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Path integrals for coherent states and classical dynamics on a homogeneous Kähler manifold

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0. Introduction

The logical structure of quantum mechanics and the relation to the classical dynamics are clearly explained by path integrals. Originally R. P. Feynman formulated his approach with a Lagrangian [3]. Afterward he and the successors discovered phase space path integrals or Hamiltonian path integrals [2] [4] [5]. In particular J. R. Klauder [7] used coherent states in path integration, and S. S. Schweber [14] studied a path integral based on a Hilbert space of holomorphic functions on C^n which is a representation space of the Heisenberg group. This work is considered as a quantization of a flat Kähler manifold C^n. Concerning a curved space, H. Kuratsuji and T. Suzuki [9] found that a classical Hamiltonian on a phase space CP^1 appears in a path integral expression for a matrix element of an irreducible representation of $SU(2)$. Also C. C. Gerry and S. Silverman [6] showed that a matrix element of the holomorphic discrete series of $SU(1, 1)$ is represented by a path integral with a classical Hamiltonian on the Poincaré disc. As we shall observe below, these phenomena happen in a general situation.

To clarify necessary assumption, we start with a brief review of the path integral using coherent states. Let G be a Lie group, K its closed subgroup, and assume that the homogeneous space G/K has an invariant measure μ. Let $(\pi, \mathscr{H})$ be an irreducible unitary representation of G with a unit vector v_0 such that $k \cdot v_0 \propto v_0$ for all $k \in K$ and let a matrix element $\langle v_0 | g \cdot v_0 \rangle$ belong to $L^2(G/K, \mu)$. Let M be an open dense subset of G/K and $g(z)$ a smooth section: $M \to G$ of the principal fiber bundle $G \to G/K$. Following A. M. Perelomov [11], we define a coherent state by

$$|z\rangle := g(z) \cdot v_0 .$$

Then an integral operator on $\mathscr{H}$

$$\int_M \mu(dz) |z\rangle \langle z|$$

is bounded and commutative with the action of G. Therefore we can assume that the operator above equals 1. Then for $z,\ w \in M$ and $X \in \mathfrak{g}$ (the Lie algebra of G),

$$\langle z|e^{-tX}|w\rangle = \int \mu(dz_1)\cdots\mu(dz_{n-1}) \prod_{j=1}^{n} \langle z_j|e^{-\varepsilon X}|z_{j-1}\rangle\,,$$

$$z_n = z,\ z_0 = w,\ \text{and}\ \varepsilon = t/n\,.$$

Moreover we assume that the $O(\varepsilon^2)$-term is well-behaved, and we use the following approximation:

$$\langle z_j|e^{-\varepsilon X}|z_{j-1}\rangle \approx \langle z_j|1-\varepsilon X|z_{j-1}\rangle = \langle z_j|z_{j-1}\rangle\{1-\varepsilon\langle z_j|X|z_{j-1}\rangle/\langle z_j|z_{j-1}\rangle\}$$
$$\approx \exp\{\varepsilon\varepsilon^{-1}\log\langle z_j|z_{j-1}\rangle - \varepsilon\langle z_j|X|z_{j-1}\rangle\}\,.$$

Let γ_s be a path on G satisfying $\gamma_0 K = w$ and $\gamma_t K = z$. We set $z_j = \gamma_{jt/n}K$. Then

$$\log\langle z_j|z_{j-1}\rangle = \log\langle z_j|\gamma_{(j-1)t/n}\gamma_{jt/n}^{-1}|z_j\rangle\,.$$

Also we note that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\log\langle z_s|\gamma_{s-\varepsilon}\gamma_s^{-1}|z_s\rangle = -\langle z_s|\partial_s\gamma_s\gamma_s^{-1}|z_s\rangle = -\langle v_0|Ad(\gamma_s^{-1})\gamma_s^{-1}\partial_s\gamma_s|v_0\rangle\,.$$

