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Oscillation criteria for a class of perturbed Schrödinger equations

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I. Introduction

Recently there has been a growing interest towards qualitative analysis of partial differential equations with or without deviating arguments. In particular, the oscillatory behaviour of the solutions of the elliptic equations and hyperbolic equations has been investigated in [1–6].

It seems that the nonexistence of positive solutions of nonlinear second order differential inequality

$$(q(t)(p(t)y(t))')' + h(t, y(t)) \leq r(t) \tag{1.1}$$

plays an important role in the above subject [see 2, 3, 4]. Uup to now, Kusano and Naito's result [2] on the nonexistence of positive solutions of (1.1) is the only one in the literature.

In this short note, we try to obtain some new criteria for the nonexistence of positive solution of (1.1). Then we apply these results to the perturbed Schrödinger equations and obtain some new results.

II. Nonexistence of positive solutions

The following conditions for (1.1) are always assumed to hold:

(a) $p, q \in C(R_+, R_+ \setminus \{0\})$, $r \in C(R_+, R)$, $R_+ = [0, \infty)$.

(b) $h \in C(R_+ \times R_+ \setminus \{0\}, R_+ \setminus \{0\})$ and is nondecreasing in the second variable.

Now we consider two possible cases for q.

Case 1: $\displaystyle\int_a^\infty \frac{dt}{q(t)} = \infty$.

THEOREM 2.1. *Assume that*

(i) *There exists a function* $\delta \in C^2([a, \infty), R)$ *such that*

$$(q(t)(p(t)\delta(t))')' = r(t) \tag{2.1}$$

and $\delta(t)$ is of alternate sign.

(ii) *Set* $Q(t, s) = \int_s^t \frac{dx}{q(x)}$, $t > s \in [a, \infty)$

$$\liminf_{t \to \infty} \frac{1}{Q(t, T)} \int_T^t Q(t, s)(r(s) - h(s, \delta_+(s)))ds = -\infty \tag{2.2}$$

for all sufficiently large $T > a$, *where* $\delta_+(s) = \max(\delta(s), 0)$. *Then inequality* (1.1) *has no solution which is positive on* $[t_0, \infty)$ *for any* $t_0 > a$.

PROOF. If not, let $y(t)$ be a positive solution of (1.1) on $[t_0, \infty)$. Then

$$[q(t)(p(t)y(t))' - q(t)(p(t)\delta(t))']' \le 0,$$

which implies that either $(p(t)y(t) - p(t)\delta(t))' \le 0$ or $(p(t)y(t) - p(t)\delta(t))' \ge 0$ eventually. Thus there are two possible cases: either $p(t)y(t) - p(t)\delta(t) \ge 0$ or $p(t)y(t) - p(t)\delta(t) \le 0$ eventually. Since $\delta(t)$ is oscillatory and $y(t)$ is positive, we should have

$$y(t) \ge \delta_+(t). \tag{2.3}$$

In view of condition (b), we have

$$h(t, y(t)) \ge h(t, \delta_+(t)). \tag{2.4}$$

Substituting (2.4) into (1.1) we have

$$p(t)y(t) \le c_1 + c_2 Q(t, t_0) + \int_{t_0}^t Q(t, s)(r(s) - h(s, \delta_+(s)))ds. \tag{2.6}$$

Using (2.2), from (2.6) we have

$$\liminf_{t \to \infty} \frac{p(t)y(t)}{Q(t, t_0)} = -\infty,$$

a contradiction. The proof is complete.

REMARK 2.1. Condition (2.2) improves condition (9) of Theorem 2 in [2].

