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Ergodic theorems for piecewise affine Markov maps with indifferent fixed points

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We study piecewise affine Markov maps with "indifferent fixed points", which are interesting with relation to intermittency. In the case that such a map T has a Lebesgue-equivalent invariant σ-finite infinite measure μ, we give ratio ergodic theorems which describe the limit value of the ratio of the sojourn time of the trajectory $\{T^k x\}_{k=0}^n$ in an interval U_1 with $\mu(U_1) = \infty$ to that in another interval U_2 with $\mu(U_2) = \infty$ for almost every x.

0. Introduction

For an interval map T, a fixed point p is called *indifferent* if

$$\lim_{x \to p} T(x) = p \quad \text{and} \quad \lim_{x \to p} |T'(x)| = 1.$$

Maps with indifferent fixed points are related to physical type I intermittency (cf. [1], [10], [13]). Our interesting indifferent fixed point p is a source, that is, $|T'(x)| > 1$ for almost every x in the neighborhood of the fixed point p.

In Inoue's paper [4], the somewhat strange notion "weakly attracting repellors" is given, that is, a fixed point p is called the *weakly attracting repellor* of an interval map T if p is unstable ($T^k x$ does not converge to p for a.e. x) and if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = f(p) \qquad \text{a.e. } x$$

for every continuous function f on the interval. In [4] we gave some conditions for the existence of weakly atracting repellors for maps with only one indifferent fixed point.

In this paper we are going to study maps with at least one indifferent fixed point. A typical example of an interval map with indifferent fixed points is

$$T(x) = \begin{cases} x/(1-x) & \text{for } x \in [0, \frac{1}{2}) \\ (2x-1)/x & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

which is deeply related to continued fractions (see [7]). The class of piecewise affine Markov maps with indifferent fixed points which is studied in this paper contains a linearization of this example.

A piecewise affine Markov map with indifferent fixed points p_i for some i is defined in the following way.

Let $0 = c_0 < c_1 < c_2 < \cdots < c_r = 1$ be a partition of [0, 1] and let $p_1 = 0$, $p_i \in (c_{i-1}, c_i)$ for $i = 2, \cdots, r-1$, $p_r = 1$. Let $\{a_{i,n}\}$ be an increasing sequence satisfying the following conditions for $i = 2, \cdots, r$:

$$a_{i,-1} = 0,\ a_{i,0} = c_{i-1},$$
$$(a_{i,n-1} - a_{i,n-2}) - (a_{i,n} - a_{i,n-1}) > 0 \qquad \text{for large } n,$$
$$a_{i,n} \text{ converges to } p_i \text{ as } n \to \infty,$$

and let $\{b_{i,n}\}$ be a decreasing sequence satisfying the following conditions for $i = 1, \cdots, r-1$:

$$b_{i,-1} = 1,\ b_{i,0} = c_i,$$
$$(b_{i,n-2} - b_{i,n-1}) - (b_{i,n-1} - b_{i,n}) > 0 \qquad \text{for large } n,$$
$$b_{i,n} \text{ converges to } p_i \text{ as } n \to \infty.$$

Define T_i: $[c_{i-1}, c_i] \to [0, 1]$ for $i = 1, \cdots, r$ by

$$T_i(x) = \frac{a_{i,n-2} - a_{i,n-1}}{a_{i,n-1} - a_{i,n}}(x - a_{i,n-1}) + a_{i,n-2}$$

on $(a_{i,n-1}, a_{i,n}]$ for $n \geqq 1$,

$$T_i(x) = \frac{b_{i,n-2} - b_{i,n-1}}{b_{i,n-1} - b_{i,n}}(x - b_{i,n-1}) + b_{i,n-2}$$

on $[b_{i,n}, b_{i,n-1})$ for $n \geqq 1$, and $T_i(p_i) = p_i$.

Let T: $[0, 1] \to [0, 1]$ be a map such that T restricted to (c_{i-1}, c_i) is T_i for $i = 1, \cdots, r$. Then T is uniquely defined on $[0, 1] \setminus \{c_i\}$. (The values $\{T(c_i)\}$ are not needed in this paper since the set of c_i is of measure 0.) Such a map T is called a *piecewise affine Markov map*. T is called a *piecewise affine Markov map with indifferent fixed points* if

$$\frac{a_{i,n-2} - a_{i,n-1}}{a_{i,n-1} - a_{i,n}} \quad \text{or} \quad \frac{b_{i,n-2} - b_{i,n-1}}{b_{i,n-1} - b_{i,n}}$$

converges to 1 for at least one i as $n \to \infty$.

Let U_1 and U_2 be the right or the left neighborhoods of indifferent fixed points p_i and p_j respectively (not necessary $i \neq j$). The first aim of this paper is to research the limit value of the mean sojourn time of the trajectory $\{T^k x\}_{k=0}^n$ in U_1 for almost every x and the limit value of the ratio of the sojourn time of $\{T^k x\}_{k=0}^n$ in U_1 to that in U_2 for almost every x. For this purpose, it is important to study a Lebesgue-equivalent T-invariant ergodic measure. Each piecewise affine Markov map T defined above has a Lebesgue-equivalent invariant ergodic σ-finite measure, say μ. But, in some cases, this measure is not finite. See Theorem 1.1 in the present paper (cf. [4], [11], [14], [15]).

For our aim, the Birkhoff individual ergodic theorem and the Hopf ratio ergodic theorem give good information if $\mu(U_1) < \infty$ and $\mu(U_2) < \infty$.

The Birkhoff individual ergodic theorem ([16], [8]): *Let T be a measure preserving transformation on a σ-finite measure space $(X, \mathscr{F}, \mu)$ and let $f \in L^1(\mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = f^* \text{ for } \mu - a.e. \ x \text{ and } f^* \in L^1(\mu).$$

The Hopf ratio ergodic theorem ([2], [8]): *Let T be a measure preserving ergodic transformation on a σ-finite measure space $(X, \mathscr{F}, \mu)$ and let $f, g \in L^1(\mu)$ with $\int g d\mu \neq 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} f(T^k x)}{\sum_{k=0}^{n} g(T^k x)} = \frac{\int_X f d\mu}{\int_X g d\mu} \quad \text{for } \mu - a.e. \ x.$$

Set $f = 1_{U_1}$ and $g = 1_{U_2}$, where 1_U is the indicator function of U. If $\mu(U_1) < \infty$ and $\mu(U_2) < \infty$, then $f, g \in L^1(\mu)$ and we can apply these ergodic theorems for T. In fact, if $\mu(U_1) < \infty$ and $\mu(U_2) < \infty$, then the mean sojourn time in U_1 tends to 0 in the case $\mu([0, 1]) = \infty$ and to $\mu(U_1)/\mu([0, 1])$ in the case $\mu([0, 1]) < \infty$, and the ratio of the sojourn time in U_1 to that in U_2 tends to $\mu(U_1)/\mu(U_2)$. However, if $\mu(U_1) = \infty$ or $\mu(U_2) = \infty$, the previous ergodic theorems do not describe anything of this nature. In this paper we present two ratio ergodic theorems (Theorems 1.2 and 1.3) and two individual ergodic theorems (Corollaries 1.2.1 and 1.3.1) which are applicable to the case $\mu(U_1) = \infty$ or $\mu(U_2) = \infty$ under some conditions.

In this paper we also research the asymptotic measure of T (Corollaries 1.2.2 and 1.3.2). If the weak limit of

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x}$$

exists for almost every x, then its limit is called the *asymptotic measure* of T, where δ_x is the Dirac δ measure on x.

Now we state the organization of this paper. In §1 we state our main results. In §2 we confirm the definition of ergodicity and state some basic properties of the first return maps. In §3 we study piecewise affine Bernoulli maps with countable partitions, which naturally arise from the first return maps and we prove two lemmas which are important to prove our main theorems. In §4 we prove Theorem 1.1 and in §5 we prove Theorems 1.2 and 1.3 and their corollaries.

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1. Results

Before stating the main theorems in the present paper, we state the following theorem which describes some estimates of m-equivalent invariant ergodic σ-finite measures for piecewise affine Markov maps defined in §0, (in this paper m is the Lebesgue measure unless we put a particular notice).

