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The fundamental representation of the affine Lie algebra $A_{n-1}^{(1)}$ and the Feynman path integral

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ABSTRACT. We show that the fundamental representation of the affine Lie algebra $A_{n-1}^{(1)}$ is constructed by means of the Feynman path integral on the coadjoint orbits. Using the complex white noise on the coadjoint orbit and generalizing the method of our previous papers, we compute the path integral on the coadjoint orbit of the infinite dimensional Heisenberg group, which realizes a kernel function of an irreducible unitary representation. If we modify the computation of the path integral by multiplying a divergent factor, we obtain the vertex operator for the fundamental representation of $A_{n-1}^{(1)}$.

0. Introduction

Following the method given by Alekseev, Faddeev and Shatashvili [2], we tried to compute the Feynman path integrals on the coadjoint orbits of noncompact Lie groups ([7], [8], [9], [10], [14] e.t.c.). As to the Heisenberg group, we succeeded in computing the path integrals for complex polarizations as well as real polarizations. As to semisimple Lie groups, for real polarizations, we computed the path integrals for $SL(2, \boldsymbol{R})$ ([8]). This was generalized to a certain class of noncompact real semisimple Lie groups ([10]).

For complex polarizations, however, we encountered difficulty of divergence of the path integrals even for most simple Lie groups like $SL(2, \boldsymbol{R})$ ([7]). In [8], for complex polarizations we gave an idea how to regularize the path integrals for $SU(2)$ and $SU(1, 1)$ ($\simeq SL(2, \boldsymbol{R})$) and showed that the path integrals give the kernel functions of the irreducible unitary representations. In [9], we generalized this result to arbitrary connected semisimple Lie groups which contain compact Cartan subgroups and succeeded in computing the regularized Feynman path integrals which give the kernel functions of the irreducible unitary representations realized by the Borel–Weil theorem. Later it was pointed out by Dr. Hashimoto that our idea was nothing

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but the regularization of the path integral using "the normal ordering" which is well-known to physicists (cf. Chapter 13 in [15], see also [14]).

If one tries to carry out this idea for Kac–Moody Lie groups one encounters new difficulty of nonexistence of the quasi-invariant measures on the coadjoint orbits. As to the infinite dimensional Heisenberg group, for real polarizations we do not know yet how to overcome this difficulty. For complex polarizations, however, it is well-known that irreducible representations are realized using the complex white noise on the coadjoint orbit. (See [5]).

In this paper we consider the affine Kac–Moody Lie group $\widehat{LG}$ of type $\mathbf{A}_{n-1}^{(1)}$.

First we take a subgroup $\widehat{LH}$ of $\widehat{LG}$ which is an infinite dimensional Heisenberg group. Choosing a complex polarization we construct irreducible representations using the complex white noise on the coadjoint orbit. We will show how to compute the path integral for $\widehat{LH}$, making use of this complex white noise. Finally we will give an idea how to express the vertex operators by means of the path integral with a modification by multiplying a divergent factor.

1. Definitions and Notation

Let G be a connected compact Lie group and $G^{\boldsymbol{C}}$ the complexification of G. We denote by LG the loop group of G and by $LG^{\boldsymbol{C}}$ the complexification of LG so that

$$LG = \{g: S^1 \to G; C^\infty\},$$

$$LG^{\boldsymbol{C}} = \{g: S^1 \to G^{\boldsymbol{C}}; C^\infty\}.$$

Then the Lie algebras of LG and $LG^{\boldsymbol{C}}$ are given by

$$L\mathfrak{g} = \{X: S^1 \to \mathfrak{g}; C^\infty\},$$

$$L\mathfrak{g}^{\boldsymbol{C}} = \{X: S^1 \to \mathfrak{g}^{\boldsymbol{C}}; C^\infty\}.$$

