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Nonlinear eigenvalue problem for a model equation of an elastic surface

Dedicated to Professor Takaŝi Kusano on his 60th birthday

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1. Introduction

In this article we discuss the existence of nonzero weak solutions of the boundary value problem

$$-\gamma \operatorname{div}\left[\frac{(\sqrt{1+|\nabla u|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}}\nabla u\right] = \lambda f(x, u) \qquad \text{in } \Omega \tag{1.1}$$

$$u \geq 0 \qquad \text{in } \Omega \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega\,, \tag{1.3}$$

where $\gamma > 1$, Ω is a bounded domain in $\boldsymbol{R}^n$, ∇u denotes the gradient of u and λ is a positive parameter. In the case $\gamma = 1$, the equation (1.1) is the mean curvature equation or the capillary surface equation. When $n = 1$, this equation describes the equilibrium state of an elastic string yielding an exterior force $f(x, u)$. We give the derivation of the equation (1.1) for one dimensional elastic string in Section 2. The parameter λ depends on a tension of the string. The purpose of this paper is to investigate the dependence between a weak solution u_λ and parameter λ.

It is easy to see that solutions of (1.1), (1.3) correspond to critical points of the functional

$$I_\lambda[u] = \int_\Omega (\sqrt{1+|\nabla u|^2}-1)^\gamma dx - \lambda \int_\Omega F(x, u)dx \tag{1.4}$$

defined on the usual Sobolev space $W_0^{1,\gamma}(\Omega)$, where

$$F(x, u) = \int_0^u f(x, \xi)d\xi\,.$$

Under appropriate growth conditions on $F(x, u)$ we show the existence of a local minimizer of I_λ in Section 3. Next we give a proof to obtain an unstable critical point of I_λ by using the mountain pass lemma without Palais-Smale condition and the monotone operator theory in Section 4. We mention an example in the case of $f(x, u) = qu^{q-1}$ to illustrate our result in Section 5.

Throughout this paper, we assume that the measure of Ω is equal to 1 without loss of generality. Further, we use a notation $\gamma^* = n\gamma/(n-\gamma)$ when $\gamma < n$ and $\gamma^* = \infty$ otherwise, repectively.

2. Derivation of the model equation for one dimensional case

In this section we derive the model equation (1.1) for one-dimensional elastic string. Let us consider an elastic string with length ℓ in the free state. We assume a constituent law between strain force F and length of extension ξ as

$$\mathsf{F} = k(\ell)\xi^{\sigma}, \tag{2.1}$$

where σ is a positive constant and $k(\ell)$ is a constant of elasticity. In the case of $\sigma = 1$, it is known as Hooke's law. Now let us consider two strings with the length ℓ_1 and ℓ_2 in the free state. Further consider the string constituted by connecting each one edges of these two strings which are stretched with length of extension ξ_1 and ξ_2 respectively. Then the forces yielding to the strings are $k(\ell_1)\xi_1^{\sigma}$ and $k(\ell_2)\xi_2^{\sigma}$ respectively. On the other hand, since the connected string has the length of extension $\xi_1 + \xi_2$, the force which works on this string is $k(\ell_1 + \ell_2)(\xi_1 + \xi_2)^{\sigma}$. Since the strain force F is fixed at each point from the law of action and reaction, the equation

$$\mathsf{F} = k(\ell_1)\xi_1^{\sigma} = k(\ell_2)\xi_2^{\sigma} = k(\ell_1 + \ell_2)(\xi_1 + \xi_2)^{\sigma} \tag{2.2}$$

holds. Namely,

$$\xi_1 = \left(\frac{\mathsf{F}}{k(\ell_1)}\right)^{1/\sigma}, \quad \xi_2 = \left(\frac{\mathsf{F}}{k(\ell_2)}\right)^{1/\sigma}, \quad \text{and} \quad \xi_1 + \xi_2 = \left(\frac{\mathsf{F}}{k(\ell_1 + \ell_2)}\right)^{1/\sigma}. \tag{2.3}$$

By the equations (2.3), we have the relation

$$\left(\frac{1}{k(\ell_1)}\right)^{1/\sigma} + \left(\frac{1}{k(\ell_2)}\right)^{1/\sigma} = \left(\frac{1}{k(\ell_1 + \ell_2)}\right)^{1/\sigma} \tag{2.4}$$

for each $\ell_1, \ell_2 > 0$. Now we assume that $k(\ell)$ is continuous in ℓ. Then this relation shows that the function $\left(\frac{1}{k(\ell)}\right)^{1/\sigma}$ is linear, i.e.,

$$\left(\frac{1}{k(\ell)}\right)^{1/\sigma} = c\ell$$

with some constant $c > 0$. Hence we have

$$k(\ell) = \frac{\kappa}{\ell^\sigma} \qquad (\kappa\text{: constant})\,. \tag{2.5}$$

Thus the relation (2.1) reduces to

$$\mathsf{F} = \kappa\left(\frac{\xi}{\ell}\right)^\sigma. \tag{2.6}$$

Hence, when the string with length ℓ in the free state is stretched to $\ell + \xi$, the potential energy is given by

$$e = \int_0^\xi \mathsf{F} d\xi = \frac{\kappa}{\sigma+1}\left(\frac{\xi}{\ell}\right)^{\sigma+1} \ell\,. \tag{2.7}$$

Now consider a deformed string which was occupied in $(x, y) = (t, 0)$, $0 \le t \le \ell$, in the free state. Let us denote by $(x(t), y(t))$ the displacement point of the string which is at $(t, 0)$ in the free state. Then the length of extension at $(t, t + dt)$ is given by

$$d\xi = \sqrt{dx(t)^2 + dy(t)^2} - dt$$

and, from (2.7), the local potential energy caused by this extension is given by

$$dE = \frac{\kappa}{\sigma+1}\left(\frac{d\xi}{dt}\right)^{\sigma+1} dt\,.$$

Thus the potential energy of this string caused by deformation is given by

$$\begin{aligned} E &= \int_0^\ell \frac{\kappa}{\sigma+1}\left(\frac{d\xi}{dt}\right)^{\sigma+1} dt \\ &= \frac{\kappa}{\sigma+1}\int_0^\ell \left\{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} - 1\right\}^{\sigma+1} dt\,. \end{aligned} \tag{2.8}$$

In case more general nonlinear strain relation $\mathsf{F} = \phi\left(\frac{\xi}{\ell}\right)$ is considered instead of (2.6), the potential energy is given by

$$E = \int_0^\ell \Phi\left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} - 1\right) dt \tag{2.9}$$

where $\Phi(t)$ is a primitive function of $\phi(t)$.

If the curve of the deformed string is given by a non-parametrized form as $y = u(x)$, $0 \le x \le \ell$, then letting $x = t$ in (2.8) we have

$$E = E[u] = \frac{\kappa}{\sigma+1}\int_0^\ell \left\{\sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1\right\}^{\sigma+1} dx\,.$$

Hence denoting

$$\int_0^\ell F(x, u)dx$$

the potential energy caused by an exterior force, we have the total energy of the deformed string as

$$I[u] = \frac{\kappa}{\sigma + 1} \int_0^\ell \left\{ \sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1 \right\}^{\sigma+1} dx - \int_0^\ell F(x, u)dx . \tag{2.10}$$

After normalizing a constant and putting $\gamma = \sigma + 1$, we easily see that the Euler equation of (2.10) is equal to (1.1).

3. A local minimizer of the functional I_λ

As is stated in Introduction, we consider the functional

$$I_\lambda[u] = \int_\Omega (\sqrt{1 + |\nabla u|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, u)dx$$

defined on the Sobolev space $W_0^{1,\gamma}(\Omega)$, where

$$F(x, u) = \int_0^u f(x, \xi)d\xi .$$

In this section we show the existence of a minimizer in the neighborhood of the origin of this functional.