THEOREM 1.1. *Let T be a piecewise affine Markov map defined in §0. Then T has an m-equivalent invariant ergodic σ-finite measure μ which satisfies the following* (1)–(3):

(1) $$\mu(([0, 1] \setminus \textstyle\bigcup_{i=1}^{r} \mathrm{nbd}\,(p_i, \Delta)) < \infty \qquad \textit{for all } \Delta > 0,$$

where $$\mathrm{nbd}\,(p_i, \Delta) = \begin{cases} (0, \Delta) & \textit{for } i = 1 \\ (p_i - \Delta, p_i + \Delta) & \textit{for } i = 2, \cdots, r-1 \\ (1-\Delta, 1) & \textit{for } i = r. \end{cases}$$

(2) *Fix $i = 2, \cdots, r$ arbitrarily.* $\mu((p_i - \Delta, p_i)) = \infty$ *for every small $\Delta > 0$ if and only if*

$$\textstyle\sum_{n=0}^{\infty} (p_i - a_{i,n}) = \infty.$$

(3) *Fix $i = 1, \cdots, r-1$ arbitrarily.* $\mu((p_i, p_i + \Delta)) = \infty$ *for every small $\Delta > 0$ if and only if*

$$\textstyle\sum_{n=0}^{\infty} (b_{i,n} - p_i) = \infty.$$

To state our main theorems we denote the conditions (C, a, i) and (C, b, i):

(C, a, i) $$(p_i - a_{i,n}) \sim v(a, i) n^{-\kappa(a,i)},$$

(C, b, i) $$(b_{i,n} - p_i) \sim v(b, i) n^{-\kappa(b,i)},$$

where $\nu(a, i)$, $\nu(b, i)$, $\kappa(a, i)$ and $\kappa(b, i)$ are positive constants for each i (in this paper $v_n \sim w_n$ stands for $\lim_{n\to\infty}(v_n/w_n) = 1$).

$\kappa(a, i)$ and $\kappa(b, i)$ are related to the closeness of the graph of T to the diagonal line in the left neighborhood of p_i and in the right neighborhood of p_i respectively. Precisely, for example, $b_{1,n} \sim \nu(b, 1)n^{-\kappa(b,1)}$ means that

$$T(x) - x = \kappa(b, 1)\nu(b, 1)^{-1/\kappa(b,1)}x^{(1/\kappa(b,1))+1} + o(x^{(1/\kappa(b,1))+1}).$$

Denote

$$\operatorname{nbd}(p_i, \omega, \Delta) = \begin{cases} (p_i - \Delta, p_i) & \text{if } \omega = a \\ (p_i, p_i + \Delta) & \text{if } \omega = b, \text{ and} \end{cases}$$

$$p(i) = p_i.$$

In the following main theorems (Theorems 1.2 and 1.3) and their corollaries Δ, Δ_0 and Δ_1 are arbitrary small positive numbers. We state two ratio ergodic theorems. One of these is

THEOREM 1.2. *Let T be a piecewise affine Markov map with indifferent fixed points. Fix $\omega_0 = a$ or b and fix $\omega_1 = a$ or b. For $j = 0, 1$, fix i_j in such way that $2 \leqq i_j \leqq r$ if $\omega_j = a$, $1 \leqq i_j \leqq r - 1$ if $\omega_j = b$. Assume that T satisfies the conditions (C, ω_0, i_0) and (C, ω_1, i_1). If $\kappa(\omega_0, i_0) < \kappa(\omega_1, i_1)$ and $\kappa(\omega_0, i_0) \leqq 1$, then*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n} 1_{\operatorname{nbd}(p(i_1),\omega_1,\Delta_1)}(T^k x)}{\sum_{k=0}^{n} 1_{\operatorname{nbd}(p(i_0),\omega_0,\Delta_0)}(T^k x)} = 0 \qquad \textit{for } m - a.e.\ x.$$

From this theorem we can obtain the following individual ergodic theorem as a corollary.

COROLLARY 1.2.1. *Assume that T satisfies the conditions (C, ω, i) for all (ω, i). If there exists only one (ω_0, i_0) such that $\kappa(\omega_0, i_0) < \kappa(\omega, i)$ for all $(\omega, i) \neq (\omega_0, i_0)$ and that $\kappa(\omega_0, i_0) \leqq 1$, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} 1_{\operatorname{nbd}(p(i_0),\omega_0,\Delta)}(T^k x) = 1 \qquad \textit{for } m - a.\ e.\ x.$$

Concerning to the asymptotic measure, we have

COROLLARY 1.2.2. *Assume that T satisfies the conditions (C, ω, i) for all (ω, i). If there exists only one i_0 such that*

$$\min\{\kappa(a, i_0), \kappa(b, i_0)\} < \min\{\kappa(\omega, i);\ i \neq i_0,\ \omega = a, b\}$$

$$\min\{\kappa(a, i_0), \kappa(b, i_0)\} \leqq 1,$$

then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k(x)}=\delta_{p(i_0)} \qquad \textit{weakly for } m-a.e.\ x.$$

This corollary means that p_{i_0} is the weakly attracting repellor.

In Theorem 1.2 we assume that $\kappa(\omega_0, i_0)<\kappa(\omega_1, i_1)$. In the following ratio ergodic theorem we consider the case $\kappa(\omega_0, i_0)=\kappa(\omega_1, i_1)=1$.

THEOREM 1.3. *Let T be a piecewise affine Markov map with indifferent fixed points. Fix $\omega_0=a$ or b and fix $\omega_1=a$ or b. For $j=0, 1$, fix i_j in such way that $2\leqq i_j\leqq r$ if $\omega_j=a$, $1\leqq i_j\leqq r-1$ if $\omega_j=b$. Assume that T satisfies the conditions (C, ω_0, i_0) and (C, ω_1, i_1). If $\kappa(\omega_0, i_0)=\kappa(\omega_1, i_1)=1$, then*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}1_{\mathrm{nbd}(p(i_1),\omega_1,\Delta_1)}(T^kx)}{\sum_{k=0}^{n}1_{\mathrm{nbd}(p(i_0),\omega_0,\Delta_0)}(T^kx)}=\rho \qquad \textit{for } m-a.e.\ x,$$

where ρ is a positive finite constant, which is independent of Δ_0, Δ_1 and x. In particular,

(1) *If $r=2$ and $\kappa(b, 1)=\kappa(a, 2)=1$, then*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}1_{(1-\Delta_1,1)}(T^kx)}{\sum_{k=0}^{n}1_{(0,\Delta_0)}(T^kx)}=\frac{c_1\nu(a, 2)}{(1-c_1)\nu(b, 1)} \qquad \textit{for } m-a.e.\ x.$$

(2) *If $\kappa(b, i_0)=\kappa(a, i_0)=1$, then*

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}1_{(p(i_0)-\Delta_1,p(i_0))}(T^kx)}{\sum_{k=0}^{n}1_{(p(i_0),p(i_0)+\Delta_0)}(T^kx)}=\frac{\nu(a, i_0)}{\nu(b, i_0)} \qquad \textit{for } m-a.e.\ x.$$

From this theorem we can obtain the following individual ergodic theorem.

COROLLARY 1.3.1. *Assume that T satisfies the conditions (C, ω, i) for all (ω, i). If $\kappa(a, i)$, $\kappa(b, i)\geqq 1$ for all i and if $\kappa(\omega_0, i_0)=1$, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}1_{\mathrm{nbd}(p(i_0),\omega_0,\Delta)}(T^kx)=s \qquad \textit{for } m-a.e.\ x,$$

where s is a positive finite constant, which is independent of small Δ and x.

Concerning to the asymptotic measure, we have

COROLLARY 1.3.2. *Assume that T satisfies the conditions (C, ω, i) for all (ω, i). If $\kappa(a, i)$, $\kappa(b, i)\geqq 1$ for all i and if $\kappa(a, i)=1$ or $\kappa(b, i)=1$ for at least one i, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k(x)}=\sum_{i=1}^{r}s_i\delta_{p_i} \qquad \textit{weakly for } m-a.e.\ x,$$

where s_i's are constants satisfying

$$\textstyle\sum_{i=1}^{r} s_i = 1 \text{ and } s_i \geqq 0 \qquad \text{for all } i = 1, \cdots, r,$$

and further $s_i \neq 0$ if and only if $\kappa(a, i) = 1$ or $\kappa(b, i) = 1$.

In the case $\kappa(\omega_0, i_0) = \kappa(\omega_1, i_l) < 1$, the author conjectures that the limit value of the ratio ergodic theorem does not exist.

2. Preliminaries

In this section we give the definition of ergodicity and a basic corollary of the Hopf ratio ergodic theorem, and we summarize some basic properties of the first return maps. First we give the definition of ergodicity for a transformation T on a σ-finite measure space $(X, \mathscr{F}, \mu)$.

DEFINITION 2.1. (T, μ) is called *ergodic* if $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ for every $A \in \mathscr{F}$ with $T^{-1}A = A$ μ – a.e.

Now we give a basic corollary of the Hopf ratio ergodic theorem.

LEMMA 2.1. *Let T be a measure preserving ergodic transformation on a σ-finite measure space $(X, \mathscr{F}, \mu)$, let $f \in L^1(\mu)$ and let $\int g d\mu = \infty$. Then*

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n} f(T^k x)}{\sum_{k=0}^{n} g(T^k x)} = 0 \qquad \text{for } \mu - a.e. \ x.$$

PROOF. Let $g_i \uparrow g$ and $\int g_i d\mu < \infty$ for each i. Then we have

$$\left| \frac{\sum_{k=0}^{n} f(T^k x)}{\sum_{k=0}^{n} g(T^k x)} \right| \leqq \left| \frac{\sum_{k=0}^{n} f(T^k x)}{\sum_{k=0}^{n} g_i(T^k x)} \right|.$$

From the Hopf ratio ergodic theorem, it follows that

$$\limsup_{n \to \infty} \left| \frac{\sum_{k=0}^{n} f(T^k x)}{\sum_{k=0}^{n} g(T^k x)} \right| \leqq \left| \frac{\displaystyle\int_X f d\mu}{\displaystyle\int_X g_i d\mu} \right| \qquad \text{for } \mu\text{-a.e. } x.$$

The right hand side of this inequality converges to 0 as i goes to infinity. Therefore we obtain the lemma.

