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Statistical inference on some mixed MANOVA-GMANOVA models with random effects

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0. Introduction

Suppose that we have serial measurements for each of N individuals on each of p occasions, and let X be an $N \times p$ data matrix of observations. Then, the growth curve model proposed by Potthoff and Roy [17] can be written as

$$X = A\Xi B + E, \tag{0.1}$$

where A is an $N \times k$ between-individual design matrix of rank k, Ξ is an unknown $k \times q$ parameter matrix, B is a $q \times p$ within-individual design matrix of rank $q(\leq p)$, and E is an $N \times p$ unobservable matrix of random errors. It is assumed that the rows of E are independently and identically distributed as $N_p(\mathbf{0}, \Sigma)$, where Σ is an unknown $p \times p$ positive definite matrix. The model (0.1) is also called a GMANOVA model since this model is a MANOVA model in the special case $B = I_p$. The model (0.1), which is adequate for balanced data, has been studied by many authors, including Potthoff and Roy [17], Khatri [12], Rao [18], Grizzle and Allen [10], etc. An extension of this model to unbalanced data has been considered in Laird and Ware [14], Vonesh and Carter [29], etc., by assuming that Σ has certain covariance structures. For an extensive survey or a comprehensive review of the literature on these models, see, e.g., Timm [27], Woolson [31] and von Rosen [22]. An extension of the model (0.1) is given by

$$X = A_1\Xi_1 B + A_2\Xi_2 + E, \tag{0.2}$$

where A_1 and A_2 are $N \times k_1$ and $N \times k_2$ design matrices, respectively, rank $[A_1, A_2] = k_1 + k_2 \leq N - p$, and Ξ_1 and Ξ_2 are unknown $k_1 \times q$ and $k_2 \times p$ parameter matrices, respectively. The expected value of X in the model (0.2) is the sum of two matrix components. The first component is the GMANOVA portion, and the second one is the MANOVA portion. The model (0.2) may be called a mixed MANOVA-GMANOVA model. This type of models has been considered in Chinchilli and Elswick [6], Verbyla and Venables [28], Yokoyama and Fujikoshi [33], [34], etc. It may be noted (Verbyla and Venables [28]) that the model (0.2) can be applied to analysis of repeated measurements with parallel profiles or covariates.

When there is no theoretical or empirical basis for assuming special covariance structures, we need to assume that Σ is arbitrary positive definite. However, there exist some situations that certain covariance structures can be imposed for repeated measurements. Typical parsimonious covariance structures are a random-effects covariance structure (see, e.g., Rao [18], [19], Ware [30], Reinsel [20], [21], Lange and Laird [15]), a uniform covariance structure (see, e.g., Arnold [3, pp. 209–238]) and an autoregressive covariance structure (see, e.g., Hudson [11], Lee [16], Fujikoshi, Kanda and Tanimura [8]).

This paper is concerned with some mixed MANOVA-GMAVOVA models with random effects. The random-effects covariance structure, which is based on random-coefficients models proposed by Rao [18], can be naturally and reasonably introduced to a repeated measurements design and enables us to make more efficient inferences. Main inferential problems on this model are divided into two parts:

(P1) the adequacy of a random-effects covariance structure,

(P2) estimation and testing problems of unknown mean parameters under this structure.

This paper consists of two parts. Part I is concerned with a multivariate parallel profile model with random effects and consists of Sections 1 to 4. In Section 1 a multivariate parallel profile model, which is useful in analyzing multiple-response parallel growth curves of several groups, is described in detail and is reduced to a canonical form. The model is a special case of (0.2), but it has a random-effects covariance structure based on several response variables. In Section 2 the likelihood ratio (= LR) statistic for a hypothesis concerning the adequacy of a random-effects covariance structure is obtained. However, since the exact LR criterion is complicated and impractical, it is suggested to use a modified LR statistic, which is the LR criterion for a modified hypothesis. An asymptotic expansion of the null distribution of the statistic is derived. The LR criterion for the hypothesis is also discussed. In Section 3 the maximum likelihood estimators (= MLE's) of unknown mean parameters are obtained under the random-effects covariance structure, and one of the MLE's is compared with the MLE in the case when the covariance matrix has no structures. Further, two testing problems are considered. Modified LR statistics and their asymptotic null distributions are obtained. In Section 4 we discuss the single-response case, in which the exact LR criteria for testing the hypotheses in Sections 2 and 3 have been obtained by Yokoyama [32] and Yokoyama and Fujikoshi [34]. In this section asymptotic non-null distributions of the LR criteria are obtained under local alternatives.

Part II is concerned with a growth curve model with covariates and random effects and consists of Sections 5 to 8. In Section 5 a growth curve model with covariates is outlined and is reduced to a canonical form. The model is a mixed MANOVA-GMANOVA model which has random-effects covariance structures based on a single response variable. In Section 6 test statistics for a general hypothesis concerning the adequacy of a family of random-effects covariance structures are proposed. A modified LR statistic and its asymptotic null distribution are obtained. The LR criterion for the hypothesis is also discussed. In Section 7 the MLE's of unknown mean parameters are obtained under one of these covariance structures, and the efficiency of one of the MLE's is discussed. Finally, in Section 8 the results of Section 6 are exemplified by a data set of repeated measurements.

Part I. Multivariate parallel profile model with random effects

1. The model and its canonical form

Suppose that m response variables $x_1, \ldots, x_m$ have been measured at p different occasions on each of N individuals, and each individual belongs to one of k groups or treatments. Let $\boldsymbol{x}_j^{(g)}$ be an mp-vector of measurements on the j-th individual in the g-th group arranged as

$$\boldsymbol{x}_j^{(g)} = (x_{11j}^{(g)}, \ldots, x_{1mj}^{(g)}, \ldots, x_{p1j}^{(g)}, \ldots, x_{pmj}^{(g)})',$$

and assume that the $\boldsymbol{x}_j^{(g)}$ are independently distributed as $N_{mp}(\boldsymbol{\mu}^{(g)}, \Omega)$, where $j = 1, \ldots, N_g$, $g = 1, \ldots, k$. Further, we assume that profiles of k groups are parallel, i.e.,

$$\boldsymbol{\mu}^{(g)} = (\mathbf{1}_p \otimes I_m)\boldsymbol{\delta}^{(g)} + \boldsymbol{\mu}, \qquad g = 1, \ldots, k, \tag{1.1}$$

where $\mathbf{1}_p$ is a p-vector of ones, and $(\mathbf{1}_p \otimes I_m)$ defines the Kronecker product of $\mathbf{1}_p$ and the $m \times m$ identity matrix. Without loss of generality we may assume that $\boldsymbol{\delta}^{(k)} = \mathbf{0}$. In the following we shall do this. Let

$$X = [\boldsymbol{x}_1^{(1)}, \ldots, \boldsymbol{x}_{N_1}^{(1)}, \ldots, \boldsymbol{x}_1^{(k)}, \ldots, \boldsymbol{x}_{N_k}^{(k)}]'.$$

Then the model of X can be written as

$$X \sim N_{N \times mp}(A_1 \Delta(\mathbf{1}_p' \otimes I_m) + \mathbf{1}_N \boldsymbol{\mu}', \Omega \otimes I_N), \tag{1.2}$$

where $N = N_1 + \cdots + N_k$,

$$A_1 = \begin{pmatrix} \mathbf{1}_{N_1} & & O \\ & \ddots & \\ O & & \mathbf{1}_{N_{k-1}} \\ \cdots & \cdots & \cdots \\ & O & \end{pmatrix}$$

is an $N \times (k-1)$ between-individual design matrix of rank $k-1$ ($\leq N-p-1$), $\Delta = [\boldsymbol{\delta}^{(1)}, \dots, \boldsymbol{\delta}^{(k-1)}]'$ is an unknown $(k-1) \times m$ parameter matrix, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \dots, \boldsymbol{\mu}_p')'$ is an mp-vector of unknown parameters, Ω is an unknown $mp \times mp$ positive definite matrix. The model (1.2) is called a multivariate parallel profile model.

We are now interested in a random-effects covariance structure

$$\Omega = (\mathbf{1}_p \otimes I_m)\Sigma_\lambda(\mathbf{1}_p' \otimes I_m) + I_p \otimes \Sigma_e, \tag{1.3}$$

where Σ_λ and Σ_e are arbitrary $m \times m$ positive semi-definite and positive definite matrices, respectively. The random-effects covariance structure (1.3) is based on the following model:

$$\boldsymbol{x}_j^{(g)} = (\mathbf{1}_p \otimes I_m)(\boldsymbol{\delta}^{(g)} + \boldsymbol{\lambda}_j^{(g)}) + \boldsymbol{\mu} + \boldsymbol{e}_j^{(g)}, \tag{1.4}$$

where $\boldsymbol{\lambda}_j^{(g)}$ and $\boldsymbol{e}_j^{(g)}$ are independently distributed as $N_m(\mathbf{0}, \Sigma_\lambda)$ and $N_{mp}(\mathbf{0}, I_p \otimes \Sigma_e)$, respectively. Here, the $\boldsymbol{\lambda}_j^{(g)}$'s are m-vectors of latent variables and can be regarded as ones denoting variation between individuals for each group. From (1.4), we have

$$V(\boldsymbol{x}_j^{(g)}) = \Omega = (\mathbf{1}_p \otimes I_m)\Sigma_\lambda(\mathbf{1}_p' \otimes I_m) + I_p \otimes \Sigma_e.$$

Therefore, the model of X with random effects can be written as

$$X \sim N_{N \times mp}(A_1\Delta(\mathbf{1}_p' \otimes I_m) + \mathbf{1}_N\boldsymbol{\mu}', ((\mathbf{1}_p \otimes I_m)\Sigma_\lambda(\mathbf{1}_p' \otimes I_m) + I_p \otimes \Sigma_e) \otimes I_N), \tag{1.5}$$

which is an extension of the single-response case due to Yokoyama and Fujikoshi [34] to the multiple-response case.

Reinsel [20], [21] introduced certain multivariate random-effects covariance structures to a multivariate GMANOVA model. Chinchilli and Carter [5] discussed the LR test for a patterned covariance structure

$$\Omega = (\mathbf{1}_p \otimes I_m)\Sigma_\lambda(\mathbf{1}_p' \otimes I_m) + (W \otimes I_m)\Sigma_\tau(W' \otimes I_m) + I_p \otimes \Sigma_e,$$

in a multivariate GMANOVA model, where W is a known $p \times (p-1)$ matrix of rank $p-1$ such that $\mathbf{1}_p'W = \mathbf{0}$, and Σ_τ is an arbitrary $m(p-1) \times m(p-1)$ positive semi-definite matrix. We note that the model (1.5) is a multivariate mixed MANOVA-GMANOVA model with multivariate random-effects covariance structure. In Section 2 we propose test statistics for the hypothesis

$$H_0: \Omega = (\mathbf{1}_p \otimes I_m)\Sigma_\lambda(\mathbf{1}_p' \otimes I_m) + I_p \otimes \Sigma_e \quad \text{vs.} \quad H_1: \text{not } H_0 \tag{1.6}$$

under the multivariate parallel profile model (1.2). By making this stronger assumption about Ω, we can expect to have more efficient estimators. For the single-response case ($m = 1$), Srivastava [25] obtained the MLE of Δ when no special assumptions about Ω are made. For the case $m = 1$, in comparison with his result, Yokoyama and Fujikoshi [34] has shown how much gains can be obtained for the maximum likelihood estimation of Δ by assuming this covariance structure. In Section 3 we discuss the efficiency of the MLE of Δ in the multiple-response case.

