HIROSHIMA MATH. J.
25 (1995), 571–583

Oscillation criteria for hale-linear second order differential equations

Horng Jaan LI

(Received March 14, 1994)
(Revised May 31, 1994)

Abstract. Some oscillation criteria are given for second order nonlinear differential equation

$$[\Phi(u'(t))]' + c(t)\Phi(u(t)) = 0,$$

where $c(t)$ is a continuous function on $[0, \infty)$ and $\Phi: \mathbb{R} \to \mathbb{R}$ is defined by $\Phi(x) = |x|^{p-2}x$ with $p > 1$ a fixed real number. If $p = 2$, then these results improve earlier oscillation criteria of Wintner, Hartman, Kamenev and Philos.

1. Introduction

In the paper, we are concerned with the differential equation

$$\text{(E)} \qquad [\Phi(u'(t))]' + c(t)\Phi(u(t)) = 0, \qquad t \geq t_0,$$

where $c(t)$ is a continuous function on $[t_0, \infty)$ and $\Phi(s)$ is a real-valued function defined by $\Phi(s) = |s|^{p-2}s$ with $p > 1$ a fixed real number. If $p = 2$, then equation (E) reduces to the linear differential equation

$$\text{(E}_1\text{)} \qquad u''(t) + c(t)u(t) = 0.$$

By a solution of (E) we mean a function $u \in C^1[t_0, \infty)$ such that $\Phi(u') \in C^1[t_0, \infty)$ and that satisfies (E). In [5], Pino established the existence, uniqueness and extension to $[t_0, \infty)$ of solutions to the initial value problem for (E). We will say that a nontrivial solution u of (E) is oscillatory if it has arbitrary large zeros, and otherwise it is nonoscillatory. Equation (E) is oscillatory if all its solutions are oscillatory.

Wintner [6] showed that equation (E_1) is oscillatory if

$$\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} c(\xi)d\xi ds = \infty.$$

Hartman [2] prove that the limit cannot be replaced by the upper limit in the above assumption and that the condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} c(\xi)d\xi ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} c(\xi)d\xi ds \leq \infty$$

suffices for the oscillation of equation (E_1).

Kamenev [3] improved Wintner's result and proved that the condition

$$\limsup_{t\to\infty} t^{-\lambda} \int_{T}^{t} (t-s)^{\lambda} c(s)ds = \infty \qquad \text{for some } \lambda > 1,$$

is sufficient for the oscillation of (E_1). Kamenev's criterion has been extended in various directions by Philos [4], Yan [7] and Yeh [8, 9].

In this note, we give an extension of Kamenev's criterion to equation (E) by using a well-known inequality as stated in Lemma 1 below. We also extend and improve the result of Philos [4].

2. Main results

In this section, we will establish some oscillation criteria for equation (E) which generalize Philos' criteria [4].

In order to discuss our main results, we need the following well-known inequality which is due to Hardy, Littlewood and Polya [1, Theorem 41].

LEMMA 1. *If X and Y are nonnegative, then*

$$X^{\gamma} + (\gamma - 1)Y^{\gamma} - \gamma X Y^{\gamma-1} \geq 0, \qquad \gamma > 1,$$

where equality holds if and only if $X = Y$.

THEOREM 2. *Let $D_0 = \{(t, s): t > s \geq t_0\}$ and $D = \{(t, s): t \geq s \geq t_0\}$. Assume that $H \in C(D; \mathbb{R})$ satisfies the following conditions:*

(i) $H(t, t) = 0$ *for* $t \geq t_0$, $H(t, s) > 0$ *for* $t > s \geq t_0$;

(ii) *H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.*

Suppose that $h: D_0 \to \mathbb{R}$ is a continuous function such that

$$-\frac{\partial H}{\partial s}(t, s) = h(t, s)[H(t, s)]^{1/q} \qquad \textit{for all } (t, s) \in D_0$$

and

$$\int_{t_0}^{t} h^p(t, s)ds < \infty \qquad \textit{for all } t \geq t_0,$$

where $(1/p) + (1/q) = 1$. If

$$(\mathrm{C}_1) \qquad \limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left\{ H(t, s)c(s) - \left(\frac{1}{p} h(t, s) \right)^p \right\} ds = \infty,$$

then equation (E) *is oscillatory.*

PROOF. Let $u(t)$ be a nonoscillatory solution of (E). Without loss of generality, we may assume that $u(t) > 0$ on $[T_0, \infty)$, for some $T_0 \geq t_0$. Define

$$(\mathrm{E}_2) \qquad v(t) = \frac{\Phi(u'(t))}{\Phi(u(t))} \qquad \text{for all } t \geq T_0.$$