We now define $\lambda \in \mathfrak{g}^*$ by $i\lambda(X) = \langle v_0|X|v_0\rangle$ and set $\lambda_X(z) = \langle Ad^*g(z)\lambda, X\rangle$. Thus we obtain a path integral expression of a matrix element for the coherent states of π:

$$\langle z|e^{-tX}|w\rangle = \int_{\gamma_t = z,\ \gamma_0 = w} \mathscr{D}\gamma \exp iS[\gamma]\,,$$

$$S[\gamma] = \int_0^t -\lambda(\gamma^{-1}\partial_s\gamma_s) - \lambda_X(\gamma)ds\,,$$

(cf. [6], [9]). Then it is remarkable that $S[\gamma]$ is the classical action over a coadjoint orbit through $\lambda \in \mathfrak{g}^*$ with the canonical symplectic structure, which suggests that we may use path integration as a way arriving at the geometric quantization.

Our main objective in this article is to give a prescription for evaluating a path integral on a homogeneous Kähler manifold. It is closely related to the coherent states and the quantization for the Kähler manifold due to J. H. Rawnsley [12].

Let G/K be a homogeneous complex manifold and $E \to G/K$ a homogeneous holomorphic line bundle with an invariant Hermitian structure. Let ω denote the curvature form of the Hermitian connection. Also we assume that $\mu := |\omega^{\dim G/K}| \neq 0$ and $\mathscr{H} := L^2_{hol}(E, \mu) \neq 0$. Let $w,\ z \in G/K$ and γ a path jointing w and z. Then the classical action S along γ is defined by

$$\int_0^t -i\gamma^*\theta - \lambda_X(\gamma_s)ds\,,$$

where θ is the connection 1-form and λ_X is a canonical Hamiltonian function for $X \in \mathfrak{g}$. For general points w and z of the phase space G/K, there is no path subject to the classical motion $\dot{\gamma} = X_\gamma$ which joints w and z. Hence, we need to consider a variant of summation of over classical paths.

Let $\{s_i\}$ be an orthonormal basis of $\mathscr{H}$. Regarding s_i as a holomorphic function through a fixed local trivialization of E, we set $\kappa(z, w) = \sum s_i(z)\overline{s_i(w)}$. Then $\theta = -\partial \log \kappa(z, z)$. Keeping this in mind, for a path γ with $\gamma_0 = w$, $\gamma_t = z$, we employ

$$\log \kappa(z, w) - \log \kappa(w, w) \text{ instead of } \int_0^t \partial_z \log \kappa d\gamma\,,$$

and

$$t\lambda_X(z, w) \text{ instead of } \int_0^t \lambda_X(\gamma)ds\,.$$

Hence, we replace $e^{iS[\gamma]}$ by

$$p_t(z, w) = \kappa(z, w)e^{-it\lambda_X(z,w)}/\kappa(w, w)\,.$$

For a holomorphic section f of E, we set

$$P_tf(z) = \int p_t(z, w)f(w)\mu(dw)\,.$$

By the preceding argument, if

$$\lim_{n\to\infty} (P_{t/n})^n = \lim_{n\to\infty} \int \mu(dz_{n-1})\cdots\mu(dz_1)p_{t/n}(z, z_{n-1})\cdots p_{t/n}(z_1, w)$$

exists, the limit is considered as an evaluation of a path integral. This formulation is fairly correct. In fact $p_t = 1$ with $X = 0$, and we see that for the regular representation $\pi(\cdot)$ of G on E,

$$-it\lambda_X(z, w) = \log \pi(g_t^{-1})\kappa(z, w) - \log \kappa(w, w) + O(t^2), \text{ as functions of } t\,.$$