Next we state some basic properties of first return maps.

The first return map of T on A is defined as $T^{n(x)}(x)$, where $n(x)$ is $\inf\{n \geqq 1;\ T^n(x) \in A\}$. In the following three lemmas, let T be a transformation on a measure space $(X, \mathscr{F}, m)$ and $A \subset X$ a measurable set with $A \subset \bigcup_{n=1}^{\infty} T^{-n}(A)$. Then the first return map is well defined.

LEMMA 2.2. *Let R be the first return map of T on A and μ_A an R-invariant σ-finite measure. Then the measure μ defined by*

$$\mu(D) = \sum_{n=1}^{\infty} \mu_A(A_n \cap T^{-n}D) \qquad \text{for } D \in \mathscr{F} \tag{2.1}$$

is T-invariant, where $A_1 = A$ and $A_{n+1} = A_n \cap T^{-n}(A^c)$ for $n \geqq 1$.

For the proof of this, see the proof of Lemma 2 in [11].

LEMMA 2.3. *Under the same situation as Lemma 2.2, if the first return map R of T is μ_A-ergodic, then T is μ-ergodic.*

The proof of this lemma is a minor modification of the proof of Lemma 3.2 in [4].

LEMMA 2.4. *Let μ be a T-invariant ergodic σ-finite measure and let $\mu|_A$ be the restriction of μ to A. Then the first return map R of T on A is $\mu|_A$-invariant ergodic. As a consequence, if μ_A^* is a μ-absolutely continuous R-invariant σ-finite measure, then $\mu_A^* = const.\ \mu|_A$.*

The proof of this lemma is a minor modification of the proof of Lemma 1 in [11].

3. Ergodic properties for piecewise affine Bernoulli maps with countable partitions

We devote this section to investigate some ergodic properties for piecewise affine Bernoulli maps with countable partitions which naturally arise from the first return maps and to prepare important lemmas.

In the whole of this section we assume that a piecewise affine Bernoulli map $R: [v, w] \to [v, w]$ satisfies the following condition:

There exists a countable partition $\{I_{i,n}\}$ of $[v, w]$ such that $I_{i,n}$ is an interval for each i, n and that

$$R(x) = (w_{i,n} - v_{i,n})^{-1}(w - v)(x - v_{i,n}) + v$$

for x in the interior of $I_{i,n}$, where $v_{i,n}$ and $w_{i,n}$ is the left and right endpoints of $I_{i,n}$ respectively, and $w_{i,n}$ converges monotonically to w_i as $n \to \infty$ for each i.

PROPOSITION 3.1. *There exists a unique m-absolutely continuous R-invariant ergodic probability measure μ.*

This is a special case of Proposition 5.1 in [4].

PROPOSITION 3.2. *The m-absolutely continuous R-invariant ergodic probability measure μ is the normalized Lebesgue measure on $[v, w]$.*

PROOF. Let $\mathscr{L}: L^1(m) \to L^1(m)$ be the Frobenius-Perron operator associated with (R, m), which is defined by

$$\int_A \mathscr{L} f(x) m(dx) = \int_{R^{-1}(A)} f(x) m(dx) \qquad \text{for } A \in \mathscr{F},\ f \in L^1(m).$$

(Some basic properties of the Frobenius-Perron operator are found in [6] and [9]. For the proof of the proposition, it is important that $\mathscr{L} f = f$ implies that f is the Radon-Nikodym derivative of an invariant measure.) Let $R_{i,n}$ be the restriction to $I_{i,n}$ of R. Then

$$\mathscr{L} 1(x) = \frac{d}{dx} \sum_{i,n} R_{i,n}^{-1}(x) = 1.$$

Thus the Lebesgue measure is R-invariant. Therefore we obtain the proposition.

In the rest of this section, put

$$E_n = \bigcup_i ([w_i, w_{i,n}] \text{ (or } [w_{i,n}, w_i])) \text{ and}$$
$$L = \{z; \log_2(\log_2 z) \text{ is a positive integer}\},$$

and let $u: N \to N$ be a monotonic increasing function satisfying

$$(w - v)^{-1} \cdot z \cdot m(E_{u(z)}) \sim \text{const} \cdot \log_2(\log_2 z).$$

The following lemmas are important to prove Theorems 1.2 and 1.3.

LEMMA 3.3.

$$\lim_{L \ni z \to \infty} \frac{w - v}{z^2 \cdot m(E_{u(z)})^2} \sum_{k=0}^{z} 1_{E_{u(z)}}(R^k x) = 0 \qquad \textit{for m-a.e. } x.$$

PROOF. Put

$$\xi_{z,k}(x) = 1_{E_{u(z)}}(R^k(x)).$$

Let P be the normalized Lebesgue measure on $[v, w]$. Then $\xi_{z,k}$'s are random variables on the probability space $([v, w], P)$. Since R is affine on $(v_{i,n}, w_{i,n})$ and $R(v_{i,n}, w_{i,n}) = (v, w)$ for each i, n, $\xi_{z,0}, \xi_{z,1}, \cdots, \xi_{z,z}$ are independent random variables with a common distribution. Put

$$X_z = \sum_{k=0}^{z} \xi_{z,k} \quad \text{and} \quad p_z = P(E_{u(z)}).$$

Let $\varepsilon > 0$ be arbitrary. Then it follows from the Chebyshev inequality that

$$\begin{aligned} &P(\{|(zp_z)^{-2}X_z| > \varepsilon\}) = P(\{|X_z| > \varepsilon z^2 p_z^2\}) \\ &\leqq (\varepsilon z^2 p_z^2)^{-2} \int |X_z|^2 dP \\ &\leqq \text{const. } \varepsilon^{-2}(\log_2(\log_2 z))^{-2} \qquad \text{for sufficiently large } z. \end{aligned}$$

Hence

$$\sum_{z\in L} P(\{|(zp_z)^{-2}X_z| > \varepsilon\}) \leqq \text{const. } \varepsilon^{-2}\sum_{l=1}^{\infty} l^{-2} < \infty.$$

Therefore we obtain the lemma by the Borel-Cantelli lemma.

LEMMA 3.4. *If* $m(E_l) \sim$ *const.* $l^{-\kappa}$ *for some* $0 < \kappa \leqq 1$, *then*

$$\lim_{L\ni z\to\infty} \frac{w-v}{z\cdot\sum_{l=1}^{u(z)} m(E_l)} \sum_{k=0}^{z} \sum_{l=1}^{u(z)} 1_{E_l}(R^k x) = 1 \qquad \textit{for } m\textit{-a.e. } x.$$

PROOF. Put

$$\zeta_{z,k}(x) = \sum_{l=1}^{u(z)} 1_{E_l}(R^k(x)).$$

Let P be the normalized Lebesgue measure on $[v, w]$. Then $\zeta_{z,k}$'s are random variables on the probability space $([v, w], P)$. Since R is affine on $(v_{i,n}, w_{i,n})$ and $R(v_{i,n}, w_{i,n}) = (v, w)$ for each i, n, $\zeta_{z,0}, \zeta_{z,1}, \cdots, \zeta_{z,z}$ are independent random variables such that $P(\zeta_{z,k} = l) = P(E_l) - P(E_{l+1})$. Put

$$Y_z = \sum_{k=0}^{z} \zeta_{z,k}.$$

First we assume that $m(E_l) \sim \text{const. } l^{-1}$. Then

$$u(z) \sim \text{const. } z/\log_2(\log_2 z).$$

Let $\varepsilon > 0$ be arbitrary. By the Chebyshev inequality we have

$$\begin{aligned} &P(\{|(z\textstyle\sum_{l=1}^{u(z)} P(E_l))^{-1} Y_z - 1| > \varepsilon\}) \\ &\leqq (\varepsilon z \textstyle\sum_{l=1}^{u(z)} P(E_l))^{-2} \int |Y_z - z\sum_{l=1}^{u(z)} P(E_l)|^2 dP \\ &\leqq \text{const. } \varepsilon^{-2}(z\log u(z))^{-2}(zu(z)) \\ &\leqq \text{const. } \varepsilon^{-2}(\log z)^{-1} \qquad \text{for sufficiently large } z. \end{aligned}$$

If $m(E_l) \sim \text{const. } l^{-\kappa}$ $(\kappa < 1)$, then $u(z) \sim \text{const. } (z/\log_2(\log_2 z))^{1/\kappa}$. By considering the third moment we have

$$P(\{|(z\textstyle\sum_{l=1}^{u(z)} P(E_l))^{-1} Y_z - 1| > \varepsilon\}) \leqq \text{const. } (\log_2(\log_2 z))^{-2}$$

for sufficiently large z. Therefore we obtain the lemma.