EXAMPLE 2.1. Consider the inequality

$$y''(t) + ty(t) \le \sin t. \tag{2.7}$$

In our notations, $r(t) = \sin t$, $\delta(t) = -\sin t$, $p(t) = q(t) \equiv 1$. It is easy to see that

$$\liminf_{t \to \infty} \frac{1}{Q(t, T)} \int_T^t Q(t, s)r(s)ds = \liminf_{t \to \infty} \frac{1}{t - T} \int_T^t (t - s)\sin s\, ds > -\infty,$$

and

$$\liminf_{t\to\infty} p(t)\delta(t) = -1.$$

Therefore Theorems 2 and 3 in [1] can not be applied to (2.7). But

$$\liminf_{t\to\infty}\frac{1}{Q(t, T)}\int_T^t Q(t, s)(r(s) - h(s, \delta_+(s)))ds$$
$$= \liminf_{t\to\infty}\frac{1}{t-T}\int_T^t (t-s)(\sin s - s(-\sin s)_+)ds = -\infty.$$

Therefore (2.7) has no positive solution on $[a, \infty)$ according to Theorem 2.1.

Case 2: $\int_a^\infty \frac{dt}{q(t)} < \infty$.

THEOREM 2.2. *In addition to* (i) *of Theorem* 2.1, *further assume that*

$$\int_T^\infty h\left(t, \left(\frac{c\lambda(t)}{p(t)} + \delta(t)\right)_+\right)dt = \infty \qquad \text{for all } c > 0 \tag{2.8}$$

where $\lambda(t) = \int_t^\infty \frac{ds}{q(s)}$. *Then inequality* (1.1) *has no positive solution on* $[t_0, \infty)$ *for any* $t_0 > a$.

PROOF. If not, let y be a positive solution of (1.1). Set $Z(t) = y(t) - \delta(t)$, then

$$(q(t)(p(t)Z(t))')' \leq -h(t, y(t)) < 0 \qquad \text{for } t \geq T. \tag{2.9}$$

Thus

$$q(t)(p(t)Z(t))' \leq q(t_1)(p(t_1)Z(t_1))' \qquad \text{for } t \geq t_1 \geq T. \tag{2.10}$$

Dividing (2.10) by $q(t)$ and integrating it we have

$$p(t)Z(t) - p(t_1)Z(t_1) \leq q(t_1)(p(t_1)Z(t_1))'\int_{t_1}^t \frac{ds}{q(s)}, \qquad t \geq t_1 \geq T.$$

This implies that $p(t)Z(t)$ is bounded above and

$$p(t_1)Z(t_1) \geq -q(t_1)(p(t_1)Z(t_1))'\lambda(t_1) \qquad \text{for } t_1 \geq T, \tag{2.11}$$

since, by the same reason as the roof in Theorem 2.1, we must have $Z(t) > 0$ eventually.

There are two ossible cases:

First we consider the case that $(p(t)Z(t))' > 0$ for $t \geq T_2$, which with (2.9) implies that

$$\int_{T_2}^{\infty} h(t, y(t))dt < \infty. \tag{2.12}$$

Since

$$p(t)Z(t) \geq p(T_2)Z(T_2) = c > 0 \qquad \text{for } T_2,$$

we get

$$Z(t) \geq \frac{c}{p(t)}$$

and

$$y(t) = Z(t) + \delta(t) \geq \left(\frac{c}{p(t)} + \delta(t)\right)_+ . \tag{2.13}$$

Substituting (2.13) into (2.12) we have

$$\int_{T_2}^{\infty} h\left(t, \left(\frac{c}{p(t)} + \delta(t)\right)_+\right)dt < \infty. \tag{2.14}$$

Since $\lambda(t) \to 0$ as $t \to \infty$. (2.14) contradicts condition (2.8).

Now we consider the second case that $(p(t)Z(t))' < 0$ eventually. In view of (2.9) and (2.11) we have

$$p(t)Z(t) \geq -q(t)(p(t)Z(t))'\lambda(t) \geq \ell\lambda(t),$$

where ℓ is a positive constant. Therefore

$$Z(t) \geq \frac{\ell\lambda(t)}{p(t)}$$

and

$$y(t) = Z(t) + \delta(t) \geq \left(\frac{\ell\lambda(t)}{p(t)} + \delta(t)\right)_+ \tag{2.15}$$

Substituting (2.15) into (2.12) we get a contradiction. The proof of this theorem is complete.

