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On a class of second order quasilinear ordinary differential equations

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Introduction

We consider second order quasilinear ordinary differential equations of the form

$$(|y'|^{\alpha-1}y')' + q(t)|y|^{\beta-1}y = 0, \tag{A}$$

where α and β are positive constants, and $q: [a, \infty) \to \mathbb{R}$ is a continuous function. Note that (A) can be written as

$$(|y'|^{\alpha}\operatorname{sgn} y')' + q(t)|y|^{\beta}\operatorname{sgn} y = 0.$$

The purpose of this paper is twofold. First, we discuss the question of global existence and uniqueness of solutions of (A) subject to the initial condition

$$y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{B}$$

By a solution of (A) on an interval $I \subset [a, \infty)$ we mean a function $y \in C^1(I)$ which has the property $|y'| \in C^1(I)$ and satisfies the equation at all points $t \in I$. A solution is said to be global if it exists on the whole interval $[a, \infty)$. It will be shown in particular that the initial value problem (A)–(B) has a unique global solution for any given values of y_0 and y_1 provided $q(t)$ is positive and locally of bounded variation on $[a, \infty)$. Secondly, we investigate the oscillatory (and nonoscillatory) behavior of solutions of (A) which are defined in a neighborhood of infinity. Such a solution is said to be oscillatory if it has an infinite sequence of zeros clustering at infinity; otherwise it is said to be nonoscillatory. Thus a nonoscillatory solution must be eventually positive or eventually negative.

Oscillation theory of equations of the type (A) was first developed by Mirzov [10–13] and Elbert [3, 4]. A considerable amount of addition to their theory has been given in the recent papers [2, 5–8]. It has thus turned out that the oscillatory character of (A) is to a large extent in common with that of the Emden-Fowler type equation

$$y'' + q(t)|y|^{\beta-1}y = 0 \tag{C}$$

to which (A) reduces when $\alpha = 1$. Here we concentrate on the case where the function $q(t)$ in (A) may change its sign infinitely often on $[a, \infty)$, since the equation (A) with positive $q(t)$ has been studied in great detail in the above-mentioned papers. Our aim is to answer (at least partially) the question whether the delicate oscillation results of Butler [1] and Naito [14, 15] for (C) with oscillating $q(t)$ can be generalized to (A) with $\alpha \neq 1$.

Part 1. Initial value problem

1. Local existence and uniqueness of solutions

The objective of this paper is to study the question of existence and uniqueness of global solutions to the initial value problem (A)–(B). For simplicity we introduce the notation

$$\xi^{\alpha *} = |\xi|^{\alpha-1}\xi = |\xi|^{\alpha} \operatorname{sgn} \xi, \qquad \xi \in \mathbb{R},\ \alpha > 0, \tag{1.1}$$

in terms of which the equation (A) can be written as

$$((y')^{\alpha *})' + q(t) y^{\beta *} = 0.$$

We first note that the problem (A)–(B) has a local solution for any values of y_0 and y_1. This follows from the Peano theorem applied to the two-dimensional initial value problem

$$y' = z^{\frac{1}{\alpha}*}, \qquad z' = -q(t) y^{\beta *}, \tag{A'}$$

$$y(t_0) = y_0, \quad z(t_0) = y_1^{\alpha *}, \tag{B'}$$

which is equivalent to the original problem (A)–(B). Since the vector function $(f(t, y, z), g(t, y, z)) = (z^{\frac{1}{\alpha}*}, -q(t)y^{\beta *})$ satisfies a local Lipschitz condition on the set D:

$$D = [a, \infty) \times \mathbb{R} \times \mathbb{R} \text{ if } \alpha \leqq 1 \text{ and } \beta \geqq 1,$$

$$D = [a, \infty) \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \text{ if } \alpha \leqq 1 \text{ and } \beta < 1,$$

$$D = [a, \infty) \times \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \text{ if } \alpha > 1 \text{ and } \beta \geqq 1,$$

$$D = [a, \infty) \times (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \text{ if } \alpha > 1 \text{ and } \beta < 1,$$

from the Picard theorem it follows that the local solution of (A)–(B) is unique

for any $y_0 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$ if $\alpha \leqq 1$ and $\beta \geqq 1$;

for any $y_0 \in \mathbb{R} \setminus \{0\}$ and $y_1 \in \mathbb{R}$ if $\alpha \leqq 1$ and $\beta < 1$;

for any $y_0 \in \mathbb{R}$ and $y_1 \in \mathbb{R} \setminus \{0\}$ if $\alpha > 1$ and $\beta \geqq 1$; and for any $y_0 \in \mathbb{R} \setminus \{0\}$ and $y_1 \in \mathbb{R} \setminus \{0\}$ if $\alpha > 1$ and $\beta < 1$.

In order to complete the discussion of uniqueness we have to examine the following two cases: (i) $\beta < 1$ and $y_0 = 0$; and (ii) $\alpha > 1$, $\beta \geqq 1$ and $y_1 = 0$

THEOREM 1.1. *Let $\beta < 1$. The solution of the initial value problem*

$$(|y'|^{\alpha-1}y')' + q(t)|y|^{\beta-1}y = 0, \tag{A}$$

$$y(t_0) = 0, \ y'(t_0) = y_1 \neq 0 \tag{B$_1$}$$

is unique in a small neighborhood of t_0.

PROOF. Let $y_1(t)$ and $y_2(t)$ be two local solutions of (A)–(B$_1$). Integrating (A) with $y = y_i$ twice from t_0 to $t \in \operatorname{dom}(y_1) \cap \operatorname{dom}(y_2)$, we have

$$y_i'(t) = \left(y_1^{\alpha*} - \int_{t_0}^{t} q(s)(y_i(s))^{\beta*} ds\right)^{\frac{1}{\alpha}*},$$

and

$$y_i(t) = \int_{t_0}^{t} \left(y_1^{\alpha*} - \int_{t_0}^{s} q(r)(y_i(r))^{\beta*} dr\right)^{\frac{1}{\alpha}*} ds, \quad i = 1, 2,$$

which implies

$$y_1(t) - y_2(t) = \int_{t_0}^{t} [(y_1^{\alpha*} - I_1(s))^{\frac{1}{\alpha}*} - (y_1^{\alpha*} - I_2(s))^{\frac{1}{\alpha}*}] ds, \quad i = 1, 2,$$

where

$$I_i(t) = \int_{t_0}^{t} q(s)(y_i(s))^{\beta*} ds, \qquad i = 1, 2.$$

By the mean value theorem we then have

$$y_1(t) - y_2(t) = \frac{1}{\alpha}\int_{t_0}^{t} |\eta(s)|^{\frac{1-\alpha}{\alpha}}(I_2(s) - I_1(s))\, ds, \tag{1.2}$$

where $\eta(t)$ lies between $y_1^{\alpha*} - I_1(t)$ and $y_1^{\alpha*} - I_2(t)$.

Let $t \geqq t_0$. Since $y_1^* - I_i(t) \to y_1^* \neq 0$ as $t \to t_0$, there is a $\delta > 0$ such that

$$\frac{1}{2}|y_1|^{\alpha} \leqq |\eta(t)| \leqq \frac{3}{2}|y_1|^{\alpha} \qquad \text{for } t \in [t_0, t_0 + \delta]. \tag{1.3}$$

Combining (1.2) with (1.3) and using the notation

$$M(y_1) = \frac{3}{2}|y_1|^\alpha \quad \text{for} \quad \alpha \leqq 1, \qquad M(y_1) = \frac{1}{2}|y_1|^\alpha \quad \text{for} \quad \alpha > 1,$$

we obtain for $t \in [t_0, t_0 + \delta]$

$$\begin{aligned} |y_1(t) - y_2(t)| &\leqq \frac{1}{\alpha} M(y_1)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^t |I_1(s) - I_2(s)|\, ds \\ &\leqq \frac{1}{\alpha} M(y_1)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^t \left(\int_{t_0}^s |q(r)|\, |(y_1(r))^{\beta*} - (y_2(r))^{\beta*}|\, dr \right) ds \qquad (1.4) \\ &= \frac{1}{\alpha} M(y_1)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^t (t-s)\, |q(s)|\, |(y_1(s))^{\beta*} - (y_2(s))^{\beta*}|\, ds. \end{aligned}$$

Define the continuous functions $z_i(t)$ by

$$z_i(t) = \frac{y_i(t)}{t - t_0} \quad \text{for} \quad t \neq t_0, \; z_i(t_0) = y_1, \; i = 1, 2.$$

Then we see from (1.4) that

$$\begin{aligned} |z_1(t) - z_2(t)| &\leqq \frac{1}{\alpha} M(y_1)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^t \frac{t-s}{t-t_0} (s - t_0)^\beta |q(s)|\, |(z_1(s))^{\beta*} - (z_2(s))^{\beta*}|\, ds \\ &\leqq \frac{1}{\alpha} M(y_1)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^t (s - t_0)^\beta |q(s)|\, |(z_1(s))^{\beta*} - (z_2(s))^{\beta*}|\, ds \qquad (1.5) \end{aligned}$$

for $t \in [t_0, t_0 + \delta]$. We now observe that a constant $\delta' < \delta$ can be chosen so that

$$|(z_1(t))^{\beta*} - (z_2(t))^{\beta*}| \leqq \beta \left(\frac{|y_1|}{2} \right)^{\beta - 1} |z_1(t) - z_2(t)| \qquad (1.6)$$

for $t \in [t_0, t_0 + \delta']$. This follows from the relation

$$|(z_1(t))^{\beta*} - (z_2(t))^{\beta*}| \leqq \beta |\zeta(t)|^{\beta-1} |z_1(t) - z_2(t)|,$$

$\zeta(t)$ being a number lying between $z_1(t)$ and $z_2(t)$, and the fact that, since $z_i(t) \to y_1 \neq 0$ as $t \to t_0$, $i = 1, 2$, $\zeta(t)$ can be made to satisfy $|y_1|/2 \leqq |\zeta(t)| \leqq 3|y_1|/2$ if t is taken sufficiently close to t_0. It follows from (1.5) and (1.6) that

$$|z_1(t) - z_2(t)| \leqq \frac{\beta}{\alpha} M(y_1)^{\frac{1-\alpha}{\alpha}} \left(\frac{|y_1|}{2} \right)^{\beta - 1} \int_{t_0}^t (s - t_0)^\beta |q(s)|\, |z_1(s) - z_2(s)|\, ds$$

for $t \in [t_0, t_0 + \delta']$. Applying Gronwall's lemma, we conclude that $z_1(t) \equiv z_2(t)$ on $[t_0, t_0 + \delta']$, which clearly implies that $y_1(t) \equiv y_2(t)$ on $[t_0, t_0 + \delta']$. That $y_1(t)$ and $y_2(t)$ coincide in a small left neighborhood of t_0 can be verified in a similar fashion. This completes the proof.

THEOREM 1.2. *Let $\alpha > 1$ and $\beta \geqq 1$. If $q(t_0) \neq 0$, then the solution of the initial value problem*

$$(|y'|^{\alpha-1}y')' + q(t)|y|^{\beta-1}y = 0, \tag{A}$$

$$y(t_0) = y_0 \neq 0, \ y'(t_0) = 0 \tag{B$_2$}$$

is unique in a small neighborhood of t_0.