4. Proof of Theorem 1.1

We are going to prove Theorem 1.1 using the first return maps defined in §2.

In this section we assume that T is a piecewise affine Markov map defined in §0. Let S_i be the first return map of T on $[p_i, c_i]$. Then S_i: $[p_i, c_i] \to [p_i, c_i]$ satisfies the following condition:

There exists a countable partition $\{I_j | j \in J\}$ of $[p_i, c_i]$ such that the restriction of S_i to the interior of each I_j is a monotonic continuous function and that S_{ij} maps the closure of I_j onto $[p_i, c_i]$ for each j, where S_{ij} is the continuous extension to the closure of I_j of the restriction of S_i to the interior of I_j.

Let j_p be the index such that $I_{j_p} = [p_i, b_{i,1}]$. Put

$$\alpha_n = S_{ij_p}^{-n}(b_{i,1}) \qquad \text{for } n \geqq 0 \text{ and}$$
$$\beta_{jn} = S_{ij}^{-1}(\alpha_n) \qquad \text{for } j \in J \setminus \{j_p\} \text{ and } n \geqq 0.$$

Now we consider the first return map R_i of S_i on $[b_{i,1}, c_i]$. Then R_i can be represented in the following form. For $j \in J \setminus \{j_p\}$,

$$R_i(x) = S_i(x) \quad \text{if} \quad S_{ij}(x) > b_{i,1}$$
$$R_i(x) = S_i^{n+1}(x) \quad \text{if} \quad S_{ij}(x) \in (\alpha_n, \alpha_{n-1}), \qquad \text{for } n \geqq 1.$$

It is clear that $R_i(x)$ is defined except on the set of the endpoints of the countable partition of $[b_{i,1}, c_i]$. Thus, it follows from Proposition 3.2 that R_i has an invariant probability measure μ_{Ai} whose density is a constant. Let μ_i be the measure on $[p_i, c_i]$ defined by

$$\mu_i(D) = \textstyle\sum_{n=1}^{\infty} \mu_{Ai}(A_n \cap S_i^{-n}D) \tag{4.1}$$

for any measurable set $D \subset [p_i, c_i]$, where $A_1 = [b_{i,1}, c_i]$ and $A_{n+1} = A_n \cap S_i^{-n}([p_i, b_{i,1}))$ for $n \geqq 1$.

LEMMA 4.1. *μ_i defined by* (4.1) *is an m-equivalent S_i-invariant ergodic σ-finite measure. Further, μ_i satisfies the following* (1)–(3):

(1) $\mu_i([p_i + \Delta, c_i]) < \infty$ *for every* $\Delta > 0$.

(2) *If*

$$\textstyle\sum_{n=0}^{\infty} (\alpha_n - p_i) = \infty,$$

then $\mu_i([p_i, c_i]) = \infty$.

(3) *If*

$$\textstyle\sum_{n=0}^{\infty} (\alpha_n - p_i) < \infty,$$

then $\mu_i([p_i, c_i]) < \infty$.

PROOF. The proof is similar to that of Lemma 4.2 in [3] and to that of Lemma 3.3 in [4]. In the proof of this lemma we omit index i for the simplicity of notations. It follows from Lemmas 2.2 and 2.3 that μ is an m-equivalent S-invariant ergodic σ-finite measure. Let A_n $(n = 1, 2, \cdots)$ be as in the expression (4.1). First we prove (1). Let k be an integer with $\alpha_k < p + \Delta$. Then it is easy to see that

$$A_n \cap S^{-n}[\alpha_k, c] = \bigcup_{k \in J - \{j_p\}} S_j^{-1}(\alpha_{k+n-1}, \alpha_{n-2}) \ m\text{-a.e.} \qquad \text{for } n \geqq 2.$$

From this it follows that

$$\mu([p + \Delta, c]) \leqq \sum_{j \in J - \{j_p\}} (\mu_A(S_j^{-1}[\alpha_k, c]) + \sum_{n=0}^{k} \mu_A(S_j^{-1}(p, \alpha_n))) < \infty.$$

Next we prove (2). Fix one j and let a_j be the left endpoint of I_j. Then, it is easy to see that

$$A_n \supset [a_j, \beta_{j,n-2}] \qquad \text{for } n \geqq 2.$$

Thus

$$\mu([p, c]) = \sum_{n=1}^{\infty} \mu_A(A_n) \geqq \sum_{n=2}^{\infty} \mu_A([a_j, \beta_{j,n-2}]) \geqq \gamma \sum_{n=2}^{\infty} |a_j - \beta_{j,n}|,$$

where γ is $(c - b_{i,1})^{-1}$. Therefore $\mu([p, c]) = \infty$.
Finally we prove (3). It is easy to see that

$$A_n = \bigcup_{j \in J - \{j_p\}} [a_j, \beta_{j,n-2}] \qquad \text{for } n \geqq 2.$$

Thus

$$\begin{aligned}\mu([p, c]) &= \sum_{n=1}^{\infty} \mu_A(A_n) \leqq \sum_{j \in J - \{j_p\}} \sum_{n=2}^{\infty} \mu_A([a_j, \beta_{j,n-2}]) + 1 \\ &\leqq \gamma \sum_{j \in J - \{j_p\}} \sum_{n=2}^{\infty} |a_j - \beta_{j,n}| + 1 \\ &\leqq \gamma \sum_{n=2}^{\infty} (\alpha_n - p) + 1 < \infty.\end{aligned}$$

PROOF OF THEOREM 1.1. Let μ be the measure on [0, 1] defined by

$$\mu(D) = \sum_{n=1}^{\infty} \mu_1(A_n \cap T^{-n} D) \qquad \text{for any measurable set } D,$$

where $A_1 = [p_1, c_1]$ $A_{n+1} = A_n \cap S^{-n}([p_1, c_1]^c)$ for $n \geqq 1$. Then, it follows from Lemmas 2.2 and 2.3 that μ is an m-equivalent T-invariant ergodic σ-finite measure. By Lemma 2.4, μ restricted to $[p_i, c_i]$ is invariant under the first return map on $[p_i, c_i]$ of T for each i and this measure equals to μ_i multiplied by constant. Thus, by Lemma 4.1 we obtain a half of (1) and (3) of Theorem 1.1. The rest of Theorem 1.1 is similarly proved.

5. Proof of Theorems 1.2 and 1.3

In this section we are going to prove Theorems 1.2 and 1.3.

We assume that T is a piecewise affine Markov map with indifferent fixed points defined in §0. Fix $i = 1, \cdots, r-1$ and fix $j = 2, \cdots, r$. We allow all the cases $i < j$, $i = j$ or $i > j$. Let $\Delta_0, \Delta_1 > 0$ be arbitrary. First we consider the ratio of the sojourn time of the trajectory $\{T^k x\}_{k=0}^n$ in $[p_i, p_i + \Delta_0)$ to that in $(p_j - \Delta_1, p_j]$ for almost every x. Put $p = p_i$, $c = c_i$, $d = c_{j-1}$ and $q = p_j$. Let S be the first return map of T on $B = [p, c] \cup [d, q]$. Then $S: B \to B$ satisfies the following condition:

There exists a countable partition $\{I_\lambda : \lambda \in \Lambda\}$ of B such that the restriction of S to the interior of I_λ is a monotonic continuous function and that $S_\lambda(I_\lambda^{cl})$ is either $[p, c]$ or $[d, q]$ for each λ, where I_λ^{cl} is the closure of I_λ and S_λ is the continuous extension to I_λ^{cl} of the restriction of S to the interior of I_λ.

Let Λ_c be the set of indices λ satisfying $S_\lambda(I_\lambda^{cl}) = [p, c]$ and let Λ_d be the set of indices λ satisfying $S_\lambda(I_\lambda^{cl}) = [d, q]$. Let λ_p and λ_q be indices such that

$$I_{\lambda_p} = [p, b_{i,1}] \quad \text{and} \quad I_{\lambda_q} = [a_{j,1}, q].$$

Put

$$\beta_{\lambda,n} = S_\lambda^{-1}(b_{i,n}) \text{ for } \lambda \in \Lambda_c \setminus \{\lambda_p\} \text{ and } n \geqq 0, \text{ and}$$
$$\alpha_{\lambda,n} = S_\lambda^{-1}(a_{j,n}) \text{ for } \lambda \in \Lambda_d \setminus \{\lambda_q\} \text{ and } n \geqq 0.$$

Then the first return map R of S on $A = [b_{i,1}, c] \cup [d, a_{j,1}]$ can be represented in the following form. For $n \geqq 1$,

$$R(x) = S^n(x) \text{ if } x \in \left(\bigcup_{\lambda \in \Lambda_d - \{\lambda_q\}} (\alpha_{\lambda,n-1}, \alpha_{\lambda,n})\right) \cup \left(\bigcup_{\lambda \in \Lambda_c - \{\lambda_p\}} (\beta_{\lambda,n}, \beta_{\lambda,n-1})\right).$$

It is clear that $R(x)$ is defined except on the set of the endpoints of the countable partition of A.