At first we put assumptions on $f(x, \xi)$ as follows:

(A1) $f: \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous,

(A2) $f(x, \xi) > 0$ on $\Omega \times (0, \infty)$, $f(x, \xi) = 0$ on $\Omega \times (-\infty, 0]$,

(A3) there exists a constant q with $1 \le q < \gamma^*$ and the inequality

$$f(x, \xi) \le d_1 \xi^{q-1} + d_2$$

holds on $\Omega \times [0, \infty)$ with some positive constants d_1, d_2.

In the beginning we see that a solution of (1.1) satisfies a weak maximum principle, and owing to the assumption (A2), weak solutions of (1.1) with (1.3) are necessarily nonnegative.

THEOREM 3.1. *Let the function* $f(x, \xi)$ *satisfy the assumptions* (A1), (A2), (A3). *If* u *in* $W_0^{1,\gamma}(\Omega)$ *satisfies* (1.1) *weakly, that is, the equality*

$$\gamma \int_\Omega \frac{(\sqrt{1 + |\nabla u|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla u|^2}} \nabla u \nabla \phi dx = \lambda \int_\Omega f(x, u)\phi dx \tag{3.1}$$

holds for any $\phi \in C_0^\infty$, *then* $u(x) \ge 0$ *almost everywhere in* Ω.

PROOF. From the assumption (A3) and the Sobolev imbedding theorem, it is easy to see that $f(x, u)$ belongs to $W_0^{1,\gamma}(\Omega)^*$ (the adjoint space of $W_0^{1,\gamma}(\Omega)$). Further the linear functional

$$\phi \to \gamma \int_\Omega \frac{(\sqrt{1+|\nabla u|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}} \nabla u \nabla \phi dx$$

is continuously extended to the space $W_0^{1,\gamma}(\Omega)$. Thus the equality

$$\gamma \int_\Omega \frac{(\sqrt{1+|\nabla u|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}} \nabla u \nabla v dx = \lambda \int_\Omega f(x, u) v dx \tag{3.2}$$

holds for any $v(x) \in W_0^{1,\gamma}(\Omega)$. Since $f(x, u) \geq 0$ in Ω, the right hand in (3.2) is nonpositive for any $v(x) \leq 0$ in Ω. Put $v(x) = \min\{u(x), 0\}$. Then v is nonpositive and belongs to $W_0^{1,\gamma}(\Omega)$, since $u \in W_0^{1,\gamma}(\Omega)$. Thus we have

$$\begin{aligned} 0 &\geq \int_\Omega \frac{(\sqrt{1+|\nabla u|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}} \nabla u \nabla v dx \\ &= \int_{\Omega \backslash \Omega^-} \frac{(\sqrt{1+|\nabla u|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}} \nabla u \nabla v dx + \int_{\Omega^-} \frac{(\sqrt{1+|\nabla u|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla u|^2}} \nabla u \nabla v dx \\ &= \int_{\Omega^-} \frac{(\sqrt{1+|\nabla v|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla v|^2}} |\nabla v|^2 dx = \int_\Omega \frac{(\sqrt{1+|\nabla v|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla v|^2}} |\nabla v|^2 dx , \end{aligned}$$

where $\Omega^- \equiv \{x \in \Omega | u(x) < 0\}$. Hence $\nabla v = 0$ almost everywhere in Ω. Noting $v \in W_0^{1,\gamma}(\Omega)$ and Poincaré's inequality $\|v\|_{L^\gamma(\Omega)} \leq c\|\nabla v\|_{L^\gamma(\Omega)}$, we see $v = 0$ in Ω. This implies that $u \geq 0$ almost everywhere in Ω.

Now we take weak solutions (1.1) as critical points of the functional I_λ. Noting that $W_0^{1,\gamma}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ $(1 \leq q < \gamma^*)$, we easily see that the functional I_λ is continuously differentiable on $W_0^{1,\gamma}(\Omega)$ under the assumptions (A1), (A2), (A3). Needless to say, a local minimizer u of I_λ is a critical point, and hence it satisfies the Euler equation weakly which is equal to (1.1).

THEOREM 3.2. *In addition to the assumptions* (A1), (A2), (A3), *let us assume* $1 \leq q < 2\gamma$ *in* (A3) *and the following.*
(A4) *There exists a constant* r *with* $1 < r < 2\gamma$ *and the inequality*

$$f(x, \xi) \geq d_3 \xi^{r-1} \tag{3.3}$$

holds on $\Omega \times [0, \xi_0)$ *with some constants* $d_3 > 0$ *and* $\xi_0 > 0$.

Then there exists a positive constant λ^* *such that, for any* $0 < \lambda < \lambda^*$, *there exists a nonnegative, nonzero local minimizer* u_λ. *Further*

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)} \to 0 \qquad as\ \lambda \to 0 .$$

PROOF. By Poincaré's inequality we may adopt the norm

$$\|\nabla u\|_{L^\gamma(\Omega)} = \left(\int_\Omega |\nabla u|^\gamma dx\right)^{1/\gamma}$$

as the one in $W_0^{1,\gamma}(\Omega)$. Note the function $\phi(X) = (\sqrt{1 + X^{2/\gamma}} - 1)^\gamma$ is convex, since

$$\phi''(X) = \frac{X^{2/\gamma - 2}}{(1 + X^{2/\gamma})^{3/2}}(\sqrt{1 + X^{2/\gamma}} - 1)^{\gamma-1}\left\{\left(1 - \frac{1}{\gamma}\right)\sqrt{1 + X^{2/\gamma}} + \frac{1}{\gamma}\right\} > 0 .$$

Further, since we assume meas $(\Omega) = 1$, Jensen's inequality

$$\int_\Omega \phi(X(x))dx \ge \phi\left(\int_\Omega X(x)dx\right) \tag{3.4}$$

holds for any $X \in L^1(\Omega)$. Putting $X = |\nabla u|^\gamma$ in (3.4) leads to the inequality

$$\int_\Omega (\sqrt{1 + |\nabla u|^2} - 1)^\gamma dx \ge \left\{\sqrt{1 + \left(\int_\Omega |\nabla u|^\gamma dx\right)^{2/\gamma}} - 1\right\}^\gamma \tag{3.5}$$

for any $u \in W_0^{1,\gamma}(\Omega)$. From the assumptions (A3) the inequality

$$\int_\Omega |F(x, u)|\, dx \le c_1 \int_\Omega (|u| + |u|^q)dx \tag{3.6}$$

holds. And further the right hand in (3.6) is dominated as

$$c_1 \int_\Omega (|u| + |u|^q)dx \le c_2\left\{\left(\int_\Omega |\nabla u|^\gamma dx\right)^{1/\gamma} + \left(\int_\Omega |\nabla u|^\gamma dx\right)^{q/\gamma}\right\}$$

with some positive constant c_2 by the Poincaré-Sobolev inequality. Hence we have

$$I_\lambda[u] \ge (\sqrt{1 + \rho^2} - 1)^\gamma - c_2\lambda(\rho + \rho^q) \tag{3.7}$$

on the sphere $\|\nabla u\|_{L^\gamma(\Omega)} = \rho$. If we take $\rho = \rho_\lambda \equiv \lambda^\alpha$ with a constant α satisfying $0 < \alpha < 1/(2\gamma - 1)$, then

$$I_\lambda[u] \ge (\sqrt{1 + \lambda^{2\alpha}} - 1)^\gamma - c_2\lambda(\lambda^\alpha + \lambda^{q\alpha}) \tag{3.8}$$

on $\|\nabla u\|_{L^\gamma(\Omega)} = \rho_\lambda$. Noting $2\gamma\alpha < \alpha + 1 \le q\alpha + 1$ from $1 \le q < 2\gamma$ and comparing the orders of λ of the first and second parts of the right hand in (3.8) as $\lambda \to 0$, we can choose a constant $\lambda^* > 0$ such that the right hand of (3.8) is positive for any $0 < \lambda \leq \lambda^*$. Hence $I_\lambda[u] > 0$ on the sphere $\|\nabla u\|_{L^\gamma(\Omega)} = \rho_\lambda$ for any $0 < \lambda \leq \lambda^*$.

