Hiroshima Math. J.
25 (1995), 313–319

Nonhomogeneity of Picard dimensions for negative radial densities

Dedicated to Professor Fumi-Yuki Maeda on his 60th birthday

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(Received November 26, 1993)

Consider the punctured open unit ball $\{0 < |x| < 1\}$ in the punctured Euclidean m-space $R^m \setminus \{0\}$ $(m \geq 2)$ in which we regard the origin $x = 0$ as an ideal boundary component of $R^m \setminus \{0\}$. For each s in $(0, 1]$ we set $U_s = \{0 < |x| < s\}$, which is also an ideal boundary neighbourhood of the ideal boundary component $x = 0$ in $R^m \setminus \{0\}$, so that Γ_s: $|x| = s$ is the relative boundary of U_s and the relative closure $\bar{U}_s$ of U_s in $R^m \setminus \{0\}$ is $U_s \cup \Gamma_s$. We set $U_1 = U$ and $\Gamma_1 = \Gamma$. A *density* $P(x)$ on U_s is a locally Hölder continuous function defined on $\bar{U}_s$. Consider the time independent Schrödinger equation

$$L_P u(x) \equiv -\Delta u(x) + P(x)u(x) = 0 \tag{1}$$

defined on $\bar{U}_s$, where Δ is the Laplacian $\Delta = \sum_{i=1}^m \partial^2/\partial x_i^2$. We are interested in the class $P(U_s, P)$ of nonnegative solutions of (1) in U_s with vanishing boundary values on Γ_s. Let $r\omega$ be the polar coordinate expression of x, where $r = |x|$ and $\omega = (x/|x|) \in \Gamma$. We set

$$l(u) \equiv -\frac{s}{\omega_m} \int_\Gamma \left[\frac{\partial}{\partial r} u(r\omega)\right]_{r=s} d\omega,$$

where $d\omega$ is the area element on Γ, ω_m the area of Γ and $\partial/\partial r$ the outer normal derivative on Γ_s considered in $\bar{U}_s$. It is convenient to consider the subclass $P_1(U_s, P) \equiv \{u \in P(U_s, P); l(u) = 1\}$. Since $P_1(U_s, P)$ is convex, we can consider the set $ex.P_1(U_s, P)$ of extreme points of $P_1(U_s, P)$ and the cardinal number $\#(ex.P_1(U_s, P))$ of $ex.P_1(U_s, P)$ which will be referred to as the *Picard dimension* of (U_s, P) at $x = 0$, $\dim(U_s, P)$ in notation:

$$\dim(U_s, P) = \#(ex.P_1(U_s, P)).$$

There exists a t in $(0, 1]$ such that $\dim P(U_s, P) = \dim P(U_t, P)$ for any s in $(0, t]$ ([8], [7], [9]). Hence we can define the *Picard dimension* of P at $x = 0$, $\dim P$ in notation, by

$$\dim P = \lim_{s \downarrow 0} \dim(U_s, P).$$

We showed in [5] that there exists a radial density P on U_s with $0<s<1$ such that $\dim P=1$ but $\dim(cP)=0$ for every $c>1$. Here a density P is radial, by definition, if $P(x)$ depends only on $|x|$. It is asked in [9] whether or not there exists a radial density P on U such that $\dim P=0$ but $\dim(cP)=1$ holds for every c in (0, 1). Consider the negative radial density P given by

$$(2)\qquad P(x)\equiv -\frac{1}{4|x|^2}\left\{(m-2)^2+\frac{1}{\left(\log\frac{\eta}{|x|}\right)^2}+\frac{1+a^2}{\left(\log\frac{\eta}{|x|}\cdot\log\log\frac{\eta}{|x|}\right)^2}\right\}$$

where a is any fixed positive constant and η is any fixed constant with $\eta>e^e$. The purpose of this paper is to show the following result which settles the above problem in the positive.

THEOREM. *The density P given by* (2) *satisfies*

$$\dim P=0\quad but\quad \dim(cP)=1$$

for any c in (0, 1).

The above density P given in (2) is also considered in [2] for the case $m=2$ in the study of the existence and asymptotic behaviors of positive solutions of (1) near the point at infinity; it is also shown that there exists a density P for every dimension $m\geq 2$ such that $\dim P=0$ (i.e. the nonexistence of positive solutions) and $\dim(cP)\geq 1$ (i.e. the mere existence of positive solutions) for every $0<c<1$ stated in our formulation.