PROOF. Let $y_1(t)$ and $y_2(t)$ be two solutions of (A)–(B_2). Integration of (A) with $y = y_i$ gives

$$y_i(t) = y_0 + \int_{t_0}^{t} \left(- \int_{t_0}^{s} q(r)(y_i(r))^{\beta *} dr \right)^{\frac{1}{\alpha}*} ds, \qquad i = 1, 2.$$

Defining

$$J_i(t) = \frac{-1}{t - t_0} \int_{t_0}^{t} q(s)(y_i(s))^{\beta *} ds \quad \text{for} \quad t \neq t_0, \ J_i(t_0) = -q(t_0)y_0^{\beta *}, \qquad i = 1, 2,$$

we have for $t \in \operatorname{dom}(y_1) \cap \operatorname{dom}(y_2)$

$$y_1(t) - y_2(t) = \int_{t_0}^{t} (s - t_0)^{\frac{1}{\alpha}*} [(J_1(s))^{\frac{1}{\alpha}*} - (J_2(s))^{\frac{1}{\alpha}*}] ds,$$

which implies

$$y_1(t) - y_2(t) = \frac{1}{\alpha} \int_{t_0}^{t} (s - t_0)^{\frac{1}{\alpha}*} |\eta(s)|^{\frac{1-\alpha}{\alpha}} (J_1(s) - J_2(s)) ds, \tag{1.7}$$

where $\eta(t)$ is a number lying between $J_1(t)$ and $J_2(t)$. Let $t \geqq t_0$ and choose $\delta > 0$ such that

$$\frac{1}{2}|q(t_0)||y_0|^{\beta} \leqq |\eta(t)| \leqq \frac{3}{2}|q(t_0)||y_0|^{\beta} \qquad \text{for } t \in [t_0, t_0 + \delta], \tag{1.8}$$

which is possible because of the fact that $J_i(t) \to -q(t_0)y_0^{\beta *} \neq 0$ as $t \to t_0$, $i = 1, 2$. From (1.7) and (1.8) we see that

$$|y_1(t) - y_2(t)| \leqq \frac{1}{\alpha} K(y_0)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^{t} (s - t_0)^{\frac{1}{\alpha}} |J_1(s) - J_2(s)| ds$$

$$
\begin{aligned}
&\leqq \frac{1}{\alpha} K(y_0)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^{t} (s-t_0)^{\frac{1-\alpha}{\alpha}} \left(\int_{t_0}^{s} |q(r)|\,|(y_1(r))^{\beta*} - (y_2(r))^{\beta*}|\,dr \right) ds \\
(1.9) \qquad &\leqq \frac{1}{\alpha} K(y_0)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^{t} \left(\int_{t_0}^{s} (r-t_0)^{\frac{1-\alpha}{\alpha}} |q(r)|\,|(y_1(r))^{\beta*} - (y_2(r))^{\beta*}|\,dr \right) ds \\
&\leqq \frac{1}{\alpha} K(y_0)^{\frac{1-\alpha}{\alpha}} \int_{t_0}^{t} (t-s)(s-t_0)^{\frac{1-\alpha}{\alpha}} |q(s)|\,|(y_1(s))^{\beta*} - (y_2(s))^{\beta*}|\,ds
\end{aligned}
$$

for $t \in [t_0, t_0 + \delta]$, where $K(t_0) = |q(t_0)|\,|y_0|^{\beta}/2$. In view of the fact that $y_i(t) \to y_0 \neq 0$ as $t \to t_0$, $i = 1, 2$, we can choose, by the mean value theorem, a $\delta' < \delta$ so that

$$
|(y_1(t))^{\beta*} - (y_2(t))^{\beta*}| \leqq \beta \left(\frac{3}{2} |y_0| \right)^{\beta-1} |y_1(t) - y_2(t)|, \qquad t \in [t_0, t_0 + \delta'].
$$

This inequality combined with (1.9) gives

$$
\begin{aligned}
&|y_1(t) - y_2(t)| \\
&\quad \leqq \frac{\beta}{\alpha} K(y_0)^{\frac{1-\alpha}{\alpha}} \left(\frac{3}{2} |y_0| \right)^{\beta-1} \int_{t_0}^{t} (t-s)(s-t_0)^{\frac{1-\alpha}{\alpha}} |q(s)|\,|y_1(s) - y_2(s)|\,ds
\end{aligned}
$$

for $t \in [t_0, t_0 + \delta']$, from which it follows that

$$
v(t) \leqq \frac{\beta}{\alpha} K(y_0)^{\frac{1-\alpha}{\alpha}} \left(\frac{3}{2} |y_0| \right)^{\beta-1} \int_{t_0}^{t} (t-s)(s-t_0)^{\frac{1-\alpha}{\alpha}} |q(s)|\,ds \cdot v(t)
$$

where $v(t) = \max\{|y_1(s) - y_2(s)| : t_0 \leqq s \leqq t\}$, $t \in [t_0, t_0 + \delta']$. This gives a contradiction in the limit as $t \to t_0$ unless $v(t) \equiv 0$ on $[t_0, t_0 + \delta']$. Similarly it can be shown that $v(t) \equiv 0$ on an interval of the form $[t_0 - \delta', t_0,]$. This completes the proof.

It remains to deal with the case where $\beta < 1$ and $y_0 = y_1 = 0$. The question is therefore whether zero is the only solution of (A) with $\beta < 1$ that satisfies $y(t_0) = y'(t_0) = 0$. An answer to this question is the following.

THEOREM 1.3. *Let $\beta < 1$. Suppose that $q(t)$ is positive and locally of bounded variation on $[a, \infty)$. Then $y(t) \equiv 0$ is the only solution of the initial value problem*

$$
\text{(A)} \qquad (|y'|^{\alpha-1} y')' + q(t)|y|^{\beta-1} y = 0,
$$

$$
(\mathrm{B}_3) \qquad y(t_0) = 0, \; y'(t_0) = 0.
$$

The proof of this theorem requires an approach different from those of the foregoing theorems. The details will be given in the next section.

2. Global existence and uniqueness of solutions

2.1. The case where $q(t)$ is of class $C^1[a, \infty)$

For a solution $y(t)$ of (A) defined on an interval $I \subset [a, \infty)$ we define the functions $V[y](t)$ and $W[y](t)$ by

$$V[y](t) = \frac{\alpha}{\alpha+1}|y'(t)|^{\alpha+1} + \frac{q(t)}{\beta+1}|y(t)|^{\beta+1}, \qquad t \in I, \tag{2.1}$$

and

$$W[y](t) = \frac{\alpha}{\alpha+1}\frac{|y'(t)|^{\alpha+1}}{q(t)} + \frac{1}{\beta+1}|y(t)|^{\beta+1}, \qquad t \in I. \tag{2.2}$$

A simple computation shows that

$$\frac{d}{dt}V[y](t) = \frac{q'(t)}{\beta+1}|y(t)|^{\beta+1}, \qquad t \in I,$$

and

$$\frac{d}{dt}W[y](t) = -\frac{\alpha}{\alpha+1}\frac{q'(t)}{(q(t))^2}|y'(t)|^{\alpha+1}, \qquad t \in I.$$

It follows that

$$-\frac{q'_-(t)}{q(t)}V[y](t) \leqq \frac{d}{dt}V[y](t) \leqq \frac{q'_+(t)}{q(t)}V[y](t), \qquad t \in I, \tag{2.3}$$

and

$$-\frac{q'_+(t)}{q(t)}W[y](t) \leqq \frac{d}{dt}W[y](t) \leqq \frac{q'_-(t)}{q(t)}W[y](t), \qquad t \in I, \tag{2.4}$$

where $q'_+(t) = \max\{q'(t), 0\}$ and $q'_-(t) = \max\{-q'(t), 0\}$.

Integrating the first order linear differential inequalities (2.3) and (2.4), we have the following basic inequalities for $V[y](t)$ and $W[y](t)$.

LEMMA 2.1. *Suppose that $q \in C^1[a, \infty)$ and $q(t) > 0$ on $[a, \infty)$. Let $y(t)$ be a solution of* (A) *on I. Then we have*

(i) $$V[y](t) \leqq V[y](t_0) \exp\left(\int_{t_0}^{t} \frac{q'_+(s)}{q(s)} ds\right), \qquad t \geqq t_0,\ t \in I;$$

(ii) $$V[y](t) \geqq V[y](t_0) \exp\left(-\int_{t_0}^{t} \frac{q'_-(s)}{q(s)} ds\right), \qquad t \geqq t_0,\ t \in I;$$

(iii) $$V[y](t) \leqq V[y](t_0) \exp\left(-\int_{t_0}^{t} \frac{q'_-(s)}{q(s)} ds\right), \qquad t \leqq t_0,\ t \in I;$$

(iv) $$V[y](t) \geqq V[y](t_0) \exp\left(\int_{t_0}^{t} \frac{q'_+(s)}{q(s)} ds\right), \qquad t \leqq t_0,\ t \in I.$$

LEMMA 2.2. *Suppose that* $q \in C^1[a, \infty)$ *and* $q(t) > 0$ *on* $[a, \infty)$. *Let* $y(t)$ *be a solution of* (A) *on* I. *Then we have*

(i) $$W[y](t) \leqq W[y](t_0) \exp\left(\int_{t_0}^{t} \frac{q'_-(s)}{q(s)} ds\right), \qquad t \geqq t_0,\ t \in I;$$

(ii) $$W[y](t) \geqq W[y](t_0) \exp\left(-\int_{t_0}^{t} \frac{q'_+(s)}{q(s)} ds\right), \qquad t \geqq t_0,\ t \in I;$$

(iii) $$W[y](t) \leqq W[y](t_0) \exp\left(-\int_{t_0}^{t} \frac{q'_+(s)}{q(s)} ds\right), \qquad t \leqq t_0,\ t \in I;$$

(iv) $$W[y](t) \geqq W[y](t_0) \exp\left(\int_{t_0}^{t} \frac{q'_-(s)}{q(s)} ds\right), \qquad t \leqq t_0,\ t \in I.$$

From Lemmas 2.1 and 2.2 it readily follows that zero is the only solution of the problem (A)–(B_3), that is, the conclusion of Theorem 1.3 holds (for any values of α and β) if $q(t)$ is supposed to be of class C^1 and positive on $[a, \infty)$. Another important implication of the above lemmas is the global existence of solutions of (A). In fact, a standard argument shows that, for any given y_0 and y_1, a local solution of the initial value problem (A)–(B) can b continued to the entire interval $[a, \infty)$ provided $q(t)$ is of class C^1 and positive on $[a, \infty)$.

Combining the above observations with the results of Section 1 we have the following theorem.

THEOREM 2.1. *Suppose that* $q \in C^1[a, \infty)$ *and* $q(t) > 0$ *on* $[a, \infty)$. *Then, for any* y_0 *and* y_1, *the solution of the initial value problem* (A)–(B) *exists on* $[a, \infty)$ *and is unique.*

2.2 The case where $q(t)$ is locally of bounded variation

Our aim here is to show that the global existence and uniqueness of solutions of the problem (A)–(B) can be eatablished under less restrictive assumptions on $q(t)$. More specifically, we want to prove several theorems including the following.

THEOREM 2.2. *Suppose that $q \in C[a, \infty)$, $q(t) > 0$ on $[a, \infty)$ and $q(t)$ is locally of bounded variation on $[a, \infty)$. Then, for any y_0 and y_1, the solution of the problem* (A)–(B) *exists on $[a, \infty)$ and is unique.*

We use the symbols

$$\int_a^b d^+ f(t), \quad \int_a^b d^- f(t) \text{ and } \int_a^b |df(t)| \tag{2.5}$$

to denote, respectively, the positive variation, the negative variation and the total variation for a function $f(t)$ defined on a finite interval $[a, b]$.