It follows from (E) that

$$v'(t) = -(p-1)|v(t)|^q - c(t) \qquad \text{for all } t \geq T_0.$$

Since

$$\int_{t_0}^{t} h^p(t, s) ds < \infty \qquad \text{for all } t \geq t_0,$$

then for all $t \geq T \geq T_0$,

$$\begin{aligned}
&\int_T^t H(t, s)c(s) ds \\
&= H(t, T)v(T) - \int_T^t \left(-\frac{\partial H}{\partial s}(t, s) \right) v(s) ds - (p-1) \int_T^t H(t, s)|v(s)|^q ds \\
&= H(t, T)v(T) - \int_T^t \{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1) H(t, s)|v(s)|^q \} ds \\
&= H(t, T)v(T) - \int_T^t \left\{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1) H(t, s)|v(s)|^q \right. \\
&\qquad \left. + \left(\frac{1}{p} h(t, s) \right)^p \right\} ds + \int_T^t \left(\frac{1}{p} h(t, s) \right)^p ds.
\end{aligned}$$

Hence, for all $t \geq T \geq T_0$, we have

$$\begin{aligned}
(1) \qquad &\int_T^t \left\{ H(t, s)c(s) - \left(\frac{1}{p} h(t, s) \right)^p \right\} ds \\
&= H(t, T)v(T) - \int_T^t \left\{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1) H(t, s)|v(s)|^q \right. \\
&\qquad \left. + \left(\frac{1}{p} h(t, s) \right)^p \right\} ds
\end{aligned}$$

Since $q > 1$, then, by Lemma 1,

$$h(t, s)[H(t, s)]^{1/q}v(s) + (p-1)H(t, s)|v(s)|^q + \left(\frac{1}{p}h(t, s)\right)^p \geq 0$$

for all $t > s \geq T_0$.

This implies that for every $t \geq T_0$,

$$\int_{T_0}^{t}\left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\}ds$$
$$\leq H(t, T_0)v(T_0) \leq H(t, T_0)|v(T_0)| \leq H(t, t_0)|v(T_0)|.$$

Therefore,

$$\int_{t_0}^{t}\left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\}ds$$
$$= \int_{t_0}^{T_0}\left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\}ds + \int_{T_0}^{t}\left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\}ds$$
$$\leq H(t, t_0)\int_{t_0}^{T_0}|c(s)|\,ds + H(t, t_0)|v(T_0)|$$
$$= H(t, t_0)\left\{\int_{t_0}^{T_0}|c(s)|\,ds + |v(T_0)|\right\}$$

for all $t \geq T_0$. This gives

$$\limsup_{t\to\infty}\frac{1}{H(t, t_0)}\int_{t_0}^{t}\left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\}ds \leq \int_{t_0}^{T_0}|c(s)|\,ds + |v(T_0)|,$$

which contradicts (C_1). This completes the proof of the theorem.

THEOREM 3. *Let H and h be as in Theorem 2, and let*

$$(C_2) \qquad 0 < \inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t, s)}{H(t, t_0)}\right\} \leq \infty$$

and

$$(C_3) \qquad \limsup_{t\to\infty}\frac{1}{H(t, t_0)}\int_{t_0}^{t}h^p(t, s)\,ds < \infty.$$

Suppose there exists a function $A \in C[t_0, \infty)$ such that

$$(C_4) \qquad \int_{t_0}^{\infty}A_+^q(s)ds = \infty$$

and

$$(C_5) \qquad \limsup_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \left\{ H(t, s)c(s) - \left(\frac{1}{p} h(t, s) \right)^p \right\} ds \geq A(T)$$

for every $T \geq t_0$,

where $A_+(s) = \max\{A(s), 0\}$, $s \geq t_0$. *Then equation* (E) *is oscillatory.*