Also, if G is the Heisenberg group $C^n \times R$ and K is the center $0 \times R$, we have for $X = (\xi, \tau) \in \mathfrak{g}$ and $0 < \lambda \in \mathfrak{k}^* = R$,

$$p_t(z, w) = e^{-\lambda|\xi t|^2/2}\pi(g_t^{-1})\kappa(z, w)/\kappa(w, w)\,, \qquad \kappa(z, w) = e^{\lambda z\bar{w}}\,.$$

By using the fact that $\kappa(z, w)$ is a reproducing kernel of $L^2_{hol}(C^n, \kappa(z, z)^{-1}d^{2n})$, we see that the operator $(P_{t/n})^n$ has a kernel function

$$e^{-\lambda|\xi t|^2(n-1)/2n^2}\pi(g_{t(1-1/n)}^{-1})\kappa(z, w)/\kappa(w, w)\,.$$

Hence $\lim_{n\to\infty}(P_{t/n})^n = \pi(g_t^{-1})$ (cf. [10], [14]).

Generally the above $p_t(z, w)$ does not coincide with the kernel function of $\pi(g_t^{-1})$. As seen in §2, the extraordinary term can be removed by a formal exchange of 'exp'. The our formula is closely related to factors of automorphy [13].

1. Hamiltonians

Let G be a Lie group and K a closed subgroup. We assume that G/K is a homogeneous complex manifold. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote Lie algebras of G and K, respectively. Fix $\lambda \in \mathfrak{g}^*$ and assume that the restriction of $i\lambda$ to $\mathfrak{k}$ lifts on K. Let E denote a homogeneous complex line bundle with unitary structure $G \times_{i\lambda} C$ and ∇ an invariant unitary connection defined by $i\lambda$ in E. We denote by ω the curvature form of ∇ and assume that ω is a $(1, 1)$-form on G/K. Then it is well known that E becomes a holomorphic line bundle with a Hermitian connection ∇. Furthermore we assume that the real 2-form $i\omega$ is nondegenerate, that is, $(G/K, i\omega)$ is a symplectic manifold. Then clearly, the universal covering of G/K is isomorphic with the universal covering of the coadjoint orbit through $\lambda \in \mathfrak{g}^*$, as homogeneous symplectic manifolds. Observe the following diagram:

$$\begin{array}{ccccc} M \times C = E \otimes E^* & \xleftarrow{\ H\ } & E \otimes \bar{E} & \longrightarrow & E \boxtimes \bar{E} \\ & & \downarrow & & \downarrow \\ & & M & \xrightarrow{\ \Delta\ } & M \times \bar{M}\,, \end{array}$$

where H is an isomorphism defined by the Hermitian metric on E and Δ is the diagonal mapping. Let $1 \in \Gamma(M, Hom(E, E))$ and let $\kappa(z, w)$ denote a unique holomorphic extension of $H^{-1}(1)$ on a neighbourhood of ΔM.

PROPOSITION 1.1. *Let $\pi(\cdot)$ denote the left regular representation of G on $\Gamma(M, E)$. For $X \in \mathfrak{g}$, we set $g_t = \exp tX$ and*

$$iH_X(z, w) = H(\partial_t|_{t=0}\pi(g_t^{-1}) \otimes 1\, \kappa(z, w))\,.$$

Then

$$-H_X(z, z) = \lambda_X(z) := \langle Ad^* g_z \lambda, X\rangle, \text{ with } g_z K = z\,.$$

PROOF. Let U be a subset of M and fix a nonvanishing holomorphic section s of $E|U$. Then $H^{-1}(1) = s \otimes s/(s, s)$, where $(\ ,\ )$ denotes the invariant Hermitian metric on E. Let $h(z, w)$ be a unique holomorphic extension of

(s, s) on a neighbourhood of ΔU in $M \times \overline{M}$. Then $\kappa(z, w) = s(z) \otimes s(w)/h(z, w)$. For $g \in G$, $z \in U$, we define $j(g, z) \in C^{\times}$ by $g^{-1}s(gz) = s(z)j(g, z)$, and also for $X \in \mathfrak{g}$ we set $j(X, z) = \partial_t|_{t=0} j(g_t, z)$.