LEEMA 2.3. *Let $y(t)$ be a solution of* (A) *on a finite interval $[t_1, t_2]$. Suppose that $q(t)$ is continuous and positive on $[t_1, t_2]$.*

(i) *If $\int_{t_1}^{t_2} d^+ \log q(t)$ exists, then*

$$V[y](t_2) \leqq V[y](t_1) \exp\left(\int_{t_1}^{t_2} d^+ \log q(t)\right). \tag{2.6}$$

(ii) *If $\int_{t_1}^{t_2} d^- \log q(t)$ exists, then*

$$V[y](t_2) \geqq V[y](t_1) \exp\left(-\int_{t_1}^{t_2} d^- \log q(t)\right). \tag{2.7}$$

PROOF. Let $\varepsilon > 0$ be given arbitrarily. Since $q(t)$ is uniformly continuous, there is a $\delta > 0$ such that

$$|q(\tau') - q(\tau'')| < \varepsilon \text{ if } |\tau' - \tau''| < \delta, \ \tau', \tau'' \in [t_1, t_2].$$

Let Δ denote the partition of $[t_1, t_2]$

$$\Delta : t_1 = \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n = t_2$$

such that $\max\{\tau_i - \tau_{i-1} : 1 \leqq i \leqq n\} < \delta$. We rewrite $V[y](t_i)$ as

$$V[y](\tau_i) = \frac{\alpha}{\alpha+1}(|y'(\tau_i)|^{\alpha+1} - |y'(\tau_{i-1})|^{\alpha+1}) + \frac{\alpha}{\alpha+1}|y'(\tau_{i-1})|^{\alpha+1}$$
$$+ \frac{q(\tau_i)}{\beta+1}(|y(\tau_i)|^{\beta+1} - |y(\tau_{i-1})|^{\beta+1}) + \frac{q(\tau_i)}{\beta+1}|y(\tau_{i-1})|^{\beta+1}$$

and use the fact that

$$|y'(\tau_i)|^{\alpha+1} - |y'(\tau_{i-1})|^{\alpha+1} = -\frac{\alpha+1}{\alpha}\int_{\tau_{i-1}}^{\tau_i} q(s)(y(s))^{\beta *} y'(s)ds$$

and

$$|y(\tau_i)|^{\beta+1} - |y(\tau_{i-1})|^{\beta+1} = (\beta+1)\int_{\tau_{i-1}}^{\tau_i} (y(s))^{\beta *} y'(s)ds.$$

We then have

$$V[y](\tau_i) = \int_{\tau_{i-1}}^{\tau_i} [q(\tau_i) - q(s)](y(s))^{\beta *} y'(s)ds$$

(2.8)
$$+ \frac{\alpha}{\alpha+1}|y'(\tau_{i-1})|^{\alpha+1} + \frac{q(\tau_i)}{\beta+1}|y(\tau_{i-1})|^{\beta+1}.$$

Since $\log q(\tau_i) - \log q(\tau_{i-1}) \leqq \int_{\tau_{i-1}}^{\tau_i} d^+ \log q(t)$, it follows that

$$V[y](\tau_i) \leqq \varepsilon \int_{\tau_{i-1}}^{\tau_i} |y(s)|^{\beta} |y'(s)| ds + \frac{\alpha}{\alpha+1}|y'(\tau_{i-1})|^{\alpha+1}$$
$$+ \frac{q(\tau_{i-1})}{\beta+1}|y(\tau_{i-1})|^{\beta+1} \exp(\log q(\tau_i) - \log(\tau_{i-1}))$$
$$\leqq \varepsilon \int_{\tau_{i-1}}^{\tau_i} |y(s)|^{\beta} |y'(s)| ds + \frac{\alpha}{\alpha+1}|y'(\tau_{i-1})|^{\alpha+1}$$
$$+ \frac{q(\tau_{i-1})}{\beta+1}|y(\tau_{i-1})|^{\beta+1} \exp\left(\int_{\tau_{i-1}}^{\tau_i} d^+ \log q(t)\right).$$

Multiplying the above by $\exp\left(-\int_{t_1}^{\tau_i} d^+ \log q(t)\right)$, we see that

$$V[y](\tau_i)\exp\left(-\int_{t_1}^{\tau_i} d^+\log q(t)\right) \leqq \varepsilon\int_{\tau_{i-1}}^{\tau_i} |y(s)|^\beta |y'(s)|\, ds$$

$$+\frac{\alpha}{\alpha+1}|y'(\tau_{i-1})|^{\alpha+1}\exp\left(-\int_{t_1}^{\tau_i} d^+\log q(t)\right)$$

$$+\frac{q(\tau_{i-1})}{\beta+1}|y(\tau_{i-1})|^{\beta+1}\exp\left(-\int_{t_1}^{\tau_{i-1}} d^+\log q(t)\right)$$

$$\leqq \varepsilon\int_{\tau_{i-1}}^{\tau_i} |y(s)|^\beta |y'(s)|\, ds + V[y](\tau_{i-1})\exp\left(-\int_{t_1}^{\tau_i} d^+\log q(t)\right).$$

Addition of the above inequalities with respect to i then yields

$$V[y](t_2)\exp\left(-\int_{t_1}^{t_2} d^+\log q(t)\right) \leqq \varepsilon\int_{t_1}^{t_2} |y(s)|^\beta |y'(s)|\, ds + V[y](t_1),$$

which, in view of the arbitrariness of ε, establishes (2.6). In order to verify (2.7) it suffices to derive from (2.8) that

$$V[y](\tau_i) \geqq -\varepsilon\int_{\tau_{i-1}}^{\tau_i} |y(s)|^\beta |y'(s)|\, ds + \frac{\alpha}{\alpha+1}|y'(\tau_{i-1})|^{\alpha+1}$$

$$+\frac{q(\tau_{i-1})}{\beta+1}|y(\tau_{i-1})|^{\beta+1}\exp\left(\log q(\tau_i) - \log q(\tau_{i-1})\right)$$

and use the relation

$$\log q(\tau_i) - \log q(\tau_{i-1}) \geqq -\int_{\tau_{i-1}}^{\tau_i} d^-\log q(t).$$

This completes the proof.

LEMMA 2.4. *Let $y(t)$ be a solution of* (A) *on* $[t_1, t_2]$. *Suppose that $q(t)$ is continuous and positive on* $[t_1, t_2]$.

(i) *If $\int_{t_1}^{t_2} d^-\log q(t)$ exists, then*

$$W[y](t_2) \leqq W[y](t_1)\exp\left(\int_{t_1}^{t_2} d^-\log q(t)\right).$$

(ii) *If* $\int_{t_1}^{t_2} d^+ \log q(t)$ *exists, then*

$$W[y](t_2) \geqq W[y](t_1) \exp\left(-\int_{t_1}^{t_2} d^+ \log q(t)\right).$$

The proof is essentially the same as that of Lemma 2.3, and so it is left to the reader.

On the basis of Lemmas 2.3 and 2.4 one can easily prove the following uniqueness theorem which is more general than Theorem 1.3.

THEOREM 2.2. *Suppose that $q(t)$ is continuous and positive on $[a, \infty)$. Let $y(t)$ be a solution* (A) *satisfying the initial condition $y(t_0) = y'(t_0) = 0$.*

(i) *If* $\int_{t_0}^{t_0+\delta} d^+ \log q(t)$ *or* $\int_{t_0}^{t_0+\delta} d^- \log q(t)$ *exists for some* $\delta > 0$, then

$y(t) \equiv 0$ *on* $[t_0, t_0 + \delta]$.

(ii) *If* $\int_{t_0-\delta}^{t_0} d^+ \log q(t)$ *or* $\int_{t_0-\delta}^{t_0} d^- \log q(t)$ *exists for some* $\delta > 0$, *then*

$y(t) \equiv 0$ *on* $[t_0 - \delta, t_0]$.

As is easily seen, Lemmas 2.3 and 2.4 can be used to prove the continuability to the right or to the left of a local solution of the initial value problem (A)–(B).

THEOREM 2.3. *Suppose that $q(t)$ is continuous and positive on $[a, \infty)$. Let $y(t)$ be a solution of the initial value problem* (A)–(B). *Then the following statements hold for any y_0 and y_1.*

(i) *If* $\int_{t_0}^{t_0+\delta} d^+ \log q(t)$ *or* $\int_{t_0}^{t_0+\delta} d^- \log q(t)$ *exists for some* $\delta > 0$, *then*

$y(t)$ *exists on* $[t_0, t_0 + \delta]$.

(ii) *If* $\int_{t_0-\delta}^{t_0} d^+ \log q(t)$ *or* $\int_{t_0-\delta}^{t_0} d^- \log q(t)$ *exists for some* $\delta > 0$, *then*

$y(t)$ *exists on* $[t_0 - \delta, t_0]$.

From Theorems 2.2 and 2.3 and what was observed in Section 1 we obtain the following theorem which is one of the main results of this paper, by using the fact that if $q(t) > 0$ has finite positive (or negative) variation, then so does

$\log q(t)$.

THEOREM 2.4. *Suppose that $q(t)$ is continuous and positive on $[a, \infty)$ and that $q(t)$ has either finite positive variation or finite negative variation on any compact subinterval of $[a, \infty)$. Then, for any y_0 and y_1, there exists a unique solution of the initial value problem* (A)–(B) *defined on the entire interval $[a, \infty)$.*

It is clear that Theorem 2.4 covers Theorem 2.2 stated at the beginning of this subsection.

Part 2. Oscillation and nonoscillation of solutions

3. An oscillation theorem

We now turn to the study of the oscillatory and nonoscillatory behavior of solutions of the equation (A) defined in a neighborhood of $t = \infty$. Such a solution $y(t)$ is said to be proper if $\sup\{|y(t)|: t \geqq T\} > 0$ for any $T \in \operatorname{dom}(y)$. As was stated in the introduction, we focus our attention on the case where the coefficient function $q(t)$ may be oscillating in the sense that it may change its sign infinitely often on $[a, \infty)$.

Out first result in this part is the following theorem which generalizes Waltman's oscillation theorem [16] for the Emden-Fowler equation $y'' + q(t)|y|^{\beta-1}y = 0$.

THEOREM 3.1. *All proper solutions of* (A) *are oscillatory if*

$$\int_a^\infty q(t)dt = \lim_{T\to\infty}\int_a^T q(t)\,dt = \infty. \tag{3.1}$$

LEMMA 3.1. *Let $y \in C^1[t_0, \infty)$ be a function such that $y(t) \neq 0$ on $[t_0, \infty)$. Then, for any $\alpha > 0$, $\beta > 0$ and $T > t_0$,* we have

$$\limsup_{t\to\infty}\left\{\frac{(y'(t))^{\alpha*}}{(y(t))^{\beta*}} + \beta\int_T^t \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds\right\} \geqq 0. \tag{3.2}$$

PROOF OF LEMMA 3.1. Suppose that (3.2) is false for some α, β and T. Then, there exist constants $k > 0$ and $T' > T$ such that

$$\frac{(y'(t))^{\alpha*}}{(y(t))^{\beta*}} + \beta\int_{T'}^t \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds \leqq -k, \qquad t \geqq T',$$

or equivalently

$$k + \beta\int_{T'}^t \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds \leqq -\frac{(y'(t))^{\alpha*}}{(y(t))^{\beta*}}, \qquad t \geqq T'. \tag{3.3}$$

This implies that $y(t)y'(t) < 0$ for $t \geqq T'$. Divide (3.3) by its left-hand side and multiply it by $-y'(t)/y(t)$. We then have

$$-\frac{y'(t)}{y(t)} \leqq \frac{|y'(t)|^{\alpha+1}/|y(t)|^{\beta+1}}{k+\beta\int_{T'}^{t}[|y'(s)|^{\alpha+1}/|y(s)|^{\beta+1}]\,ds}, \qquad t \geqq T'.$$

An integration of the above inequality on $[T', t]$ yields

$$k\left|\frac{y(T')}{y(t)}\right|^{\beta} \leqq k+\beta\int_{T'}^{t}\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds, \qquad t \geqq T'. \tag{3.4}$$

From (3.3) and (3.4) we see that

$$k\left|\frac{y(T')}{y(t)}\right|^{\beta} \leqq -\frac{(y'(t))^{\alpha*}}{(y(t))^{\beta*}}, \qquad t \geqq T',$$

which implies

$$|y'(t)| \geqq k^{\frac{1}{\alpha}}|y(T')|^{\frac{\beta}{\alpha}}, \qquad t \geqq T'. \tag{3.5}$$

Integrating (3.5), we obtain a contradition:

$$\lim_{t\to\infty} y(t) = -\infty \text{ if } y(t) > 0; \quad \lim_{t\to\infty} y(t) = \infty \text{ if } y(t) < 0.$$

Therefore (3.2) must be true.

PROOF OF THEOREM 3.1. Assume to the contrary that (A) has a nonoscillatory solution $y(t)$. Suppose that $y(t) \neq 0$ for $t \geqq t_0$. Dividing (A) by $(y(t))^{\beta*}$ and integrating it over $[t_0, t]$, we obtain

$$\frac{(y'(t))^{\alpha*}}{(y(t))^{\beta*}} + \beta\int_{t_0}^{t}\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds = \frac{(y'(t_0))^{\alpha*}}{(y(t_0))^{\beta*}} - \int_{t_0}^{t} q(s)\,ds, \qquad t \geqq t_0, \tag{3.6}$$

which implies because of (3.1) that

$$\lim_{t\to\infty}\left\{\frac{(y'(t))^{\alpha*}}{(y(t))^{\beta*}} + \beta\int_{t_0}^{t}\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds\right\} = -\infty.$$

This, however, contradicts Lemma 3.1, and the proof is complete.