Next let us put

$$\inf_{u \in B(\rho_\lambda)} I_\lambda[u] = I^\lambda ,$$

where $B(\rho_\lambda) = \{u \in W_0^{1,\gamma}(\Omega) | \|\nabla u\|_{L^\gamma(\Omega)} \leq \rho_\lambda\}$. Since the inequality (3.7) holds, $I_\lambda \neq -\infty$. Let us take a sequence $\{u_n\}$ in $B(\rho_\lambda)$ such that

$$I_\lambda[u_n] \to I^\lambda \qquad \text{as } n \to \infty .$$

Since $\|\nabla u_n\|_{L^\gamma(\Omega)} \leq \rho_\lambda$, the sequence $\{u_n\}$ is bounded in $W_0^{1,\gamma}(\Omega)$. Noting that $W_0^{1,\gamma}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ by the assumption $1 \leq q < \gamma^*$, we can find out a subsequence $\{u_{n_k}\}$ and $u_\lambda \in W_0^{1,\gamma}(\Omega)$ such that $u_{n_k} \to u_\lambda$ weakly in $W_0^{1,\gamma}(\Omega)$ and strongly in $L^q(\Omega)$ as $k \to \infty$. Since the norm of $W_0^{1,\gamma}(\Omega)$ is weakly lower semicontinuous, the inequality

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)} \leq \liminf_{k \to \infty} \|\nabla u_{n_k}\|_{L^\gamma(\Omega)} \leq \rho_\lambda$$

holds. Thus u_λ belongs to $B(\rho_\lambda)$. As is stated formerly, the functional

$$\int_\Omega (\sqrt{1 + |\nabla u|^2} - 1)^\gamma dx$$

is convex and continuous on $W_0^{1,\gamma}(\Omega)$. Hence this functional is weakly lower semicontinuous on $W_0^{1,\gamma}(\Omega)$. Here we used the fact that a convex and lower semicontinuous functional defined on a Banach space is weakly lower semicontinuous. For the proof see e.g. Dacorogna [9, Theorem 1.2 in Chap. 3]. Thus the inequality

$$\liminf_{k \to \infty} \int_\Omega (\sqrt{1 + |\nabla u_{n_k}|^2} - 1)^\gamma dx \geq \int_\Omega (\sqrt{1 + |\nabla u_\lambda|^2} - 1)^\gamma dx$$

holds. Further, since the functional $\int_\Omega F(x, u) dx$ is continuous on $L^\gamma(\Omega)$, we have

$$\begin{aligned}
&\liminf_{k \to \infty} \left\{ \int_\Omega (\sqrt{1 + |\nabla u_{n_k}|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, u_{n_k}) dx \right\} \\
&= \liminf_{k \to \infty} \int_\Omega (\sqrt{1 + |\nabla u_{n_k}|^2} - 1)^\gamma dx - \lambda \lim_{k \to \infty} \int_\Omega F(x, u_{n_k}) dx \\
&\geq \int_\Omega (\sqrt{1 + |\nabla u_\lambda|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, u_\lambda) dx = I_\lambda[u_\lambda] .
\end{aligned}$$

Namely, $I_\lambda[u_\lambda] = I^\lambda$. Hence this limit function u_λ is a minimizer of I_λ in

$B(\rho_\lambda)$. Next, in order to show that u_λ is an interior point of $B(\rho_\lambda)$, let us choose a nonzero function $\varphi(x)$ in $C_0^\infty(\Omega)\cap B(\rho_\lambda)$ satisfying $0\le\varphi(x)<\xi_0$ in Ω, and put $u=\varepsilon\varphi$ in $I_\lambda[u]$. Then from the assumption (A4), we have

$$\begin{aligned} I_\lambda[\varepsilon\varphi] &= \int_\Omega (\sqrt{1+\varepsilon^2|\nabla\varphi|^2}-1)^\gamma dx - \lambda\int_\Omega F(x,\varepsilon\varphi(x))dx \\ &\le \frac{\varepsilon^{2\gamma}}{2^\gamma}\int_\Omega |\nabla\varphi|^{2\gamma}dx - c_3\lambda\varepsilon^r\int_\Omega \varphi(x)^r dx \end{aligned}$$

for any $0<\varepsilon<1$ with some constant $c_3>0$. Since $1<r<2\gamma$ by the assumption, $I_\lambda[\varepsilon\varphi]<0$ holds for sufficiently small $\varepsilon>0$. This implies $I^\lambda<0$, and hence u_λ is nonzero. And further, since $I_\lambda[u]$ is positive on the boundary of $B(\rho_\lambda)$ (i.e. $\|\nabla u\|_{L^\gamma(\Omega)}=\rho_\lambda$) when $0<\lambda\le\lambda^*$ as is stated formerly, the minimizer u_λ is an interior point of the set $B(\rho_\lambda)$. Thus u_λ is a local minimizer of I_λ in $W_0^{1,\gamma}(\Omega)$. Finally, $\|\nabla u_\lambda\|_{L^\gamma(\Omega)}\le\rho_\lambda=\lambda^\alpha\to 0$ as $\lambda\to 0$.

THEOREM 3.3. *If $1\le q<\gamma$ instead of the assumption $1\le q<2\gamma$ in Theorem* 3.2, *then the results in Theorem* 3.2 *holds as $\lambda^*=\infty$ and u_λ in Theorem* 3.2 *is a global minimizer of I_λ.*

Further we assume the following:

(A5) *There exists a constant $s>1$ which satisfies the inequality*

$$f(x,\xi)\ge d_4\xi^{s-1}-d_5 \qquad \text{on } \Omega\times(0,\infty) \tag{3.9}$$

with some constants d_4, $d_5>0$.

Then, this minimizer u_λ satisfies

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)}\to\infty \qquad \text{as } \lambda\to\infty\,. \tag{3.10}$$

PROOF. The first half in the theorem is clear. We have only to notice that $I_\lambda[u]\to\infty$ as $\|\nabla u\|_{L^\gamma(\Omega)}\to\infty$. Hence we only show that $\|\nabla u_\lambda\|_{L^\gamma(\Omega)}\to\infty$ as $\lambda\to\infty$ under the assumption (A5). First let us take $\varphi\in C_0^\infty(\Omega)$, $\varphi(x)\ge 0$ in Ω with $\|\nabla\varphi\|_{L^\gamma(\Omega)}=1$. In the assumption (A5), we may assume $s<\gamma$. Then, using the assumption (A5), and noting the inequality

$$\sqrt{1+|p|^2}-1=\frac{|p|^2}{\sqrt{1+|p|^2}+1}\le|p|$$

for $p\in\boldsymbol{R}^n$, we have

$$\begin{aligned} &\inf\{I_\lambda[u]\,|\,\|\nabla u\|_{L^\gamma(\Omega)}=\rho\} \\ &\le I_\lambda[\rho\varphi] \\ &\le \int_\Omega |\nabla(\rho\varphi)|^\gamma dx - c_4\lambda\int_\Omega |\rho\varphi|^s dx + c_5\lambda\int_\Omega |\rho\varphi|dx \\ &= \rho^\gamma - c_6\lambda\rho^s + c_7\lambda\rho\,, \end{aligned}$$

where c_4 and c_5 are positive constants and

$$c_6 = c_4 \int_\Omega \varphi^s dx\,, \qquad c_7 = c_5 \int_\Omega \varphi dx\,.$$

Hence,

$$I^\lambda \equiv \min_{u \in W_0^{1,\gamma}(\Omega)} I_\lambda[u] \le \inf_{\rho > 0} (\rho^\gamma - c_6 \lambda \rho^s + c_7 \lambda \rho)\,.$$