We now reduce the model (1.2) to a canonical form, which implies that the problem of obtaining the LR test under the model (1.2) can be reduced to the one of obtaining the LR test under a GMANOVA model. Let $G = [p^{-1/2}\mathbf{1}_p, g_2^{(1)},\dots,g_2^{(p-1)}] = [p^{-1/2}\mathbf{1}_p, G_2]$ be an orthogonal matrix of order p. Then

$$\begin{aligned} Q &= G \otimes I_m \\ &= [Q_1, Q_2^{(1)},\dots,Q_2^{(p-1)}] = [Q_1, Q_2] \end{aligned}$$

is an orthogonal matrix of order mp. Further, let $H = [N^{-1/2}\mathbf{1}_N, H_2]$ be an orthogonal matrix of order N. Consider the transformation from X to

$$\begin{bmatrix} z' \\ Y \end{bmatrix} = \begin{bmatrix} N^{-1/2}\mathbf{1}_N' \\ H_2' \end{bmatrix} XQ.$$

Then, letting $Y = [Y_1, Y_2^{(1)},\dots,Y_2^{(p-1)}] = [Y_1, Y_2]$, we have

$$\begin{bmatrix} & z' & \\ Y_1 & & Y_2 \end{bmatrix} \sim N_{N\times mp}\left(\begin{bmatrix} & \theta' & \\ \tilde{A}_1\Gamma & & O \end{bmatrix}, \Psi \otimes I_N\right), \tag{1.7}$$

where $\theta' = N^{-1/2}\mathbf{1}_N' A_1[\Gamma, O] + N^{1/2}\mu' Q$, $\tilde{A}_1 = H_2' A_1$, $\Gamma = p^{1/2}\Delta$ and

$$\Psi = Q'\Omega Q = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}.$$

We can express the hypothesis (1.6) as

$$H_0: \Psi = \begin{pmatrix} p\Sigma_\lambda + \Sigma_e & O \\ O & I_{p-1} \otimes \Sigma_e \end{pmatrix} \quad \text{vs.} \quad H_1: \text{not } H_0. \tag{1.8}$$

2. Tests for random-effects covariance structure

In order to examine whether or not the model (1.5) can be assumed, we consider the LR test for the hypothesis (1.6) under the model (1.2). This is equivalent to considering the LR test for the hypothesis (1.8) under the model (1.7). Let $L(\boldsymbol{\theta}, \Gamma, \Psi)$ be the likelihood function of $[z, Y']$. Then we have

$$\begin{aligned} g(\Gamma, \Psi) &= -2\log L(\hat{\boldsymbol{\theta}}, \Gamma, \Psi) \\ &= N\log|\Psi| + \mathrm{tr}\,\Psi^{-1}[Y_1 - \tilde{A}_1\Gamma \quad Y_2]'[Y_1 - \tilde{A}_1\Gamma \quad Y_2], \end{aligned}$$

where $\hat{\boldsymbol{\theta}} = z$. The minimum of $g(\Gamma, \Psi)$ when Γ and Ψ are unrestricted, which has been obtained by Khatri [12] and Gleser and Olkin [9], is given by

$$\min g(\Gamma, \Psi) = N\log\left(\left|\frac{1}{N}S_{11\cdot2}\right| \times \left|\frac{1}{N}Y_2'Y_2\right|\right) + Nmp, \tag{2.1}$$

where $S_{11\cdot2} = S_{11} - S_{12}S_{22}^{-1}S_{21}$ and

$$S = Y'(I_{N-1} - \tilde{A}_1(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1')Y = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

As is seen later on, the minimum of $g(\Gamma, \Psi)$ under H_0 in (1.8) is complicated. For simplicity, we consider the LR test for a modified hypothesis

$$\tilde{H}_0\colon \Psi = \begin{pmatrix} \Psi_{11} & O \\ O & I_{p-1}\otimes\Sigma_e \end{pmatrix}, \tag{2.2}$$

where Ψ_{11} is an arbitrary $m \times m$ positive definite matrix. We note that the difference between H_0 and $\tilde{H}_0$ is whether or not Ψ_{11} satisfies a restriction $\Psi_{11} \geq \Sigma_e$. It is easily seen that

(2.3)

$$\min_{\tilde{H}_0} g(\Gamma, \Psi) = N\log\left|\frac{1}{N}S_{11}\right| + N(p-1)\log\left|\frac{1}{N(p-1)}\sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)}\right| + Nmp.$$

Therefore, from (2.1) we can obtain the LR test statistic

$$\tilde{\Lambda} = \frac{|S|}{|S_{11}||S_{22}|} \times \frac{|Y_2'Y_2|}{\left|\dfrac{1}{p-1}\sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)}\right|^{p-1}} \tag{2.4}$$

for testing $\tilde{H}_0$ vs. H_1, which may be also used for testing H_0 vs. H_1. In order to express the statistic (2.4) in terms of the original observations, let

$$S_t = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(x_j^{(g)} - \bar{x})(x_j^{(g)} - \bar{x})', \quad S_w = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(x_j^{(g)} - \bar{x}^{(g)})(x_j^{(g)} - \bar{x}^{(g)})',$$

where

$$\bar{x} = \frac{1}{N}\sum_{g=1}^{k}\sum_{j=1}^{N_g} x_j^{(g)} \quad \text{and} \quad \bar{x}^{(g)} = \frac{1}{N_g}\sum_{j=1}^{N_g} x_j^{(g)}.$$

Noting that

$$(\tilde{A}_1'\tilde{A}_1)^{-1} = \operatorname{diag}\left(\frac{1}{N_1},\ldots,\frac{1}{N_{k-1}}\right) + \frac{1}{N_k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}', \tag{2.5}$$

we have

$$\begin{aligned} &S_{11} = \frac{1}{p}(\mathbf{1}_p' \otimes I_m)S_w(\mathbf{1}_p \otimes I_m), \quad S_{11\cdot 2} = \left(\frac{1}{p}(\mathbf{1}_p' \otimes I_m)S_w^{-1}(\mathbf{1}_p \otimes I_m)\right)^{-1}, \\ &Y_2^{(i)\prime}Y_2^{(i)} = (g_2^{(i)\prime} \otimes I_m)S_t(g_2^{(i)} \otimes I_m), \quad |Y_2'Y_2| = \left|\frac{1}{p}(\mathbf{1}_p' \otimes I_m)S_t^{-1}(\mathbf{1}_p \otimes I_m)\right||S_t|. \end{aligned} \tag{2.6}$$

The statistic (2.4) can be decomposed as

$$\tilde{\Lambda} = \Lambda^{(1)}\Lambda^{(2)}, \tag{2.7}$$

where

$$\Lambda^{(1)} = \frac{|S|}{|S_{11}||S_{22}|} = \frac{|S_{11\cdot 2}|}{|S_{11\cdot 2} + S_{12}S_{22}^{-1}S_{21}|} \quad \text{and} \quad \Lambda^{(2)} = \frac{|Y_2'Y_2|}{\left|\frac{1}{p-1}\sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)}\right|^{p-1}}.$$

The statistics $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are the LR statistics for $\Psi_{12} = O$ and $\Psi_{22} = I_{p-1} \otimes \Sigma_e$, respectively. Further, we can decompose $\Lambda^{(2)}$, which is the LR statistic for testing multivariate sphericity, as

$$\Lambda^{(2)} = \Lambda_1^{(2)}\Lambda_2^{(2)}, \tag{2.8}$$

where

$$\Lambda_1^{(2)} = \frac{|Y_2'Y_2|}{\prod_{i=1}^{p-1}|Y_2^{(i)\prime}Y_2^{(i)}|} \quad \text{and} \quad \Lambda_2^{(2)} = \frac{\prod_{i=1}^{p-1}|Y_2^{(i)\prime}Y_2^{(i)}|}{\left|\frac{1}{p-1}\sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)}\right|^{p-1}}.$$

The statistic $\Lambda_1^{(2)}$ is the LR statistic for testing the hypothesis that Ψ_{22} is block diagonal, and the statistic $\Lambda_2^{(2)}$ is the one for testing equality of diagonal matrices given that Ψ_{22} is block diagonal. It is easy to verify that under H_0, $S_{11\cdot 2}$, $S_{12}S_{22}^{-1}S_{21}$ and $Y_2'Y_2$ are independent,

$$S_{11\cdot 2} \sim W_m(N - k - m(p-1), \Psi_{11}), \quad S_{12}S_{22}^{-1}S_{21} \sim W_m(m(p-1), \Psi_{11}),$$

$$Y_2'Y_2 \sim W_{m(p-1)}(N-1, I_{p-1} \otimes \Sigma_e) \quad \text{and} \quad \sum_{i=1}^{p-1} Y_2^{(i)\prime} Y_2^{(i)} \sim W_m((N-1)(p-1), \Sigma_e).$$

Therefore, the statistics $\Lambda^{(1)}$ and $\Lambda^{(2)}$ in (2.7) are independent. The h-th moment of $\Lambda^{(1)}$ is obtained from that $\Lambda^{(1)}$ is distributed as a lambda distribution $\Lambda_{m,m(p-1),N-k-m(p-1)}$ and is given by

$$(2.9) \qquad E(\Lambda^{(1)h}) = \frac{\Gamma_m\left(\frac{1}{2}\{N-k-m(p-1)\}+h\right)\Gamma_m\left(\frac{1}{2}(N-k)\right)}{\Gamma_m\left(\frac{1}{2}\{N-k-m(p-1)\}\right)\Gamma_m\left(\frac{1}{2}(N-k)+h\right)},$$

where $\Gamma_m(\cdot)$ is the multivariate gamma function defined by $\Gamma_m(n/2) = \pi^{m(m-1)/4}\prod_{j=1}^{m}\Gamma((n-j+1)/2)$. On the other hand, it is known (Boik [4]) that under H_0, the statistics $\Lambda_1^{(2)}$ and $\Lambda_2^{(2)}$ in (2.8) are independent, and the h-th moment of $\Lambda^{(2)}$ is given by

$$(2.10) \qquad E(\Lambda^{(2)h}) = (p-1)^{m(p-1)h}\frac{\Gamma_{m(p-1)}\left(\frac{1}{2}N+h\right)\Gamma_m\left(\frac{1}{2}N(p-1)\right)}{\Gamma_{m(p-1)}\left(\frac{1}{2}N\right)\Gamma_m\left(\frac{1}{2}N(p-1)+(p-1)h\right)}.$$

From these null moments of $\tilde{\Lambda}$, we can obtain an asymptotic expansion of the null distribution of $-N\rho\log\tilde{\Lambda}$ by expanding its characteristic function. For the method, see, e.g., Anderson [2].