4. Nonoscillatory solutions

In what follows we assume that $q(t)$ is (conditionally) integrable on $[a, \infty)$, that is,

$$\int_a^\infty q(t)\,dt = \lim_{T\to\infty} \int_a^T q(t)\,dt \quad \text{exists and is finite,} \tag{4.1}$$

in which case the function

$$Q(t) = \int_t^\infty q(s)\,ds \tag{4.2}$$

is well-defined on $[a, \infty)$.

LEMMA 4.1. *If $y(t)$ is a nonoscillatory solution of* (A) *such that $y(t) \neq 0$ on $[t_0, \infty)$, then*

$$\frac{(y'(t))^{\alpha *}}{(y(t))^{\beta *}} = Q(t) + \beta \int_t^\infty \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds, \qquad t \geqq t_0. \tag{4.3}$$

PROOF. Let $t \geqq t_0$ be fixed arbitrarily and integrate the equation (A) divided by $(y(t))^{\beta *}$ over $[t, \tau]$. Then we have

$$\frac{(y'(\tau))^{\alpha *}}{(y(\tau))^{\beta *}} - \frac{(y'(t))^{\alpha *}}{(y(t))^{\beta *}} + \beta \int_t^\tau \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds + \int_t^\tau q(s)\,ds = 0, \qquad \tau > t \geqq t_0. \tag{4.4}$$

We claim that

$$\int_t^\infty \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds < \infty, \qquad t \geqq t_0. \tag{4.5}$$

In fact, if (4.5) does not hold, then there is $T > t$ such that

$$-\frac{(y'(t))^{\alpha *}}{(y(t))^{\beta *}} + \beta \int_t^T \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds + \int_t^\tau q(s)\,ds \geqq 1, \qquad \tau \geqq T.$$

It then follows from (4.4) that

$$\frac{(y'(\tau))^{\alpha *}}{(y(\tau))^{\beta *}} + \beta \int_T^\tau \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds = \frac{(y'(t))^{\alpha *}}{(y(t))^{\beta *}} - \beta \int_t^T \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds - \int_t^\tau q(s)\,ds \leqq -1, \qquad \tau \geqq t,$$

which contradicts Lemma 3.1. Consequently, (4.5) holds as claimed. Letting $\tau \to \infty$ in (4.4), we see that the finite limit $\eta = \lim_{\tau\to\infty} [(y'(\tau))^{\alpha *}/(y(\tau))^{\beta *}]$ exists and

$$\frac{(y'(t))^{\alpha *}}{(y(t))^{\beta *}} = \eta + Q(t) + \beta \int_t^\infty \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}\,ds, \qquad t \geqq t_0. \tag{4.6}$$

To establish (4.3) it suffices to prove that $\eta = 0$ in (4.6). Suppose the contrary: either $\eta > 0$ or $\eta < 0$. Let $\eta > 0$. Then, (4.6) shows that there is $T_1 > t_0$ such that

$$y(t)y'(t) > 0 \quad \text{and} \quad |y'(t)|^{\alpha}/|y(t)|^{\beta} \geqq \eta/2 \qquad \text{for } t \geqq T_1. \tag{4.7}$$

Integrating the inequality $|y'(t)| \geqq (\eta/2)^{1/\alpha}|y(T_1)|^{\beta/\alpha}$, $t \geqq T_1$, following from (4.7), we see that $\lim_{t\to\infty} |y(t)| = \infty$. It follows that

$$\int_{T_1}^{t} \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}} ds \geqq \frac{\eta}{2} \int_{T_1}^{t} \left|\frac{y'(s)}{y(s)}\right| ds = \frac{\eta}{2} \log \left|\frac{y(t)}{y(T_1)}\right| \to \infty \quad \text{as} \quad t \to \infty,$$

which contradicts (4.5). Now let $\eta < 0$. Take $T_2 > t_0$ so that

$$-\frac{(y'(t))^{\alpha *}}{(y(t))^{\beta *}} + \int_{t}^{\tau} q(s)\, ds \geqq -\frac{\eta}{2}, \qquad \tau \geqq t \geqq T_2.$$

In view of (4.4) we then have

$$\frac{(y'(\tau))^{\alpha *}}{(y(\tau))^{\beta *}} + \beta \int_{t}^{\tau} \frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}} ds \leqq \frac{\eta}{2} < 0, \qquad \tau \geqq t \geqq T_2,$$

a contradiction to Lemma 3.1. Thus we must have $\eta = 0$. This completes the proof.

REMARK 4.1. Lemma 4.1 is a generalization of a useful identity obtained by Kwong and Wong [9] for the Emden-Fowler equation (C).

We now prove a theorem which provides information about the possible asymptotic behavior of nonoscillatory solutions of the equation (A).

THEOREM 4.1. *Suppose that $Q(t) \geqq 0$ for $t \geqq a$. If $y(t)$ is a nonoscillatory solution of* (A), *then there exist positive constants c_1, c_2 and $t_0 \geqq a$ such that*

$$c_1 \leqq |y(t)| \leqq c_2 t, \qquad t \geqq t_0. \tag{4.8}$$

PROOF. We may assume that $y(t)$ is eventually positive: $y(t) > 0$ for $t \geqq t_0$. Since $Q(t) \geqq 0$ by hypothesis, Lemma 4.1 implies that $y'(t) \geqq 0$, and hence $y(t) \geqq y(t_0)$ for $t \geqq t_0$. Thus, the first inequality $y(t) \geqq c_1$, $t \geqq t_0$, in (4.8) holds with $c_1 = y(t_0)$.

To prove the second inequality in (4.8) we first observe from (4.3) that $(y'(t))^{\alpha}/(y(t))^{\beta} \geqq Q(t)$, $t \geqq t_0$, i.e.,

$$(y'(t))^{\alpha} \geqq Q(t)(y(t))^{\beta}, \qquad t \geqq t_0. \tag{4.9}$$

We now integrate (A) over $[t, \tau]$, $\tau \geqq t \geqq t_0$, obtaining

$$(4.10) \qquad (y'(t))^\alpha = (y'(\tau))^\alpha - Q(\tau)(y(\tau))^\beta + Q(t)(y(t))^\beta + \beta \int_t^\tau Q(s)(y(s))^{\beta-1} y'(s)\,ds$$

for $\tau \geqq t \geqq t_0$. Note that $\int_t^\infty Q(s)(y(s))^{\beta-1} y'(s)\,ds < \infty$, since otherwise it would follow from (4.10) that $(y'(\tau))^\alpha - Q(\tau)(y(\tau))^\beta \to -\infty$ as $\tau \to \infty$, which is inconsistent with (4.9). Therefore, letting $\tau \to \infty$ in (4.10), we obtain

$$(4.11) \qquad (y'(t))^\alpha = \eta + Q(t)(y(t))^\beta + \beta \int_t^\infty Q(s)(y(s))^{\beta-1} y'(s)\,ds, \qquad t \geqq t_0,$$

where η denotes the finite limit

$$\eta = \lim_{\tau \to \infty} \{(y'(\tau))^\alpha - Q(\tau)(y(\tau))^\beta\} \geqq 0.$$

Let us define the functions $K_1(t)$ and $K_2(t)$ by

$$(4.12) \qquad K_1(t) = \int_t^\infty Q(s)(y(s))^{\beta-1} y'(s)\,ds, \qquad t \geqq t_0,$$

$$(4.13) \qquad K_2(t) = \int_t^\infty (Q(s))^{1+\frac{1}{\alpha}} (y(s))^{(1+\frac{1}{\alpha})\beta - 1}\,ds, \qquad t \geqq t_0.$$

From (4.9) and (4.12) we see that $K_1(t) \geqq K_2(t)$, so that $K_2(t)$ is well-defined for $t \geqq t_0$.

In order to estimate $y(t)$ from above we derive the inequality

$$y'(t) \leqq (3\eta)^{\frac{1}{\alpha}} + 3^{\frac{1}{\alpha}}(Q(t))^{\frac{1}{\alpha}}(y(t))^{\frac{\beta}{\alpha}} + (3\beta)^{\frac{1}{\alpha}}(K_1(t))^{\frac{1}{\alpha}}, \qquad t \geqq t_0,$$

from (4.11) and integrate it over $[t_0, t]$:

$$(4.14) \qquad y(t) \leqq y(t_0) + (3\eta)^{\frac{1}{\alpha}}(t - t_0) + 3^{\frac{1}{\alpha}} \int_{t_0}^t (Q(s))^{\frac{1}{\alpha}}(y(s))^{\frac{\beta}{\alpha}}\,ds + (3\beta)^{\frac{1}{\alpha}} \int_{t_0}^t (K_1(s))^{\frac{1}{\alpha}}\,ds, \qquad t \geqq t_0.$$

Using the inequalities

$$\int_{t_0}^t (K_1(s))^{\frac{1}{\alpha}}\,ds \leqq (K_1(t_0))^{\frac{1}{\alpha}}(t - t_0), \qquad t \geqq t_0,$$

which follows from the decreasing property of $K_1(t)$, and

$$\int_{t_0}^t (Q(s))^{\frac{1}{\alpha}}(y(s))^{\frac{\beta}{\alpha}}ds \leqq \left(\int_{t_0}^t (Q(s))^{1+\frac{1}{\alpha}}(y(s))^{(1+\frac{1}{\alpha})\beta-1}ds\right)^{\frac{1}{\alpha+1}}\left(\int_{t_0}^t (y(s))^{\frac{1}{\alpha}}ds\right)^{\frac{\alpha}{\alpha+1}}$$

$$\leqq (K_2(t_0))^{\frac{1}{\alpha+1}}(y(t))^{\frac{1}{\alpha+1}}(t-t_0)^{\frac{\alpha}{\alpha+1}}, \qquad t \geqq t_0,$$

which is a consequence of Hölder's inequality, we find from (4.14) that

$$\begin{aligned} y(t) &\leqq y(t_0) + [(3\eta)^{\frac{1}{\alpha}} + (K_1(t_0))^{\frac{1}{\alpha}}](t-t_0) + (K_2(t_0))^{\frac{1}{\alpha+1}}(y(t))^{\frac{1}{\alpha+1}}(t-t_0)^{\frac{\alpha}{\alpha+1}} \\ &\leqq y(t_0) + [(3\eta)^{\frac{1}{\alpha}} + (K_1(t_0))^{\frac{1}{\alpha}}]\,t + (K_2(t_0))^{\frac{1}{\alpha+1}}(y(t))^{\frac{1}{\alpha+1}}\,t^{\frac{\alpha}{\alpha+1}} \end{aligned}$$

for $t \geqq t_0$. It follows that

$$\frac{y(t)}{t} \leqq \frac{y(t_0)}{t_0} + (3\eta)^{\frac{1}{\alpha}} + (K_1(t_0))^{\frac{1}{\alpha}} + (K_2(t_0))^{\frac{1}{\alpha+1}}\left(\frac{y(t)}{t}\right)^{\frac{1}{\alpha+1}}, \qquad t \geqq t_0,$$

which means that the function $z(t) = y(t)/t$ satisfies the inequality $z(t) \leqq A + B(z(t))^{1/(\alpha+1)}$, $t \geqq t_0$, for some positive constants A and B. It is elementary to verify that the values of $z(t)$ is bounded from above by a positive constant depending on A and B, or equivalently that $y(t) \leqq c_2 t$, $t \geqq t_0$, for some positive constant c_2. This is the desired inequality, and the proof is complete.

REMARK 4.2. Theorem 4.1 is a partial generalization of a result of Naito [15] for the Emden-Fowler equation which asserts that, under the condition $Q(t) \geqq 0$, each nonoscillatory solution $y(t)$ of (C) has one of the following three types of asymptotic behavior as $t \to \infty$: (I) $\lim_{t\to\infty} y(t) = \text{const} \neq 0$; (II) $\lim_{t\to\infty} y(t) = \pm\infty$, $\lim_{t\to\infty} [y(t)/t] = 0$; (III) $\lim_{t\to\infty} [y(t)/t] = \text{const} \neq 0$. We conjecture that the same is true of the nonoscillatory solutions of (A).