EXAMPLE 2.2. Consider the differential inequality

$$(t^2 y')' + y \leq 2t \cos t - t^2 \sin t. \tag{2.16}$$

In our notations, $q(t) = t^2$, $p(t) = 1$, $h(t, y) = y$, $r(t) = 2t\cos t - t^2 \sin t$, $\delta(t) = \sin t$ and $\lambda(t) = 1/t$. It is clear that the results in [2] can not be applied to (2.16). But

$$\int_T^\infty h\left(t, \left(\frac{c\lambda(t)}{p(t)} + \delta(t)\right)_+\right)dt = \int_T^\infty \left(\frac{c}{t} + \sin t\right)_+ dt = \infty.$$

Therefore (2.16) has no positive solution on $[t_0, \infty)$ according to Theorem 2.2.

REMARK 2.2. In Theorem 2 in [2], the criterion does not include the term h. Thus it loses much information. In Theorem 3 in [2], the condition

$$\liminf_{t\to\infty} [p(t)\delta(t)] = 0$$

and the condition that the unperturbed inequality (10) has no positive solution are required. In our results these are not necessary.

III. Perturbed Schrödinger equations

Consider the second order perturbed Schrödinger equation of the form

$$\Delta u + C(x, u) = f(x), \qquad x \in E \tag{3.1}$$

where Δ is the Laplace operator in n-dimensional Euclidean space R^n, E is an exterior domain in R^n, and $C\colon E \times R \to R$ and $f\colon E \to R$ are continuous.

A function $V\colon E \to R$ is called oscillatory in E if $V(x)$ has arbitrarily large zeros, that is, the set $\{x \in E\colon V(x) = 0\}$ is unbounded.

The following Lemma is taken from [2].

LEMMA 3.1. *Assume that*

(i) $C(x, -u) = -C(x, u)$ *for* $x \in E$ *and* $u > 0$

(ii) $C(x, u) \geq H(|x|)\phi(u)$, *for* $x \in E$ *and* $u > 0$

where $H \in C(R_+, R_+ \setminus \{0\})$, $\phi\colon (0, \infty) \to (0, \infty)$ *is continuous and convex.*

Let $F(t)$ *be the spherical mean of* $f(x)$ *over* $S_t = \{x \in R^n; |x| = t\}$, *i.e.*

$$F(t) = \frac{1}{\sigma_n t^{n-1}} \int_{|x|=t} f(x)dS, \qquad t > 0,$$

where σ_n *is the surface area of the unit sphere in* R^n.

Then equation (3.1) *is oscillatory in E if the ordinary differential inequalities*

$$(t^{n-1}y')' + t^{n-1}H(t)\phi(y) \leq t^{n-1}F(t) \tag{3.2}$$

and

(3.3) $$(t^{n-1}y')' + t^{n-1}H(t)\phi(y) \leq -t^{n-1}F(t)$$

are oscillatory at $t = \infty$ *in the sense that neither* (3.2) *nor* (3.3) *has a solution which is positive on* $[t_0, \infty)$ *for any* $t_0 > 0$.

It was shown in [2], (3.2) and (3.3) are equivalent to the following inequalities

(3.4) $$(t^{3-n}(t^{n-2}y)')' + tH(t)\phi(y) \leq tF(t)$$

and

(3.5) $$(t^{3-n}(t^{n-2}y)')' + tH(t)\phi(y) \leq -tF(t).$$

Using Theorem 2.1 to (3.1) we have the following result.

THEOREM 3.1. *Suppose that* $C(x, u)$ *satisfies the conditions in Lemma* 3.1, *and further assume that there exists a function* $\tilde{f} \in C^2$ *such that*

(3.6) $$tF(t) = (t^{3-n}(t^{n-2}\tilde{f})')'$$

and $\tilde{f}$ *is oscillatory. Set*

(3.7) $$\begin{cases} G_1(t) = tF(t) - tH(t)\phi(\tilde{f}_+(t)) \\ G_2(t) = -tF(t) - tH(t)\phi(\tilde{f}_-(t)) \end{cases}$$

and

(3.8) $$\liminf_{t\to\infty} \int_T^t \left(1 - \frac{\log s}{\log t}\right) G_1(s)ds = -\infty, \quad \liminf_{t\to\infty} \int_T^t \left(1 - \frac{\log s}{\log t}\right) G_2(s)ds = -\infty \qquad \text{for } n = 2,$$