If we take

$$\rho = \left(\frac{c_6 s}{\gamma} \lambda\right)^{1/(\gamma - s)},$$

then we have

$$I^\lambda \le -\delta \lambda^{\gamma/(\gamma - s)} + c_8 \lambda^{1/(\gamma - s) + 1}\,, \tag{3.11}$$

where

$$\delta = \left(\frac{c_6 s}{\gamma}\right)^{\gamma/(\gamma - s)} \left(\frac{\gamma}{s} - 1\right) > 0$$

and

$$c_8 = c_7 \left(\frac{c_6 s}{\gamma}\right)^{1/(\gamma - s)}.$$

On the other hand, from (3.7) the inequality

$$I_\lambda[u] \ge (\sqrt{1 + \rho^2} - 1)^\gamma - c_2 \lambda (\rho + \rho^q) \tag{3.12}$$

holds on $\|\nabla u\|_{L^\gamma(\Omega)} = \rho$. The right hand in (3.12) is monotone decreasing on the interval $(0, (c_2 \lambda / \gamma)^{1/(\gamma - 1)})$ for each fixed $\lambda > 0$. Let us take a constant α with $0 < \alpha < 1/(\gamma - 1)$, then

$$\lambda^\alpha \le \left(\frac{c_2 \lambda}{\gamma}\right)^{1/(\gamma - 1)} \quad \text{if} \quad \lambda \ge \left(\frac{c_2}{\gamma}\right)^{-1/(1 - \alpha(\gamma - 1))} \quad (\equiv \lambda_0)\,.$$

Thus, if $\lambda \ge \lambda_0$, then the right hand in (3.12) is monotone decreasing on $(0, \lambda^\alpha)$. And hence

$$\begin{aligned} I_\lambda[u] &\ge (\sqrt{1 + \rho^2} - 1)^\gamma - c_2 \lambda (\rho + \rho^q) \\ &\ge (\sqrt{1 + \lambda^{2\alpha}} - 1)^\gamma - c_2 \lambda (\lambda^\alpha + \lambda^{\alpha q}) \end{aligned}$$

holds on $\|\nabla u\|_{L^\gamma(\Omega)} = \rho$ with $0 < \rho < \lambda^\alpha$. This shows that

$$I_\lambda[u] \ge (\sqrt{1 + \lambda^{2\alpha}} - 1)^\gamma - c_2 \lambda (\lambda^\alpha + \lambda^{\alpha q}) \tag{3.13}$$

holds on the ball $B(\lambda^\alpha) = \{u \in W_0^{1,\gamma}(\Omega) \mid \|\nabla u\|_{L^\gamma(\Omega)} \le \lambda^\alpha\}$ when $\lambda \ge \lambda_0$. Next take $\alpha > 0$ over again so small that $1 + q\alpha < \gamma/(\gamma - s)$. Then by comparing the orders of λ as $\lambda \to \infty$ in the right hands in (3.11) and (3.13), we see that the inequality

$$I_\lambda[u] \ge (\sqrt{1 + \lambda^{2\alpha}} - 1)^\gamma - c_2\lambda(\lambda^\alpha + \lambda^{\alpha q})$$
$$> -\delta\lambda^{\gamma/(\gamma-s)} + c_8\lambda^{1/(\gamma-s)+1}$$

holds on $B(\lambda^\alpha)$ for $\lambda \ge \lambda_1$ with some large constant λ_1 greater than λ_0. This shows that $u_\lambda \notin B(\lambda^\alpha)$, i.e. $\|\nabla u_\lambda\|_{L^\gamma(\Omega)} > \lambda^\alpha$ for any $\lambda \ge \lambda_1$. Hence we have

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)} > \lambda^\alpha \to \infty \qquad \text{as } \lambda \to \infty .$$

4. Existence and asymptotic behavior of the secondary solution

In order to find out the second critical point of I_λ, we put a further assumption on $f(x, \xi)$.

(A6) There exist constants m greater than γ and $\xi_1 > 0$ such that the inequality

$$f(x, \xi)\xi \ge m \int_0^\xi f(x, \eta)d\eta \tag{4.1}$$

holds on $\Omega \times [\xi_1, \infty)$.

Then we have

THEOREM 4.1. *Let the assumptions* (A1), (A2), (A3), (A6) *be satisfied. Further, if* $\gamma < q < \gamma^*$ *in* (A3), *then there exists a postive constant* λ_* *such that, for any* $0 < \lambda < \lambda_*$, *there exists a nonnegative, nonzero critical point* v_λ *of* I_λ. *Further, this critical point* v_λ *satisfies*

$$\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to \infty \qquad \text{as } \lambda \to 0 .$$

REMARK. The asymptotic behavior $\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to \infty$ as $\lambda \to 0$ implies that this solution v_λ is different from the one obtained in Section 3. In fact we show the existence of v_λ by using the mountain pass lemma without Palais-Smale condition and the monotone operator method. This suggests that v_λ is an unstable critical point, while u_λ obtained in Section 3 is a local minimizer, i.e. a stable solution.

Prior to giving the proof of Theorem 4.1, recall the Ambrosetti-Rabinowitz mountain pass lemma without Palais-Smale condition.

LEMMA 4.1. *Let* I *be a* C^1*-function on a Banach space* E. *Suppose there exists a neighborhood* U *of* 0 *in* E *and a constant* α *which satisfy the following*:

i) *$I[u] \geq \alpha$ on the boundary of U,*
ii) *$I[0] < \alpha$,*
iii) *there exists a $w_0 \notin U$ satisfying $I[w_0] < \alpha$.*
Then, for the constant

$$\mu \equiv \inf_{\gamma \in \Gamma} \max_{w \in \gamma} I[w] \qquad (\geq \alpha), \tag{4.2}$$

where Γ denotes the class of paths joining 0 to w_0, there exists a sequence $\{u_j\}$ in E such that $I[u_j] \to \mu$ and $I'[u_j] \to 0$ in E^*.

The proof of this lemma was given by Aubin and Ekeland [3], which relies on Ekeland's minimization principle. The brief proof is given in [5] by Brezis.

Now we verify that Lemma 4.1 is applicable in our situation, namely the functional I_λ on $W_0^{1,\gamma}(\Omega)$ satisfies the hypotheses i), ii), iii).

Let us put

$$\lambda_\rho = \frac{(\sqrt{1+\rho^2}-1)^\gamma}{c_2(\rho+\rho^q)} \qquad (>0), \tag{4.3}$$

where c_2 is the constant in the inequality (3.7). Then the inequality (3.7) implies

$$\begin{aligned} I_\lambda[u] &\geq (\sqrt{1+\rho^2}-1)^\gamma - c_2\lambda(\rho+\rho^q) \\ &= c_2(\lambda_\rho - \lambda)(\rho+\rho^q) \end{aligned}$$

on $\|\nabla u\|_{L^\gamma(\Omega)} = \rho$. Thus, by taking $\alpha = c_2(\lambda_\rho - \lambda)(\rho+\rho^q)$ and $U = \{u \in W_0^{1,\gamma}(\Omega) \mid \|\nabla u\|_{L^\gamma(\Omega)} < \rho\}$, which are denoted by α_ρ and U_ρ respectively, the hypothesis i) holds for these α_ρ and U_ρ. Since $I_\lambda[0] = 0$, the hypothesis ii) is valid if $\alpha_\rho > 0$, i.e., $0 < \lambda < \lambda_\rho$. Finally we check the hypothesis iii) for these constant α_ρ and neighborhood U_ρ. From the inequality (4.1), the inequality

$$F(x,\xi) \equiv \int_0^\xi f(x,\eta)d\eta \geq F(x,\xi_1)\left(\frac{\xi}{\xi_1}\right)^m$$

holds for $\xi \geq \xi_1$. Since $f(x,\xi)$ is positive for $\xi > 0$, there exist an $x_0 \in \Omega$ and a neighborhood D of x_0 such that the inequality

$$F(x,\xi) \geq d_9\xi^m - c_{10} \tag{4.4}$$

holds on $D \times (0,\infty)$ with some positive constants c_9, c_{10}.