5. Existence of nonoscillatory solutions

This section is concerned with the existence of nonoscillatory solutions of (A). More specifically, we want to construct (i) nonoscillatory solutions which behave like nonzero constants as $t \to \infty$, and (ii) those which behave like constant multiples of t as $t \to \infty$. In view of Theorem 4.1 a solution of type (i) [respectively type (ii)] may be referred to as a *minimal* [respectively *maximal*] nonoscillatory solution of (A).

We first give a criterion for the existence of minimal nonoscillatory solution of (A).

THEOREM 5.1. *Suppose that $Q(t) \geqq 0$ for $t \geqq a$. A necessary and sufficient condition for* (A) *to have a nonoscillatory solution which tends to a nonzero constant as $t \to \infty$ is that*

$$\int_a^\infty (Q(t))^{\frac{1}{\alpha}} dt < \infty \quad and \quad \int_a^\infty \left(\int_t^\infty (Q(s))^{1+\frac{1}{\alpha}} ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{5.1}$$

PROOF. (Necessity) Let $y(t)$ be a nonoscillatory solution of (A). We may suppose that $y(t) > 0$ for $t \geqq t_0$. Then, by Lemma 4.1, $y'(t) \geqq 0$, $t \geqq t_0$, and

$$\frac{(y'(t))^\alpha}{(y(t))^\beta} = Q(t) + \beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds, \qquad t \geqq t_0. \tag{5.2}$$

This shows that $(y'(t))^\alpha/(y(t))^\beta \geqq Q(t)$, $t \geqq t_0$, or $y'(t)/(y(t))^{\beta/\alpha} \geqq (Q(t))^{1/\alpha}$, $t \geqq t_0$. Integrating the last inequality, we have

$$\int_{t_0}^t (Q(s))^{\frac{1}{\alpha}} ds \leqq \int_{t_0}^t \frac{y'(s)}{(y(s))^{\beta/\alpha}} ds \leqq \int_{y(t_0)}^{y(t)} \frac{dv}{v^{\beta/\alpha}}, \qquad t \geqq t_0,$$

which, in view of the boundedness of $y(t)$, implies that

$$\int_{t_0}^\infty (Q(t))^{\frac{1}{\alpha}} dt < \infty. \tag{5.3}$$

We now use the inequality

$$\frac{(y'(t))^\alpha}{(y(t))^\beta} \geqq \beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds, \qquad t \geqq t_0, \tag{5.4}$$

which also follows from (5.2). From (5.4) and the inequality

$$\begin{aligned} \frac{(y'(t))^{\alpha+1}}{(y(t))^{\beta+1}} &= \frac{(y'(t))^\alpha}{(y(t))^\beta} \cdot \frac{y'(t)}{(y(t))^{\beta/\alpha}} \cdot (y(t))^{\frac{\beta}{\alpha}-1} \\ &\geqq Q(t) \cdot (Q(t))^{\frac{1}{\alpha}} \cdot c = c(Q(t))^{1+\frac{1}{\alpha}}, \qquad t \geqq t_0, \end{aligned}$$

c being a positive constant, we see that

$$\beta c \int_t^\infty (Q(s))^{1+\frac{1}{\alpha}} ds \leqq \frac{(y'(t))^\alpha}{(y(t))^\beta}, \qquad t \geqq t_0,$$

or equivalently

$$(\beta c)^{\frac{1}{\alpha}}\left(\int_t^\infty (Q(s))^{1+\frac{1}{\alpha}}\,ds\right)^{\frac{1}{\alpha}} \leqq \frac{y'(t)}{(y(t))^{\beta/\alpha}}, \qquad t \geqq t_0. \tag{5.5}$$

Integrating (5.5) from t_0 to t and letting $t \to \infty$, we conclude that

$$\int_{t_0}^\infty \left(\int_t^\infty (Q(s))^{1+\frac{1}{\alpha}}\,ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{5.6}$$

The inequalities (5.3) and (5.6) clearly imply (5.1).

(Sufficiency) Assume that (5.1) holds. Define

$$R(t) = \int_t^\infty (Q(s))^{1+\frac{1}{\alpha}}\,ds, \qquad t \geqq a. \tag{5.7}$$

Note that

$$\int_t^\infty Q(s)(R(s))^{\frac{1}{\alpha}}\,ds \leqq R(t)\left(\int_t^\infty (R(s))^{\frac{1}{\alpha}}\,ds\right)^{\frac{1}{\alpha+1}}, \qquad t \geqq a. \tag{5.8}$$

In fact, by Hölder's inequality, we have

$$\begin{aligned}\int_t^\infty Q(s)(R(s))^{\frac{1}{\alpha}}\,ds &\leqq \left(\int_t^\infty (Q(s))^{1+\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha+1}}\left(\int_t^\infty (R(s))^{1+\frac{1}{\alpha}}\,ds\right)^{\frac{1}{\alpha+1}} \\ &\leqq (R(t))^{\frac{\alpha}{\alpha+1}}\left(R(t)\int_t^\infty (R(s))^{\frac{1}{\alpha}}\,ds\right)^{\frac{1}{\alpha+1}} \\ &= R(t)\left(\int_t^\infty (R(s))^{\frac{1}{\alpha}}\,ds\right)^{\frac{1}{\alpha+1}}, \qquad t \geqq a.\end{aligned}$$

Take any constant $\lambda > 0$ and let it be fixed. Put

$$\mu = \left(\frac{3}{2}\lambda\right)^\beta;\ \tilde{\lambda} = \frac{3}{2}\lambda \text{ if } \beta \geqq 1 \text{ and } \tilde{\lambda} = \frac{\lambda}{2} \text{ if } \beta < 1.$$

Choose $\nu > 0$ so that

$$\beta\tilde{\lambda}^{\beta-1}(2\mu)^{\frac{1}{\alpha}} \leqq \frac{\nu}{2} \tag{5.9}$$

and let $T > a$ be large enough so that

$$(2\mu)^{\frac{1}{\alpha}} \int_T^\infty (Q(s))^{\frac{1}{\alpha}} ds \leqq \frac{\lambda}{4}, \tag{5.10}$$

$$(2\nu)^{\frac{1}{\alpha}} \int_T^\infty (R(s))^{\frac{1}{\alpha}} ds \leqq \frac{\lambda}{4}, \tag{5.11}$$

and

$$\beta \tilde{\lambda}^{\beta-1} (2\nu)^{\frac{1}{\alpha}} \left(\int_T^\infty (R(s))^{\frac{1}{\alpha}} ds \right)^{\frac{1}{\alpha+1}} \leqq \frac{\nu}{2}. \tag{5.12}$$

We now define Y to be the set of functions $y \in C[T, \infty)$ satisfying

$$|y(t) - \lambda| \leqq \frac{\lambda}{2}, \qquad t \geqq T, \tag{5.13}$$

and

$$|y(t_1) - y(t_2)| \leqq (\mu \tilde{Q}(T) + \nu R(T))^{\frac{1}{\alpha}} |t_1 - t_2|, \qquad t_1, t_2 \geqq T, \tag{5.14}$$

where $\tilde{Q}(T) = \sup \{Q(t) : t \geqq T\}$, and define Z to be the set of functions $z \in C[T, \infty)$ satisfying

$$|z(t)| \leqq \mu Q(t) + \nu R(t), \qquad t \geqq T. \tag{5.15}$$

Let F_1 and F_2 denote the mappings from $Y \times Z$ to $C[T, \infty)$ defined by

$$F_1(y, z)(t) = \lambda - \int_t^\infty (z(s))^{\frac{1}{\alpha}*} ds, \qquad t \geqq T, \tag{5.16}$$

and

$$F_2(y, z)(t) = Q(t)(y(t))^\beta + \beta \int_t^\infty Q(s)(y(s)^{\beta-1}(z(s))^{\frac{1}{\alpha}*} ds, \qquad t \geqq T, \tag{5.17}$$

respectively. Finally we define the mapping $F: Y \times Z \to C[T, \infty) \times C[T, \infty)$ by

$$F(y, z) = (F_1(y, z), F_2(y, z)), \qquad (y, z) \in Y \times Z. \tag{5.18}$$

It can be shown that F maps $Y \times Z$ continuously into a relatively compact subset of $Y \times Z$.

(i) We first show that F maps $Y \times Z$ into itself. Let $(y, z) \in Y \times Z$. From (5.15) we see that

$$|z(t)|^{\frac{1}{\alpha}} \leqq (2\mu)^{\frac{1}{\alpha}}(Q(t))^{\frac{1}{\alpha}} + (2\nu)^{\frac{1}{\alpha}}(R(t))^{\frac{1}{\alpha}}, \qquad t \geqq T,$$

so that, using (5.10) and (5.11), we have

$$\int_t^\infty (z(s))^{\frac{1}{\alpha}*} ds \leqq (2\mu)^{\frac{1}{\alpha}} \int_t^\infty (Q(s))^{\frac{1}{\alpha}} ds + (2\nu)^{\frac{1}{\alpha}} \int_t^\infty (R(s))^{\frac{1}{\alpha}} ds \leqq \frac{\lambda}{2}, \qquad t \geqq T.$$

In view of (5.16), it follows that $|F_1(y, z) - \lambda| \leqq \lambda/2$ for $t \geqq T$. Also,

$$|F_1(y, z)(t_1) - F_1(y, z)(t_2)| = \left| \int_{t_1}^{t_2} (z(s))^{\frac{1}{\alpha}*} ds \right|$$

$$\leqq \left| \int_{t_1}^{t_2} (\mu Q(s) + \nu R(s))^{\frac{1}{\alpha}} ds \right| \leqq (\mu \tilde{Q}(T) + \nu R(T))^{\frac{1}{\alpha}} |t_1 - t_2|$$

for $t_1, t_2 \geqq T$. This implies that $F_1(y, z) \in Y$. Next, using (5.17), we compute:

$$|F_2(y, z)(t)| \leqq \left(\frac{3}{2}\lambda\right)^\beta Q(t) + \beta \tilde{\lambda}^{\beta-1} \int_t^\infty Q(s)(\mu Q(s) + \nu R(s))^{\frac{1}{\alpha}} ds$$

$$\leqq \mu Q(t) + \beta \tilde{\lambda}^{\beta-1}(2\mu)^{\frac{1}{\alpha}} \int_t^\infty (Q(s))^{1+\frac{1}{\alpha}} ds + \beta \tilde{\lambda}^{\beta-1}(2\nu)^{\frac{1}{\alpha}} \int_t^\infty Q(s)(R(s))^{\frac{1}{\alpha}} ds$$

$$\leqq \mu Q(t) + \frac{\nu}{2} R(t) + \beta \tilde{\lambda}^{\beta-1}(2\nu)^{\frac{1}{\alpha}} \left(\int_t^\infty (R(s))^{\frac{1}{\alpha}} ds \right)^{\frac{1}{\alpha+1}} R(t)$$

$$\leqq \mu Q(t) + \frac{\nu}{2} R(t) + \frac{\nu}{2} R(t) = \mu Q(t) + \nu R(t), \qquad t \geqq T,$$

where (5.8), (5.9) and (5.12) have been used. This shows that $F_2(y, z) \in Z$. Therefore, by (5.18), we conclude that $F(y, z) \in Y \times Z$ as desired.