THEOREM 2.1. *When the hypothesis H_0 is true, the distribution function of $-N\rho\log\tilde{\Lambda}$ can be expanded for large $M = N\rho$ as*

$$P(-N\rho\log\tilde{\Lambda} \le x)$$

$$= P(\chi_f^2 \le x) + \frac{\gamma_2}{M^2}\{P(\chi_{f+4}^2 \le x) - P(\chi_f^2 \le x)\} + O(M^{-3}),$$

where $f = \frac{1}{2}m(mp^2+p-2m-2)$,

$$\gamma_2 = \frac{1}{48}m\Big\{2m(p-1)\{6u^2 - 6(mp+1)u + m(p-1)(2mp+m+3)$$
$$+ 2m^2+3m-1\} + 6\{(p-1)\{m(p-1)+1\} - (m+1)\}(u+k)^2$$
$$- 2\left\{(p-1)\{2m^2(p-1)^2+3m(p-1)-1\} - \frac{1}{p-1}(2m^2+3m-1)\right\}(u+k)$$
$$+ (p-1)\{m(p-1)-1\}\{m(p-1)+1\}\{m(p-1)+2\}$$

$$- \frac{1}{(p-1)^2}(m-1)(m+1)(m+2)\Big\},$$

where $u = N(1-\rho) - k$, *and* ρ *is defined by*

$$fN(1-\rho) = \frac{1}{12} m \Big\{ 6m(p-1)(2k+mp+1)$$
$$+ (p-1)\{2m^2(p-1)^2 + 3m(p-1) - 1\} - \frac{1}{p-1}(2m^2+3m-1)\Big\}.$$

We now consider the exact LR criterion $\Lambda^{N/2}$ for H_0 vs. H_1. Let

$$\hat{\Psi}_{11} = \frac{1}{N} S_{11}, \quad \hat{\Sigma}_e = \frac{1}{N(p-1)} \sum_{i=1}^{p-1} Y_2^{(i)\prime} Y_2^{(i)}. \tag{2.11}$$

If it holds that

$$\hat{\Psi}_{11} - \hat{\Sigma}_e \geq O, \tag{2.12}$$

the LR statistic Λ is equal to $\tilde{\Lambda}$. However, if (2.12) does not hold, we need to solve the problem of minimizing

$$g^*(\Sigma_\lambda, \Sigma_e) = \log|p\Sigma_\lambda + \Sigma_e| + \mathrm{tr}\,(p\Sigma_\lambda + \Sigma_e)^{-1}\hat{\Psi}_{11} + (p-1)(\log|\Sigma_e| + \mathrm{tr}\,\Sigma_e^{-1}\hat{\Sigma}_e),$$

which is equal to the quantity obtained by minimizing $g(\Gamma, \Psi)/N$ under H_0 with respect to Γ. The problem is not simple and is left as a future problem. Here, we give a bound. Let $\delta_1 \geq \cdots \geq \delta_m$ and $\delta_1^* \geq \cdots \geq \delta_m^*$ be the characteristic roots of $p\Sigma_\lambda + \Sigma_e$ and Σ_e, respectively, and let $t_1 > \cdots > t_m$ and $t_1^* > \cdots > t_m^*$ be ones of $\hat{\Psi}_{11}$ and $\hat{\Sigma}_e$, respectively. Then, from Anderson [1] we can get a lower bound given by

$$\min_{\Sigma_\lambda \geq 0, \Sigma_e > 0} g^*(\Sigma_\lambda, \Sigma_e) \geq \min_{\omega} \sum_{i=1}^{m} \left\{ \log \delta_i + \frac{t_i}{\delta_i} + (p-1)\left(\log \delta_i^* + \frac{t_i^*}{\delta_i^*} \right) \right\}, \tag{2.13}$$

where $\omega = \{\delta_1 \geq \cdots \geq \delta_m, \delta_1^* \geq \cdots \geq \delta_m^*, \delta_i \geq \delta_i^* > 0, i = 1, \ldots, m\}$. It is easily seen that the values of δ_i's and δ_i^*'s which minimize the right-hand side of (2.13) are

$$\hat{\delta}_i = \begin{cases} t_i, & \text{if } t_i \geq t_i^*, \\ \frac{1}{p}\{t_i + (p-1)t_i^*\}, & \text{if } t_i^* > t_i, \end{cases}$$

(2.14)

$$\hat{\delta}_i^* = \begin{cases} t_i^*, & \text{if } t_i \geq t_i^*, \\ \frac{1}{p}\{t_i + (p-1)t_i^*\}, & \text{if } t_i^* > t_i \end{cases}$$

for $i = 1,\dots,m$. Therefore, we obtain the following bound for Λ:

$$\bar{\Lambda} = \begin{cases} \tilde{\Lambda}, & \text{if } \hat{\Psi}_{11} - \hat{\Sigma}_e \geq O, \\ R, & \text{elsewhere,} \end{cases} \tag{2.15}$$

where

$$R = \frac{|S_{11\cdot 2}|\,|Y_2'Y_2|}{\left|\frac{1}{p}(S_{11} + \sum_{i=1}^{p-1} Y_2^{(i)\prime} Y_2^{(i)})\right|^p}.$$

Since the statistic R is obtained by letting $\delta_i = \delta_i^* = \{t_i + (p-1)t_i^*\}/p$ for all $i = 1,\dots,m$ in the right-hand side of (2.13), we have

$$\bar{\Lambda} \leq \Lambda \leq \tilde{\Lambda}. \tag{2.16}$$

We note that the LR statistic Λ agrees with $\bar{\Lambda}$ in the case $m = 1$.

3. The MLE's and LR tests

3.1. The MLE's of unknown mean parameters

In this section we obtain the MLE's of unknown mean parameters in the multivariate parallel profile model (1.5) and consider the efficiency of the MLE of Δ. Reinsel [20], [21] discussed some aspects of estimation and hypothesis testing in a multivariate GMANOVA model with multivariate random-effects covariance structure. It may be noted that the covariance matrix of the model (1.5) has a multivariate random-effects covariance structure, but the mean structure is a multivariate mixed MANOVA-GMANOVA model. The canonical form of the model (1.5) is the same as the model (1.7) except the covariance matrix, i.e.,

$$\begin{bmatrix} & z' & \\ Y_1 & & Y_2 \end{bmatrix} \sim N_{N\times mp}\left(\begin{bmatrix} & \theta' & \\ \tilde{A}_1\Gamma & & O \end{bmatrix}, \Psi \otimes I_N\right) \tag{3.1}$$

with

$$\Psi = \begin{pmatrix} p\Sigma_\lambda + \Sigma_e & O \\ O & I_{p-1}\otimes\Sigma_e \end{pmatrix}.$$

It is easily seen that the MLE's of $\boldsymbol{\theta}$ and Γ are given by

$$\hat{\boldsymbol{\theta}} = \boldsymbol{z}, \tag{3.2}$$
$$\hat{\Gamma} = (\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'Y_1, \text{ i.e., } \hat{\Delta} = p^{-1/2}(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'Y_1.$$

Hence the MLE of $\boldsymbol{\mu}$ is given by

$$\hat{\boldsymbol{\mu}} = N^{-1/2}Q\boldsymbol{z} - N^{-1}Q\begin{pmatrix} Y_1'\tilde{A}_1(\tilde{A}_1'\tilde{A}_1)^{-1}A_1'\mathbf{1}_N \\ \mathbf{0} \end{pmatrix}. \tag{3.3}$$

We now express the MLE's given in (3.2) and (3.3) in terms of the original observations. Noting that

$$\tilde{A}_1'Y_1 = \operatorname{diag}(N_1,\dots,N_{k-1})\{[\bar{\boldsymbol{x}}^{(1)},\dots,\bar{\boldsymbol{x}}^{(k-1)}]' - \mathbf{1}_{k-1}\bar{\boldsymbol{x}}'\}\frac{1}{\sqrt{p}}(\mathbf{1}_p \otimes I_m),$$

from (2.5) it can be shown that

$$(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'Y_1 = \frac{1}{\sqrt{p}}[\bar{\boldsymbol{x}}^{(1)} - \bar{\boldsymbol{x}}^{(k)},\dots,\bar{\boldsymbol{x}}^{(k-1)} - \bar{\boldsymbol{x}}^{(k)}]'(\mathbf{1}_p \otimes I_m).$$

Therefore, from these results we have the following theorem.

THEOREM 3.1. *The MLE's of* Δ *and* $\boldsymbol{\mu}$ *under the multivariate parallel profile model* (1.5) *are given as follows*:

$$\hat{\Delta} = \frac{1}{p}[\bar{\boldsymbol{x}}^{(1)} - \bar{\boldsymbol{x}}^{(k)},\dots,\bar{\boldsymbol{x}}^{(k-1)} - \bar{\boldsymbol{x}}^{(k)}]'(\mathbf{1}_p \otimes I_m),$$

$$\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{x}} - \frac{1}{p}(\mathbf{1}_p\mathbf{1}_p' \otimes I_m)(\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}^{(k)}).$$

On the other hand, the MLE of Δ when Ω has no structures, i.e., is arbitrary positive definite is given by

$$\tilde{\Delta} = [\bar{\boldsymbol{x}}^{(1)} - \bar{\boldsymbol{x}}^{(k)},\dots,\bar{\boldsymbol{x}}^{(k-1)} - \bar{\boldsymbol{x}}^{(k)}]'S_w^{-1}(\mathbf{1}_p \otimes I_m)((\mathbf{1}_p' \otimes I_m)S_w^{-1}(\mathbf{1}_p \otimes I_m))^{-1}.$$

The result, which is an extension of Srivastava [25] to a multivariate case, follows by rewriting a general expression in Chinchilli and Elswick [6]. The estimators $\hat{\Delta}$ and $\tilde{\Delta}$ have the following properties.

THEOREM 3.2. *Under the multivariate parallel profile model* (1.5) *it holds that both the estimators* $\hat{\Delta}$ *and* $\tilde{\Delta}$ *are unbiased, and*

$$V(\operatorname{vec}(\hat{\Delta})) = \frac{1}{p}(p\Sigma_\lambda + \Sigma_e) \otimes M,$$

$$V(\mathrm{vec}(\tilde{\Delta})) = \frac{1}{p}\left\{1 + \frac{m(p-1)}{N-k-m(p-1)-1}\right\}(p\Sigma_\lambda + \Sigma_e)\otimes M,$$

where $M = \mathrm{diag}(N_1^{-1},\dots,N_{k-1}^{-1}) + N_k^{-1}\mathbf{1}_{k-1}\mathbf{1}_{k-1}'$.

PROOF. Since $\hat{\Delta} = p^{-1/2}(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1' Y_1$, we have

$$E(\hat{\Delta}) = \Delta \quad \text{and} \quad V(\mathrm{vec}(\hat{\Delta})) = \frac{1}{p}(p\Sigma_\lambda + \Sigma_e)\otimes(\tilde{A}_1'\tilde{A}_1)^{-1},$$

which imply the result on $\hat{\Delta}$. By an argument similar to the one in Srivastava [25], it can be shown that for any positive definite covariance matrix Ω, $E(\tilde{\Delta}) = \Delta$ and

$$V(\mathrm{vec}(\tilde{\Delta})) = \left\{1 + \frac{m(p-1)}{N-k-m(p-1)-1}\right\}((\mathbf{1}_p'\otimes I_m)\Omega^{-1}(\mathbf{1}_p\otimes I_m))^{-1}\otimes M.$$

Under the assumption of $\Omega = (\mathbf{1}_p\otimes I_m)\Sigma_\lambda(\mathbf{1}_p'\otimes I_m) + I_p\otimes\Sigma_e$, it holds that

$$((\mathbf{1}_p'\otimes I_m)\Omega^{-1}(\mathbf{1}_p\otimes I_m))^{-1} = \frac{1}{p}(p\Sigma_\lambda + \Sigma_e),$$

which proves the desired result.

From Theorem 3.2, we obtain

$$V(\mathrm{vec}(\tilde{\Delta})) - V(\mathrm{vec}(\hat{\Delta})) = \frac{m(p-1)}{p\{N-k-m(p-1)-1\}}(p\Sigma_\lambda + \Sigma_e)\otimes M > O,$$

which implies that under the model (1.5) $\hat{\Delta}$ is more efficient than $\tilde{\Delta}$. This shows that we can get a more efficient estimator for Δ by assuming a random-effects covariance structure. For the case $m = 1$, this result agrees with the one in Theorem 2.2 in Yokoyama and Fujikoshi [34].