Now let us take a nonnegative function ϕ in $C_0^\infty(\Omega)$ satisfying $\phi(x) \geq 1$ on D. Then, from (4.4) the inequality

$$F(x, r\phi(x)) \geq c_9 r^m \phi(x)^m - c_{10} \tag{4.5}$$

holds on U for $r \geq \xi_1$. Hence

$$\begin{aligned} I_\lambda[r\phi] &= \int_\Omega (\sqrt{1 + r^2|\nabla\phi|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, r\phi)dx \\ &\leq \int_\Omega (\sqrt{1 + r^2|\nabla\phi|^2} - 1)^\gamma dx - \lambda \int_D F(x, r\phi)dx \\ &\leq r^\gamma \int_\Omega |\nabla\phi|^\gamma dx - \lambda c_9 r^m \int_D \phi^m dx + \lambda c_{10} \\ &\to -\infty \qquad \text{as } r \to \infty\,. \end{aligned} \tag{4.6}$$

Therefore $I_\lambda[r\phi] < 0$ for large r, and hence we have check d the hypothesis i), ii), iii) by taking $\alpha = \alpha_\rho$ and $U = U_\rho$ in our problem when $0 < \lambda < \lambda_\rho$.

Let $0 < \lambda < \lambda_\rho$. Then Lemma 4.1 asserts that there exists a sequence $\{u_j\}$ in $W_0^{1,\gamma}(\Omega)$ such that $I_\lambda[u_j] \to \mu_\lambda$, which is the constant μ defined by (4.2) for the functional I_λ, the neighborhood $U = U_\rho$ and $w_0 = R\phi$ for sufficently large R, and $I'_\lambda[u_j] \to 0$ in $W_0^{1,\gamma}(\Omega)^*$. This sequence satisfies the following.

LEMMA 4.2. *Let the hypotheses in Theorem* 4.1 *be satisfied,* $0 < \lambda < \lambda_\rho$, *and* $\{u_j\}$ *be the sequence in* $W_0^{1,\gamma}(\Omega)$ *obtained in Lemma* 4.1. *Namely it satisfies* $I_\lambda[u_j] \to \mu_\lambda$ *and* $I'_\lambda[u_j] \to 0$ *in* $W_0^{1,\gamma}(\Omega)^*$. *Then this sequence* $\{u_j\}$ *is bounded in* $W_0^{1,\gamma}(\Omega)$.

PROOF. The conditions $I_\lambda[u_j] \to \mu_\lambda$ and $I'_\lambda[u_j] \to 0$ mean

$$\int_\Omega (\sqrt{1 + |\nabla u_j|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, u_j)dx = \mu_\lambda + o(1)\,, \tag{4.7}$$

$$-\gamma \operatorname{div}\left[\frac{(\sqrt{1 + |\nabla u_j|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla u_j|^2}} \nabla u_j\right] - \lambda f(x, u_j) = \zeta_j \tag{4.8}$$

and $\|\zeta_j\|_{W_0^{1,\gamma}(\Omega)^*} = o(1)$ as $j \to \infty$. Here we put $I'_\lambda[u_j] = \zeta_j$. Operating the equality (4.8) to $u_j \in W_0^{1,\gamma}(\Omega)$, we have

$$\gamma \int_\Omega \frac{(\sqrt{1 + |\nabla u_j|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla u_j|^2}} |\nabla u_j|^2 dx - \lambda \int_\Omega f(x, u_j)u_j dx = \langle \zeta_j, u_j \rangle\,, \tag{4.9}$$

where $\langle \zeta_j, u_j \rangle$ denotes the action of $\zeta_j \in W_0^{1,\gamma}(\Omega)^*$ to $u_j \in W_0^{1,\gamma}(\Omega)$. From the equalities (4.7) and (4.9), we have

$$\begin{aligned} &\lambda \int_\Omega \{f(x, u_j)u_j - \gamma F(x, u_j)\}dx \\ &= \gamma \int_\Omega \left[\frac{(\sqrt{1 + |\nabla u_j|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla u_j|^2}} |\nabla u_j|^2 dx - (\sqrt{1 + |\nabla u_j|^2} - 1)^\gamma\right] dx \\ &\quad - \langle \zeta_j, u_j \rangle + \gamma\mu_\lambda + o(1)\,. \end{aligned} \tag{4.10}$$

Since the integrand in the right hand in (4.10) is nonnegative, the inequality

$$\lambda \int_\Omega \{f(x, u_j)u_j - \gamma F(x, u_j)\}dx \le -\langle \zeta_j, u_j \rangle + \gamma\mu_\lambda + o(1) \tag{4.11}$$

holds. From the assumption (A6), the left hand in (4.10) is estimated as

$$\lambda \int_\Omega \{f(x, u_j)u_j - \gamma F(x, u_j)\}dx \ge \lambda(m - \gamma) \int_\Omega F(x, u_j)dx - c_{11} \tag{4.12}$$

with some constant c_{11}. Combining (4.7), (4.11), (4.12), and noting $m - \gamma > 0$ by the assumption, we have

$$\begin{aligned}\int_\Omega (\sqrt{1 + |\nabla u_j|^2} - 1)^\gamma dx &= \lambda \int_\Omega F(x, u_j)dx + \mu_\lambda + o(1) \\ &\le \frac{\lambda}{m - \gamma}\left[\int_\Omega \{f(x, u_j)u_j - \gamma F(x, u_j)\}dx + c_{11}\right] + \mu_\lambda + o(1) \\ &\le \frac{1}{m - \gamma}\{-\langle \zeta_j, u_j \rangle + \gamma\mu_\lambda + c_{11}\} + \mu_\lambda + o(1) \\ &\le \frac{1}{m - \gamma}\{\|\zeta_j\|_{W_0^{1,\gamma}(\Omega)^*}\|\nabla u_j\|_{L^\gamma(\Omega)} + \gamma\mu_\lambda + c_{11}\} + \mu_\lambda + o(1) \\ &\le c_{12}\|\nabla u_j\|_{L^\gamma(\Omega)} + c_{13}\,.\end{aligned}$$

with some positive constants c_{12} and c_{13} which are independent on u_j. Putting $\|\nabla u_j\|_{L^\gamma(\Omega)} = \rho_j$ and using Jensen's inequality, we have

$$\begin{aligned}(\sqrt{1 + \rho_j^2} - 1)^\gamma &\le \int_\Omega (\sqrt{1 + |\nabla u_j|^2} - 1)^\gamma dx \\ &\le c_{12}\rho_j + c_{13}\,.\end{aligned}$$

Since $\gamma > 1$, $\{\rho_j\}$ is bounded in j, that is, the sequence $\{u_j\}$ is bounded in $W_0^{1,\gamma}(\Omega)$.

Hence we can find out $v \in W_0^{1,\gamma}(\Omega)$ and a subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, such that $u_j \to v$ strongly in $L^r(\Omega)$ for any $1 \le r < \gamma^*$, weakly in $W_0^{1,\gamma}(\Omega)$, and almost everywhere.

Put

$$I_\lambda[u] = K[u] - \lambda \int_\Omega F(x, u)dx\,,$$

where

$$K[u] = \int_\Omega (\sqrt{1 + |\nabla u|^2} - 1)^\gamma dx\,.$$

Then each function u_j of the sequence satisfies

$$K'[u_j] = I'_\lambda[u_j] + \lambda f(x, u_j) \tag{4.13}$$

in the sense of $W_0^{1,\gamma}(\Omega)^*$ and $I'_\lambda[u_j] \to 0$ in $W_0^{1,\gamma}(\Omega)^*$. Noting the assumption (A3), we have

$$f(x, u_j) \to f(x, v) \qquad \text{in } L^{q/(q-1)}(\Omega),$$

and hence in $W_0^{1,\gamma}(\Omega)^*$. By taking the limit $j \to \infty$ in (4.13), the right hand side converges to $\lambda f(x, v)$ in $W_0^{1,\gamma}(\Omega)^*$.