1. We begin with some definitions. A function u is a solution of (1) in U_s if u is a C^2 function on U_s which satisfies (1) in U_s. A lower semicontinuous, lower finite function v on U_s is a supersolution of (1) in U_s if $v(x)\geq u(x)$ in B whenever $v(x)\geq u(x)$ on the boundary ∂B of B for any ball B in U_s and for any solution $u(x)$ of (1) in B continuous in $\bar{B}$. If $v(x)$ is a C^2 function on U_s, then $v(x)$ is a supersolution of (1) on U_s if and only if $L_P v(x)\geq 0$ on U_s. A *potential* p of (1) on U_s is a nonnegative supersolution of (1) in U_s such that, if $p\geq u$ holds on U_s for some solution u of (1) in U_s, then $u\leq 0$ on U_s. We take any point y fixed in U_s. By the *Green's function* $G_s(x, y)$ of (1) on U_s (with its pole y) we mean, if it exists, the potential of (1) on U_s satisfying $L_P G_s(x, y)=\delta_y(x)$ on U_s, where $\delta_y(x)$ is the Dirac measure at y. The pair $(U_s, \mathscr{H}_P)$ with the sheaf $\mathscr{H}_P$ of solutions of (1) on U_s is a Brelot's harmonic space. There exists a positive potential of (1) on U_s if and only if there exists the Green's function $G_s(x, y)$ of (1) on U_s ([3], [6], etc.).

Consider the density $Q(x)$ on U given by

$$(3)\qquad Q(x)\equiv-\frac{1}{4|x|^2}\left\{(m-2)^2+\frac{1}{\left(\log\frac{\eta}{|x|}\right)^2}+\frac{1}{\left(\log\frac{\eta}{|x|}\cdot\log\log\frac{\eta}{|x|}\right)^2}\right\}.$$

To find linearly independent solutions of

$$(4)\qquad -\left(\frac{d^2}{dr^2}u(r)+\frac{m-1}{r}\frac{d}{dr}u(r)\right)+Q(r)u(r)=0$$

where $Q(r)$ is given in (3) with $r=|x|$, we set $\log_2 r=\log\log r$ and $\log_3 r=\log\log_2 r$. Take the function $p_{\alpha,\beta}(r)\equiv r^\alpha(\log(\eta/r)\log_2(\eta/r))^\beta$, where α and β are arbitrarily given constants which are determined later. Then by the direct computation we can easily see that

$$\frac{\frac{d}{dr}p_{\alpha,\beta}(r)}{p_{\alpha,\beta}(r)}=\frac{\alpha}{r}-\frac{\beta}{r\log\frac{\eta}{r}}-\frac{\beta}{r\log\frac{\eta}{r}\log_2\frac{\eta}{r}}$$

and hence

$$\frac{d^2}{dr^2}p_{\alpha,\beta}(r)=\left\{\frac{\alpha(\alpha-1)}{r^2}+\frac{\beta(1-2\alpha)}{r^2\log\frac{\eta}{r}}+\frac{\beta^2-\beta}{\left(r\log\frac{\eta}{r}\right)^2}\right.$$
$$\left.+\frac{\beta(1-2\alpha)}{r^2\log\frac{\eta}{r}\log_2\frac{\eta}{r}}+\frac{\beta(2\beta-1)}{r^2\left(\log\frac{\eta}{r}\right)^2\log_2\frac{\eta}{r}}+\frac{\beta^2-\beta}{\left(r\log\frac{\eta}{r}\log_2\frac{\eta}{r}\right)^2}\right\}p_{\alpha,\beta}(r)$$

on (0, 1]. Therefore we have the following identity:

$$\frac{d^2}{dr^2}p_{\alpha,\beta}(r)+\frac{m-1}{r}\frac{d}{dr}p_{\alpha,\beta}(r)=\left\{\frac{\alpha(\alpha+m-2)}{r^2}-\frac{(2\alpha+m-2)\beta}{r^2\log\frac{\eta}{r}}+\frac{\beta^2-\beta}{\left(r\log\frac{\eta}{r}\right)^2}\right.$$
$$\left.-\frac{(2\alpha+m-2)\beta}{r^2\log\frac{\eta}{r}\log_2\frac{\eta}{r}}+\frac{\beta(2\beta-1)}{\left(r\log\frac{\eta}{r}\right)^2\log_2\frac{\eta}{r}}+\frac{\beta^2-\beta}{\left(r\log\frac{\eta}{r}\log_2\frac{\eta}{r}\right)^2}\right\}p_{\alpha,\beta}(r).$$