3.2. LR tests for two hypotheses

In this section we consider two testing problems under the multivariate parallel profile model (1.5). First we consider the LR test for

$$\text{(3.4)} \qquad H_{01}: \Sigma_\lambda = O \quad \text{vs.} \quad H_{11}: \text{not } H_{01}$$

under the model (1.5). For testing the hypothesis (3.4), we may start from the model (3.1). Let $L(\boldsymbol{\theta}, \Gamma, \Sigma_\lambda, \Sigma_e)$ be the likelihood function of $[\mathbf{z}, Y']$. Then we have

$$\text{(3.5)} \qquad g_1(\Sigma_\lambda, \Sigma_e) = -2\log L(\hat{\boldsymbol{\theta}}, \hat{\Gamma}, \Sigma_\lambda, \Sigma_e)$$
$$= N\left\{\log|p\Sigma_\lambda + \Sigma_e| + \mathrm{tr}(p\Sigma_\lambda + \Sigma_e)^{-1}\frac{1}{N}S_{11}\right.$$

$$+ (p-1)\left\{\log|\Sigma_e| + \operatorname{tr}\Sigma_e^{-1}\frac{1}{N(p-1)}\sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)}\right\}\Bigg\},$$

where $\hat{\theta}$ and $\hat{\Gamma}$ are defined in (3.2), and

$$S_{11} = Y_1'(I_{N-1} - P_{\tilde{A}_1})Y_1, \quad P_{\tilde{A}_1} = \tilde{A}_1(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'.$$

Then the minimum of (3.5) under H_{01} is given by

$$\min_{H_{01}} g_1(\Sigma_\lambda, \Sigma_e) = Np\log\left|\frac{1}{Np}(S_{11} + \sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)})\right| + Nmp. \tag{3.6}$$

Since the minimum of (3.5) under H_{11} is complicated (see (2.13), (2.14)), we consider the minimization of $g_1(\Psi_{11}, \Sigma_e)$ under the model (3.1) with the same modified covariance matrix as in (2.2). Therefore, $\min g_1(\Psi_{11}, \Sigma_e)$ is equal to (2.3). From (2.3) and (3.6), we can suggest a test statistic

$$\tilde{\Lambda}_1 = \frac{|S_{11}|\left|\frac{1}{p-1}\sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)}\right|^{p-1}}{\left|\frac{1}{p}(S_{11} + \sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)})\right|^{p}} \tag{3.7}$$

for testing H_{01} vs. H_{11}. The statistic (3.7) can be expressed in terms of the original observations, using (2.6).

LEMMA 3.1. *When the hypothesis H_{01} is true, the h-th moment of $\tilde{\Lambda}_1$ is*

$$E(\tilde{\Lambda}_1^h) = \frac{p^{mph}}{(p-1)^{m(p-1)h}} \frac{\Gamma_m\left(\frac{1}{2}(N-k)+h\right)}{\Gamma_m\left(\frac{1}{2}(N-k)\right)}$$
$$\times \frac{\Gamma_m\left(\frac{1}{2}(N-1)(p-1)+(p-1)h\right)\Gamma_m\left(\frac{1}{2}\{(N-1)p-(k-1)\}\right)}{\Gamma_m\left(\frac{1}{2}(N-1)(p-1)\right)\Gamma_m\left(\frac{1}{2}\{(N-1)p-(k-1)\}+ph\right)}.$$

PROOF. It is easy to verify that under H_{01},

$$S_{11} \sim W_m(N-k, \Sigma_e) \quad \text{and} \quad \sum_{i=1}^{p-1} Y_2^{(i)\prime}Y_2^{(i)} \sim W_m((N-1)(p-1), \Sigma_e).$$

Further, these two statistics are independent. Therefore, the h-th moment of $\tilde{\Lambda}_1$ can be written as

$$E(\tilde{\Lambda}_1^h) = \frac{p^{mph}}{(p-1)^{m(p-1)h}}$$

$$\times \frac{\Gamma_m\left(\frac{1}{2}(N-k+2h)\right)\Gamma_m\left(\frac{1}{2}\{(N-1)(p-1)-2h\}\right)}{\Gamma_m\left(\frac{1}{2}(N-k)\right)\Gamma_m\left(\frac{1}{2}(N-1)(p-1)\right)} E\left[\left(\frac{|W_2|}{|W_1+W_2|}\right)^{ph}\right],$$

where W_1 and W_2 are independently distributed, $W_1 \sim W_m(\nu_1, I_m)$, $W_2 \sim W_m(\nu_2, I_m)$, and $\nu_1 = N-k+2h$, $\nu_2 = (N-1)(p-1)-2h$. The desired result follows from that

$$\frac{|W_2|}{|W_1+W_2|} \sim \Lambda_{m,\nu_1,\nu_2}.$$

Using Lemma 3.1, we can obtain an asymptotic expansion of the null distribution of $-N\rho_1 \log \tilde{\Lambda}_1$ by expanding its characteristic function.

THEOREM 3.3. *When the hypothesis H_{01} is true, the distribution function of $-N\rho_1 \log \tilde{\Lambda}_1$ can be expanded for large $M = N\rho_1$ as*

$$P(-N\rho_1 \log \tilde{\Lambda}_1 \le x)$$
$$= P(\chi^2_{f_1} \le x) + \frac{\gamma_2}{M^2}\{P(\chi^2_{f_1+4} \le x) - P(\chi^2_{f_1} \le x)\} + O(M^{-3}),$$

where $f_1 = \frac{1}{2}m(m+1)$,

$$\gamma_2 = \frac{1}{48}m\Bigg\{6(m+1)u^2 - 2\Bigg\{\left\{1+\frac{1}{p(p-1)}\right\}(2m^2+3m-1)$$
$$-\frac{6}{p}(k-1)\{m+1-(p-1)(k-1)\}\Bigg\}u + \left\{1+\frac{2p-1}{p^2(p-1)^2}\right\}(m-1)(m+1)(m+2)$$
$$-\frac{2}{p^2}(2p-1)(k-1)\left\{\frac{1}{p-1}(2m^2+3m-1) - 3(m+1)(k-1) + 2(p-1)(k-1)^2\right\}\Bigg\},$$

where $u = N(1-\rho_1) - k$, *and* ρ_1 *is defined by*

$$f_1 N(1-\rho_1) = \frac{1}{12}m\Bigg\{6(m+1)k + \left\{1+\frac{1}{p(p-1)}\right\}(2m^2+3m-1)$$
$$-\frac{6}{p}(k-1)\{m+1-(p-1)(k-1)\}\Bigg\}.$$

We now consider the exact LR criterion $\Lambda_1^{N/2}$ for H_{01} vs. H_{11}. If (2.12) holds, the LR statistic Λ_1 is equal to $\tilde{\Lambda}_1$. However, if it is not the case, the LR statistic Λ_1 is complicated. As a simple bound for Λ_1, we can suggest

$$\bar{\Lambda}_1 = \begin{cases} \tilde{\Lambda}_1, & \text{if } \hat{\Psi}_{11} - \hat{\Sigma}_e \geq O, \\ 1, & \text{elsewhere} \end{cases} \tag{3.8}$$

such that

$$\tilde{\Lambda}_1 \leq \Lambda_1 \leq \bar{\Lambda}_1. \tag{3.9}$$

We note that the LR statistic Λ_1 agrees with $\bar{\Lambda}_1$ in the case $m = 1$.

Next we consider the LR test for

$$H_{02}: \Delta = O \quad \text{vs.} \quad H_{12}: \text{not } H_{02} \tag{3.10}$$

under the multivariate parallel profile model (1.5). For simplicity, we consider again the model (3.1) with the modified covariance matrix given in (2.2). Let

$$\begin{aligned} g_2(\Psi_{11}, \Sigma_e) = N\Bigg\{&\log|\Psi_{11}| + \operatorname{tr} \Psi_{11}^{-1} \frac{1}{N} Y_1' Y_1 \\ &+ (p-1)\Bigg\{\log|\Sigma_e| + \operatorname{tr} \Sigma_e^{-1} \frac{1}{N(p-1)} \sum_{i=1}^{p-1} Y_2^{(i)\prime} Y_2^{(i)}\Bigg\}\Bigg\}. \end{aligned} \tag{3.11}$$

This function is defined by the same way as $g_1(\Psi_{11}, \Sigma_e)$, i.e., by considering the maximization of the likelihood function $L(\boldsymbol{\theta}, \Delta, \Psi_{11}, \Sigma_e)$ under H_{02} with respect to $\boldsymbol{\theta}$. Then we have

$$\begin{aligned} \min g_2(\Psi_{11}, \Sigma_e) = N \log &\left|\frac{1}{N} Y_1' Y_1\right| \\ &+ N(p-1) \log \left|\frac{1}{N(p-1)} \sum_{i=1}^{p-1} Y_2^{(i)\prime} Y_2^{(i)}\right| + Nmp. \end{aligned} \tag{3.12}$$

Therefore, from (2.3) we can suggest a test statistic

$$\tilde{\Lambda}_2 = \frac{|S_{11}|}{|Y_1' Y_1|} = \frac{|S_{11}|}{|S_{11} + Y_1' P_{\tilde{A}_1} Y_1|} \tag{3.13}$$

for testing H_{02} vs. H_{12}, where $Y_1' Y_1 = \frac{1}{p}(\mathbf{1}_p' \otimes I_m) S_t (\mathbf{1}_p \otimes I_m)$. It is easy to verify that under H_{02},

$$S_{11} \sim W_m(N - k, \Psi_{11}) \quad \text{and} \quad Y_1' P_{\tilde{A}_1} Y_1 \sim W_m(k - 1, \Psi_{11}).$$

Further, these two statistics are independent. Therefore, the h-th moment of $\tilde{\Lambda}_2$ is obtained from that $\tilde{\Lambda}_2$ is distributed as a lambda distribution $\Lambda_{m,k-1,N-k}$

and is given by

$$E(\tilde{\Lambda}_2^h) = \frac{\Gamma_m\left(\frac{1}{2}(N-k)+h\right)\Gamma_m\left(\frac{1}{2}(N-1)\right)}{\Gamma_m\left(\frac{1}{2}(N-k)\right)\Gamma_m\left(\frac{1}{2}(N-1)+h\right)}.$$

For an asymptotic expansion of the null distribution of $-N\rho_2 \log \tilde{\Lambda}_2$, see, e.g., Siotani, Hayakawa and Fujikoshi [24, p. 250].