(3.9) $$\liminf_{t\to\infty} \int_T^t \left(1 - \left(\frac{s}{t}\right)^{n-2}\right) G_1(s)ds = -\infty, \quad \liminf_{t\to\infty} \int_T^t \left(1 - \left(\frac{s}{t}\right)^{n-2}\right) G_2(s)ds = -\infty \qquad \text{for } n \geq 3,$$

for all large T. Then equation (3.1) *is oscillatory in E.*

REMARK 3.1. Condition (3.8) and (3.9) improve conditions (20)–(23) respectively in [2] by withdrawing the information from the second term of (3.1).

PROOF OF THEOREM 3.1. Use Theorem 2.1 to (3.4) and (3.5) with $q(t) = t^{3-n}$, $h(t, y) = tH(t)\phi(y)$, $r(t) = tF(t)$ [or $-tF(t)$] and note that $\int_T^\infty \frac{ds}{q(s)} = \infty$ and

$$Q(t,s)=\begin{cases}\log t-\log s, & n=2\\ \dfrac{1}{n-2}(t^{n-2}-s^{n-2}), & n\geq 3.\end{cases} \tag{3.10}$$

Conditions (3.8) and (3.9) follow from (2.2) in Theorem 2.1.

In view of Theorem 2.2 we can study (3.2) and (3.3) directly. In (3.2) and (3.3), $q(t)=t^{n-1}$, $p(t)=1$. For $n\geq 3$

$$\int_T^\infty \frac{ds}{q(s)}=\int_T^\infty \frac{ds}{s^{n-1}}<\infty,\quad \lambda(t)=\int_t^\infty \frac{ds}{q(s)}=\frac{t^{2-n}}{n-2}.$$

The following result follows from Theorem 2.2.

THEOREM 3.2. *Suppose $n\geq 3$ and there exists a $\delta\in C^2$ such that*

$$(t^{n-1}\delta'(t))'=t^{n-1}F(t)$$

and $\delta(t)$ is oscillatory and

$$\begin{cases}\displaystyle\int_T^\infty t^{n-1}H(t)\phi((ct^{2-n}+\delta(t))_+)dt=\infty\\ \displaystyle\int_T^\infty t^{n-1}H(t)\phi((ct^{2-n}+\delta(t))_-)dt=\infty.\end{cases} \tag{3.11}$$

Then the equation (3.1) *is oscillatory in E.*

EXAMPLE 3.1. Consider the equation

$$\Delta u+\frac{2}{|x|^2}u=\frac{1}{|x|}\cos|x|-\sin|x| \tag{3.12}$$

in $E=\{x\in R^3: |x|\geq 1\}$. In view of Lemma 3.1, equation (3.12) is oscillatory if

$$\begin{cases}(t^2y')'+2y\leq t^2F(t)\\ (t^2y')'+2y\leq -t^2F(t)\end{cases} \tag{3.13}$$

are oscillatory at $t=\infty$, where

$$F(t)=\frac{1}{t}\cos t-\sin t. \tag{3.14}$$

Theorem 5 in [2] is not applicable to (3.12). In our notation, $\delta(t)=\sin t$ for (3.14). Condition (3.11) becomes

$$(3.15)\qquad \begin{cases} \displaystyle\int_T^\infty t^{n-1}H(t)\phi((ct^{2-n}+\delta(t))_+)dt = \int_T^\infty \left(\frac{c}{t}+\sin t\right)_+ dt = \infty \\ \displaystyle\int_T^\infty t^{n-1}H(t)\phi((ct^{2-n}+\delta(t))_-)dt = \int_T^\infty \left(\frac{c}{t}+\sin t\right)_- dt = \infty. \end{cases}$$

According to Theorem 3.2, equation (3.12) is oscillatory in E.

REMARK 3.2. Using our results Theorems 2.1 and 2.2, some results in [3, 4] can be improved immediately.

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