Setting $\alpha=-(m-2)/2$ and $\beta=1/2$, it follows that the function

$$p(r)=r^{-\frac{m-2}{2}}\left\{\log\frac{\eta}{r}\log_2\frac{\eta}{r}\right\}^{\frac{1}{2}}$$

is a solution of (4) in (0, 1]. Observe that the function $q(r) \equiv \log_3 (\eta/r)$ is a solution of

$$\frac{d^2}{dr^2} v(r) + \left\{ \frac{m-1}{r} + 2\frac{\frac{d}{dr}p(r)}{p(r)} \right\} \frac{d}{dr} v(r) = 0 \tag{5}$$

in (0, 1]. Hence $p(r)$ and $p(r)q(r)$ are linearly independent solutions of (4) in (0, 1]. It is also evident that $p(|x|)$ and $p(|x|)q(|x|)$ are solutions of

$$L_Q u(x) \equiv (-\Delta + Q(x))u(x) = 0 \tag{6}$$

in U continuous in $\bar{U}$.

Choose any s in (0, 1] and take any t fixed in $(0, s)$. We set

$$h(x) = \frac{q(|x|) - q(s)}{q(t) - q(s)} p(|x|)$$

which is a solution of (6) in U which coincides with $p(t)$ on Γ_t and 0 on Γ_s. Observe that

$$\frac{q(|x|) - q(s)}{q(t) - q(s)} > 1 \qquad (< 1, \text{ resp.})$$

for $|x| < t$ ($> t$, resp.). In view of this we see that

$$h(x) > p(|x|) \qquad (h(x) < p(|x|), \text{ resp.})$$

for $|x| < t$ ($> t$, resp.). Consider the function $v(x)$ given by $h(x)$ on $U_s \setminus \bar{U}_t$ and $p(|x|)$ on $\bar{U}_t$. Since

$$v(x) = \min (h(x), p(|x|)) \qquad (x \in U_s),$$

$v(x)$ is a positive supersolution of (6) on U_s. The unicity theorem assures that $v(x)$ is not a solution of (6) on U_s by virtue of the fact that $h(x) \neq p(|x|)$ on U_s. Hence by the Riesz decomposition theorem (cf., e.g. [1], [6]) there exists a positive potential and thus the Green's function of (6) on U_s. Observe that $Q(x) = O(|x|^{-2})$ as $|x| \to 0$. It is known ([4]) that $\dim (U_s, Q) = 1$ whenever $Q(x) = O(|x|^{-2})$ as $|x| \to 0$ and there exists the Green's function of (6) on U_s. Since s is arbitrary in (0, 1], we have shown:

ASSERTION 1. $\dim Q = 1$ *for the density* Q *given by* (3).

2. We next consider the Schrödinger equation

$$L_P u(x) \equiv (-\Delta + P(x))u(x) = 0 \tag{7}$$

on $\bar{U}_s$ with $0 < s \le 1$, where $P(x)$ is the density given by (2). We set

$$f(r) = p(r)\sin\left(\frac{a}{2}q(r)\right) \quad \text{and} \quad g(r) = p(r)\cos\left(\frac{a}{2}q(r)\right).$$

