all sequences $\{d_i\}$ and $HDer^m(R)$.

Theorem A also establishes the following assertion. Let I be an ideal of R and let $\{d_i\}_1^m$ corresponds to $D(\in HDer^m(R))$. Then I is D-differential if and only if I is d_i-differential for all i.

THEOREM B [5, Proposition (1.4)]. *Let I be an ideal of a ring R, P a minimal prime ideal of I, Q the P-primary component of I and $D \in HDer^m(R)$ $(1 \leqq m \leqq \infty)$. Assume I is D-differential. Then Q is also D-differential.*

2. Associated prime ideals of differential ideals

To prove Theorem 1, we shall show the following lemma by making use of the Krull intersection theorem for Laskerian rings (cf. [7]).

LEMMA 1. *Let R be a Laskerian ring containing a copy of the rational numbers. Let d be a derivation of R and x be in the Jacobian radical of R such that $d(x) = 1$. Then x is not a zero-divisor.*

PROOF. Let A be the annihilator of x. Then we have $A \subset Rx^n$ for any $n \geqq 1$. By [7, Corollary 3.2], $\bigcap_n Rx^n = (0)$, and thus $A = (0)$.

PROPOSITION 1. *Let R be a Laskerian ring containing a copy of the rational numbers and let I be an ideal of R. Let $\underline{D}$ be a subset of $Der(R)$ such that I is $\underline{D}$-differential. Then the associated prime ideals of I are $\underline{D}$-differential.*

PROOF. If an associated prime ideal P of I is not $\underline{D}$-differential, then there are $d \in \underline{D}$ and $x \in P$ such that $d(x) \notin P$. By passage $R_p/(I)$, we assume without loss of generality that $I = (0)$ and $d(x) = 1$. By Lemma 1, x is not a zero-divisor. On the other hand x is a zero divisor by [1, Proposition 4.7]. Therefore we get a contradiction.

PROPOSITION 2. *Let R be a Laskerian ring containing a copy of the rational numbers and let I be an ideal of R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m \leqq \infty)$ such that I is $\underline{H}$-differential. Then the associated prime ideals are also $\underline{H}$-differential.*

PROOF. For each $D = (D_0, D_1, \dots, D_m) \in \underline{H}$, we can construct the set $\underline{D} = \{d_i \in Der(R) | i = 1, 2, \dots, m\}$, where $d_1 = D_1$, $d_2 = D_2 - (1/2)D_1^2$, ... by Theorem A. Since I is $\underline{D}$-differential, the associated prime ideals of I are $\underline{D}$-differential by Proposition 1, and hence these are $\underline{H}$-differential.

THEOREM 1. *Let R be a Laskerian ring of characteristic 0, I an ideal of R and let $\{P_1, \dots, P_t\}$ be the associated prime ideals of I such that $P_i \cap Z = (0)$, where Z is a copy of the rational integers which is contained in R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m \leqq \infty)$ such that I is $\underline{H}$-differential. Then $P_1, \dots, P_t$ are also $\underline{H}$-differential.*

PROOF. We form the quotient ring $S^{-1}R$ with respect to $S = Z - (0)$. Put $S^{-1}R = R'$. Then R' contains a copy of the rational numbers. For each $D \in \underline{H}$, D can be uniquely extended to the higher derivation D' of R'. Put $\underline{H}' = \{D' \in HDer^m(R') | D \in \underline{H}\}$. Then IR' is $\underline{H}'$-differential and $P_1R', \dots, P_tR'$ are the associated prime ideals of IR'. By Proposition 2, $P_1R', \dots, P_tR'$ are $\underline{H}'$-differential. Since $P_i = P_iR' \cap R$, $P_1, \dots, P_t$ are $\underline{H}$-differential.

The following Proposition 3 was proved in [4, Theorem 1] under the assumption that the ring R is Noetherian. But by making use of [3, Theorem 8], the following fact is established in the proof of Theorem 1 of [4].

PROPOSITION 3 ([4, Theorem 1]). *Let R be a strongly Laskerian ring* *and let I be an ideal of R. Let $\underline{H}$ be a subset of $HDer^{\infty}(R)$ such that I is $\underline{H}$-differential. Then the associated prime ideals of I are $\underline{H}$-differential.*

In Theorem 1, the assumption "$char(R) = 0$" and "$P_i \cap Z = (0)$" can not be omitted as we show in the following examples.

EXAMPLES. (1) Let $R = k[X]$ be a polynomial ring over a field k of characteristic $p \neq 0$. Put $I = (X^p)$ and $P = (X)$. Then P is the associated prime ideal of I. Define $D = (D_0, D_1) \in HDer^1(R)$ by $D_1(X) = 1$ and $D_1(k) = 0$. Then I is D-differential. But P is not D-differential.