Let R_a be the first return map on $[d, a_{j,1}]$ of R and let R_b be the first return map on $[b_{i,1}, c]$ of R. Then it follows from Proposition 3.1 that there exist an R_a-invariant measure on $[d, a_{j,1}]$ and an R_b-invariant measure on $[b_{i,1}, c]$, which are m-absolutely continuous ergodic probability measures. Thus, by Lemmas 2.2, 2.3 and 2.4 there exists an m-absolutely continuous R-invariant ergodic probability measure μ_A on A. Let μ_B be the m-absolutely continuous S-invariant ergodic σ-finite measure as in Lemma 2.2.

Now we prepare some notations.

Notations:

$$F(T, t, x) := \frac{\sum_{k=0}^{t} 1_{(d,q)}(T^k x)}{\sum_{k=0}^{t} 1_{(p,c)}(T^k x)}.$$

$$F(S, t, x) := \frac{\sum_{k=0}^{t} 1_{(d,q)}(S^k x)}{\sum_{k=0}^{t} 1_{(p,c)}(S^k x)}.$$

$$E_{\alpha_n} := E(\alpha_n) := \bigcup_{\lambda \in \Lambda_d - \{\lambda_q\}} \{x \in I_\lambda ;\ x \geqq \alpha_{\lambda,n}\}.$$

$$E_{\beta_n} := E(\beta_n) := \bigcup_{\lambda \in \Lambda_c - \{\lambda_p\}} \{x \in I_\lambda ;\ x \leqq \beta_{\lambda,n}\}.$$

$$\tau_t := \max \{\tau ;\ \textstyle\sum_{k=1}^{\tau} 1_A(S^k x) = t\}.$$

t^*: the integer satisfying $\log_2(\log_2 t^*) = [\log_2(\log_2 t)] + 1$,

where [ ] is the Gaussian symbol.

$n(t) := n_t$: the maximal integer satisfying

$$\log_2(\log_2 t^*) \leqq t^*(\mu_A(E(\alpha_{n_t})) + \mu_A(E(\beta_{n_t}))).$$

$$H_d(t, x) := \textstyle\sum_{k=0}^{\tau_t} 1_{(d, a_{jn(t)}]}(S^k x).$$

$$H_c(t, x) := \textstyle\sum_{k=0}^{\tau_t} 1_{[b_{in(t)}, c)}(S^k x).$$

$$H_q(t, x) := \textstyle\sum_{k=0}^{\tau_t} 1_{(a_{jn(t)}, q)}(S^k x).$$

$$H_p(t, x) := \textstyle\sum_{k=1}^{\tau_t} 1_{(p, b_{in(t)})}(S^k x).$$

$$\eta(t, x) := \frac{H_d(t, x)}{H_c(t, x)}.$$

REMARK 5.1. For m-a.e. $x \in A$ we have

$$H_d(t, x) = \textstyle\sum_{k=0}^{\tau_t} \sum_{l=1}^{n_t} 1_{(a_{j,l-1}, a_{j,l}]}(S^k x) = \sum_{k=0}^{t} \sum_{l=1}^{n_t} 1_{E(\alpha_{l-1})}(R^k x),$$

$$H_c(t, x) = \textstyle\sum_{k=0}^{t} \sum_{l=1}^{n_t} 1_{E(\beta_{l-1})}(R^k x).$$

The following lemma means that it is essential for our purpose to calculate the limit value of $F(S, t, x)$.

LEMMA 5.1. *Assume that*

$$\mu_B((p, p+\varepsilon)) = \infty,\ \mu_B((q-\varepsilon, q)) = \infty \ \textit{and}$$

$$\mu_B((p+\varepsilon, c) \cup (d, q-\varepsilon)) < \infty \qquad \textit{for every small } \varepsilon > 0$$

and that $\lim_{t\to\infty} F(S, t, x)$ *exists. Then*

$$\lim_{t\to\infty} F(S, t, x) = \lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_{(q-\Delta_1, q)}(T^k x)}{\sum_{k=0}^{t} 1_{(p, p+\Delta_0)}(T^k x)}$$

for μ_B*-a.e.* x *and any small* $\Delta_0, \Delta_1 > 0$.

PROOF. If the limit value of $F(S, t, x)$ exists, then obviously,

$$\lim_{t\to\infty} F(T, t, x) = \lim_{t\to\infty} F(S, t, x).$$

By Lemma 2.1 we have

$$\lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_{(d,q-\Delta_1]}(T^k x)}{\sum_{k=0}^{t} 1_{(p,p+\Delta_0)}(T^k x)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_{[p+\Delta_0,c)}(T^k x)}{\sum_{k=0}^{t} 1_{(p,p+\Delta_0)}(T^k x)} = 0$$

for μ_B-a.e. x and any small $\Delta_0, \Delta_1 > 0$. Thus, we easily have

$$\begin{aligned}
\lim_{t\to\infty} F(S, t, x) &= \lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_{(d,q)}(T^k x)}{\sum_{k=0}^{t} 1_{(p,c)}(T^k x)} \\
&= \lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_{(q-\Delta_1,q)}(T^k x) + \sum_{k=0}^{t} 1_{(d,q-\Delta_1]}(T^k x)}{\sum_{k=0}^{t} 1_{(p,p+\Delta_0)}(T^k x) + \sum_{k=0}^{t} 1_{[p+\Delta_0,c)}(T^k x)} \\
&= \lim_{t\to\infty} \frac{\sum_{k=0}^{t} 1_{(q-\Delta_1,q)}(T^k x)}{\sum_{k=0}^{t} 1_{(p,p+\Delta_0)}(T^k x)}
\end{aligned}$$

for μ_B-a.e. x.

In the lemmas in the rest of this section we assume the conditions (C, a, j) and (C, b, i). The following lemma is important to determine the limit value of $F(S, t, x)$.

LEMMA 5.2. *Assume that* $\kappa(b, i) \leqq \kappa(a, j) \leqq 1$. *Then*: (I)

$$\mu_A(E_{\alpha_n}) = a'(q - a_{j,n}) \sim a'' n^{-\kappa(a,j)},$$

$$\mu_A(E_{\beta_n}) = b'(b_{i,n} - p) \sim b'' n^{-\kappa(b,i)},$$

(5.1)
$$\rho = \lim_{n\to\infty} \frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})}$$

exists and $0 \leqq \rho < \infty$, *where* a', a'', b' *and* b'' *are positive finite constants. Further,*

(1) ρ *is* 0 *if* $\kappa(b, i) < \kappa(a, j)$.

(2) ρ *is positive and finite if* $\kappa(b, i) = \kappa(a, j)$.

(3) $\rho = \dfrac{\nu(a, j)(c - p)}{\nu(b, i)(q - c)}$
if $r = 2, i = 1, j = 2$ *and* $\kappa(b, i) = \kappa(a, j) = 1$.

(4) $\rho = \dfrac{\nu(a, j)}{\nu(b, i)}$ *if* $j = i$ *and* $\kappa(b, i) = \kappa(a, j) = 1$.

(II) ρ *in* (5.1) *satisfies the following property*: *for any* $\varepsilon > 0$ *and for* μ_A-*a.e.* x *there exists an integer* t_0 *such that for* $t \geqq t_0$

$$\left| \frac{\sum_{l=1}^{n_t} \sum_{k=0}^{t} 1_{E(\alpha_l)}(R^k x)}{\sum_{l=1}^{n_t} \sum_{k=0}^{t} 1_{E(\beta_l)}(R^k x)} - \rho \right| < \varepsilon.$$

As a consequence $\eta(t, x)$ *converges to* ρ *for* μ_A*-a.e.* x.