Since

$$H(\pi(g_t^{-1}) \otimes 1\, \kappa(z, z)) = j(g_t, z)h(z, z)/h(g_t z, z)\,,$$

we have

$$iH_X(z, z) = j(X, z) - \partial_X h(z, z)/h(z, z)\,.$$

We now represent s as $s = \phi f$ with a local section ϕ *of* $G \to G/K$ and a C-valued function f. Then by definition

$$\nabla s = \phi\{df + i\lambda(\phi^{-1}d\phi)f\}\,,$$

here $d\phi$ is a section of $T^*M \otimes_R C \otimes \phi^{-1}TG$. We define $J(g, z) \in K$ for $g \in G$, $z \in U$ by $g^{-1}\phi(gz) = \phi(z)J(g, z)$. Since

$$j(g, z) = J(g, z)^{i\lambda} f(gz)/f(z)\,, \text{ and } \partial_t|_{t=0} J(g_t, z) = -X + d_X\phi\,,$$

where X denotes the differential of the left action, we have

$$j(X, z) = -i\lambda(Ad\phi^{-1}X) + i\lambda(\phi^{-1}d\phi) + d_X f/f\,.$$

Also since $h = |f|^2$, we have

$$j(X, z) - \partial_X h/h = -i\lambda(Ad\phi^{-1}X) + i\lambda(\phi^{-1}d\phi) + d_X f/f - \partial_X f/f - \partial_X \bar{f}/\bar{f}\,.$$

The holomorphicity of s means that

$$i\lambda(\phi^{-1}\bar{\partial}_X\phi) + \bar{\partial}_X f/f = 0\,,$$

$$i\lambda(\phi^{-1}\partial_X\phi) - \partial_X\bar{f}/\bar{f} = -conj.\{i\lambda(\phi^{-1}\bar{\partial}_X\phi) + \bar{\partial}_X f/f\} = 0\,.$$

Thus $iH_X(z, z) = -\lambda_X(z)$.

2. Totally complex polarization

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{g}_c$ its complexification. Let $\lambda \in \mathfrak{g}^*$ and let $\mathfrak{p} \subset \mathfrak{g}_c$ be a totally complex polarization for λ [1]. We denote by G_c the 1-connected Lie group with the Lie algebra $\mathfrak{g}_c$. Let P and $\bar{P}$ be analytic subgroups generated by $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, respectively. Moreover we assume that $i\lambda$ holomorphically lifts on P. Also we set $q^{i\lambda} = conj.(\bar{q}^{-i\lambda})$ for $q \in \bar{P}$. Since these characters coincide on $P \cap \bar{P}$, we can define a holomorphic function $i\lambda$ on $P\bar{P}$ by

$$(xy)^{i\lambda} := x^{i\lambda}y^{i\lambda} \text{ for } x \in P\,,\ y \in \bar{P}\,.$$

Let G be an analytic subgroup generated by $\mathfrak{g}$ and set $K = G \cap \bar{P}$. We define a holomorphic line bundle over $G_c/\bar{P}$ by $E := G_c \times_{i\lambda} C$. Since the natural mapping $G/K \to G_c/\bar{P}$ is an open embedding, we can consider G/K as a complex manifold. Let U be an open subset and $\psi : U \to P$ a holomorphic mapping, and assume that the following diagram is commutative:

$$\begin{array}{ccc} \psi : U & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ G/K & \longrightarrow & G_c/\bar{P}. \end{array}$$

Let (α, β) be a local holomorphic section of $P \times \bar{P} \to P\bar{P}$ and set

$$\alpha(z, w) = \alpha(\overline{\psi(w)}^{-1}\psi(z)) \text{ and } \beta(z, w) = \beta(\overline{\psi(w)}^{-1}\psi(z)).$$

We can now state a main observation, which gives a group theoretic description of a Hamiltonian function for $X \in \mathfrak{g}$.