(2) Let $R = Z[X, Y]$ be a polynomial ring over the rational integers Z. Consider the ideal $I = (pY^p, X^pY^p)$ of R, where p is a prime number. Define $D = (D_0, D_1) \in HDer^1(R)$ by $D_1(X) = 1$ and $D_1(Y) = Y$. Then I is D-differential. Put $Q_1 = (Y^p)$, $Q_2 = (p, X^p)$, $P_1 = (Y)$ and $P_2 = (p, X)$. Then $I = Q_1 \cap Q_2$ is an irredundant primary decomposition of I and P_1, P_2 are the associated prime ideals of I. It is easy to see that Q_1, Q_2, P_1 are D-differential but P_2 is not D-differential.

3. Primary decomposition of differential ideals

We first consider a ring of characteristic $q \neq 0$. To prove Theorem 2, we need the following lemma.

LEMMA 2. *Let R be a ring of characteristic $q \neq 0$ and let $D = (D_0, \dots, D_m) \in HDer^m(R)$ $(1 \leqq m < \infty)$. Put $t = (m!)q$. Then $D_n(r^t) = 0$ for all $r \in R$ and $n = 1, 2, \dots, m$.*

PROOF. Let $f: R \to R[X]/(X^{m+1})$ $(f(r) = \sum D_n(r)X^n \ mod(X^{m+1}))$ be the ring-homomorphism associated to D (cf. [8]). Then we get that

$$f(r^t) = f(r)^t = (r + XY)^t \ mod(X^{m+1}),$$

where $Y = \sum D_i(r)X^{i-1}$. Therefore

$$f(r^t) = r^t + \sum {}_tC_i r^{t-i}(YX)^i \ mod(X^{m+1}),$$

where ${}_tC_i = (t!)/(i!)(t-i)!$. Since ${}_tC_i$ is multiple of q for any $i (1 \leqq i \leqq m)$, $f(r^t) = r^t \ mod(X^{m+1})$. Hence we get that $D_n(r^t) = 0$ for all $n \geqq 1$.

We can now prove the following theorem.

THEOREM 2. *Let R be a ring of characteristic $q \neq 0$ and let I be a decomposable ideal of R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m < \infty)$ such that I is $\underline{H}$-differential. Then I can be represented as an irredundant intersection of a finite number of $\underline{H}$-differential primary ideals of R.*

PROOF. Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition and let P_i be the radical of Q_i. Let $Q_i^{(t)}$ be the ideal generated by the set $\{x^t | x \in Q_i\}$, where $t = (m!)q$. Then, by Lemma 2, it is easy to check that $Q_i^{(t)}$ are $\underline{H}$-differential, and furthermore $I + Q_i^{(t)}$ are $\underline{H}$-differential. Since P_i is a minimal prime ideal of $I + Q_i^{(t)}$, the P_i-primary component Q_i' of $I + Q_i^{(t)}$ is $\underline{H}$-differential by Theorem B and $Q_i \supset Q_i' \supset I$. Hence $I = Q_i' \cap \cdots \cap Q_n'$.

We next show that a differential ideal of R can be written as an intersection of a finite number of differential primary ideals under the assumption that the ring R is strongly Laskerian.

PROPOSITION 4. *Let R be a strongly Laskerian ring and let I be an ideal of R. Let $\underline{H}$ be a subset of $HDer^{\infty}(R)$ such that I is $\underline{H}$-differential. Then I can be written as an irredundant intersection of a finite number of primary ideals which are $\underline{H}$-differential.*

PROOF. Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition and let P_i be the radical of Q_i. Then P_i are $\underline{H}$-differential by Proposition 3. Since R is strongly Laskerian, there is an integer t such that $P_i^t \subset Q_i$. It is straightforward that $I + P_i^t$ $(i = 1, \ldots, n)$ are $\underline{H}$-differential and P_i is a minimal prime ideal of $I + P_i^t$. Suppose that Q_i' is the P_i-primary component of $I + P_i^t$. Then Q_i' is $\underline{H}$-differential by Theorem B and $Q_i \supset Q_i' \supset I$. Therefore $I = Q_i' \cap \cdots \cap Q_n'$.

PROPOSITION 5. *Let R be a strongly Laskerian ring containing a copy of the rational number and let I be an ideal of R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m < \infty)$ such that I is $\underline{H}$-differential. Then I can be represented as an irredundant intersection of a finite number of $\underline{H}$-differential primary ideals of R.*

PROOF. Since the associated prime ideals of I are $\underline{H}$-differential by Proposition 2, we can obtain the proof in almost the same way as Proposition 4. Therefore we shall omit the proof.

We show the following proposition without the assumption that the ring R is strongly Laskerian.