Since $\Delta_r(u(r)v(r)) = (\Delta_r u(r))v(r) + 2\nabla_r u(r)\cdot\nabla_r v(r) + u(r)(\Delta_r v(r))$ for any functions $u(r)$ and $v(r)$, where $\Delta_r = \partial^2/\partial r^2 + (m-1)r^{-1}\partial/\partial r$ and $\nabla_r = \partial/\partial r$, it is easy to see that

$$\Delta_r f(r) = \left\{\frac{\Delta_r p(r)}{p(r)} - \frac{a^2}{4}\left\{\frac{d}{dr}q(r)\right\}^2\right\} f(r) + \frac{a}{2}\left\{2\frac{\frac{d}{dr}p(r)}{p(r)}\frac{d}{dr}q(r) + \Delta_r q(r)\right\} g(r)$$

and

$$\Delta_r g(r) = \left\{\frac{\Delta_r p(r)}{p(r)} - \frac{a^2}{4}\left\{\frac{d}{dr}q(r)\right\}^2\right\} g(r) - \frac{a}{2}\left\{2\frac{\frac{d}{dr}p(r)}{p(r)}\frac{d}{dr}q(r) + \Delta_r q(r)\right\} f(r).$$

Then (5) with $v = q$ yields

$$2\frac{\frac{d}{dr}p(r)}{p(r)}\frac{d}{dr}q(r) + \Delta_r q(r) = 0$$

and also (4) with $u = p$ implies $\Delta_r p(r)/p(r) = Q(r)$. Therefore functions $f(r)$ and $g(r)$ are solutions of

$$(8) \qquad -\left(\frac{d^2}{dr^2}u(r) + \frac{m-1}{r}\frac{d}{dr}u(r)\right) + P(r)u(r) = 0$$

in $(0, 1]$, where $P(r)$ is given in (2) with $r = |x|$. Also the Wronskian of $f(r)$ and $g(r)$ does not vanish in $(0, 1]$ so that $f(r)$ and $g(r)$ are linearly independent solutions of (8) in $(0, 1]$.

Suppose that there exists a nonzero function $h(x) = h(r\omega)$ in $P(U_s, P)$ for some s in $(0, 1]$. The function

$$h^*(r) = \int_\Gamma h(r\omega)d\omega$$

is a positive solution of (8) in $(0, s)$ so that it is a linear combination of $f(r)$ and $g(r)$. Hence we have

$$h^*(r) = kp(r)\sin\left(\frac{a}{2}q(r) + \rho\right)$$

in $(0, s)$ for some constant $k>0$ and ρ with $2\pi>\rho\geq 0$. This is a contradiction since h^* is not of constant sign in $(0, s)$. Therefore $P(U_s, P)=\{0\}$ for any s in $(0,1]$ and we have:

ASSERTION 2. $\dim P=0$ *for the density* P *given by* (2).

3. *Proof of Theorem.* Since we have shown $\dim P=0$ in the above assertion 2, we only have to show that $\dim(cP)=1$ for any c in $(0, 1)$. Let $S(x)$ and $T(x)$ be any densities on U. We write $S(x)\prec T(x)$ if there exists an s in $(0, 1]$ such that $S(x)<T(x)$ in U_s. We consider the Schrödinger equation given by

$$L_{cP}u(x)\equiv(-\Delta+cP(x))u(x)=0 \tag{9}$$

where c is any constant in $(0, 1)$. We observe that the following relation is valid for any c in $(0, 1)$:

$$4|x|^2\left(\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|}\right)^2(cP(x)-Q(x))+ca^2=$$

$$(1-c)(m-2)^2\left(\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|}\right)^2+(1-c)\left(\log_2\frac{\eta}{|x|}\right)^2+(1-c)\succ ca^2.$$

Therefore we have $Q(x)\prec cP(x)$ for any c in $(0, 1)$. We recall that $u(x)\equiv p(|x|)$ is a positive solution of (6) in U: $L_Q u(x)=0$ on U. On the other hand, since we have

$$L_{cP}u(x)=L_Q u(x)+(cP(x)-Q(x))u(x)=(cP(x)-Q(x))u(x)\succ 0,$$

there exists a $t\in(0, 1)$ such that $L_{cP}u(x)>0$ on U_s for any $s\in(0, t)$ so that $u(x)$ is a positive supersolution but not a solution of (9) in U_s. Hence, again by the Riesz decomposition theorem, there exists a positive potential of (9) in U_s and thus the Green's function of (9) in U_s. It is clear that $cP(x)=O(|x|^{-2})$ as $|x|\to 0$. Hence again by [4] we have $\dim(U_s, cP)=1$ for any s in $(0, t)$ and a fortiori

$$\dim cP=\lim_{s\downarrow 0}\dim(U_s, cP)=1$$

for any c in $(0, 1)$. The proof of Theorem is herewith complete.

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