THEOREM 2.1. $H_X(z, w) = -\langle Ad^*\psi(z)\beta(z, w)^{-1}\lambda, X\rangle$.

PROOF. We write ψ as $\psi = \phi f$ with a local section ϕ of $G \to G/K$ and a smooth mapping $f : U \to \bar{P}$. Then $s := \psi \cdot 1 = \phi \cdot f^{i\lambda}$ is a holomorphic section of $E|U$. Since $\psi = \phi f$ and $\bar{\psi} = \phi \bar{f}$, we see that

$$h(z, z) := (s, s) = f^{i\lambda}\overline{f^{i\lambda}} = (\bar{f}^{-1})^{i\lambda} f^{i\lambda} = \{\overline{\psi(z)}^{-1}\psi(z)\}^{i\lambda}.$$

Thus

$$h(z, w) = \alpha(z, w)^{i\lambda}\beta(z, w)^{i\lambda}.$$

For $g \in G$, $z \in U$, we define $J(g, z) \in \bar{P}$ by $g^{-1}\psi(gz) = \psi(z)J(g, z)$. Let $j(g, z) = J(g, z)^{i\lambda}$. Then $g^{-1}s(gz) = s(z)j(g, z)$. Since $\overline{\psi(w)}^{-1}\psi(gz) = \overline{\psi(w)}^{-1}g\psi(z)J(g, z)$, we obtain

$$h(gz, w)j(g, z)^{-1} = \{\overline{\psi(w)}^{-1}g\psi(z)\}^{i\lambda}.$$

Let $g_t := \exp tX \in G$, $a_t \in P$ and $b_t \in \bar{P}$ satisfy $\overline{\psi(w)}^{-1}g_t\psi(z) = a_t b_t$. Then

$$-iH_X(z, w) = \partial_t|_{t=0} \log (a_t^{i\lambda} b_t^{i\lambda}),$$

$$a_0^{-1}\overline{\psi(w)}^{-1}X\psi(z)b_0^{-1} = a_0^{-1}\dot{a}_0 + b_0^{-1}\dot{b}_0.$$

Hence $H_X(z, w) = -Ad^*\psi b_0^{-1}\lambda(X)$.

In our situation, if $G_c \neq P$, the linear form $i\lambda$ does not define a character of G_c. Keeping this in mind, we consider

$$A_t(z, w) = \{Ad\beta(z, w)\psi(z)^{-1} \cdot \exp tX\}^{-i\lambda}$$

as a substitution for $e^{itH_X(z,w)} = \exp t\langle -i\lambda, Ad\beta(z, w)\psi(z)^{-1}X\rangle$. Since

$$\{\alpha^{-1}\overline{\psi(w)}^{-1}g_t\psi(z)\beta^{-1}\}^{-i\lambda} = \{\overline{\psi(w)}^{-1}g_t\psi(z)\}^{-i\lambda}(\alpha\beta)^{i\lambda},$$

we obtain

PROPOSITION 2.2. $\Lambda_t(z, w) = j(g_t, z)h(z, w)/h(g_t z, w)$.

Finally we supplement the case $L^2_{hol}(E, \mu) \neq 0$. Let $\{s_i\}$ be an orthonormal basis of $L^2_{hol}(E, \mu)$ and set $k(z, w) = \sum f_i(z)\overline{f_i(w)}$ with $s_i = sf_i$. Since $H(\sum s_i \otimes s_i)$ is G-invariant, the irreducibility [8] implies that $k(z, w)h(z, w)$ is a constant c. Hence, employing μ/c as an invariant measure on G/K, we may assume that $k(z, w)h(z, w) = 1$. Then

$$k(z, w)\Lambda_t(z, w)/k(w, w) = j(g_t, z)k(g_t z, w)h(w, w)$$

is a kernel function for the regular representation $\pi(g_t^{-1})$.

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