PROOF. Let $u(t)$ be a nonoscillatory solution of (E). Without loss of generality, we may assume that $u(t) > 0$ on $[T_0, \infty)$. Define $v(t)$ as in (E_2) for $t \geq T_0$. As in the proof of Theorem 2, (1) holds. Then

$$(2) \qquad \frac{1}{H(t, T)} \int_T^t \left\{ H(t, s)c(s) - \left(\frac{1}{p} h(t, s) \right)^p \right\} ds$$

$$= v(T) - \frac{1}{H(t, T)} \int_T^t \left\{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1)H(t, s)|v(s)|^q + \left(\frac{1}{p} h(t, s) \right)^p \right\} ds$$

for $t > T > T_0$. Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \left\{ H(t, s)c(s) - \left(\frac{1}{p} h(t, s) \right)^p \right\} ds$$

$$= v(T) - \liminf_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \left\{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1)H(t, s)|v(s)|^q + \left(\frac{1}{p} h(t, s) \right)^p \right\} ds$$

for all $T \geq T_0$. Thus, by (C_5),

$$v(T) \geq A(T) + \liminf_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \left\{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1)H(t, s)|v(s)|^q + \left(\frac{1}{p} h(t, s) \right)^p \right\} ds$$

for all $T \geq T_0$. This shows that

$$(3) \qquad v(T) \geq A(T)$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \left\{ h(t, s)[H(t, s)]^{1/q} v(s) + (p-1)H(t, s)|v(s)|^q \right.$$

$$+\left(\frac{1}{p}h(t,s)\right)^p\Bigg\}ds<\infty$$

for all $T \geq T_0$. Let

$$f(t)=\frac{p-1}{H(t,T_0)}\int_{T_0}^t H(t,s)|v(s)|^q\,ds$$

and

$$g(t)=\frac{1}{H(t,T_0)}\int_{T_0}^t h(t,s)[H(t,s)]^{1/q}v(s)\,ds$$

for all $t > T_0$. Then

$$\begin{aligned}(4)\qquad &\liminf_{t\to\infty}[f(t)+g(t)]\\ &=\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\{(p-1)H(t,s)|v(s)|^q+h(t,s)[H(t,s)]^{1/q}v(s)\}\,ds\\ &\leq\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\Bigg\{(p-1)H(t,s)|v(s)|^q+h(t,s)[H(t,s)]^{1/q}v(s)\\ &\qquad\qquad+\left(\frac{1}{p}h(t,s)\right)^p\Bigg\}ds\\ &<\infty.\end{aligned}$$

Now, we claim that

$$(5)\qquad \int_{T_0}^\infty |v(s)|^q\,ds<\infty.$$

Suppose to the contrary that

$$(6)\qquad \int_{T_0}^\infty |v(s)|^q\,ds=\infty.$$

By (C_2), there is a positive constant ξ satisfying

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}>\xi>0.$$

Let μ be any arbitrary positive number. Then it follows from (6) that there exists a $T_1 > T_0$ such that

$$\int_{T_0}^t |v(s)|^q\,ds\geq\frac{\mu}{\xi}\qquad\text{for all } t\geq T_1.$$

Therefore,

$$\begin{aligned} f(t) &= \frac{p-1}{H(t, T_0)} \int_{T_0}^{t} H(t, s) d\left(\int_{T_0}^{s} |v(\tau)|^q d\tau\right) \\ &= \frac{p-1}{H(t, T_0)} \int_{T_0}^{t} \left(\int_{T_0}^{s} |v(\tau)|^q d\tau\right)\left(-\frac{\partial H}{\partial s}(t, s)\right) ds \\ &\geq \frac{p-1}{H(t, T_0)} \int_{T_1}^{t} \left(\int_{T_0}^{s} |v(\tau)|^q d\tau\right)\left(-\frac{\partial H}{\partial s}(t, s)\right) ds \\ &\geq \frac{(p-1)\mu}{\xi H(t, T_0)} \int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t, s)\right) ds \\ &= (p-1) \frac{\mu H(t, T_1)}{\xi H(t, T_0)} \end{aligned}$$

for all $t \geq T_1$. By (7), there is a $T_2 \geq T_1$ such that

$$\frac{H(t, T_1)}{H(t, t_0)} \geq \xi \qquad \text{for all } t \geq T_2,$$

which implies

$$f(t) \geq (p-1)\mu \qquad \text{for all } t \geq T_2.$$

Since μ is arbitrary, then

$$\lim_{t \to \infty} f(t) = \infty. \tag{8}$$

Next, let us consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0, ∞) with $\lim_{n \to \infty} t_n = \infty$ satisfying

$$\lim_{n \to \infty} [f(t_n) + g(t_n)] = \liminf_{t \to \infty} [f(t) + g(t)].$$