PROPOSITION 6. *Let R be a ring of characteristic* 0, *Z a copy of the rational integers which is contained in R and let I be a decomposable ideal of R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m < \infty)$ such that I is $\underline{H}$-differential. If $I \cap Z \neq (0)$, I can be represented as an intersection of finite number of $\underline{H}$-differential primary ideals of R.*

PROOF. Let $I \cap Z = (q)$ $(q \neq 0)$. Then the residue ring R/I contains the ring $Z/(q)$ of characteristic q. For each $D \in \underline{H}$, D induces a higher derivation $D' = (D_0', D_1', \ldots, D_m') \in HDer^m(R/I)$ in the natural way, i.e., $D_n'(r + I) = D_n(r) +$

$I(r \in R)$. Thus we may assume that $char(R) = q$ and $I = (0)$, and thus Proposition 6 is proved by Theorem 2.

We can now state the theorem.

THEOREM 3. *Let R be a strongly Laskerian ring of characteristic 0 and let I be an ideal of R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m < \infty)$ such that I is $\underline{H}$-differential. Then I can be represented as an irredundant intersection of finite number of $\underline{H}$-differential primary ideals of R.*

PROOF. By Proposition 6, we may assume that $I \cap Z = (0)$. Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition such that $Q_i \cap Z = (0)$ $(i = 1, \ldots, t)$ and $Q_i \cap Z \neq (0)$ $(i = t+1, \ldots, n)$. Put $I_1 = Q_1 \cap \cdots \cap Q_t$ and $I_2 = Q_{t+1} \cap \cdots \cap Q_n$. Then $I = I_1 \cap I_2$ and $I_2 \cap Z \neq (0)$.

First we form the quotient ring $S^{-1}R := R_1$ with respect to $S = Z - (0)$. Then R_1 contains a copy of the rational numbers and $IR_1 = I_1 R_1$. For each $D \in \underline{H}$, D can be uniquely extended to the higher derivation of R_1, denoted by D'. Put $\underline{H}' = \{D' \in HDer^m(R_1) | D \in \underline{H}\}$. Since IR_1 is $\underline{H}'$-differential, IR_1 can be written as an intersection $Q'_1 \cap \cdots \cap Q'_t$ of primary ideals which are $\underline{H}'$-differential by Proposition 5. Put $Q_i^* = f^{-1}(Q'_i)$ $(i = 1, \ldots, t)$, where $f: R \to R_1$ is the natural mapping. It is easy to check that $Q_1^*, \ldots, Q_t^*$ are $\underline{H}$-differential and $I_1 = f^{-1}(IR_1) = Q_1^* \cap \cdots \cap Q_t^*$.

We next form R/I. Put $R_2 = R/I$, $J_1 = I_1/I$ and $J_2 = I_2/I$. Then we have that $char(R_2) = 0$, $(0) = J_1 \cap J_2$ and $J_2 \cap Z = (q)$ for some non zero integer q. Since I is $\underline{H}$-differential, for each $D \in \underline{H}$, D induces a higher derivation D'' of rank m of R_2 in the natural way. Put $\underline{H}'' = \{D'' \in HDer^m(R_2) | D \in \underline{H}\}$. Furthermore we put $J'_2 = qR_2$. Then the ideal J'_2 is $\underline{H}''$-differential and $J_2 \supset J'_2$. Thus $(0) = J_1 \cap J_2 \supset J_1 \cap J'_2$, and hence $(0) = J_1 \cap J'_2$. By Proposition 6, J'_2 can be written as an intersection $Q''_1 \cap \cdots \cap Q''_s$ of $\underline{H}''$-differential primary ideals of R_2. Put $Q_i^{**} = g^{-1}(Q''_i)$ $(i = 1, \ldots, s)$, where $g: R \to R_2$ is the natural mapping. Then we have that $Q_1^{**}, \ldots, Q_s^{**}$ are $\underline{H}$-differential and $g^{-1}(J_1) = I_1$. Thus $I = g^{-1}(J_1) \cap g^{-1}(J'_2) = I_1 \cap Q_1^{**} \cap \cdots \cap Q_s^{**}$, and hence $I = Q_1^* \cap \cdots \cap Q_t^* \cap Q_1^{**} \cap \cdots \cap Q_s^{**}$ is an intersection of $\underline{H}$-differential primary ideals.

By Theorems 2, 3 and Proposition 4, we have the following theorem.

THEOREM 4. *Let R be a strongly Laskerian ring and let I be an ideal of R. Let $\underline{H}$ be a subset of $HDer^m(R)$ $(1 \leqq m \leqq \infty)$ such that I is $\underline{H}$-differential. Then I can be represented as an irredundant intersection of a finite number of $\underline{H}$-differential primary ideals of R.*

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