In general it is hopeless to obtain $K'[v] = \lambda f(x, v)$ because of the nonlinearity of $K'[v]$. But fortunately we can show this in our case owing to the convexity of K.

LEMMA 4.3. *For the limit function v of the sequence $\{u_j\}$ stated above, the equality $K'[v] = \lambda f(x, v)$ holds in the sense of $W_0^{1,\gamma}(\Omega)^*$.*

The proof of this lemma is given by using the monotonicity method of Minty and Browder. See e.g. Lions [13, Chap. 2] and Saaty [19, pp. 58–59].

Let us take $\lambda_* = \sup_{\rho > 0} \lambda_\rho$. Then, noting

$$K'[v] = -\gamma \operatorname{div}\left[\frac{(\sqrt{1 + |\nabla v|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla v|^2}} \nabla v\right],$$

we easily see that the limit function v given above, which we denote by v_λ from now on, is the required solution stated in Theorem 4.1.

Now we show the latter part in Theorem 4.1, namely $\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to \infty$ as $\lambda \to 0$. Recall

$$\lambda_\rho = \frac{(\sqrt{1 + \rho^2} - 1)^\gamma}{c_2(\rho + \rho^q)}$$

and $\lambda_\rho \to 0$ as $\rho \to 0$ and ∞, respectively, from the assumption $\gamma < q$. Hence λ_ρ attains its maximum λ_* at some point $\rho = \rho_0 > 0$. Let us take the sequence $\{u_j\}$ in Lemma 4.2 which converges to the solution v_λ. Then, for any λ with $0 < \lambda < \lambda_*$, taking $\rho = \rho_0$ in the definition of α_ρ and noting that $F(x, \xi)$ is nonnegative, we have

$$\begin{aligned}
\alpha_{\rho_0} &= (\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q) \\
&\le \mu_\lambda = \lim_{j\to\infty} I_\lambda[u_j] \\
&= \lim_{j\to\infty}\left\{\int_\Omega (\sqrt{1 + |\nabla u_j|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, u_j)dx\right\} \\
&\le \lim_{j\to\infty} \int_\Omega \frac{(\sqrt{1 + |\nabla u_j|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla u_j|^2}} |\nabla u_j|^2 dx\,.
\end{aligned} \tag{4.14}$$

Further, from (4.9) and the fact that $\|\zeta_j\|_{W_0^{1,\gamma}(\Omega)^*} \to 0$ as $j \to \infty$, the right hand in (4.14) is equal to

$$\frac{\lambda}{\gamma} \lim_{j\to\infty} \int_\Omega f(x, u_j)u_j dx = \frac{\lambda}{\gamma} \int_\Omega f(x, v_\lambda)v_\lambda dx . \tag{4.15}$$

Hence we have, from the assumption (A3),

$$(\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q) \le \frac{\lambda}{\gamma} \int_\Omega f(x, v_\lambda)v_\lambda dx$$
$$\le \frac{\lambda}{\gamma}\left\{c_{14} \int_\Omega |v_\lambda|^q dx + c_{15}\right\}$$

with some positive constants c_{14} and c_{15}. This implies that

$$\int_\Omega |v_\lambda|^q dx \ge \frac{\gamma(\lambda_* - \lambda)c_2(\rho_0 + \rho_0^q)}{\lambda c_{14}} - \frac{c_{15}}{c_{14}} \to \infty$$

as $\lambda \to 0$. Noting the Poincaré-Sobolev inequality, we complete the proof of Theorem 4.1.

In Theorem 3.3 we have shown the existence of global minimizer for any $\lambda > 0$ and given the asymptotic behavior of this minimizer as $\lambda \to \infty$ under the appropriate assumption on $f(x, \xi)$. Corresponding to this, we can give the existence and asymptotic behavior of unstable solution for large λ if $f(x, \xi)$ satisfies a certain behavior on ξ.

THEOREM 4.2. *Let the assumptions* (A1), (A2) *and* (A6) *be satisfied. Further let us assume*
(A3′) *there exist constants* p *and* q *with* $2\gamma < p \le q < \gamma^*$ *and the inequality*

$$f(x, \xi) \le d_6(\xi^{p-1} + \xi^{q-1})$$

holds on $\Omega \times [0, \infty)$ *with some constant* $d_6 > 0$.

Then there exists a nonzero and nonnegative critical point v_λ *for any* $\lambda > 0$.

Besides these assumptions, if (A6) *is valid for* $m \ge 2\gamma$ *with* $\xi_1 = 0$, *then* v_λ *satisfies*

$$\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to 0 \qquad \text{as } \lambda \to \infty .$$

PROOF. From the assumption (A3′), the inequality

$$I_\lambda[u] \ge (\sqrt{1 + \rho^2} - 1)^\gamma - c_{16}\lambda(\rho^p + \rho^q)$$

holds on $\|\nabla u\|_{L^\gamma(\Omega)} = \rho$ with some constant $c_{16} > 0$. Hence if we put

$$\lambda_\rho = \frac{(\sqrt{1 + \rho^2} - 1)^\gamma}{c_{16}(\rho^p + \rho^q)}$$

instead of (4.3), then we can find out a critical point v_λ of I_λ for any λ with $0 < \lambda < \sup_{\rho>0} \lambda_\rho$. The proof is just the same. Noting $\sup_{\rho>0} \lambda_\rho = \infty$, we see that there exists a critical point for any $\lambda > 0$. Next let us take the function ϕ stated in checking the hypothesis iii) in Lemma 4.1, namely a nonegative function $\phi \in C_0^\infty(\Omega)$ with $\phi(x) \geq 1$ on D where $F(x, \xi)$ satisfies the inequality (4.4) for any $\xi \geq 0$. Then we have

$$\begin{aligned} I_\lambda[r\phi] &= \int_\Omega (\sqrt{1 + r^2 |\nabla\phi|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, r\phi) dx \\ &\leq \left(\int_\Omega |\nabla\phi|^\gamma dx \right) r^\gamma - \lambda \int_\Omega F(x, r\phi) dx \\ &\equiv Ar^\gamma - \lambda H(r), \end{aligned} \tag{4.16}$$

where $A = \int_\Omega |\nabla\phi|^\gamma dx$ and $H(r) = \int_\Omega F(x, r\phi) dx$. It is easy to see that $H(r)$ is continuous in $r \geq 0$, positive on $r > 0$ and $H(0) = 0$. Further, as was seen in (4.5),

$$H(r) \geq Br^m - c_{10} \qquad \text{for } r \geq \xi_1 ,$$

where $B = c_9 \int_D \phi(x)^m dx > 0$. Since $m > \gamma$, there exists R (> 0) which is independent on λ such that

$$Ar^\gamma - \lambda H(r) \leq 0$$

for $\lambda \geq 1$ and $r \geq R$.

Take the function $R\phi$ as the w_0 stated in Lemma 4.1 as before, and consider the straight line from the origin to $R\phi$ among all paths which connect these two points. Then from Lemma 4.1, (4.16) and the positivity of $H(r)$ for $r > 0$,

$$\begin{aligned} 0 \leq \mu_\lambda &\leq \max_{0 \leq r \leq R} I_\lambda[r\phi] \\ &\leq \max_{0 \leq r \leq R} [Ar^\gamma - \lambda H(r)] \to 0 \qquad \text{as } \lambda \to \infty , \end{aligned}$$

where μ_λ denotes the μ stated in Lemma 4.1.