It follows from (4) that there exists a constant M such that

$$f(t_n) + g(t_n) \leq M \qquad \text{for } n = 1, 2, 3, \cdots. \tag{9}$$

Furthermore, (8) guarantees that

$$\lim_{n \to \infty} f(t_n) = \infty \tag{10}$$

and hence (9) gives

$$\lim_{n \to \infty} g(t_n) = -\infty. \tag{11}$$

Then, by (9) and (10),

$$1+\frac{g(t_n)}{f(t_n)}\le\frac{M}{f(t_n)}<\frac{1}{2}\qquad\text{for } n \text{ large enough.}$$

Thus,

$$\frac{g(t_n)}{f(t_n)}<-\frac{1}{2}\qquad\text{for } n \text{ large enough.}$$

This and (11) imply that

$$\lim_{n\to\infty}\frac{|g(t_n)|^q}{f(t_n)}=\infty. \tag{11}$$

On the other hand, by the Hölder inequality, we have

$$\begin{aligned}
|g(t_n)|^q &= \left|\frac{1}{H(t_n, T_0)}\int_{T_0}^{t_n} h(t_n, s)[H(t_n, s)[H(t_n, s)]^{1/q}v(s)ds\right|^q\\
&\le \frac{1}{p-1}\left\{\frac{1}{H(t_n, T_0)}\int_{T_0}^{t_n} h^p(t_n, s)ds\right\}^{q-1}\left\{\frac{p-1}{H(t_n, T_0)}\int_{T_0}^{t_n} H(t_n, s)|v(s)|^q ds\right\}\\
&\le \frac{f(t_n)}{p-1}\left\{\frac{1}{H(t_n, T_0)}\int_{t_0}^{t_n} h^p(t_n, s)ds\right\}^{q-1}
\end{aligned}$$

for any positive integer n. Consequently,

$$\frac{|g(t_n)|^q}{f(t_n)}\le\frac{1}{p-1}\left\{\frac{1}{H(t_n, T_0)}\int_{t_0}^{t_n} h^p(t_n, s)ds\right\}^{q-1}\qquad\text{for } n \text{ large enough.}$$

But, (7) guarantees that

$$\liminf_{t\to\infty}\frac{H(t, T_0)}{H(t, t_0)}>\xi$$

and hence there exists a $T_3\ge T_0$ such that

$$\frac{H(t, T_0)}{H(t, t_0)}\ge\xi\qquad\text{for every } t\ge T_3.$$

Thus,

$$\frac{H(t_n, T_0)}{H(t_n, t_0)}\ge\xi\qquad\text{for } n \text{ large enough}$$

and therefore

$$\frac{|g(t_n)|^q}{f(t_n)}\le\frac{1}{p-1}\left\{\frac{1}{\xi H(t_n, t_0)}\int_{t_0}^{t_n} h^p(t_n, s)ds\right\}^{q-1}\qquad\text{for } n \text{ large enough.}$$

It follows from (12) that

$$\lim_{n\to\infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} h^p(t_n, s)ds = \infty. \tag{13}$$

This gives

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} h^p(t, s)ds = \infty,$$

which contradicts (C_3). Then (5) holds. Hence, by (3),

$$\int_{T_0}^{\infty} A_+^q(s)ds \le \int_{T_0}^{\infty} |v(s)|^q ds < \infty,$$

which contradicts (C_4). This completes the proof of the theorem.

THEOREM 4. *Let H and h be as in Theorem 2 and suppose that*

$$\liminf_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)c(s)ds < \infty \tag{C$_6$}$$

and (C_2) hold. If there exists a continuous function A on $[t_0, \infty)$ such that (C_4) and

$$\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\} ds \ge A(T) \tag{C$_7$}$$

hold for every $T \ge t_0$, then equation (E) is oscillatory.