(ii) We then show that F is continuous. Let $\{(y_k, z_k)\}$ be a sequence in $Y \times Z$ converging to $(y, z) \in Y \times Z$ in the topology of $C[T, \infty) \times C[T, \infty)$. From the inequalities

$$|F_1(y_k, z_k)(t) - F_1(y, z)(t)| \leqq \int_T^\infty |(z_k(s))^{\frac{1}{\alpha}*} - (z(s))^{\frac{1}{\alpha}*}| ds$$

and

$$|F_2(y_k, z_k)(t) - F_2(y, z)(t)| \leqq Q(t)|(y_k(t))^\beta - (y(t))^\beta|$$

$$+ \beta \int_T^\infty Q(s)|(y_k(s))^{\beta-1}(z_k(s))^{\frac{1}{\alpha}*} - (y(s))^{\beta-1}(z(s))^{\frac{1}{\alpha}*}| ds,$$

holding for $t \geqq T$, it follows by the Lebesgue convergence theorem that $\{F_1(y_k, z_k)(t)\}$ converges to $F_1(y, z)(t)$ uniformly on $[T, \infty)$ and that $\{F_2(y_k, z_k)(t)\}$ converges to $F_2(y, z)(t)$ uniformly on any compact subinterval of $[T, \infty)$. The applicability of the Lebesgue convergence theorem follows from the inequalities

$$|(z_k(t))^{\frac{1}{\alpha}*} - (z(t))^{\frac{1}{\alpha}*}| \leqq 2[(2\mu)^{\frac{1}{\alpha}}(Q(t))^{\frac{1}{\alpha}} + (2\nu)^{\frac{1}{\alpha}}(R(s))^{\frac{1}{\alpha}}],$$

$$Q(t)|(y_k(t))^{\beta-1}(z_k(t))^{\frac{1}{\alpha}*} - (y(t))^{\beta-1}(z(t))^{\frac{1}{\alpha}*}|$$
$$\leqq 2\tilde{\lambda}^{\beta-1}Q(t)[(2\mu)^{\frac{1}{\alpha}}(Q(t))^{\frac{1}{\alpha}} + (2\nu)^{\frac{1}{\alpha}}(R(t))^{\frac{1}{\alpha}}].$$

This shows that $\{F(y_k, z_k)\}$ converges to $F(y, z)$ in $C[T, \infty) \times C[T, \infty)$, establishing the continuity of F.

(iii) We finally show that $F(Y \times Z)$ is relatively compact in $C[T, \infty) \times C[T, \infty)$. For this purpose it suffices to prove that $F_1(Y \times Z)$ and $F_2(Y \times Z)$ are relatively compact in $C[T, \infty)$. The equicontinuity of $F_1(Y \times Z)$ on $[T, \infty)$ is a consequence of the inequality

$$\left|\frac{d}{dt}F_1(y, z)\right| \leqq (\mu\tilde{Q}(T) + \nu R(T))^{\frac{1}{\alpha}}, \qquad t \geqq T,$$

holding for all $(y, z) \in Y \times Z$. The local equicontinuity of $F_2(Y \times Z)$ follows from the inequalities holding for all $(y, z) \in Y \times Z$:

$$|Q(t_1)(y(t_1))^\beta - Q(t_2)(y(t_2))^\beta| = |Q(t_1)((y(t_1))^\beta - (y(t_2))^\beta) + (Q(t_1) - Q(t_2))(y(t_2))^\beta)|$$
$$\leqq \beta\tilde{\lambda}^{\beta-1}\tilde{Q}(T)|y(t_1) - y(t_2)| + \left(\frac{3}{2}\lambda\right)^\beta \left|\int_{t_1}^{t_2} q(s)ds\right|$$
$$\leqq \beta\tilde{\lambda}^{\beta-1}\tilde{Q}(T)(\mu\tilde{Q}(T) + \nu R(T))^{\frac{1}{\alpha}}|t_1 - t_2| + \mu\left|\int_{t_1}^{t_2} q(s)\,ds\right|, \qquad t_1, t_2 \geqq T,$$

and

$$\left|\frac{d}{dt}\beta\int_t^\infty Q(s)(y(s))^{\beta-1}(z(s))^{\frac{1}{\alpha}*}ds\right| \leqq \beta Q(t)(y(t))^{\beta-1}|(z(t)|^{\frac{1}{\alpha}}$$
$$\leqq \beta\tilde{\lambda}^{\beta-1}\tilde{Q}(T)(\mu\tilde{Q}(T) + \nu R(T))^{\frac{1}{\alpha}}, \quad t \geqq T.$$

Since $F_1(Y \times Z)$ and $F_2(Y \times Z)$ are uniformly bounded on $[T, \infty)$, we conclude that they are both relatively compact in $C[T, \infty)$.

All the hypotheses of the Schauder-Tychonoff fixed point theorem have thus been verified, and so there exists an element $(y, z) \in Y \times Z$ such that

$(y, z) = F(y, z)$. By (5.18), (5.16) and (5.17) the functions $y(t)$ and $z(t)$ satisfy for $t \geqq T$

$$y(t) = \lambda - \int_t^\infty (z(s))^{\frac{1}{\alpha}*} ds, \tag{5.19}$$

$$z(t) = Q(t)(y(t))^\beta + \beta \int_t^\infty Q(s)(y(s))^{\beta-1}(z(s))^{\frac{1}{\alpha}*} ds. \tag{5.20}$$

Differentiating (5.19) and (5.20), we see that $y'(t) = (z(t))^{1/\alpha *}$ or $(y'(t))^{\alpha *} = z(t)$, and

$$\begin{aligned} z'(t) &= -q(t)(y(t))^\beta + \beta Q(t)(y(t))^{\beta-1} y'(t) - \beta Q(t)(y(t))^{\beta-1}(z(t))^{\frac{1}{\alpha}*} \\ &= -q(t)(y(t))^\beta, \qquad t \geqq T, \end{aligned}$$

which shows that $y(t)$ is a solution of the equation (A) on $[T, \infty)$. Since (5.19) implies $\lim_{t\to\infty} y(t) = \lambda > 0$, the construction of a minimal nonoscillatory solution of (A) has been complete. This finishes the proof of Theorem 5.1.

REMARK 5.1. Theorem 5.1 generalizes a theorem of Naito [14] regarding the Emden-Fowler equation (C).

A criterion for the existence of a maximal nonoscillatory solution of (A) is given in the following theorem.

THEOREM 5.2. *Suppose that* $Q(t) \geqq 0$ *for* $t \geqq a$. *A necessary and sufficient condition for* (A) *to have a nonoscillatory solution* $y(t)$ *such that*

$$c_1 t \leqq |y(t)| \leqq c_2 t, \qquad t \geqq t_0, \tag{5.21}$$

for some positive constants c_1, c_2 *and* $t_0 \geqq a$ *is that*

$$\int_a^\infty t^{\beta-1} Q(t) dt < \infty \quad \text{and} \quad \int_a^\infty t^{(1+\frac{1}{\alpha})\beta - 1} (Q(t))^{1+\frac{1}{\alpha}} dt < \infty. \tag{5.22}$$

PROOF. (Necessity) Assume that (A) has a nonoscillatory solution $y(t)$ satisfying (5.21). We may suppose that $y(t)$ is positive, so that $c_1 t \leqq y(t) \leqq c_2 t$ for $t \geqq t_0$. We will combine the equation

$$(y'(t))^\alpha = Q(t)(y(t))^\beta + \beta (y(t))^\beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds, \qquad t \geqq t_0, \tag{5.23}$$

which follows from (4.3) with the equation

$$(y'(t))^\alpha = (y'(t_0))^\alpha - Q(t_0)(y(t_0))^\beta + Q(t)(y(t))^\beta - \beta \int_{t_0}^{t} Q(s)(y(s))^{\beta-1} y'(s) ds,$$

$$t \geqq t_0,$$

which is an integrated version of (A). We then obtain

(5.24)

$$\beta(y(t))^\beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds = (y'(t_0))^\alpha - Q(t_0)(y(t_0))^\beta - \beta \int_{t_0}^{t} Q(s)(y(s))^{\beta-1} y'(s) ds$$

for $t \geqq t_0$. The second inequality in (5.22) is an immediate consequence of (5.23) and (5.24). In fact, we obtain

$$y'(t) \geqq (Q(t))^{\frac{1}{\alpha}}(y(t))^{\frac{\beta}{\alpha}}, \qquad t \geqq t_0,$$

and

$$\int_{t_0}^{\infty} Q(s)(y(s))^{\beta-1} y'(s) ds < \infty, \tag{5.25}$$

from (5.23) and (5.24), respectively, and so

$$\int_{t_0}^{\infty} (Q(s))^{1+\frac{1}{\alpha}} (y(s))^{(1+\frac{1}{\alpha})\beta - 1} ds < \infty. \tag{5.26}$$

The second inequality in (5.22) then follows from (5.26) and (5.21).

To derive the first inequality in (5.22) we first observe from (5.24) that there exists a finite limit

$$\eta = \lim_{t\to\infty} \beta(y(t))^\beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds \geqq 0. \tag{5.27}$$

An integration of the equation

$$y'(t) = \left(Q(t)(y(t))^\beta + \beta(y(t))^\beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds \right)^{\frac{1}{\alpha}}, \qquad t \geqq t_0,$$

(cf. (5.23)) gives

$$y(t) - y(\tau) = \int_\tau^t \left(Q(s)(y(s))^\beta + \beta(y(s))^\beta \int_s^\infty \frac{(y'(r))^{\alpha+1}}{(y(r))^{\beta+1}} dr \right)^{\frac{1}{\alpha}} ds, \qquad t \geqq \tau \geqq t_0,$$

which, in view of (5.21), leads to

(5.28)
$$c_1 t - y(\tau) \leqq 2^{\frac{1}{\alpha}} \int_\tau^t (Q(s))^{\frac{1}{\alpha}} (y(s))^{\frac{\beta}{\alpha}} ds + 2^{\frac{1}{\alpha}} \int_\tau^t \left(\beta (y(s))^\beta \int_s^\infty \frac{(y'(r))^{\alpha+1}}{(y(r))^{\beta+1}} dr \right)^{\frac{1}{\alpha}} ds$$

for $t \geqq \tau \geqq t_0$. From (5.26) we have

$$\liminf_{t\to\infty} \frac{2^{\frac{1}{\alpha}}}{t} \int_\tau^t (Q(s))^{\frac{1}{\alpha}} (y(s))^{\frac{\beta}{\alpha}} ds$$

(5.29)
$$\geqq \lim_{t\to\infty} \frac{c_1 t - y(\tau)}{t} - \lim_{t\to\infty} \frac{2^{\frac{1}{\alpha}}}{t} \int_\tau^t \left(\beta (y(s))^\beta \int_s^\infty \frac{(y'(r))^{\alpha+1}}{(y(r))^{\beta+1}} dr \right)^{\frac{1}{\alpha}} ds$$

$$= c_1 - 2^{\frac{1}{\alpha}} \lim_{t\to\infty} \left(\beta (y(t))^\beta \int_t^\infty \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds \right)^{\frac{1}{\alpha}} = c_1 - (2\eta)^{\frac{1}{\alpha}},$$

where (5.27) has been used. On the other hand, using the inequality

$$\int_\tau^t (Q(s))^{\frac{1}{\alpha}} (y(s))^{\frac{\beta}{\alpha}} ds \leqq \left(\int_\tau^t (Q(s))^{1+\frac{1}{\alpha}} (y(s))^{(1+\frac{1}{\alpha})\beta - 1} ds \right)^{\frac{1}{\alpha+1}} \left(\int_\tau^t (y(s))^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha+1}},$$

$$t \geqq \tau,$$

(cf. the proof of Theorem 4.1), (5.21) and (4.13), we find

$$\frac{2^{\frac{1}{\alpha}}}{t} \int_\tau^t (Q(s))^{\frac{1}{\alpha}} (y(s))^{\frac{\beta}{\alpha}} ds \leqq \frac{2^{\frac{1}{\alpha}}}{t} \left(\int_\tau^\infty (Q(s))^{1+\frac{1}{\alpha}} (y(s))^{(1+\frac{1}{\alpha})\beta - 1} ds \right)^{\frac{1}{\alpha+1}} \left(\int_\tau^t (c_2 s)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha+1}}$$

$$= \frac{2^{\frac{1}{\alpha}}}{t} (K_2(\tau))^{\frac{1}{\alpha+1}} \left[c_2^{\frac{1}{\alpha}} \frac{\alpha}{\alpha+1} \left(t^{\frac{\alpha+1}{\alpha}} - \tau^{\frac{\alpha+1}{\alpha}} \right) \right]^{\frac{\alpha}{\alpha+1}}$$

$$\leqq \ell (K_2(\tau))^{\frac{1}{\alpha+1}}, \qquad t \geqq \tau \geqq t_0$$

where $\ell = (2c_2)^{1/\alpha} (\alpha/(\alpha+1))^{\alpha/(\alpha+1)}$, and hence

(5.30)
$$\limsup_{t\to\infty} \frac{2^{\frac{1}{\alpha}}}{t} \int_\tau^t (Q(s))^{\frac{1}{\alpha}} (y(s))^{\frac{\beta}{\alpha}} ds \leqq \ell (K_2(\tau))^{\frac{1}{\alpha+1}}.$$

From (5.29) and (5.30) it follows that $c_1 - (2\eta)^{\frac{1}{\alpha}} \leqq 0$, that is, the constant η defined by (5.27) is positive. Letting $t \to \infty$ in (5.23), we see that

$\liminf_{t\to\infty}(y'(t))^\alpha \geqq \eta$, so that there is $t_1 > t_0$ such that $y'(t) \geqq \eta/2$ for $t \geqq t_1$. Combining the last inequality with (5.25) shows that the first inequality in (5.22) is true. Thus the proof of the necessity of (5.22) is complete.