PROOF. Let g be the Radon-Nikodym derivative of μ_A with respect to the Lebesgue measure m. By the virtue of Proposition 3.2 we have

$$g(x) = \begin{cases} \gamma_c & \text{for a.e. } x \in [b_{i,1}, c] \\ \gamma_d & \text{for a.e. } x \in [d, a_{j,1}], \end{cases}$$

where γ_c and γ_d are positive finite constants. Put

$$K_{bcc} = [b_{i,1}, c] \cap (\textstyle\bigcup_{\lambda \in \Lambda_c - \{\lambda_p\}} I_\lambda), \quad K_{bcd} = [b_{i,1}, c] \cap (\textstyle\bigcup_{\lambda \in \Lambda_d - \{\lambda_q\}} I_\lambda),$$

$$K_{dac} = [d, a_{j,1}] \cap (\textstyle\bigcup_{\lambda \in \Lambda_c - \{\lambda_p\}} I_\lambda) \quad \text{and} \quad K_{dad} = [d, a_{j,1}] \cap (\textstyle\bigcup_{\lambda \in \Lambda_d - \{\lambda_q\}} I_\lambda).$$

Since

$$m(E_{\beta_n} \cap K_{bcc}) = \frac{(b_{i,n} - p)m(K_{bcc})}{c - p},$$

$$m(E_{\beta_n} \cap K_{dac}) = \frac{(b_{i,n} - p)m(K_{dac})}{c - p},$$

$$m(E_{\alpha_n} \cap K_{bcd}) = \frac{(q - a_{j,n})m(K_{bcd})}{q - d} \quad \text{and}$$

$$m(E_{\alpha_n} \cap K_{dad}) = \frac{(q - a_{j,n})m(K_{dad})}{q - d},$$

we have

$$\begin{aligned} \mu_A(E_{\alpha_n}) &= \mu_A(E_{\alpha_n} \cap K_{bcd}) + \mu_A(E_{\alpha_n} \cap K_{dad}) \\ &= \gamma_c m(E_{\alpha_n} \cap K_{bcd}) + \gamma_d m(E_{a_n} \cap K_{dad}) \\ &= (q - d)^{-1}(q - a_{j,n})(\gamma_c m(K_{bcd}) + \gamma_d m(K_{dad})) \end{aligned}$$

and

$$\mu_A(E_{\beta_n}) = (c - p)^{-1}(b_{i,n} - p)(\gamma_c m(K_{bcc}) + \gamma_d m(K_{dac})),$$

which show the first two assertions in (I). Since

$$\mu_A(K_{bcc} \cup K_{bcd}) = \mu_A(R^{-1}(K_{bcc} \cup K_{bcd})) = \mu_A(K_{bcc} \cup K_{dac}),$$

$$K_{bcc} \cap K_{bcd} = \emptyset \quad \text{and} \quad K_{bcc} \cap K_{dac} = \emptyset,$$

we have

$$\gamma_c m(K_{bcd}) = \gamma_d m(K_{dac}).$$

Therefore

$$\frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})} = \frac{m(K_{bcd}) \cdot (m(K_{dac}) + m(K_{dad})) \cdot (c - p) \cdot (q - a_{j,n})}{m(K_{dac}) \cdot (m(K_{bcc}) + m(K_{bcd})) \cdot (q - d) \cdot (b_{i,n} - p)}. \tag{5.2}$$

Clearly, $m(K_{bcd}) \cdot (m(K_{dac}) + m(K_{dad})) \cdot (c - p)$ and $m(K_{dac}) \cdot (m(K_{bcc}) + m(K_{bcd})) \cdot (q - d)$ are positive and finite. Thus, the limit (5.1) exists and we obtain (1) and (2). Next, we show (3). If $r = 2$ and $j = i + 1$, it is obvious that $m(K_{bcc}) = m(K_{dad}) = 0$. Hence,

$$\frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})} = \frac{c \cdot (1 - a_{j,n})}{(1 - c) \cdot b_{i,n}}.$$

In addition, if $\kappa(b, i) = \kappa(a, j) = 1$,

$$\lim_{n\to\infty} \frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})} = \frac{c \cdot v(a, j)}{(1 - c) \cdot v(b, i)}.$$

This implies (3). Finaly we show (4). If $j = i$, it is obvious that

$$m(K_{bcc})/m(K_{bcd}) = (c - p)/(p - d) \text{ and}$$
$$m(K_{dac})/m(K_{dad}) = (c - p)/(p - d).$$

Thus, from the trivial equalities $K_{bcc} \cup K_{bcd} = [b_{i,1}, c]$, $K_{dac} \cup K_{dad} = [d, a_{j,1}]$, $K_{bcc} \cap K_{bcd} = \emptyset$ and $K_{dac} \cap K_{dad} = \emptyset$, we have

$$\frac{m(K_{bcd})}{m(K_{dac})} = \frac{(c - b_{i,1})(p - d)}{(a_{j,1} - d)(c - p)}.$$

Hence it follows from (5.2) that

$$\frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})} = \frac{(p - a_{j,n})}{(b_{i,n} - p)}.$$

In addition, if $\kappa(b, i) = \kappa(a, i) = 1$,

$$\lim_{n\to\infty} \frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})} = \frac{v(a, j)}{v(b, i)}.$$

This implies (4).

Next we prove (II). Let $t_a = [t \cdot \mu_A([d, a_{j,1}])]$ and let $t_b = [t \cdot \mu_A([b_{i,1}, c])]$. Let $\varepsilon > 0$ be arbitrary. Notice that

$$\lim_{n\to\infty} \frac{\sum_{l=1}^{n} \mu_A(E_{\alpha_l})}{\sum_{l=1}^{n} \mu_A(E_{\beta_l})} = \lim_{n\to\infty} \frac{\mu_A(E_{\alpha_n})}{\mu_A(E_{\beta_n})} = \rho,$$

since $\sum_{l=1}^{\infty}\mu_A(E_{\alpha_l})=\sum_{l=1}^{\infty}\mu_A(E_{\beta_l})=\infty$. Then, by applying Lemma 3.4 for R_a and R_b, it is easy to see that for μ_A-a.e. x there exists an integer t_1 such that

$$\left|\frac{\sum_{l=1}^{n_t}\sum_{k=0}^{t_a}1_{E(\alpha_l)}(R_a^k x)}{\sum_{l=1}^{n_t}\sum_{k=0}^{t_b}1_{E(\beta_l)}(R_b^k x)}-\rho\right|\leqq 2^{-1}\varepsilon \qquad \text{for } t\geqq t_1.$$

It follows from the Birkhoff individual ergodic theorem that there exists an integer t_2 such that

$$\left|\frac{\sum_{l=1}^{n_t}\sum_{k=0}^{t}1_{E(\alpha_l)}(R^k x)}{\sum_{l=1}^{n_t}\sum_{k=0}^{t}1_{E(\beta_l)}(R^k x)}-\frac{\sum_{l=1}^{n_t}\sum_{k=0}^{t_a}1_{E(\alpha_l)}(R_a^k x)}{\sum_{l=1}^{n_t}\sum_{k=0}^{t_b}1_{E(\beta_l)}(R_b^k x)}\right|\leqq 2^{-1}\varepsilon \qquad \text{for } t\geqq t_2.$$

Therefore ρ satisfies the property in the lemma. It follows from Remark 5.1 that $\eta(t, x)$ converges to ρ for μ_A-a.e. x. This completes the proof of the lemma.

For ρ in Lemma 5.2 we are going to prove the following proposition.

PROPOSITION 5.3. *$F(S, t, x)$ converges to ρ for μ_B-a.e. x if $\kappa(b, i)<\kappa(a, j)\leqq 1$ or $\kappa(b, i)=\kappa(a, j)=1$.*

In order to prove this, we show the following lemma.

LEMMA 5.4. *If $\kappa(b, i)<\kappa(a, j)\leqq 1$ or $\kappa(b, i)=\kappa(a, j)=1$, then*

$$\mu_A\left(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x;\,|F(S, \tau_t, x)-\rho|>\varepsilon\}\right)=0 \qquad \textit{for any } \varepsilon>0.$$

In order to prove Lemma 5.4 we prepare the following lemma.

LEMMA 5.5. *If $\kappa(a, j)=1$,*

$$H_d(t, x)\sim a'' t\log n_t,$$

where a'' is the constant in Lemma 5.2 (I). A similar relation holds if b and c replace a and d respectively.

PROOF. Let $t_a=t\cdot\mu_A([d, a_{j,1}])$ and let R_a be the first return map of R on $[d, a_{j,1}]$. Then, by Remark 5.1 and the Birkhoff individual ergodic theorem, we get

$$H_d(t, x)\sim\sum_{l=1}^{n_t}\sum_{k=0}^{t_a}1_{E(\alpha_l)}(R_a^k x).$$

It follows from Lemma 3.4 that

$$\sum_{l=1}^{n_t}\sum_{k=0}^{t_a}1_{E(\alpha_l)}(R_a^k x)\sim t\sum_{l=1}^{n_t}\mu_A(E(\alpha_l))\sim a'' t\sum_{l=1}^{n_t}\frac{1}{l}\sim a'' t\log n_t.$$

This completes the proof.

PROOF OF LEMMA 5.4. Put

$$E_{n_t} = E(n_t) = E(\alpha_{n_t}) \cup E(\beta_{n_t}).$$

Let R_{n_t} be the first return map on E_{n_t} of R and let $\mu_{E(n_t)}$ be the m-equivalent R_{n_t}-invariant ergodic probability measure.

First we consider the case $\kappa(b, i) = \kappa(a, j) = 1$.

We remark that

$$F(S, \tau_t, x) = \frac{\sum_{k=0}^{\tau_t} 1_{(d,q)}(S^k x)}{\sum_{k=0}^{\tau_t} 1_{(p,c)}(S^k x)} = \frac{H_d(t, x) + H_q(t, x)}{H_c(t, x) + H_p(t, x)}.$$

Roughly speaking, Lemma 5.2 (II) implies that $\eta(t, x)$ converges to ρ as $t \to \infty$, and hence it is sufficient to prove that $H_d(t, x)$ and $H_c(t, x)$ are much greater than $H_q(t, x)$ and $H_p(t, x)$ for sufficiently large t.