Next let $\{u_j\}$ be the sequence stated in Lemma 4.2. Namely,

$$\begin{aligned} &I_\lambda[u_j] \to \mu_\lambda , \\ &u_j \to v_\lambda \qquad \text{weakly in } W_0^{1,\gamma}(\Omega) \text{ and strongly in } L^q(\Omega) . \end{aligned}$$

Since I_λ is weakly lower semicontinuous in $W_0^{1,\gamma}(\Omega)$, the inequality

$$\mu_\lambda \geq I_\lambda \equiv \int_\Omega (\sqrt{1 + |\nabla u_\lambda|^2} - 1)^\gamma dx - \lambda \int_\Omega F(x, v_\lambda) dx \tag{4.17}$$

holds. From the assumption (A6) with $\xi_1 = 0$, we have

$$\int_\Omega F(x, v_\lambda)dx \le \frac{1}{m}\int_\Omega f(x, v_\lambda)v_\lambda dx\,. \tag{4.18}$$

Further, since $v_\lambda \in W_0^{1,\gamma}(\Omega)$ is a weak solution of (1.1), the equality

$$\gamma\int_\Omega \frac{(\sqrt{1+|\nabla v_\lambda|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla v_\lambda|^2}}|\nabla u_\lambda|^2 dx = \lambda\int_\Omega f(x, v_\lambda)v_\lambda dx \tag{4.19}$$

holds. From (4.17), (4.18) and (4.19),

$$\begin{aligned}\mu_\lambda &\ge \int_\Omega (\sqrt{1+|\nabla v_\lambda|^2}-1)^\gamma dx - \frac{\lambda}{m}\int_\Omega f(x, v_\lambda)v_\lambda dx \\ &= \int_\Omega \left[(\sqrt{1+|\nabla v_\lambda|^2}-1)^\gamma - \frac{\gamma}{m}\frac{(\sqrt{1+|\nabla v_\lambda|^2}-1)^{\gamma-1}}{\sqrt{1+|\nabla v_\lambda|^2}}|\nabla v_\lambda|^2\right]dx\,. \end{aligned}\tag{4.20}$$

Now we put

$$\varphi(X) = (\sqrt{1+X^{2/\gamma}}-1)^\gamma - \frac{\gamma}{m}\frac{(\sqrt{1+X^{2/\gamma}}-1)^{\gamma-1}}{\sqrt{1+X^{2/\gamma}}}X^{2/\gamma}\,.$$

Noting $m \ge 2\gamma$, we easily see that

$$\varphi(X) \sim \begin{cases} \dfrac{1}{2\gamma}\left(1-\dfrac{2\gamma}{m}\right)X^2 & (m > 2\gamma) \\[2ex] \dfrac{1}{2^{\gamma+2}}X^{2+2/\gamma} & (m = 2\gamma) \end{cases}$$

as $X \to 0$ and

$$\varphi(X) \sim \left(1-\frac{\gamma}{m}\right)X \qquad \text{as } X \to \infty\,.$$

Further $\varphi(X) > 0$ for $X > 0$. Thus there exists a monotone increasing convex function $g(X)$ on $[0, \infty)$ such that $\varphi(X) \ge g(X) > 0$ for $X > 0$ with $g(0) = 0$. For example, we may put

$$g(X) = \begin{cases} \varepsilon X^{2+2/\gamma} & (0 \le X < 1) \\[2ex] \varepsilon\left(2+\dfrac{2}{\gamma}\right)X - \varepsilon\left(1+\dfrac{2}{\gamma}\right) & (X \ge 1) \end{cases}$$

with small $\varepsilon > 0$. From (4.20) and by using Jensen's inequality, we have

$$\mu_\lambda \geq \int_\Omega \varphi(|\nabla v_\lambda|^\gamma)dx \geq \int_\Omega g(|\nabla v_\lambda|^\gamma)dx$$

$$\geq g\left(\int_\Omega |\nabla v_\lambda|^\gamma dx\right) \geq 0 . \tag{4.21}$$

Letting $\lambda \to \infty$ in (4.21) and noting that $\mu_\lambda \to 0$ as $\lambda \to \infty$, we see that

$$g\left(\int_\Omega |\nabla v_\lambda|^\gamma dx\right) \to 0 \qquad \text{as } \lambda \to \infty .$$

Since g is monotone increasing and $g(0) = 0$, we have

$$\int_\Omega |\nabla v_\lambda|^\gamma dx \to 0 \qquad \text{as } \lambda \to \infty .$$

5. For the case when $f(x, u) = qu^{q-1}$

To illustrate our result presented in the preceding section, we consider the boundary value problem

$$-\gamma \operatorname{div}\left[\frac{(\sqrt{1 + |\nabla u|^2} - 1)^{\gamma-1}}{\sqrt{1 + |\nabla u|^2}}\nabla u\right] = \lambda q u^{q-1} \qquad \text{in } \Omega \tag{5.1}$$

$$u \geq 0 \qquad \text{in } \Omega \tag{5.2}$$

$$u = 0 \qquad \text{on } \partial\Omega , \tag{5.3}$$

where $1 < q < \gamma^*$. In this case $f(x, u) = qu_+^{q-1}$ and $F(x, u) = u_+^q$, for which solutions of (5.1), (5.2), (5.3) correspond to critical points of the functional

$$I_\lambda[u] = \int_\Omega (\sqrt{1 + |\nabla u|^2} - 1)^\gamma dx - \lambda \int_\Omega u_+^q dx \tag{5.4}$$

defined on the Sobolev space $W_0^{1,\gamma}(\Omega)$, where $u_+ = \max\{u, 0\}$. Then assumptions (A1), (A2), (A3), (A5) are satisfied. Further assumptions (A4), (A6) and (A3′) hold for $1 < q < 2\gamma$, $\gamma < q$ and $2\gamma < q < \gamma^*$ respectively.

Thus, it follows that

(i) *When* $1 < q < \gamma$, *there exist nontrivial weak solutions* $\{u_\lambda\}$, $\lambda > 0$, *such that*

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)} \to 0 \qquad \text{as } \lambda \to 0 , \tag{5.5}$$

and

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)} \to \infty \qquad \text{as } \lambda \to \infty . \tag{5.6}$$

(ii) *When* $\gamma < q < 2\gamma$, *there exist nontrivial weak solutions* $\{u_\lambda\}$, $0 < \lambda < \lambda^*$, *and* $\{v_\lambda\}$, $0 < \lambda < \lambda_*$, *such that*

$$\|\nabla u_\lambda\|_{L^\gamma(\Omega)} \to 0 \qquad as\ \lambda \to 0\,, \tag{5.7}$$

and

$$\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to \infty \qquad as\ \lambda \to 0\,, \tag{5.8}$$

respectively.

(iii) *When* $2\gamma < q < \gamma^*$, *there exist nontrivial weak solutions* $\{v_\lambda\}$, $\lambda > 0$, *such that*

$$\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to \infty \qquad as\ \lambda \to 0\,, \tag{5.9}$$

and

$$\|\nabla v_\lambda\|_{L^\gamma(\Omega)} \to 0 \qquad as\ \lambda \to \infty\,. \tag{5.10}$$

On the other hand, to mention the non-existence of nontrivial solutions in problem (5.1)–(5.3) for $q > 2n\gamma/(n - 2\gamma)$ and $n > 2\gamma$, we recall here a general variational identity obtained by Pucci and Serrin [17] corresponding to critical points of the functional

$$I[u] = \int_\Omega \mathscr{F}(x, u, \nabla u)dx \tag{5.11}$$

where $x = (x_1, \cdots, x_n)$, $u = u(x_1, \cdots, x_n)$ and $\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$. We consider integrands $\mathscr{F} = \mathscr{F}(x, u, p)$, $p = (p_1, \cdots, p_n)$, which are of class C^1 on the domain $\bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n$, and the vector function

$$\mathscr{F}_p(x, u, p) = \left(\frac{\partial \mathscr{F}}{\partial p_1}, \cdots, \frac{\partial \mathscr{F}}{\partial p_n}\right)(x, u, p) \tag{5.12}$$

is of class C^1 on $\bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n$. Critical points of (5.11), which are of class $C^2(\Omega)$, satisfy the Euler equation

$$\operatorname{div}\{\mathscr{F}_p(x, u, \nabla u)\} = \mathscr{F}_u(x, u, \nabla u)\,, \qquad x \in \Omega\,, \tag{5.13}$$

where $\mathscr{F}_u = \frac{\partial \mathscr{F}}{\partial u}$.