(Sufficiency) Assume that (5.22) holds. Define

$$S(t) = \max\left\{\int_t^\infty s^{\beta-1}Q(s)\,ds,\ \int_t^\infty s^{(1+\frac{1}{\alpha})\beta-1}(Q(s))^{1+\frac{1}{\alpha}}ds\right\},\qquad t \geqq a. \tag{5.31}$$

Observe that for $t \geqq T \geqq a$

$$\begin{aligned}\int_T^t s^{\frac{\beta}{\alpha}}(Q(s))^{\frac{1}{\alpha}}ds &\leqq \left(\int_T^t s^{(1+\frac{1}{\alpha})\beta-1}(Q(s))^{1+\frac{1}{\alpha}}ds\right)^{\frac{1}{\alpha+1}}\left(\int_T^t s^{\frac{1}{\alpha}}ds\right)^{\frac{\alpha}{\alpha+1}}\\ &\leqq \left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}(S(T))^{\frac{1}{\alpha+1}}t.\end{aligned} \tag{5.32}$$

(i) Suppose first that $\alpha \geqq 1$. Let $\lambda > 0$ be fixed arbitrarily, and put

$$\mu = \left(\frac{3}{2}\lambda^{\frac{1}{\alpha}}\right)^\beta,\ \tilde{\lambda} = \frac{3}{2}\lambda^{\frac{1}{\alpha}} \text{ for } \beta \geqq 1 \text{ and } \tilde{\lambda} = \frac{1}{2}\lambda^{\frac{1}{\alpha}} \text{ for } \beta < 1.$$

Take a constant $\nu > 0$ such that

$$\beta\tilde{\lambda}^{\beta-1}(\lambda^{\frac{1}{\alpha}} + \mu^{\frac{1}{\alpha}}) \leqq \frac{\nu}{2} \tag{5.33}$$

and choose $T > a$ so large that

$$2\mu^{\frac{1}{\alpha}}\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}(S(T))^{\frac{1}{\alpha+1}} \leqq \frac{1}{4}\lambda^{\frac{1}{\alpha}}, \tag{5.34}$$

$$2\nu^{\frac{1}{\alpha}}(S(T))^{\frac{1}{\alpha}} \leqq \frac{1}{4}\lambda^{\frac{1}{\alpha}}, \tag{5.35}$$

and

$$\beta\tilde{\lambda}^{\beta-1}\nu^{\frac{1}{\alpha}}(S(T))^{\frac{1}{\alpha}} \leqq \frac{\nu}{2}. \tag{5.36}$$

For these λ, μ, ν and T define Y to be the set of functions $y \in C[T, \infty)$ such that

$$|y(t) - \lambda^{\frac{1}{\alpha}}t| \leqq \frac{1}{2}\lambda^{\frac{1}{\alpha}}t,\qquad t \geqq T, \tag{5.37}$$

and

(5.38) $$|y(t_1) - y(t_2)| \leqq \int_{t_1}^{t_2} (\lambda + \mu s^\beta Q(s) + \nu S(s))^{\frac{1}{\alpha}} ds, \qquad t_2 \geqq t_1 \geqq T,$$

and define Z to be the set of functions $z \in C[T, \infty)$ such that

(5.39) $$|z(t) - \lambda| \leqq \mu t^\beta Q(t) + \nu S(t), \qquad t \geqq T.$$

We now define the mapping $F\colon Y \times Z \to C[T, \infty) \times C[T, \infty)$ by

(5.40) $$F(y, z) = (F_1(y, z), F_2(y, z)), \qquad (y, z) \in Y \times Z,$$

where

(5.41) $$F_1(y, z)(t) = \lambda^{\frac{1}{\alpha}} T + \int_T^t (z(s))^{\frac{1}{\alpha}*} ds, \qquad t \geqq T,$$

and

(5.42) $$F_2(y, z)(t) = \lambda + Q(t)(y(t))^\beta + \beta \int_t^\infty Q(s)(y(s))^{\beta-1}(z(s))^{\frac{1}{\alpha}*} ds, \qquad t \geqq T.$$

Let $(y, z) \in Y \times Z$. Then, using (5.41), (5.42), (5.32)–(5.36), and the inequalities

$$|\xi^{\theta*} - \eta^{\theta*}| \leqq |\xi - \eta|^\theta \qquad \text{for } 0 < \theta \leqq 1 \text{ and } \xi\eta \geqq 0,$$
$$|\xi^{\theta*} - \eta^{\theta*}| \leqq 2|\xi - \eta|^\theta \qquad \text{for } 0 < \theta \leqq 1 \text{ and } \xi\eta < 0,$$

we obtain

$$\begin{aligned}
|F_1(y, z)(t) - \lambda^{\frac{1}{\alpha}} t| &\leqq \int_T^t |(z(s))^{\frac{1}{\alpha}*} - \lambda^{\frac{1}{\alpha}}| ds \\
&\leqq 2\int_T^t |z(s) - \lambda|^{\frac{1}{\alpha}} ds \leqq 2\int_T^t (\mu s^\beta Q(s) + \nu S(s))^{\frac{1}{\alpha}} ds \\
&\leqq 2\mu^{\frac{1}{\alpha}} \int_T^t s^{\frac{\beta}{\alpha}} (Q(s))^{\frac{1}{\alpha}} ds + 2\nu^{\frac{1}{\alpha}} \int_T^t (S(s))^{\frac{1}{\alpha}} ds \\
&\leqq 2\mu^{\frac{1}{\alpha}} \left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}} (S(T))^{\frac{1}{\alpha+1}} t + 2\nu^{\frac{1}{\alpha}} (S(T))^{\frac{1}{\alpha}} t \\
&\leqq \frac{1}{4}\lambda^{\frac{1}{\alpha}} t + \frac{1}{4}\lambda^{\frac{1}{\alpha}} t = \frac{1}{2}\lambda^{\frac{1}{\alpha}} t, \qquad t \geqq T,
\end{aligned}$$

$$|F_1(y, z)(t_1) - F_1(y, z)(t_2)| \leqq \int_{t_1}^{t_2} |z(s)|^{\frac{1}{\alpha}} ds$$

$$\leqq \int_{t_1}^{t_2} (\lambda + \mu s^\beta Q(s) + \nu S(s))^{\frac{1}{\alpha}} ds, \qquad t_2 \geqq t_1 \geqq T, \tag{5.43}$$

and

$$|F_2(y, z)(t) - \lambda| \leqq \left(\frac{3}{2}\lambda^{\frac{1}{\alpha}}\right)^\beta t^\beta Q(t) + \beta\tilde{\lambda}^{\beta-1} \int_t^\infty s^{\beta-1} Q(s)(\lambda + \mu s^\beta Q(s) + \nu S(s))^{\frac{1}{\alpha}} ds$$

$$\leqq \mu t^\beta Q(t) + \beta\tilde{\lambda}^{\beta-1}\lambda^{\frac{1}{\alpha}} \int_t^\infty s^{\beta-1} Q(s)\, ds + \beta\tilde{\lambda}^{\beta-1}\mu^{\frac{1}{\alpha}} \int_t^\infty s^{(1+\frac{1}{\alpha})\beta-1} (Q(s))^{1+\frac{1}{\alpha}} ds$$

$$+ \beta\tilde{\lambda}^{\beta-1}\nu^{\frac{1}{\alpha}} \int_t^\infty s^{\beta-1} Q(s)(S(s))^{\frac{1}{\alpha}} ds$$

$$\leqq \mu t^\beta Q(t) + \beta\tilde{\lambda}^{\beta-1}\lambda^{\frac{1}{\alpha}} S(t) + \beta\tilde{\lambda}^{\beta-1}\mu^{\frac{1}{\alpha}} S(t) + \beta\tilde{\lambda}^{\beta-1}\nu^{\frac{1}{\alpha}} (S(T))^{\frac{1}{\alpha}} \int_t^\infty s^{\beta-1} Q(s)\, ds$$

$$\leqq \mu t^\beta Q(t) + \beta\tilde{\lambda}^{\beta-1}(\lambda^{\frac{1}{\alpha}} + \mu^{\frac{1}{\alpha}}) S(t) + \beta\tilde{\lambda}^{\beta-1}\nu^{\frac{1}{\alpha}} (S(T))^{\frac{1}{\alpha}} S(t)$$

$$\leqq \mu t^\beta Q(t) + \nu S(t), \qquad t \geqq T.$$

Therefore, $(y, z) \in Y \times Z$ implies $F_1(y, z) \in Y$ and $F_2(y, z) \in Z$, so that F defined by (5.40) sends $Y \times Z$ into itself. Furthermore, as in the proof of Theorem 5.1 it can be shown that F is continuous and $F(Y \times Z)$ is relatively compact in the topology of $C[T, \infty) \times C[T, \infty)$. Consequently, the Schauder-Tychonoff theorem is applicable to F and there exists an element $(y, z) \in Y \times Z$ such that $(y, z) = F(y, z)$, i.e.,

$$\begin{aligned} y(t) &= \lambda^{\frac{1}{\alpha}} T + \int_T^t (z(s))^{\frac{1}{\alpha}*} ds, \qquad t \geqq T, \\ z(t) &= \lambda + Q(t)(y(t))^\beta + \beta \int_t^\infty Q(s)(y(s))^{\beta-1}(z(s))^{\frac{1}{\alpha}*} ds, \qquad t \geqq T. \end{aligned} \tag{5.44}$$

It is easily verified by differentiation of (5.44) that $y(t)$ is a solution of the equation (A) on $[T, \infty)$. That $y(t)$ satisfies (5.21) follows from (5.37).

(ii) Next suppose that $\alpha < 1$. Let λ, μ and $\tilde{\lambda}$ be as in the case (i). Take a constant $\nu > 0$ such that

$$\beta\tilde{\lambda}^{\beta-1}[(3\lambda)^{\frac{1}{\alpha}}+(3\mu)^{\frac{1}{\alpha}}]\leqq\frac{\nu}{2}.$$

Choose $T>a$ large enough so that

$$\frac{1}{\alpha}(2\lambda)^{\frac{1}{\alpha}-1}(\mu+\nu)S(T)\leqq\frac{1}{6}\lambda^{\frac{1}{\alpha}},$$

$$\frac{1}{\alpha}(4\mu)^{\frac{1}{\alpha}}\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}(S(T))^{\frac{1}{\alpha+1}}\leqq\frac{1}{6}\lambda^{\frac{1}{\alpha}},$$

$$\frac{1}{\alpha}(4\nu)^{\frac{1}{\alpha}}(S(T))^{\frac{1}{\alpha}}\leqq\frac{1}{6}\lambda^{\frac{1}{\alpha}},$$

and

$$\beta\tilde{\lambda}^{\beta-1}(3\nu)^{\frac{1}{\alpha}}(S(T))^{\frac{1}{\alpha}}\leqq\frac{\nu}{2},$$

and let the sets Y, Z and the mapping F be defined exactly as in the case (i) (cf. (5.37)–(5.41)). Using the inequality

$$|\xi^{\theta*}-\eta^{\theta*}|\leqq\theta[\max\{|\xi|,|\eta|\}]^{\theta-1}|\xi-\eta|,\qquad\theta>1,$$

we see that if $z(t)$ satisfies (5.39), then for $t\geqq T$

$$(5.45)\quad\begin{aligned}|(z(t))^{\frac{1}{\alpha}*}-\lambda^{\frac{1}{\alpha}}|&\leqq\frac{1}{\alpha}(\lambda+\mu t^{\beta}Q(t)+\nu S(t))^{\frac{1}{\alpha}-1}(\mu t^{\beta}Q(t)+\nu S(t))\\&\leqq\frac{1}{\alpha}[(2\lambda)^{\frac{1}{\alpha}-1}+(2\mu t^{\beta}Q(t)+2\nu S(t))^{\frac{1}{\alpha}-1}](\mu t^{\beta}Q(t)+\nu S(t))\\&\leqq\frac{1}{\alpha}(2\lambda)^{\frac{1}{\alpha}-1}(\mu t^{\beta}Q(t)+\nu S(t))+\frac{1}{\alpha}(2\mu t^{\beta}Q(t)+2\nu S(t))^{\frac{1}{\alpha}}\\&\leqq\frac{1}{\alpha}(2\lambda)^{\frac{1}{\alpha}-1}(\mu t^{\beta}Q(t)+\nu S(t))+\frac{1}{\alpha}[(4\mu)^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}(Q(t))^{\frac{1}{\alpha}}+(4\nu)^{\frac{1}{\alpha}}(S(t))^{\frac{1}{\alpha}}].\end{aligned}$$