4. Parallel profile analysis for single-response case

For the single-response case ($m = 1$), we can obtain the exact LR criteria for two testing problems in Sections 2 and 3. Asymptotic null distributions of the LR criteria have been derived by Yokoyama [32] and Yokoyama and Fujikoshi [34]. In this section we derive asymptotic non-null distributions of the LR criteria under local alternatives. First we consider the LR criterion for the hypothesis (1.6) under the model (1.2) in the case $m = 1$, i.e., for

(4.1) $$H_0: \Omega = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p \quad \text{vs.} \quad H_1: \text{not } H_0$$

under

(4.2) $$X \sim N_{N\times p}(A_1 \boldsymbol{\delta} \mathbf{1}_p' + \mathbf{1}_N \boldsymbol{\mu}', \Omega \otimes I_N),$$

where $\lambda^2 \geq 0$ and $\sigma^2 > 0$. Then, the canonical forms (1.7) and (1.8) can be expressed as

(4.3) $$\begin{bmatrix} \multicolumn{2}{c}{\boldsymbol{z}'} \\ \boldsymbol{y}_1 & Y_2 \end{bmatrix} \sim N_{N\times p}\left(\begin{bmatrix} \multicolumn{2}{c}{\boldsymbol{\theta}'} \\ \tilde{A}_1\gamma & O \end{bmatrix}, \Psi \otimes I_N\right)$$

and

(4.4) $$H_0: \Psi = \begin{pmatrix} p\lambda^2 + \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 I_{p-1} \end{pmatrix} \quad \text{vs.} \quad H_1: \text{not } H_0,$$

respectively. The MLE's of λ^2 and σ^2 under H_0 can be obtained by a well-known technique in a variance components model (see, e.g., Arnold [3, p. 251]) and are given by

(4.5) $$\begin{aligned} \hat{\lambda}^2 &= \max\left\{\frac{1}{p}\left(\frac{1}{N}s_{11} - \frac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2\right), 0\right\}, \\ \hat{\sigma}^2 &= \min\left\{\frac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2, \frac{1}{Np}(s_{11} + \operatorname{tr} Y_2'Y_2)\right\}, \end{aligned}$$

respectively. Therefore, we can write the LR criterion as

$$\Lambda^{N/2} = \begin{cases} R_1, & \text{if } s_{11}/N \geq \operatorname{tr} Y_2'Y_2/\{N(p-1)\}, \\ R_2, & \text{if } s_{11}/N < \operatorname{tr} Y_2'Y_2/\{N(p-1)\}, \end{cases} \tag{4.6}$$

where

$$R_1 = \frac{\left(\frac{1}{N} s_{11\cdot 2}\right)^{\frac{N}{2}} \left|\frac{1}{N} Y_2'Y_2\right|^{\frac{N}{2}}}{\left(\frac{1}{N} s_{11}\right)^{\frac{N}{2}} \left\{\frac{1}{N(p-1)} \operatorname{tr} Y_2'Y_2\right\}^{\frac{N(p-1)}{2}}}, \qquad R_2 = \frac{\left(\frac{1}{N} s_{11\cdot 2}\right)^{\frac{N}{2}} \left|\frac{1}{N} Y_2'Y_2\right|^{\frac{N}{2}}}{\left\{\frac{1}{Np}(s_{11} + \operatorname{tr} Y_2'Y_2)\right\}^{\frac{Np}{2}}},$$

and

$$s_{11} = \frac{1}{p} \mathbf{1}_p' S_w \mathbf{1}_p, \quad s_{11\cdot 2} = \left(\frac{1}{p} \mathbf{1}_p' S_w^{-1} \mathbf{1}_p\right)^{-1},$$

$$\operatorname{tr} Y_2'Y_2 = \operatorname{tr} S_t - \frac{1}{p} \mathbf{1}_p' S_t \mathbf{1}_p, \quad |Y_2'Y_2| = \frac{1}{p} \mathbf{1}_p' S_t^{-1} \mathbf{1}_p |S_t|.$$

THEOREM 4.1. *Let $\Lambda^{N/2}$ be the LR criterion for testing H_0 vs. H_1. Then, under the sequence of local alternatives*

$$H_1^{(N)}: \Psi = \Psi_0 + \sqrt{\frac{2}{N}}\, \eta,$$

it holds that

$$\lim_{N\to\infty} P(-N \log \Lambda \leq c)$$

$$= \begin{cases} P(\chi_f^2(\delta_1) \leq c), & \text{if } \lambda^2 > 0, \\ \Phi(\delta^*) P(\chi_f^2(\delta_1) \leq c) + \{1 - \Phi(\delta^*)\} P(\chi_{f+1}^2(\delta_2) \leq c), & \text{if } \lambda^2 = 0, \end{cases}$$

where $\Psi_0 = \operatorname{diag}(\tau^2, \sigma^2, \ldots, \sigma^2)$, $\tau = (p\lambda^2 + \sigma^2)^{1/2}$, *$\eta$ is a fixed matrix partitioned as*

$$\eta = \begin{pmatrix} \eta_{11} & \boldsymbol{\eta}_{12}' \\ \boldsymbol{\eta}_{21} & \eta_{22} \end{pmatrix}, \ \eta_{11}: 1 \times 1,$$

$$f = \frac{1}{2}(p^2 + p - 4),$$

$$\delta_1 = \frac{2}{\sigma^2\tau^2} \boldsymbol{\eta}_{12}' \boldsymbol{\eta}_{21} + \frac{1}{\sigma^4} \operatorname{tr} \eta_{22}^2 - \frac{1}{(p-1)\sigma^4} (\operatorname{tr} \eta_{22})^2,$$

$$\delta_2 = \frac{1}{\sigma^4}\left\{\mathrm{tr}\,\eta^2 - \frac{1}{p}(\mathrm{tr}\,\eta)^2\right\},\ \delta^* = \frac{1}{\sigma^2}\sqrt{\frac{p-1}{p}}\left(\eta_{11} - \frac{1}{p-1}\mathrm{tr}\,\eta_{22}\right),$$

$\chi_f^2(\delta_1)$ denotes a χ^2 variate with f degrees of freedom and noncentrality parameter δ_1, and $\Phi(x)$ denotes the distribution function of the standard normal distribution.

PROOF. From the definition of $\Lambda^{N/2}$ we have

$$P(-N\log\Lambda \le c) = P(-2\log R_1 \le c,\ s_{11}/N \ge \mathrm{tr}\ Y_2'Y_2/\{N(p-1)\}) + P(-2\log R_2 \le c,\ s_{11}/N < \mathrm{tr}\ Y_2'Y_2/\{N(p-1)\}).$$

Let

$$\frac{1}{\sqrt{2N}}(\Psi^{-1/2}S\Psi^{-1/2} - NI_p) = U.$$

Then all the different elements of U are asymptotically independent, and the limiting distributions of u_{ii} and u_{ij} are $N(0, 1)$ and $N(0, 1/2)$, respectively, where $1 \le i, j \le p$, $i \ne j$. Under $H_1^{(N)}$, $N^{-1}S$ can be expressed as

$$\frac{1}{N}S = \Psi_0 + \sqrt{\frac{2}{N}}V,$$

where

$$V = \eta + \Psi^{1/2}U\Psi^{1/2} = \begin{pmatrix} v_{11} & \boldsymbol{v}_{12}' \\ \boldsymbol{v}_{21} & V_{22} \end{pmatrix}.$$

Here we note that the limiting distribution of V is the same as the one of $\eta + \Psi_0^{1/2}U\Psi_0^{1/2}$. Then, by the same way as in Yokoyama [32], we can expand $-2\log R_1$ as

$$-2\log R_1 = Z_1 + O_p(N^{-1/2}),$$

where

$$Z_1 = \frac{2}{\sigma^2\tau^2}\boldsymbol{v}_{12}'\boldsymbol{v}_{21} + \frac{1}{\sigma^4}\mathrm{tr}\ V_{22}^2 - \frac{1}{(p-1)\sigma^4}(\mathrm{tr}\ V_{22})^2.$$

It is easily seen that the limiting distribution of Z_1 is a noncentral χ^2 distribution with f degrees of freedom and noncentrality parameter δ_1. When $\tau^2 > \sigma^2$, we have

$$\lim_{N\to\infty} P\left(\frac{1}{N}s_{11} \ge \frac{1}{N(p-1)}\mathrm{tr}\ Y_2'Y_2\right) = 1$$

and hence

$$\lim_{N\to\infty} P(-N\log\Lambda \le c) = \lim_{N\to\infty} P(-2\log R_1 \le c)$$

$$= P(\chi_f^2(\delta_1) \le c).$$

When $\tau^2 = \sigma^2$, we can expand $-2\log R_2$ as

$$-2\log R_2 = Z_2 + O_p(N^{-1/2}),$$

where

$$Z_2 = \frac{1}{\sigma^4}\left\{\mathrm{tr}\, V^2 - \frac{1}{p}(\mathrm{tr}\, V)^2\right\}.$$

Let

$$Z^* = \frac{1}{\sigma^2}\sqrt{\frac{p-1}{p}}\left(v_{11} - \frac{1}{p-1}\mathrm{tr}\, V_{22}\right).$$

Then the lmiting distribution of Z^* is $N(\delta^*, 1)$. It is easy to verify that Z^* and Z_1 are independent, $Z_2 = Z^{*2} + Z_1$. Therefore, the limiting distribution of Z_2 is $\chi_{f+1}^2(\delta_2)$, $\delta_2 = \delta^{*2} + \delta_1$. Since $s_{11}/N \ge \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}$ is equivalent to $Z^* \ge 0$, it holds that

$$\lim_{N\to\infty} P(-N\log\Lambda \le c) = \lim_{N\to\infty} \{P(Z_1 \le c, Z^* \ge 0) + P(Z_2 \le c, Z^* < 0)\}$$

$$= \Phi(\delta^*)P(\chi_f^2(\delta_1) \le c) + \{1 - \Phi(\delta^*)\}P(\chi_{f+1}^2(\delta_2) \le c),$$

which proves the desired result.

Next we consider the LR criterion for the hypothesis (3.4) under the model (1.5) in the case $m = 1$, i.e., for

(4.7) $$H_{01}: \lambda^2 = 0 \quad \text{vs.} \quad H_{11}: \text{not } H_{01}$$

under

(4.8) $$X \sim N_{N\times p}(A_1\boldsymbol{\delta}\mathbf{1}_p' + \mathbf{1}_N\boldsymbol{\mu}', (\lambda^2\mathbf{1}_p\mathbf{1}_p' + \sigma^2 I_p)\otimes I_N).$$

It is easily seen that the MLE's of λ^2 and σ^2 under H_{01} is given by

(4.9) $$\hat{\lambda}_0^2 = 0 \quad \text{and} \quad \hat{\sigma}_0^2 = \frac{1}{Np}(s_{11} + \mathrm{tr}\, Y_2'Y_2),$$

respectively. Therefore, from (4.5) we can write the LR criterion as

(4.10) $$\Lambda_1^{N/2} = \begin{cases} R_3, & \text{if } s_{11}/N \ge \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}, \\ 1, & \text{if } s_{11}/N < \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}, \end{cases}$$

where

$$R_3 = \frac{\left(\frac{1}{N}s_{11}\right)^{\frac{N}{2}}\left\{\frac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2\right\}^{\frac{N(p-1)}{2}}}{\left\{\frac{1}{Np}(s_{11}+\operatorname{tr} Y_2'Y_2)\right\}^{\frac{Np}{2}}}.$$

THEOREM 4.2. *Let $\Lambda_1^{N/2}$ be the LR criterion for testing H_{01} vs. H_{11}. Then, under the sequence of local alternatives*

$$H_{11}^{(N)}: \lambda^2 = \sqrt{\frac{2}{N}}\,\alpha^2,$$

it holds that

$$\lim_{N\to\infty} P(-N\log\Lambda_1 \le c) = 1 - \Phi(\delta_1^*) + \Phi(\delta_1^*)P(\chi_1^2(\delta_1^{*2}) \le c),$$

where α^2 is a constant, and $\delta_1^* = \sqrt{p(p-1)}\,\alpha^2/\sigma^2$.