LEMMA 5.1. *Let* $u = u(x)$ *be of class* $C^2(\Omega)$ *and* $p_i = \frac{\partial u}{\partial x_i}$, $i = 1, \cdots, n$. *Then, the following identity holds in* Ω:

$$- \operatorname{div} [\{(x \cdot p) + au\}\mathscr{F}_p(x, u, p) - x\mathscr{F}(x, u, p)]$$

$$= n\mathscr{F}(x, u, p) + x \cdot \mathscr{F}_x(x, u, p) - (1 + a)p \cdot \mathscr{F}_p(x, u, p)$$

$$- au \operatorname{div} \{\mathscr{F}_p(x, u, p)\} - (x \cdot p)(\operatorname{div} \{\mathscr{F}_p(x, u, p)\} - \mathscr{F}_u(x, u, p)), \quad (5.14)$$

where a is an arbitrary constant and $\mathscr{F}_x = \left(\frac{\partial \mathscr{F}}{\partial x_1}, \cdots, \frac{\partial \mathscr{F}}{\partial x_n}\right)$.

PROOF. It is easy to see that

$$\operatorname{div} \{(x \cdot p)\mathscr{F}_p(x, u, p)\} = \sum_i \frac{\partial}{\partial x_i} \left\{\sum_j x_j p_j \frac{\partial \mathscr{F}}{\partial p_i}(x, u, p)\right\}$$

$$= p \cdot \mathscr{F}_p(x, u, p) + \sum_{i,j} x_j \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial \mathscr{F}}{\partial p_i}(x, u, p) + (x \cdot p) \operatorname{div} \{\mathscr{F}_p(x, u, p)\}, \quad (5.15)$$

$$\operatorname{div} \{u\mathscr{F}_p(x, u, p)\} = \sum_i \frac{\partial}{\partial x_i} \left\{u \frac{\partial \mathscr{F}}{\partial p_i}(x, u, p)\right\}$$

$$= p \cdot \mathscr{F}_p(x, u, p) + u \operatorname{div} \{\mathscr{F}_p(x, u, p)\}, \quad (5.16)$$

$$\operatorname{div} \{x\mathscr{F}(x, u, p)\} = \sum_j \frac{\partial}{\partial x_j} \{x_j \mathscr{F}(x, u, p)\}$$

$$= n\mathscr{F}(x, u, p) + x \cdot \mathscr{F}_x(x, u, p) + (x \cdot p)\mathscr{F}_u(x, u, p)$$

$$+ \sum_{i,j} x_j \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial \mathscr{F}}{\partial p_i}(x, u, p), \quad (5.17)$$

and, subtracting (5.15) and a times (5.16) from (5.17) implies (5.14).

LEMMA 5.2. *Let $u \in C^2(\Omega)$ be a solution of* (5.13) *with $u = 0$ on $\partial\Omega$ and a be an arbitrary constant. Let*

$$P(x, p) = p \cdot \mathscr{F}_p(x, 0, p) - \mathscr{F}(x, 0, p) \quad (5.18)$$

$$Q(x, u, p) = n\mathscr{F}(x, u, p) + x \cdot \mathscr{F}_x(x, u, p)$$

$$- (1 + a)p \cdot \mathscr{F}_p(x, u, p) - au\mathscr{F}_u(x, u, p) \quad (5.19)$$

then the identity

$$- \int_{\partial\Omega} P(x, \nabla u)(x \cdot \nu) ds = \int_{\Omega} Q(x, u, \nabla u) dx \quad (5.20)$$

holds, where ν is the outer normal vector on $\partial\Omega$.

PROOF. Since u satisfies (5.13) on Ω and $\nabla u = \frac{\partial u}{\partial \nu} \nu$ on $\partial\Omega$,

$$(x \cdot \nabla u)(\mathscr{F}_p(x, u, \nabla u) \cdot \nu) = (x \cdot \nu)(\mathscr{F}_p(x, u, \nabla u) \cdot \nabla u) \qquad \text{on } \partial\Omega . \tag{5.21}$$

The identities (5.13), (5.14), (5.21) and the divergence theorem imply

$$\begin{aligned} -\int_{\partial\Omega} (x \cdot \nu)\{(\mathscr{F}_p(x, u, \nabla u) \cdot \nabla u) - \mathscr{F}(x, u, \nabla u)\} ds \\ = \int_{\Omega} \{n\mathscr{F}(x, u, \nabla u) + x \cdot \mathscr{F}_x(x, u, \nabla u) \\ - (1 + a)\nabla u \cdot \mathscr{F}_p(x, u, \nabla u) - au\mathscr{F}_u(x, u, \nabla u)\} dx . \end{aligned} \tag{5.22}$$

This shows (5.20).

LEMMA 5.3 (Pucci and Serrin [17, Theorem 1]). *Assume Ω is bounded and star-shaped with respect to the origin. Suppose also that*

$$P(x, p) \geq 0 \qquad \textit{for all } (x, p) \in \partial\Omega \times \boldsymbol{R}^n , \tag{5.23}$$

and that there exists a real number a such that

$$Q(x, u, p) \geq 0 \qquad \textit{for all } (x, u, p) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n , \tag{5.24}$$

where P and Q are defined by (5.18) *and* (5.19), *respectively. Assume finally that either $u = 0$ or $p = 0$ whenever the equality in* (5.24) *holds. Then the variational equation* (5.13) *has no nontrivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ which vanishes on $\partial\Omega$.*

PROOF. Since $x \cdot \nu \geq 0$ on $\partial\Omega$, (5.20), (5.23), (5.24) imply $Q(x, u, p) = 0$ and hence $u = 0$ or $p = 0$ in Ω. It follows that $u \equiv 0$ in Ω.

Now we apply Lemma 5.3 to the problem (5.1)–(5.3). In this case,

$$\mathscr{F}(x, u, p) = (\sqrt{1 + |p|^2} - 1)^\gamma - \lambda u^q ,$$

for which the relation

$$\frac{1}{2} \frac{|p|^2}{\sqrt{1 + |p|^2}} \leq \sqrt{1 + |p|^2} - 1 \leq \frac{|p|^2}{\sqrt{1 + |p|^2}}$$

implies that

$$\begin{aligned} P(x, p) &= p \cdot \mathscr{F}_p(x, 0, p) - \mathscr{F}(x, 0, p) \\ &= \gamma(\sqrt{1 + |p|^2} - 1)^{\gamma-1} \frac{|p|^2}{\sqrt{1 + |p|^2}} - (\sqrt{1 + |p|^2} - 1)^\gamma \\ &\geq (\gamma - 1)(\sqrt{1 + |p|^2} - 1)^{\gamma-1} \geq 0 , \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} Q(x, u, p) &= n\mathscr{F}(x, u, p) + x\cdot\mathscr{F}_x(x, u, p) \\ &\quad - (1+a)p\cdot\mathscr{F}_p(x, u, p) - au\mathscr{F}_u(x, u, p) \\ &= n\{(\sqrt{1+|p|^2}-1)^\gamma - \lambda u^q\} \\ &\quad - (1+a)\gamma(\sqrt{1+|p|^2}-1)^{\gamma-1}\frac{|p|^2}{\sqrt{1+|p|^2}} + \lambda aqu^q \\ &\geq (n-2(1+a)\gamma)(\sqrt{1+|p|^2}-1)^\gamma + \lambda(aq-n)u^q\,. \end{aligned} \tag{5.26}$$

Taking $a = n/(2\gamma) - 1$ the assumption of Lemma 5.3 is satisfied for $a > 0$ and $q > n/a$. Hence, there exist no nontrivial solutions for $n > 2\gamma$ and $q > 2n\gamma/(n-2\gamma)$.

We have no result about our problem concerning the existence of solutions in the case $\gamma^* \leq q \leq 2n\gamma/(n-2\gamma)$.

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