Put

$$C_t = \{x \in A;\ \textstyle\sum_{l=1}^{t^*} 1_{E(n_t)}(R^l x) > (\log_2(\log_2 t^*))^2\},$$

$$D_{n_t} = \{x \in E_{n_t};\ R_{n_t}^l x \in (E_{n_t + M_t})^C \quad \text{for all } 1 \leqq l \leqq (\log_2(\log_2 t^*))^2\},$$

where $M_t = \left[\dfrac{t^*(\log n_t)^{1/2}}{(\log_2(\log_2 t^*))^2}\right]$,

$$E_{n_t}^1 = E_{n_t},\ E_{n_t}^h = R^{-h+1} E_{n_t} \setminus (\textstyle\bigcup_{k=1}^{h-1} E_{n_t}^k) \qquad \text{for } h \geqq 2,$$

$$D_{n_t}^1 = D_{n_t} \text{ and } D_{n_t}^h = R^{-h+1} D_{n_t} \setminus (\textstyle\bigcup_{k=1}^{h-1} E_{n_t}^k) \qquad \text{for } h \geqq 2.$$

Then we have

$$\begin{aligned}
(5.3)\qquad & \mu_A(\textstyle\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \{x \in A;\ H_p(t^*, x) + H_q(t^*, x) \geqq t^*(\log n_t)^{1/2}\}) \\
&= \mu_A(\textstyle\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \bigcup_{h=1}^{\infty} \{x \in E_{n_t}^h;\ H_p(t^*, x) + H_q(t^*, x) \geqq t^*(\log n_t)^{1/2}\}) \\
&\leqq \mu_A(\textstyle\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \bigcup_{h=1}^{\infty} (E_{n_t}^h \setminus D_{n_t}^h)) + \mu_A(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} C_t).
\end{aligned}$$

Since R is an affine map on each $(\beta_{\lambda,n}, \beta_{\lambda,n-1})$ and on each $(\alpha_{\lambda,n-1}, \alpha_{\lambda,n})$ and R maps $(\alpha_{\lambda,n-1}, \alpha_{\lambda,n})$ and $(\beta_{\lambda,n}, \beta_{\lambda,n-1})$ onto $(d, a_{j,1})$ and $(b_{i,1}, c)$ respectively, we have

$$\begin{aligned}
\mu_A(\textstyle\bigcup_{h=1}^{\infty} (E_{n_t}^h \setminus D_{n_t}^h)) &= \textstyle\sum_{h=1}^{\infty} \mu_A(E_{n_t}^h \setminus D_{n_t}^h) \\
&= \textstyle\sum_{h=1}^{\infty} (\mu_A(E_{n_t}^h) \cdot (\mu_A(E_{n_t}^h \setminus D_{n_t}^h)/\mu_A(E_{n_t}^h))) \\
&= \textstyle\sum_{h=1}^{\infty} \mu_A(E_{n_t}^h) \cdot (\mu_A(E_{n_t} \setminus D_{n_t})/\mu_A(E_{n_t})) \\
&= \mu_A(E_{n_t} \setminus D_{n_t})/\mu_A(E_{n_t}) = \mu_{E(n_t)}(D_{n_t}^C).
\end{aligned}$$

Hence

(5.4) $$\mu_A(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{h=1}^{\infty}(E_{n_t}^h\setminus D_{n_t}^h))\leqq \lim_{k\to\infty}\sum_{t\in L_k}\mu_{E(n_t)}(D_{n_t}^C),$$

where $L_k=\{t\geqq k;\log_2(\log_2 t)\text{ is an integer}\}$. Since Lemma 3.3 implies that

$$\mu_A(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}C_t)=0,$$

it follows from (5.3) and (5.4) that

$$\mu_A(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x\in A;\,H_p(t^*,x)+H_q(t^*,x)\geqq t^*(\log n_t)^{1/2}\})$$
$$\leqq \lim_{k\to\infty}\sum_{t\in L_k}\mu_{E(n_t)}(D_{n_t}^C).$$

Since $t\leqq t^*$, by the above inequality we have

$$\mu_A(\bigcap_{k=0}^{\infty}\bigcup_{t=k}^{\infty}\{x\in A;\,H_p(t,x)+H_q(t,x)\geqq t^*(\log n_t)^{1/2}\})$$
$$\leqq \lim_{k\to\infty}\sum_{t\in L_k}\mu_{E(n_t)}(D_{n_t}^C).$$

Now we estimate $\mu_{E(n_t)}(E_{n_t+M_t})$. Since

$$n_t+M_t\sim\frac{t^*v+(\log_2(\log_2 t^*))^{-1}t^*(\log(t^*v/\log_2(\log_2 t^*)))^{1/2}}{\log_2(\log_2 t^*)}$$

holds for $v=a''+b''$, we have

$$\mu_{E(n_t)}(E_{n_t+M_t})=\frac{\mu(E_{n_t+M_t})}{\mu(E_{n_t})}\sim\frac{\log_2(\log_2 t^*)}{v}\left(\log\left(\frac{vt^*}{\log_2(\log_2 t^*)}\right)\right)^{-1/2}.$$

Thus

$$\mu_{E(n_t)}(D_{n_t}^C)=1-\mu_{E(n_t)}(D_{n_t})=1-(1-\mu_{E(n_t)}(E_{n_t+M_t}))^{\log_2(\log_2 t^*)}$$
$$\sim\frac{(\log_2(\log_2 t^*))^2}{v(\log(t^*v/\log_2(\log_2 t^*)))^{1/2}}.$$

Therefore we obtain

$$\lim_{k\to\infty}\sum_{t\in L_k}\mu_{E(n_t)}(D_{n_t}^C)=0.$$

Hence

$$\lim_{t\to\infty}\frac{H_p(t,x)+H_q(t,x)}{t^*\log n_t}=0\qquad\text{for }\mu_A\text{-a.e. }x,$$

which implies the lemma in this case by Lemma 5.2 (II) and Lemma 5.5.

Next we consider the case $\kappa(b,i)<\kappa(a,j)\leqq 1$.
Roughly speaking, Lemma 5.2 (II) implies that $\eta(t,x)$ converges to 0 as $t\to\infty$, and hence it is sufficient to prove that

$$H_q(t,x)/(H_p(t,x)+H_q(t,x))\longrightarrow 0\qquad(t\longrightarrow\infty).$$

Put

$$G(\alpha_{n_t}) = \{x \in E_{n_t};\ R^l_{n_t}(x) \in E(\alpha_{n_t}) \qquad \text{for some } 1 \leqq l \leqq (\log_2 (\log_2 t^*))^2\}.$$

If we show

$$\lim_{k\to\infty} \sum_{t\in L_k} \mu_{E(n_t)}(G(\alpha_{n_t})) = 0, \tag{5.5}$$

we obtain the lemma in this case in a way similar to the first case. So, we are going to prove (5.5). By Lemma 5.2 (I) we have

$$\mu(E(\alpha_{n_t})) \sim \theta_1 (n_t)^{-\kappa(a,j)} \sim \theta_2 \left(\frac{\log_2 (\log_2 t^*)}{t^*} \right)^{\kappa(a,j)/\kappa(b,i)},$$

where θ_1 and θ_2 are some positive constants. Thus we have

$$\begin{aligned}\mu_{E(n_t)}(G(\alpha_{n_t})) &= \frac{\mu(G(\alpha_{n_t}))}{\mu(E(n_t))} \leqq \frac{(\log_2(\log_2 t^*))^2 \mu(E(\alpha_{n_t}))}{\mu(E(n_t))} \\ &\sim (\log_2 (\log_2 t^*) \mu(E(\alpha_{n_t}))t^* \\ &\sim \theta_2 \log_2 (\log_2 t^*)^{1+(\kappa(a,j)/\kappa(b,i))} (t^*)^{1-(\kappa(a,j)/\kappa(b,i))}.\end{aligned}$$

Since $\kappa(b, i) < \kappa(a, j) \leqq 1$, we obtain (5.5).

Thus the proof of the lemma is complete.

Proof of Proposition 5.3. Let V be an arbitrary set with $\mu_A(V) = 0$. Since

$$\bigcup_{n=0}^{\infty} S^{-n} A = B \quad \text{and} \quad \mu_B(\bigcup_{n=0}^{\infty} S^{-n} V) = 0,$$

it is sufficient to show that

$$\mu_A(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \{x \in A;\ |F(S, t, x) - \rho| > \varepsilon\}) = 0 \qquad \text{for any } \varepsilon > 0.$$

Obviously, this is equivalent to

$$\mu_A(\bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \{x \in A;\ |F(S, \tau_t, x) - \rho| > \varepsilon\}) = 0 \qquad \text{for any } \varepsilon > 0.$$

Thus the proposition follows from Lemmas 5.2 and 5.4.