PROOF. From the definition of $\Lambda_1^{N/2}$ we have

$$P(-N\log\Lambda_1 \le c) = P(-2\log R_3 \le c,\ s_{11}/N \ge \operatorname{tr} Y_2'Y_2/\{N(p-1)\}) + P(s_{11}/N < \operatorname{tr} Y_2'Y_2/\{N(p-1)\}).$$

Let

$$\frac{1}{\sqrt{2N}}\left(\frac{1}{\sigma^2 + p\sqrt{2/N}\,\alpha^2}s_{11} - N\right) = U_1,$$

$$\frac{1}{\sqrt{2N(p-1)}}\left(\frac{1}{\sigma^2}\operatorname{tr} Y_2'Y_2 - N(p-1)\right) = U_2.$$

Then U_1 and U_2 are independent, and the limiting distribution of U_i is $N(0, 1)$, $i = 1, 2$. Let

$$U_1^* = U_1 + \frac{p\alpha^2}{\sigma^2}.$$

Then, by the same way as in Yokoyama and Fujikoshi [34], we can expand $-2\log R_3$ as

$$-2\log R_3 = Z_1^{*2} + O_p(N^{-1/2}),$$

where

$$Z_1^* = \sqrt{\frac{p-1}{p}}\,U_1^* - \sqrt{\frac{1}{p}}\,U_2.$$

Here we note that the limiting distribution of Z_1^* is $N(\delta_1^*, 1)$. Since $s_{11}/N \geq \mathrm{tr}\, Y_2' Y_2/\{N(p-1)\}$ is equivalent to $Z_1^* \geq 0$, it holds that

$$\lim_{N\to\infty} P(-N\log \Lambda_1 \leq c) = \lim_{N\to\infty} \{P(Z_1^{*2} \leq c,\ Z_1^* \geq 0) + P(Z_1^* < 0)\},$$

which implies the desired result.

We note that under the null hypotheses H_0 and H_{01}, the limiting distributions of the LR criteria in Theorems 4.1 and 4.2 agree with the results in Yokoyama [32] and Yokoyama and Fujikoshi [34].

Part II. Growth curve model with covariates and random effects

5. The model and its canonical form

An important application of the mixed MANOVA-GMANOVA model (0.2) arises in the growth curve model with covariates. In the model (0.1), suppose that we can use the observations of r covariates for the N individuals. Let C be the $N \times r$ observation matrix of r covariates. Then the model (0.1) can be extended to a case of (0.2), and the model of X can be written as

$$X \sim N_{N\times p}(A\Xi B + C\Theta,\ \Sigma \otimes I_N), \tag{5.1}$$

where Θ is an unknown $r \times p$ parameter matrix. It is assumed that C is fixed and rank $[A, C] = k + r \leq N - p$. Here, without loss of generality we may assume that $BB' = I_q$. In fact, if $BB' \neq I_q$, we may replace Ξ and B by $\Xi(BB')^{1/2}$ and $(BB')^{-1/2}B$, respectively. In the following we shall do this. The model (5.1) is called a growth curve model with covariates.

We are now interested in a family of covariance structures

$$\Sigma = B_s'\Delta_s B_s + \sigma_s^2 I_p (= \Sigma_s), \qquad 0 \leq s \leq q, \tag{5.2}$$

which is based on random-coefficients models with differing numbers of random effects, where Δ_s is an arbitrary $s \times s$ positive semi-definite matrix, $\sigma_s^2 > 0$, B_s is the matrix which is composed of the first s rows of B. This family is a generalization of random-effects covariance structures proposed by Rao [18]. In fact, the covariance structure (5.2) can be naturally introduced by assuming that the first s columns of Ξ are random. We note that Lange and Laird [15] has introduced the family (5.2) of covariance structures to a

GMANOVA model (0.1).

A test statistic for testing $H_{0s}: \Sigma = \Sigma_s$ vs. H_{1s}: not H_{0s} under the mixed MANOVA-GMANOVA model (5.1) has been proposed by Yokoyama and Fujikoshi [33]. In Section 6 we propose test statistics for the hypothesis

$$H_{0s}: \Sigma = \Sigma_s \quad \text{vs.} \quad H_{1t}: \Sigma = \Sigma_t \tag{5.3}$$

under the model (5.1), where $0 \leq s < t \leq q$.

We now give a canonical reduction. We define the submatrices $B_{\bar{t}}$ and $B_{\bar{s}\cap t}$ of B by $B = [B_t', B_{\bar{t}}']'$, $B_t = [B_s', B_{\bar{s}\cap t}']'$. Let $\bar{B}$ be a $(p-q) \times p$ matrix such that $\bar{B}\bar{B}' = I_{p-q}$ and $B\bar{B}' = O$, i.e.,

$$Q = \begin{bmatrix} B_s \\ B_{\bar{s}\cap t} \\ B_{\bar{t}} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

is an orthogonal matrix of order p. Further, let $H = [H_1, H_2]$ be an orthogonal matrix of order N such that H_1 is an orthonormal basis matrix on the space spanned by the column vectors of C. Consider the transformation from X to

$$\begin{bmatrix} Z \\ Y \end{bmatrix} = \begin{bmatrix} H_1' \\ H_2' \end{bmatrix} XQ'.$$

Then, letting $Y = H_2'XQ' = [Y_1, Y_2, Y_3, Y_4] = [Y_{(123)}, Y_4]$, we have

$$\begin{bmatrix} Z & \\ Y_{(123)} & Y_4 \end{bmatrix} \sim N_{N\times p}\left(\begin{bmatrix} \mu & \\ \tilde{A}\Xi & O \end{bmatrix}, \Psi \otimes I_N\right), \tag{5.4}$$

where $\mu = H_1'A[\Xi, O] + H_1'C\Theta Q'$, $\tilde{A} = H_2'A$ and $\Psi = Q\Sigma Q'$. Here we note that (Θ, Ξ) is an invertible function of (μ, Ξ). In fact, Θ can be expressed in terms of μ and Ξ as

$$\Theta = (H_1'C)^{-1}\mu Q - (H_1'C)^{-1}H_1'A\Xi B. \tag{5.5}$$

We can express the hypothesis (5.3) as

$$H_{0s}: \Psi = \begin{pmatrix} \Delta_s + \sigma_s^2 I_s & O \\ O & \sigma_s^2 I_{p-s} \end{pmatrix} \quad \text{vs.} \quad H_{1t}: \Psi = \begin{pmatrix} \Delta_t + \sigma_t^2 I_t & O \\ O & \sigma_t^2 I_{p-t} \end{pmatrix}. \tag{5.6}$$

6. Tests for ramdom-effects covariance structures

We may consider the LR test for the hypothesis (5.6) under (5.4) instead of the one for the hypothesis (5.3) under (5.1). Since the elements of μ in (5.4) are free parameters, for testing the hypothesis (5.6) we may consider the LR test formed by only the density of Y. The model of Y can be written as

$$Y \sim N_{n\times p}([\tilde{A}\Xi,\ O],\ \Psi \otimes I_n), \tag{6.1}$$

where $n = N - r$. Let $L(\Xi, \Psi)$ be the likelihood function of Y. Then we can see that the MLE of Ξ under H_{0s} or H_{1t} is given by $\hat{\Xi} = (\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}'Y_{(123)}$. Let

$$\begin{aligned} g(\Psi) &= -2\log L(\hat{\Xi}, \Psi) \\ &= n\log|\Psi| + \operatorname{tr}\Psi^{-1}[Y_{(123)} - \tilde{A}\hat{\Xi}\quad Y_4]'[Y_{(123)} - \tilde{A}\hat{\Xi}\quad Y_4]. \end{aligned}$$

As is seen later on, the minimum of $g(\Psi)$ under H_{0s} or H_{1t} can be obtained in a closed form. However, since it is complicated, we consider the LR test for a modified hypothesis

$$\tilde{H}_{0s}\colon \Psi = \begin{pmatrix} \Psi^{(s)} & O \\ O & \sigma_s^2 I_{p-s} \end{pmatrix} \quad \text{vs.} \quad \tilde{H}_{1t}\colon \Psi = \begin{pmatrix} \Psi^{(t)} & O \\ O & \sigma_t^2 I_{p-t} \end{pmatrix}, \tag{6.2}$$

where $\Psi^{(s)}$ and $\Psi^{(t)}$ are assumed to be arbitrary $s \times s$ and $t \times t$ positive definite matrices, respectively, and

$$\Psi^{(t)} = \begin{pmatrix} \Psi_{11}^{(t)} & \Psi_{12}^{(t)} \\ \Psi_{21}^{(t)} & \Psi_{22}^{(t)} \end{pmatrix}, \qquad \Psi_{11}^{(t)}: s \times s.$$

It is easily seen that

(6.3)

$$\min_{\tilde{H}_{0s}} g(\Psi) = n\log\left|\frac{1}{n}S_{11}\right| + n(p-s)\log\left\{\frac{1}{n(p-s)}(\operatorname{tr}S_{(23)(23)} + \operatorname{tr}Y_4'Y_4)\right\} + np$$

and

(6.4)

$$\min_{\tilde{H}_{1t}} g(\Psi) = n\log\left|\frac{1}{n}S_{(12)(12)}\right| + n(p-t)\log\left\{\frac{1}{n(p-t)}(\operatorname{tr}S_{33} + \operatorname{tr}Y_4'Y_4)\right\} + np,$$

where

$$S = Y'(I_n - \tilde{A}(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}')Y = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix},$$

$$S_{(12)(12)}=\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad \text{and} \quad S_{(23)(23)}=\begin{pmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{pmatrix}.$$

Therefore, we can suggest a test statistic

$$\tilde{\Lambda}_{s,t}=\frac{|S_{(12)(12)}|\left\{\frac{1}{p-t}(\mathrm{tr}\, S_{33}+\mathrm{tr}\, Y_4' Y_4)\right\}^{p-t}}{|S_{11}|\left\{\frac{1}{p-s}(\mathrm{tr}\, S_{(23)(23)}+\mathrm{tr}\, Y_4' Y_4)\right\}^{p-s}} \tag{6.5}$$

for testing H_{0s} vs. H_{1t}. In order to express the statistic (6.5) in terms of the original observations, we denote S_{ij} and $Y_4' Y_4$ in terms of

$$D=[X, C, A]'[X, C, A]=\begin{pmatrix} D_{xx} & D_{xc} & D_{xa} \\ D_{cx} & D_{cc} & D_{ca} \\ D_{ax} & D_{ac} & D_{aa} \end{pmatrix}. \tag{6.6}$$

Noting that $H_2 H_2' = I_N - P_C$, $P_C = C(C'C)^{-1}C'$, it can be shown (see Yokoyama and Fujikoshi [33]) that

$$Y_4' Y_4 = Q_4 D_{xx\cdot c} Q_4', \qquad S_{ij} = Q_i D_{xx\cdot ca} Q_j', \tag{6.7}$$

where $D_{xx\cdot c} = D_{xx} - D_{xc} D_{cc}^{-1} D_{cx}$ and

$$D_{xx\cdot ca} = D_{xx} - [D_{xc}, D_{xa}]\begin{bmatrix} D_{cc} & D_{ca} \\ D_{ac} & D_{aa} \end{bmatrix}^{-1}\begin{bmatrix} D_{cx} \\ D_{ax} \end{bmatrix},$$

which is equal to $D_{xx\cdot c} - D_{xa\cdot c} D_{aa\cdot c}^{-1} D_{ax\cdot c}$.

The statistic (6.5) can be decomposed as

$$\tilde{\Lambda}_{s,t} = \Lambda_1 \Lambda_2, \tag{6.8}$$

where

$$\Lambda_1 = \frac{|S_{11\cdot 2}|}{|S_{11}|} = \frac{|S_{11\cdot 2}|}{|S_{11\cdot 2} + S_{12} S_{22}^{-1} S_{21}|}$$

and

$$\Lambda_2 = \frac{|S_{22}|\left\{\frac{1}{p-t}(\mathrm{tr}\, S_{33}+\mathrm{tr}\, Y_4' Y_4)\right\}^{p-t}}{\left\{\frac{1}{p-s}(\mathrm{tr}\, S_{22}+\mathrm{tr}\, S_{33}+\mathrm{tr}\, Y_4' Y_4)\right\}^{p-s}}.$$

Here, the statistics Λ_1 and Λ_2 are the LR statistics for $\Psi_{12}^{(t)} = O$ and

$\Psi_{22}^{(t)} = \sigma_s^2 I_{t-s}$, respectively.