Since $|F_1(y,z)(t)-\lambda^{1/\alpha}t|\leqq\int_T^t|(z(s))^{1/\alpha*}-\lambda^{1/\alpha}|ds$ for $(y,z)\in Y\times Z$, (5.45) and the choice of λ, $\tilde{\lambda}$, μ, ν and T ensure that

$$|F_1(y,z)(t)-\lambda^{\frac{1}{\alpha}}t|\leqq\frac{1}{2}\lambda^{\frac{1}{\alpha}}t,\qquad t\geqq T.$$

It is clear that (5.42) holds, and so F_1 maps $Y\times Z$ into Y. F_2 maps $Y\times Z$

into Z; in fact, we have for $(y, z) \in Y \times Z$

$$|F_2(y, z)(t) - \lambda| \leqq \left(\frac{3}{2}\lambda^{\frac{1}{\alpha}}\right)^{\beta} t^{\beta} Q(t) + \beta\tilde{\lambda}^{\beta-1} \int_t^{\infty} s^{\beta-1} Q(s)(\lambda + \mu s^{\beta} Q(s) + \nu S(s))^{\frac{1}{\alpha}} ds$$

$$\leqq \mu t^{\beta} Q(t) + \beta\tilde{\lambda}^{\beta-1}(3\lambda)^{\frac{1}{\alpha}} \int_t^{\infty} s^{\beta-1} Q(s)\, ds$$

$$+ \beta\tilde{\lambda}^{\beta-1}(3\mu)^{\frac{1}{\alpha}} \int_t^{\infty} s^{(1+\frac{1}{\alpha})\beta-1} (Q(s))^{1+\frac{1}{\alpha}} ds + \beta\tilde{\lambda}^{\beta-1}(3\nu)^{\frac{1}{\alpha}} \int_t^{\infty} s^{\beta-1} Q(s)(S(s))^{\frac{1}{\alpha}} ds$$

$$\leqq \mu t^{\beta} Q(t) + \beta\tilde{\lambda}^{\beta-1}[(3\lambda)^{\frac{1}{\alpha}} + (3\mu)^{\frac{1}{\alpha}}] S(t) + \beta\tilde{\lambda}^{\beta-1}(3\nu)^{\frac{1}{\alpha}}(S(T))^{\frac{1}{\alpha}} S(t)$$

$$\leqq \mu t^{\beta} Q(t) + \nu S(t), \qquad t \geqq T.$$

It follows therefore that F maps $Y \times Z$ into itself. The continuity of F and the relative compactness of $F(Y \times Z)$ can be proved routinely, and so F has a fixed element $(y, z) \in Y \times Z$. As is easily seen, the first component y gives a solution of (A) on $[T, \infty)$ satisfying (5.21). This completes the proof of the sufficiency part of Theorem 5.2.

6. Oscillation criteria

We are finally interested in criteria for all proper solutions of (A) to be oscillatory.

THEOREM 6.1. *Suppose that $\alpha < \beta$ and $Q(t) \geqq 0$ for $t \geqq a$. Then, all proper solutions of* (A) *are oscillatory if and only if*

$$\int_a^{\infty} \left[(Q(t))^{\frac{1}{\alpha}} + \left(\int_t^{\infty} (Q(s))^{1+\frac{1}{\alpha}} ds\right)^{\frac{1}{\alpha}}\right] dt = \infty. \tag{6.1}$$

PROOF. The failure of (6.1) implies (5.1), so that, by Theorem 5.1, there exists a nonoscillatory solution of (A).

Conversely, suppose the existence of a nonoscillatory solution $y(t)$ of (A). We may assume that $y(t) > 0$ for $t \geqq t_0$. Then, by Lemma 4.1,

$$\frac{(y'(t))^{\alpha}}{(y(t))^{\beta}} = Q(t) + \beta \int_t^{\infty} \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds, \qquad t \geqq t_0. \tag{6.2}$$

It follows that

$$\frac{(y'(t))^{\alpha}}{(y(t))^{\beta}} \geqq Q(t) \quad \text{or} \quad \frac{y'(t)}{(y(t))^{\frac{\beta}{\alpha}}} \geqq (Q(t))^{\frac{1}{\alpha}}, \qquad t \geqq t_0, \tag{6.3}$$

and

$$(6.4) \qquad \frac{(y'(t))^{\alpha}}{(y(t))^{\beta}} \geqq \beta \int_t^{\infty} \frac{(y'(s))^{\alpha+1}}{(y(s))^{\beta+1}} ds, \qquad t \geqq t_0.$$

Integrating the second inequality in (6.3) from t_0 to t and letting $t \to \infty$, we have, because of $\alpha < \beta$,

$$(6.5) \qquad \int_{t_0}^{\infty} (Q(s))^{\frac{1}{\alpha}} ds < \infty.$$

Next, substituting (6.3) into (6.4), we obtain

$$\frac{(y'(t))^{\alpha}}{(y(t))^{\beta}} \geqq \beta \int_t^{\infty} (Q(s))^{1+\frac{1}{\alpha}} (y(s))^{\frac{\beta-\alpha}{\alpha}} ds,$$

$$\geqq \beta (y(t_0))^{\frac{\beta-\alpha}{\alpha}} \int_t^{\infty} (Q(s))^{1+\frac{1}{\alpha}} ds, \qquad t \geqq t_0,$$

which is equivalent to

$$\frac{y'(t)}{(y(t))^{\frac{\beta}{\alpha}}} \geqq [\beta (y(t_0))^{\frac{\beta-\alpha}{\alpha}}]^{\frac{1}{\alpha}} \left(\int_t^{\infty} (Q(s))^{1+\frac{1}{\alpha}} ds \right)^{\frac{1}{\alpha}}, \qquad t \geqq t_0.$$

An integration of this inequality yields

$$(6.6) \qquad \int_{t_0}^{\infty} \left(\int_t^{\infty} (Q(s))^{1+\frac{1}{\alpha}} ds \right)^{\frac{1}{\alpha}} dt < \infty.$$

The inequalities (6.5) and (6.6) thus obtained clearly contradict (6.1), and so, under the condition (6.1), all proper solutions of (A) must be oscillatory. This finishes the proof.

REMARK 6.1. Theorem 6.1 is a generalization of an oscillation theorem of Butler [1] for the superlinear Emden-Fowler equation (C) ($\beta > 1$) subject to the condition $Q(t) \geqq 0$.

Let us consider the case $\alpha > \beta$ in (A). From Theorem 5.2 we see that in order that all solutions of (A) with $Q(t) \geqq 0$ be oscillatory it is necessary that either

$$(6.7) \qquad \int_a^{\infty} t^{\beta-1} Q(t) dt = \infty \quad \text{or} \quad \int_a^{\infty} t^{(1+\frac{1}{\alpha})\beta-1} (Q(t))^{1+\frac{1}{\alpha}} dt = \infty.$$

In view of Naito's result [15] characterizing the oscillation situation for the sublinear Emden-Fowler equation (C) ($\beta < 1$) it would be natural to expect that (6.7) is also sufficient for oscillation of all solutions of (A). However, we have been unable to prove this conjecture. What we have proved so far is the sufficiency of the second integral condition in (6.7).

THEOREM 6.2. *Suppose that* $\alpha > \beta$ *and* $Q(t) \geqq 0$ *for* $t \geqq a$. *All proper solutions of* (A) *are oscillatory if*

$$\int_a^\infty t^{(1+\frac{1}{\alpha})\beta - 1} (Q(t))^{1+\frac{1}{\alpha}} dt = \infty. \tag{6.7'}$$

PROOF. Assume that (A) has a nonoscillatory solution $y(t)$. We may suppose that $y(t) > 0$ for $t \geqq t_0$. From the proof of Theorem 4.1 we see that the function $K_2(t)$ defined by (4.13) is convergent:

$$K_2(t) = \int_t^\infty (Q(s))^{1+\frac{1}{\alpha}} (y(s))^{(1+\frac{1}{\alpha})\beta - 1} ds < \infty, \qquad t \geqq t_0. \tag{6.8}$$

(i) Suppose that $\left(1 + \dfrac{1}{\alpha}\right)\beta - 1 \leqq 0$. Since $y(t) \leqq ct$, $t \geqq t_0$, for some $c > 0$ (cf. Theorem 3.3), we then conclude from (6.8) that

$$\int_t^\infty s^{(1+\frac{1}{\alpha})\beta - 1} (Q(s))^{1+\frac{1}{\alpha}} ds < \infty,$$

which contradicts (6.7').

(ii) Suppose that $\left(1 + \dfrac{1}{\alpha}\right)\beta - 1 > 0$. From (4.11), (4.12) and (4.13) it follows that

$$(y'(t))^\alpha \geqq \beta K_2(t) \quad \text{or} \quad y'(t) \geqq \beta^{\frac{1}{\alpha}} (K_2(t))^{\frac{1}{\alpha}}, \qquad t \geqq t_0.$$

Integrating the last inequality over $[t_0, t]$, we have

$$y(t) \geqq \beta^{\frac{1}{\alpha}} \int_{t_0}^t (K_2(s))^{\frac{1}{\alpha}} ds \geqq \beta^{\frac{1}{\alpha}} (K_2(t))^{\frac{1}{\alpha}} (t - t_0), \qquad t \geqq t_0,$$

so that

$$\frac{(Q(t))^{1+\frac{1}{\alpha}} (y(t))^{(1+\frac{1}{\alpha})\beta - 1}}{(K_2(t))^{(1+\frac{1}{\alpha})\frac{\beta}{\alpha} - \frac{1}{\alpha}}} \geqq \beta^{(1+\frac{1}{\alpha})\frac{\beta}{\alpha} - \frac{1}{\alpha}} (t - t_0)^{(1+\frac{1}{\alpha})\beta - 1} (Q(t))^{1+\frac{1}{\alpha}}, \qquad t \geqq t_0.$$

Since $K_2'(t) = -(Q(t))^{1+\frac{1}{\alpha}}(y(t))^{(1+\frac{1}{\alpha})\beta-1}$, the above inequality can be written as

$$(6.9)\qquad \frac{-K_2'(t)}{(K_2(t))^{(1+\frac{1}{\alpha})\frac{\beta}{\alpha}-\frac{1}{\alpha}}} \geqq \beta^{(1+\frac{1}{\alpha})\frac{\beta}{\alpha}-\frac{1}{\alpha}}(t-t_0)^{(1+\frac{1}{\alpha})\beta-1}(Q(t))^{1+\frac{1}{\alpha}}, \qquad t \geqq t_0.$$

We integrate (6.9) over $[t_0, t]$ and let $t \to \infty$. Noting that $\alpha > \beta$, we then conclude that

$$\beta^{(1+\frac{1}{\alpha})\frac{\beta}{\alpha}-\frac{1}{\alpha}} \int_{t_0}^{\infty} (t-t_0)^{(1+\frac{1}{\alpha})\beta-1}(Q(t))^{1+\frac{1}{\alpha}}\,dt \leqq \frac{(K_2(t_0))^{(1+\frac{1}{\alpha})(1-\frac{\beta}{\alpha})}}{\left(1+\frac{1}{\alpha}\right)\left(1-\frac{\beta}{\alpha}\right)},$$

which again contradicts (6.7′). This completes the proof.

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