PROOF. Without loss of generality, we may assume that (E) has a solution $u(t)$ such that $u(t) > 0$ on $[T_0, \infty)$ for some $T_0 \ge t_0$. Define $v(t)$ as in (E_2) for $t \ge T_0$. As in the proof of Theorem 3, we see that (2) holds. Then

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left\{H(t, s)c(s) - \left(\frac{1}{p}h(t, s)\right)^p\right\} ds \\
&= v(T) - \limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left\{h(t, s)[H(t, s)]^{1/q}v(s) + (p-1)H(t, s)|v(s)|^q \right.\\
&\qquad \left. + \left(\frac{1}{p}h(t, s)\right)^p\right\} ds
\end{aligned}$$

for all $T \ge T_0$. It follows from (C_7) that

$$v(T) \ge A(T) + \limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left\{h(t, s)[H(t, s)]^{1/q}v(s) + (p-1)H(t, s)|v(s)|^q\right.$$

$$+\left(\frac{1}{p}h(t,s)\right)^p\Bigg\}ds$$

for all $T \geq T_0$. Hence, (3) holds and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\Bigg\{h(t,s)[H(t,s)]^{1/q}v(s)+(p-1)H(t,s)|v(s)|^q$$
$$+\left(\frac{1}{p}h(t,s)\right)^p\Bigg\}ds<\infty$$

for all $T \geq T_0$. This implies that

$$\text{(14)}\qquad \limsup_{t\to\infty}[f(t)+g(t)]$$
$$=\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\{(p-1)H(t,s)|v(s)|^q+h(t,s)[H(t,s)]^{1/q}v(s)\}\,ds$$
$$\leq\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\Bigg\{(p-1)H(t,s)|v(s)|^q+h(t,s)[H(t,s)]^{1/q}v(s)$$
$$+\left(\frac{1}{p}h(t,s)\right)^p\Bigg\}ds$$
$$<\infty,$$

where $f(t)$ and $g(t)$ are defined as in the proof of Theorem 3. By (C_7),

$$A(t_0)\leq\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\Bigg\{H(t,s)c(s)-\left(\frac{1}{p}h(t,s)\right)^p\Bigg\}ds$$
$$\leq\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)c(s)ds-\left(\frac{1}{p}\right)^p\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t h^p(t,s)ds.$$

This and (C_6) imply that

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t h^p(t,s)ds<\infty.$$

Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0,∞) with $\lim_{n\to\infty}t_n=\infty$ satisfying

$$\text{(15)}\qquad \lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n}h^p(t_n,s)ds=\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t h^p(t,s)ds<\infty.$$

Now, suppose that (6) holds. Using the procedure of the proof of Theorem 3, we conclude that (8) is satisfied. It follows from (14) that there exists a constant M such that (9) is fulfilled. Then, as in the proof of Theorem

3, we see that (13) holds, which contradicts (15). This contradiction proves that (6) fails. The remainder of the proof proceeds as in the proof of Theorem 3, we omit the detail.

Now, define

$$H(t, s) = (t - s)^{\lambda}, \qquad t \geq s \geq t_0,$$

where $\lambda > p - 1$ is a constant. Then $H(t, s)$ is continuous on $D = \{(t, s): t \geq s \geq t_0\}$ and satisfies

$$H(t, t) = 0 \quad \text{for } t \geq t_0, \qquad H(t, s) > 0 \quad \text{for } t > s \geq t_0.$$

Moreover, H has a continuous and nonpositive partial derivative on $D_0 = \{(t, s): t > s \geq t_0\}$ with respect to the second variable. Furthermore, the function

$$h(t, s) = \lambda(t - s)^{(\lambda/p - 1)}, \qquad t > s \geq t_0,$$

is continuous and satisfies

$$-\frac{\partial H}{\partial s}(t, s) = h(t, s)[H(t, s)]^{1/q} \qquad \text{for all } (t, s) \in D_0.$$

We have that (C_2) and (C_3) hold, since for every $s \geq t_0$

$$\lim_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} = \lim_{t \to \infty} \frac{(t - s)^{\lambda}}{(t - t_0)^{\lambda}} = 1$$

and

$$\lim_{t \to \infty} t^{-\lambda} \int_{t_0}^{t} (t - s)^{\lambda - p} ds = \lim_{t \to \infty} t^{-\lambda} \left(\frac{-1}{\lambda - p + 1} (t - s)^{\lambda - p + 1} \right) \Bigg|_{t_0}^{t}$$
$$= \lim_{t \to \infty} \frac{t^{-\lambda}}{\lambda - p + 1} (t - t_0)^{\lambda - p + 1} = 0.$$

Thus, Theorem 3 leads to the following corollary.