LEMMA 6.1. *When the hypothesis* H_{0s} *is true, it holds that*

(i) Λ_1 *and* Λ_2 *are independent,*

$$\text{(ii)}\quad E(\Lambda_1^h) = \frac{\Gamma_s\left(\frac{1}{2}(n-k-t+s)+h\right)\Gamma_s\left(\frac{1}{2}(n-k)\right)}{\Gamma_s\left(\frac{1}{2}(n-k-t+s)\right)\Gamma_s\left(\frac{1}{2}(n-k)+h\right)},$$

$$\text{(iii)}\quad E(\Lambda_2^h) = \frac{(p-s)^{(p-s)h}}{(p-t)^{(p-t)h}} \frac{\Gamma_{t-s}\left(\frac{1}{2}(n-k)+h\right)}{\Gamma_{t-s}\left(\frac{1}{2}(n-k)\right)}$$

$$\times \frac{\Gamma\left(\frac{1}{2}\{n(p-t)-k(q-t)\}+(p-t)h\right)\Gamma\left(\frac{1}{2}\{n(p-s)-k(q-s)\}\right)}{\Gamma\left(\frac{1}{2}\{n(p-t)-k(q-t)\}\right)\Gamma\left(\frac{1}{2}\{n(p-s)-k(q-s)\}+(p-s)h\right)}.$$

PROOF. It is easy to verify that under H_{0s},

$S_{11\cdot 2} \sim W_s(n-k-(t-s), \Psi^{(s)})$, $S_{12}S_{22}^{-1}S_{21} \sim W_s(t-s, \Psi^{(s)})$,
$S_{22} \sim W_{t-s}(n-k, \sigma_s^2 I_{t-s})$, $S_{33} \sim W_{q-t}(n-k, \sigma_s^2 I_{q-t})$ and
$Y_4' Y_4 \sim W_{p-q}(n, \sigma_s^2 I_{p-q})$.

Further, these five statistics are independent. Therefore, Λ_1 and Λ_2 are independent. The h-th moment of Λ_1 is obtained from that Λ_1 is distributed as a lambda distribution $\Lambda_{s,t-s,n-k-(t-s)}$. The h-th moment of Λ_2 can be written as

$$E(\Lambda_2^h) = 2^{(t-s)h} \frac{(p-s)^{(p-s)h}}{(p-t)^{(p-t)h}} \frac{\Gamma_{t-s}\left(\frac{1}{2}(2h+n-k)\right)}{\Gamma_{t-s}\left(\frac{1}{2}(n-k)\right)} E\left[\frac{K_2^{(p-t)h}}{(K_1+K_2)^{(p-s)h}}\right],$$

where K_1 and K_2 are independently distributed, $K_1 \sim \chi^2_{\nu_1}$, $K_2 \sim \chi^2_{\nu_2}$, and

$$\nu_1 = (2h+n-k)(t-s), \quad \nu_2 = n(p-t) - k(q-t).$$

Here, letting $U = K_1 + K_2$ and $V = K_2/(K_1+K_2)$, it is well known that U and V are independent with

$$U \sim \chi^2_{\nu_1+\nu_2}, \quad V \sim B\left(\frac{1}{2}\nu_2, \frac{1}{2}\nu_1\right).$$

The third result (iii) follows from the above fact.

Using Lemma 6.1, we can obtain an asymptotic expansion of the null distribution of $-n\rho\log\tilde{\Lambda}_{s,t}$ by expanding its characteristic function.

THEOREM 6.1. *When the hypothesis H_{0s} is true, the distribution function of $-n\rho\log\tilde{\Lambda}_{s,t}$ can be expanded for large $M=n\rho$ as*

$$P(-n\rho\log\tilde{\Lambda}_{s,t}\le x)$$
$$=P(\chi_f^2\le x)+\frac{\gamma_2}{M^2}\{P(\chi_{f+4}^2\le x)-P(\chi_f^2\le x)\}+O(M^{-3}),$$

where $f=\dfrac{1}{2}(t-s)(t+s+1)$,

$$\gamma_2=\frac{1}{48}(t-s)\Big\{6(t+s+1)u^2-2\Big\{6(t+1)s+2(t-s)^2+3(t-s)-1$$
$$+\frac{2}{(p-t)(p-s)}\{3(p-q)^2k^2-6(p-q)k+2\}\Big\}u$$
$$+2\{(3t-s+3)s+2(t-s)^2+3(t-s)-1\}s+(t-s-1)(t-s+1)(t-s+2)$$
$$-\frac{4}{(p-t)^2(p-s)^2}(2p-t-s)(p-q)\{(p-q)k-1\}\{(p-q)k-2\}k\Big\},$$

where $u=n(1-\rho)-k$, and ρ is defined by

$$fn(1-\rho)=\frac{1}{12}(t-s)\Big\{6(t+s+1)k+6(t+1)s+2(t-s)^2+3(t-s)-1$$
$$+\frac{2}{(p-t)(p-s)}\{3(p-q)^2k^2-6(p-q)k+2\}\Big\}.$$

We now obtain the exact LR criterion $\Lambda_{s,t}^{n/2}$ for H_{0s} vs. H_{1t}, which may be obtained by starting from the distribution of Y. Let

$$\tilde{\Psi}^{(s)}=\frac{1}{n}S_{11},\quad \tilde{\sigma}_s^2=\frac{1}{n(p-s)}(\mathrm{tr}\,S_{(23)(23)}+\mathrm{tr}\,Y_4'Y_4),$$
$$\tilde{\Psi}^{(t)}=\frac{1}{n}S_{(12)(12)},\quad \tilde{\sigma}_t^2=\frac{1}{n(p-t)}(\mathrm{tr}\,S_{33}+\mathrm{tr}\,Y_4'Y_4).$$

For the case

$$\tilde{\Psi}^{(s)}-\tilde{\sigma}_s^2I_s\ge O\quad\text{and}\quad\tilde{\Psi}^{(t)}-\tilde{\sigma}_t^2I_t\ge O, \tag{6.9}$$

the LR statistic $\Lambda_{s,t}$ is equal to $\tilde{\Lambda}_{s,t}$. However, if (6.9) is not satisfied, we need to solve the problem of minimizing

(6.10)
$$\frac{1}{n} g(\Delta_s, \sigma_s^2) = \log|\Delta_s + \sigma_s^2 I_s| + \mathrm{tr}\,(\Delta_s + \sigma_s^2 I_s)^{-1} \tilde{\Psi}^{(s)} + (p-s)\left(\log \sigma_s^2 + \frac{\tilde{\sigma}_s^2}{\sigma_s^2}\right)$$

or

(6.11)
$$\frac{1}{n} g(\Delta_t, \sigma_t^2) = \log|\Delta_t + \sigma_t^2 I_t| + \mathrm{tr}\,(\Delta_t + \sigma_t^2 I_t)^{-1} \tilde{\Psi}^{(t)} + (p-t)\left(\log \sigma_t^2 + \frac{\tilde{\sigma}_t^2}{\sigma_t^2}\right)$$

under H_{0s} or H_{1t}, respectively. Let $l_1 > \cdots > l_s (> 0)$ be the eigenvalues of $\tilde{\Psi}^{(s)}$ and $W = [w_1, \ldots, w_s]$ be an orthogonal matrix such that $W'\tilde{\Psi}^{(s)}W = \mathrm{diag}\,(l_1, \ldots, l_s)$, and let

$$m_s = \max\left\{j;\, l_j \geq \frac{1}{p-j}\{l_{j+1} + \cdots + l_s + (p-s)\tilde{\sigma}_s^2\}\right\}.$$

Then the values of σ_s^2 and Δ_s which minimize (6.10) are

$$(6.12)\quad \hat{\sigma}_s^2 = \frac{1}{p-m_s}\{l_{m_s+1} + \cdots + l_s + (p-s)\tilde{\sigma}_s^2\} \quad \text{and} \quad \hat{\Delta}_s = \sum_{j=1}^{m_s} (l_j - \hat{\sigma}_s^2) w_j w_j',$$

respectively (see, Schott [23], Khatri and Rao [13], Fujikoshi [7]). Similarly we obtain the values $\hat{\sigma}_t^2$ and $\hat{\Delta}_t$ of σ_t^2 and Δ_t which minimize (6.11) under H_{1t}. Therefore, the exact LR criterion can be expressed as

$$(6.13)\qquad \Lambda_{s,t}^{n/2} = \exp\left[-\frac{1}{2}\{g(\hat{\Delta}_s, \hat{\sigma}_s^2) - g(\hat{\Delta}_t, \hat{\sigma}_t^2)\}\right].$$

Since the LR statistic $\Lambda_{s,t}$ is complicated and impractical, we propose to use the following simple bounds for $\Lambda_{s,t}$:

$$(6.14)\qquad \bar{\Lambda}_{s,t} = \begin{cases} \tilde{\Lambda}_{s,t}, & \text{if } l_s \geq \tilde{\sigma}_s^2, \\ R_s, & \text{if } \tilde{\sigma}_s^2 > l_s, \end{cases} \quad \text{and} \quad \Lambda_{s,t}^* = \begin{cases} \tilde{\Lambda}_{s,t}, & \text{if } d_t \geq \tilde{\sigma}_t^2, \\ R_t, & \text{if } \tilde{\sigma}_t^2 > d_t, \end{cases}$$

where $d_1 > \cdots > d_t (> 0)$ denote the eigenvalues of $\tilde{\Psi}^{(t)}$, and

$$R_s = \frac{|\tilde{\Psi}^{(t)}|(\tilde{\sigma}_t^2)^{p-t}}{|\tilde{\Psi}^{(s)}|\{l_s \exp(\tilde{\sigma}_s^2/l_s - 1)\}^{p-s}} \quad \text{and} \quad R_t = \frac{|\tilde{\Psi}^{(t)}|\{d_t \exp(\tilde{\sigma}_t^2/d_t - 1)\}^{p-t}}{|\tilde{\Psi}^{(s)}|(\tilde{\sigma}_s^2)^{p-s}}.$$

The statistics $\bar{\Lambda}_{s,t}^{n/2}$ and $\Lambda_{s,t}^{*n/2}$ in (6.14) are approximate LR criteria for H_{0s} vs. $\tilde{H}_{1t}$ and $\tilde{H}_{0s}$ vs. H_{1t}, respectively, and we have

(6.15) $$\bar{\Lambda}_{s,t} \le \Lambda_{s,t} \le \Lambda^*_{s,t}.$$

Then, from (6.14) and (6.15) it is easily seen that

(6.16) $$|\Lambda_{s,t} - \tilde{\Lambda}_{s,t}| \le \lambda,$$

where $\lambda = \Lambda^*_{s,t} - \bar{\Lambda}_{s,t}$.

7. The MLE's

In this section we obtain the MLE's of unknown mean parameters under the mixed MANOVA-GMANOVA model (5.1) with $\Sigma = B_s'\Delta_s B_s + \sigma_s^2 I_p (= \Sigma_s)$ and consider the efficiency of the MLE of Ξ. This model is reduced to the same canonical form as in (5.4), but the covariance matrix Ψ is given by

$$\Psi = \begin{pmatrix} \Delta_s + \sigma_s^2 I_s & O \\ O & \sigma_s^2 I_{p-s} \end{pmatrix}.$$

It is easily seen that the MLE's of μ and Ξ are given by

(7.1) $$\hat{\mu} = Z \quad \text{and} \quad \hat{\Xi} = (\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)},$$

respectively. Therefore, from (5.5) the MLE of Θ is given by

(7.2) $$\hat{\Theta} = (H_1'C)^{-1}ZQ - (H_1'C)^{-1}H_1'A(\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)}B.$$

We now express the MLE's given in (7.1) and (7.2) in terms of the original observations or the matrix D in (6.6). Noting that

$$(\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)} = (A'H_2H_2'A)^{-1}A'H_2H_2'XB', \qquad (H_1'C)^{-1}H_1' = (C'C)^{-1}C',$$

we have the following theorem.