Proof of Theorem 1.2. If $\kappa(a, i_1) > 1$, Theorem 1.1 and Lemma 2.1 imply Theorem 1.2. So we assume that $\kappa(a, i_1) \leqq 1$. Since T is a piecewise affine Markov map, Lemmas 5.1 and 5.2 and Proposition 5.3 imply

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{\mathrm{nbd}(p(i_1),a,\Delta_1)}(T^k x)}{\sum_{k=0}^{n} 1_{\mathrm{nbd}(p(i_0),b,\Delta_0)}(T^k x)} = 0 \qquad \text{for } m\text{-a.e. } x.$$

The other cases are similarly proved.

PROOF OF COROLLARY 1.2.1. This is obvious from Theorem 1.2.

PROOF OF COROLLARY 1.2.2. Let $f \in C([0, 1])$. It is sufficient to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f(p_{i_0}) \qquad \text{for } m\text{-a.e. } x.$$

Set

$$f_\varepsilon(x) = f(p_{i_0}) \text{ if } x \in \text{nbd}\,(p_{i_0}, \varepsilon) \text{ and } f_\varepsilon(x) = f(x) \text{ otherwise.}$$

Then f_ε converges to f uniformly on $[0, 1]$ as $\varepsilon \to 0$. Since f is bounded, it follows from Theorem 1.2 that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f_\varepsilon(T^k(x)) = f(p_{i_0}) \qquad \text{for } m\text{-a.e. } x.$$

Therefore, from the uniformity of convergence of f_ε it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f(p_{i_0}) \qquad \text{for } m\text{-a.e. } x.$$

This completes the proof.

PROOF OF THEOREM 1.3. Since T is a piecewise affine Markov map, Lemmas 5.1 and 5.2 and Proposition 5.3 imply

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{\text{nbd}\,(p(i_1), a, \Delta_1)}(T^k x)}{\sum_{k=0}^{n} 1_{\text{nbd}\,(p(i_0), b, \Delta_0)}(T^k x)} = \rho \qquad \text{for } m\text{-a.e. } x.$$

Lemma 5.2 (I) (2) means that ρ in the above equality is a positive finite constant. "In particular" part follows from Lemma 5.2 (I) (3) and (4). The existence of the positive finite constant ρ for the other combinations of ω_0 and ω_1 is similarly proved.

PROOF OF COROLLARY 1.3.1. This is obvious from Theorem 1.3.

PROOF OF COROLLARY 1.3.2. Remarking Theorem 1.1 and Lemma 2.1, we obtain this corollary in a way similar to Corollary 1.2.2.

6. Remarks

In this section we state some remarks related to our main results.

In a way similar to the proof of Corollaries 1.2.2 and 1.3.2, we have

REMARK 6.1.1. Under the same situation as Corollary 1.2.1, let f be a function belonging to $L^1(m)$ and satisfying the following conditions;

$\cdot f$ is bounded on $(\bigcup_{\kappa(a,i)\leqq 1}(p_i - \Delta, p_i)) \cup (\bigcup_{\kappa(b,i)\leqq 1}(p_i, p_i + \Delta))$.

$\cdot f$ is continuous on $\mathrm{nbd}(p_{i_0}, \omega_0, \Delta)$ for a small $\Delta > 0$.

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \begin{cases} \lim_{x\to p(i_0)-0} f(x) & \text{if } \omega_0 = a \\ \lim_{x\to p(i_0)+0} f(x) & \text{if } \omega_0 = b \end{cases} \qquad \text{for } m\text{-a.e. } x.$$

REMARK 6.1.2. Under the same situation as Corollary 1.3.1, let f be a function belonging to $L^1(m)$ such that f is bounded and continuous on

$$(\bigcup_{\kappa(a,i)=1}(p_i - \Delta, p_i)) \cup (\bigcup_{\kappa(b,i)=1}(p_i, p_i + \Delta)).$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \sum_{\kappa(a,i)=1} s_{a,i} \lim_{x\to p_i - 0} f(x)$$
$$+ \sum_{\kappa(b,i)=1} s_{b,i} \lim_{x\to p_i + 0} f(x)$$

for m-a.e. x, where $s_{a,i}$'s and $s_{b,i}$'s are constants, which are independent of f and satisfy

$$\sum_{\kappa(a,i)=1} s_{a,i} + \sum_{\kappa(b,i)=1} s_{b,i} = 1,$$

$s_{a,i} > 0$ for i with $\kappa(a, i) = 1$ and $s_{b,i} > 0$ for i with $\kappa(b, i) = 1$.

By an idea similar to the proof of Theorem 1.2, we can prove

REMARK 6.2. Assume that a map T: $[0, 1] \to [0, 1]$ satisfies the following conditions:

(1) There exists a partition $0 = c_0 < c_1 < \cdots < c_r = 1$ such that the restriction of T to (c_{i-1}, c_i) is a C^2 function and can be extended to $[c_{i-1}, c_i]$ as a C^2 function; let T_i be such an extension for $i = 1, \cdots, r$.

(2) $T_i(c_{i-1}, c_i) = (0, 1)$ for $i = 1, \cdots, r$.

(3) $T_i'(x) > 1$ for x with $T_i(x) \neq x$.

(4) Let $\omega = a$ or b, and let $i = 1, \cdots, r$. There exist at least two pairs of (ω, i) such that

$$T_i(p_i) = p_i, \ T_i'(p_i) = 1,$$

$$|T_i(x) - x| = \kappa(\omega, i)\nu(\omega, i)|x - p_i|^{(1/\kappa(\omega,i))+1} + o(|x - p_i|^{(1/\kappa(\omega,i))+1}),$$

where $0 < \kappa(\omega, i) \leqq 1$ and $\nu(\omega, i) > 0$ are constants.

If (ω_0, i_0) and (ω_1, i_1) satisfy the above condition (4) with $\kappa(\omega_0, i_0) < \kappa(\omega_1, i_1)$, then

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} 1_{\mathrm{nbd}(p(i_1),\omega_1,\Delta_1)}(T^k x)}{\sum_{k=0}^{n} 1_{\mathrm{nbd}(p(i_0),\omega_0,\Delta_0)}(T^k x)} = 0 \qquad \text{for } m\text{-a.e. } x,$$

where nbd (p_i, ω, Δ) is the same one defined in §1.

We can give the corollaries of this remark which correspond to Corollaries 1.2.1 and 1.2.2.

The condition (1) in Remark 6.2 implies that the right and the left derivatives of T at c_i's are finite, which is important since, without the finiteness of the derivatives, there are some cases such tht $\mu((0, \Delta)) < \infty$ for $\Delta > 0$ even if $\kappa(b, 1) \leqq 1$, where μ is an m-equivalent T-invariant ergodic σ-finite measure (cf. [4], [5]).

References

[1] P. Berge, Y. Pomeau, C. Vidal. Order within Chaos: Towards a deterministic approach to turbulence, Wiley, New York, 1984.

[2] E. Hopf. Ergodentheorie, Ergebnisse d. Math. u. Grenzgeb., **5**, Chelsea, New York, 1948.

[3] T. Inoue. Asymptotic stability of densities for piecewise convex maps, Ann. Polon. Math., **LVII** (1992), 83–90.

[4] T. Inoue. Weakly attracting repellors for piecewise convex maps, Japan J. Indust. Appl. Math., **9** (1992), 413–430.

[5] T. Inoue. A classification of cusp-type maps, in Advanced Series in Dynamical Systems, **11**, Proceedings of the RIMS Conference Structure and Bifurcations of Dynamical Systems, ed. S. Ushiki, Wold Scientific, Singapore, 1992.

[6] T. Inoue and H. Ishitani. Asymptotic periodicity of densities and ergodic properties for nonsingular systems, Hiroshima Math. J., **21** (1991), 597–620.

[7] Sh. Ito and S. Yasutomi. On continued fractions, substitutions and characteristic sequences $[nx + y] - [(n-1)x + y]$, preprint.

[8] U. Krengel. Ergodic Theorems, de Gruyter studies in Math., **6**, Walter de Gruyter, Berlin, New York, 1985.

[9] A. Lasota and M. C. Mackey. Probabilistic Properties of Deterministic Systems, Cambridge Univ. Press, 1984.

[10] P. Manneville. Intermittency, self-similarity and $1/f$ spectrum in dissipative dynamical systems, J. Physique, **41** (1980), 1235–1243.

[11] G. Pianigiani. First return map and invariant measures, Israel J. Math., **35** (1980), 32–48.

[12] A. N. Shiryayev. Probability, GTM **95**, Springer, New York, Berlin, Heidelberg, Tokyo, 1984.

[13] Y. Takahashi. Power spectrum and Fredholm determinant related to intermittent chaos, in Stochastic Processes in Physics and Engineering, 357–379, D. Reidel Publishing Company, 1988.

[14] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points, Israel J. Math., **37** (1980), 303–314.

[15] M. Thaler. Transformations on [0, 1] with infinite invariant measures, Israel J. Math., **46** (1983), 67–96.

[16] P. Walters. An Introduction to Ergodic Theory, GTM **79**, Springer, New York, Heidelberg, Berlin, 1982.

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