COROLLARY 5. *Let $\lambda > p - 1$ be a constant. Suppose that there exists a function $A \in C[t_0, \infty)$ such that* (C_4) *holds and*

$$(C_8) \qquad \limsup_{t \to \infty} t^{-\lambda} \int_{T}^{t} (t - s)^{\lambda} c(s) ds \geq A(T) \qquad \text{for all } T \geq t_0.$$

Then equation (E) *is oscillatory.*

REMARK 1. Let $\lambda > p - 1$ is a constant and suppose that

$$\limsup_{t\to\infty} t^{-\lambda}\int_{t_0}^{t}(t-s)^{\lambda}c(s)ds=\infty.$$

Then we can easily verify that

$$\limsup_{t\to\infty} t^{-\lambda}\int_{T}^{t}(t-s)^{\lambda}c(s)ds=\infty \qquad \text{for every } T\geq t_0.$$

Hence, we choose $A(T)=1$ on $[t_0, \infty)$. Then (C_4) and (C_5) hold. Thus, Kamenev's criterion [3] is contained in Corollary 5 with $p=2$. Moreover, we note that Corollary 5 improves the oscillation result of Yan [7].

REMARK 2. We can apply Theorems 2, 3 and 4 by taking

$$H(t, s)=\left(\int_{s}^{t}\frac{1}{\psi(x)}dx\right)^{\lambda}, \qquad t\geq s\geq t_0,$$

where $\lambda > p-1$ is a constant and $\psi(x)$ is a positive continuous function on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty}\frac{1}{\psi(x)}dx=\infty.$$

For example, if $\psi(x)=x$ for $x\geq t_0>0$, then

$$H(t, s)=\left(\log\frac{t}{s}\right)^{\lambda}, \qquad t\geq s\geq t_0.$$

We see that H is a continuous function satisfying

$$H(t, t)=0 \quad \text{for } t\geq t_0, \qquad H(t, s)>0 \quad \text{for } t>s\geq t_0$$

and H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable. Moreover, the function

$$h(t, s)=\frac{\lambda}{s}\left(\log\frac{t}{s}\right)^{(\lambda/p)-1}, \qquad t>s\geq t_0,$$

is a continuous function with

$$-\frac{\partial H}{\partial s}(t, s)=h(t, s)[H(t, s)]^{1/q} \qquad \text{for } t>s\geq t_0.$$

Hence, by applying Theorems 2, 3 and 4 in the special case considered, we derive three new oscillation criteria for equation (E).

ACKNOWLEDGEMENT. The author is grateful to the referee for valuable

suggestions and comments.

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, "Inequalities", 2nd edition, Cambridge University Press, 1988.

[2] P. Hartman, On nonoscillatory linear differential equations of second order, Amer. J. Math., **74** (1952), 389–400.

[3] I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, Math. Zametki., **23** (1978), 249–251.

[4] Ch. G. Philos, Oscillation theorems for linear differential equations of second order, Arch. Math., **53** (1989), 482–492.

[5] M. Del Pino, Applied Math. Engineering Thesis, Univ. of Chile.

[6] A. Wintner, A criterion of oscillatory stability, Quart. Appl. Math., **7** (1949), 115–117.

[7] J. Yan, Oscillation theorems for second order linear differential equations with damping, Proc. Amer. Math. Soc., **98** (1986), 276–282.

[8] C. C. Yeh, An oscillation criterion for second order nonlinear differential equations with functional arguments, J. Math. Anal. Appl., **76** (1980), 72–76.

[9] C. C. Yeh, Oscillation teorems for nonlinear second order differential equations with damped term, Proc. Amer. Math. Soc., **84** (1982), 397–402.

Chienkuo Junior College of Technology and Commerce
Chang-Hua, Taiwan