THEOREM 7.1. *The MLE's of Ξ and Θ under the mixed MANOVA-GMANOVA model* (5.1) *with $\Sigma = \Sigma_s$ are given as follows*:

$$\begin{aligned} \hat{\Xi} &= [A'(I_N - P_C)A]^{-1}A'(I_N - P_C)XB' \\ &= D_{aa\cdot c}^{-1}D_{ax\cdot c}B', \\ \hat{\Theta} &= (C'C)^{-1}C'X - (C'C)^{-1}C'A[A'(I_N - P_C)A]^{-1}A'(I_N - P_C)XB'B \\ &= D_{cc}^{-1}(D_{cx} - D_{ca}D_{aa\cdot c}^{-1}D_{ax\cdot c}B'B). \end{aligned}$$

We now consider the MLE of Ξ under (5.1) where Σ is an arbitrary positive definite matrix. The MLE can be obtained as the MLE of Ξ under (6.1) where Ψ is an arbitrary positive definite matrix. Since the latter model is a GMANOVA model, it is well known (see, e.g., Siotani, Hayakawa and

Fujikoshi [24, p. 312]) that the MLE of Ξ can be expressed as

$$\tilde{\Xi} = (F_{11}\tilde{A} + F_{12}Y_4')Y_{(123)} \tag{7.3}$$
$$= (\tilde{A}'\tilde{A})^{-1}\tilde{A}'H_2XS^{*-1}B'(BS^{*-1}B')^{-1},$$

where $S^* = X'H_2'(I_n - \tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{A}')H_2X$ and

$$\begin{pmatrix} \tilde{A}'\tilde{A} & \tilde{A}'Y_4 \\ Y_4'\tilde{A} & Y_4'Y_4 \end{pmatrix}^{-1} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

Chinchilli and Elswick [6] obtained an expression similar to (7.3). In terms of the submatrices of the matrix D in (6.6), we can write

$$\tilde{A}'\tilde{A} = D_{aa} - D_{ac}D_{cc}^{-1}D_{ca} = D_{aa\cdot c}, \quad \tilde{A}'H_2X = D_{ax} - D_{ac}D_{cc}^{-1}D_{cx} = D_{ax\cdot c},$$
$$S^* = D_{xx\cdot c} - D_{xa\cdot c}D_{aa\cdot c}^{-1}D_{ax\cdot c} = D_{xx\cdot ac}.$$

Therefore, we can write $\tilde{\Xi}$ as

$$\tilde{\Xi} = D_{aa\cdot c}^{-1}D_{ax\cdot c}D_{xx\cdot ac}^{-1}B'(BD_{xx\cdot ac}^{-1}B')^{-1}. \tag{7.4}$$

THEOREM 7.2. *Under the mixed MANOVA-GMANOVA model* (5.1) *it holds that both the estimators $\hat{\Xi}$ and $\tilde{\Xi}$ are unbiased, and*

(i) *if* $\Sigma = \Sigma_s$, $V(\mathrm{vec}\,(\hat{\Xi})) = \Psi_s \otimes M$,

(ii) $V(\mathrm{vec}\,(\tilde{\Xi})) = \left\{1 + \dfrac{p-q}{N-(k+r)-(p-q)-1}\right\}(B\Sigma^{-1}B')^{-1} \otimes M,$

where $M = [A'(I_N - P_C)A]^{-1}$ *and*

$$\Psi_s = \begin{pmatrix} \Delta_s + \sigma_s^2 I_s & O \\ O & \sigma_s^2 I_{q-s} \end{pmatrix}.$$

PROOF. Since $\hat{\Xi} = (\tilde{A}'\tilde{A})^{-1}\tilde{A}'Y_{(123)}$, we have

$$E(\hat{\Xi}) = \Xi \quad \text{and} \quad V(\mathrm{vec}\,(\hat{\Xi})) = \Psi_s \otimes (\tilde{A}'\tilde{A})^{-1},$$

which imply the result on $\hat{\Xi}$. The results on $\tilde{\Xi}$ are essentially obtained from a general result in a GMANOVA model (see, e.g., Grizzle and Allen [10]). Here we give a direct proof. Since $\tilde{A}'H_2X$ and S^* are independent, the unbiasedness of $\tilde{\Xi}$ is easily obtained, and the covariance matrix of $\mathrm{vec}\,(\tilde{\Xi})$ given S^* can be written as

$$V(\mathrm{vec}\,(\tilde{\Xi}) \mid S^*) = (\tilde{B}\tilde{S}^{-1}\tilde{B}')^{-1}\tilde{B}\tilde{S}^{-2}\tilde{B}'(\tilde{B}\tilde{S}^{-1}\tilde{B}')^{-1} \otimes [A'(I_N - P_C)A]^{-1},$$

where $\tilde{B} = B\Sigma^{-1/2}$ and $\tilde{S} = \Sigma^{-1/2}S^*\Sigma^{-1/2}$. Let $G = [G_1, G_2]$ be an orthogonal matrix such that $G_1 = \tilde{B}'(\tilde{B}\tilde{B}')^{-1/2}$, and let

$$W = G'\tilde{S}G = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \qquad W_{11}: q \times q.$$

Then W is distributed as $W_p(N-(k+r), I_p)$, and it holds that

$$(\tilde{B}\tilde{S}^{-1}\tilde{B}')^{-1}\tilde{B}\tilde{S}^{-2}\tilde{B}'(\tilde{B}\tilde{S}^{-1}\tilde{B}')^{-1} = (\tilde{B}\tilde{B}')^{-1/2}(I_q + W_{12}W_{22}^{-2}W_{21})(\tilde{B}\tilde{B}')^{-1/2}.$$

Noting that $W_{12}W_{22}^{-1/2}$ and W_{22} are independent and the elements of $W_{12}W_{22}^{-1/2}$ are independently distributed as $N(0, 1)$, we have

$$E[(\tilde{B}\tilde{S}^{-1}\tilde{B}')^{-1}\tilde{B}\tilde{S}^{-2}\tilde{B}'(\tilde{B}\tilde{S}^{-1}\tilde{B}')^{-1}]$$
$$= \left\{1 + \frac{p-q}{N-(k+r)-(p-q)-1}\right\}(B\Sigma^{-1}B')^{-1},$$

which implies the desired result.

It is easily checked that $(B\Sigma_s^{-1}B')^{-1} = \Psi_s$. Therefore, from Theorem 7.2, under the assumption of $\Sigma = \Sigma_s$ we obtain

$$V(\mathrm{vec}(\tilde{\Xi})) - V(\mathrm{vec}(\hat{\Xi})) = \frac{p-q}{N-(k+r)-(p-q)-1}\Psi_s \otimes M > O.$$

This shows that a more efficient estimator for Ξ can be obtained by assuming a random-effects covariance structure.

We note that the MLE's of unknown variance parameters σ_s^2 and Δ_s under the mixed MANOVA-GMANOVA model (5.1) with $\Sigma = \Sigma_s$ are given by (6.12). The MLE's are not unbiased. On the other hand, the usual unbiased estimators of σ_s^2 and Δ_s may be defined by

(7.5)

$$\tilde{\sigma}_s^2 = \frac{1}{(N-r)(p-s)}(\mathrm{tr}\, S_{(23)(23)} + \mathrm{tr}\, Y_4' Y_4) \quad \text{and} \quad \tilde{\Delta}_s = \frac{1}{N-(k+r)}S_{11} - \tilde{\sigma}_s^2 I_s,$$

respectively. However, there is the possibility that the use of $\tilde{\Delta}_s$ can lead to a nonpositive semi-definite estimate of Δ_s. The estimators in (7.5) can be expressed in terms of the original observations, using (6.7). Relating to these unbiased estimators, we may propose the estimators obtained from the MLE's by taking

$$\tilde{\Psi}^{(s)} = \frac{1}{N-(k+r)}S_{11},$$

in stead of $\hat{\Psi}^{(s)} = n^{-1}S_{11}$. The modified MLE's of σ_s^2 and Δ_s when $m_s = q$ are equal to (7.5). Such estimators are called restricted maximum likelihood estimators (= REMLE's).

8. Numerical example

We now examine the data (see, e.g., Srivastava and Carter [26, Table 7.14]) of the price indices of hand soaps packaged in 4 ways, estimated by 12 consumers. For 6 of the consumers, the packages have been labeled with a well-known brand name. For the remaining 6 consumers, no label is used. From the data, we obtain

$$\bar{x}^{(1)} = (.31667,\ .45833,\ .47500,\ .64167)',$$

$$\bar{x}^{(2)} = (.60000,\ .66667,\ .85000,\ .96667)',$$

$$\bar{x} = (.45833,\ .56250,\ .66250,\ .80417)',$$

$$S_t = \begin{pmatrix} .45917 & .32875 & .52375 & .35958 \\ & .38563 & .39813 & .41688 \\ & & .72563 & .54438 \\ & & & .61229 \end{pmatrix},$$

$$S_w = \begin{pmatrix} .21833 & .15167 & .20500 & .08333 \\ & .25542 & .16375 & .21375 \\ & & .30375 & .17875 \\ & & & .29542 \end{pmatrix}.$$

For the observation matrix X: 12×4, we assume the model (5.1) with

$$E(X) = \begin{pmatrix} \mathbf{1}_6 \\ \mathbf{0} \end{pmatrix} \xi \mathbf{1}_4' + \mathbf{1}_{12}(\theta_1, \theta_2, \theta_3, \theta_4) \tag{8.1}$$

and the random-effects covariance structure

$$V(\operatorname{vec}(X)) = (\delta^2 \mathbf{1}_4 \mathbf{1}_4' + \sigma^2 I_4) \otimes I_{12}. \tag{8.2}$$

The adequacy of the structures (8.1) and (8.2) to the data has been examined in Yokoyama [32]. Now we consider testing the hypothesis

$$H_{00}\colon \Sigma = \sigma^2 I_4 \quad \text{vs.} \quad H_{11}\colon \Sigma = \delta^2 \mathbf{1}_4 \mathbf{1}_4' + \sigma^2 I_4 \tag{8.3}$$

under this model. Since $s = 0$, $t = q = 1$, and

$$s_{22} = \frac{1}{p} \mathbf{1}_p' S_w \mathbf{1}_p = .76635, \ \operatorname{tr} Y_4' Y_4 = \operatorname{tr} S_t - \frac{1}{p} \mathbf{1}_p' S_t \mathbf{1}_p = .35130,$$

it follows from (6.5) and Theorem 6.1 that

$$\tilde{\Lambda}_{0,1} = \frac{s_{22}\left(\dfrac{1}{p-1}\operatorname{tr} Y_4' Y_4\right)^{p-1}}{\left\{\dfrac{1}{p}(s_{22} + \operatorname{tr} Y_4' Y_4)\right\}^{p}} = .20189,$$

and

$$-n\rho \log \tilde{\Lambda}_{0,1} = 15.223 > \chi_f^2(.01) = 6.635.$$

Therefore, the hypothesis H_{00} should be rejected. This shows that random effects on each individual are not absent in this example.

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