Let $\widehat{LG}$ and $\widehat{LG}^{\boldsymbol{C}}$ be the central extensions of LG and $LG^{\boldsymbol{C}}$, respectively (see [15]). Let $\widehat{L\mathfrak{g}}$ and $\widehat{L\mathfrak{g}}^{\boldsymbol{C}}$ denote the Lie algebras of $\widehat{LG}$ and $\widehat{LG}^{\boldsymbol{C}}$, respectively. Then we have

$$\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \sqrt{-1}\boldsymbol{R}\gamma \qquad (\sqrt{-1}\boldsymbol{R}\gamma : \text{center}),$$

$$\widehat{L\mathfrak{g}}^{\boldsymbol{C}} = L\mathfrak{g}^{\boldsymbol{C}} \oplus \boldsymbol{C}\gamma \qquad (\boldsymbol{C}\gamma : \text{center}).$$

For any $X \in \mathfrak{g}^{\boldsymbol{C}}$ and $k \in \boldsymbol{Z}$ we identify $X \otimes t^k \in \mathfrak{g}^{\boldsymbol{C}} \otimes \boldsymbol{C}[t, t^{-1}]$ with the mapping

$$S^1 \ni e^{i\theta} \mapsto e^{ik\theta} X \in \mathfrak{g}^{\boldsymbol{C}}.$$

Thus, $X \otimes t^k \in L\mathfrak{g}^{\boldsymbol{C}}$. Then $L\mathfrak{g}^{\boldsymbol{C}}$ is the completion of $\mathfrak{g}^{\boldsymbol{C}} \otimes \boldsymbol{C}[t, t^{-1}]$ with respect to the C^∞-topology so that $L\mathfrak{g}^{\boldsymbol{C}}$ is spanned by

$$\{X \otimes t^k; X \in \mathfrak{g}^{\boldsymbol{C}}, k \in \boldsymbol{Z}\}.$$

The bracket product of $\widehat{L\mathfrak{g}}^{\boldsymbol{C}}$ is given by:

for any $X, Y \in \mathfrak{g}^{\boldsymbol{C}}$, $k, l \in \boldsymbol{Z}$ and $\xi, \eta \in \boldsymbol{C}$

$$[X \otimes t^k + \xi\gamma, Y \otimes t^l + \eta\gamma] = [X, Y] \otimes t^{k+l} + k \operatorname{tr}(XY)\delta_{k+l,0}\gamma.$$

Now we give some notation for the affine Lie algebra $\mathbf{A}_{n-1}^{(1)}$. Throughout this paper we fix an integer $n \geq 2$. Let M_n be the algebra of all complex $n \times n$ matrices. For any $A \in M_n$ we denote by A^* the complex conjugate transposed matrix of A. Put

$$G = SU(n) = \{g \in M_n; gg^* = I_n, \det(g) = 1\},$$

where I_n denotes the unit matrix of order n. Then the Lie algebra of G is

$$\mathfrak{g} = \mathfrak{su}(n) = \{X \in M_n; X + X^* = 0, \operatorname{tr}(X) = 0\}.$$

The complexifications of G and $\mathfrak{g}$ are given by $G^{\boldsymbol{C}} = SL(n, \boldsymbol{C})$ and $\mathfrak{g}^{\boldsymbol{C}} = \mathfrak{sl}(n, \boldsymbol{C})$, respectively. We extend the map $*$ to a conjugate linear endomorphism of $L\mathfrak{g}^{\boldsymbol{C}}$ such that

$$(X \otimes t^k)^* = X^* \otimes t^{-k} \qquad (X \in \mathfrak{g}^{\boldsymbol{C}}, k \in \boldsymbol{Z}),$$

$$(\lambda\gamma)^* = \bar{\lambda}\gamma \qquad (\lambda \in \boldsymbol{C}).$$

Then it is easy to see that

$$\widehat{L\mathfrak{g}} = \{X \in \widehat{L\mathfrak{g}}^{\boldsymbol{C}}; X + X^* = 0\}.$$

Let E_{ij} denote the matrix with the (i, j) entry 1 and all the rest 0. Put

$$\Lambda = tE_{n1} + \sum_{i=1}^{n-1} E_{i\,i+1} \in L\mathfrak{g}^{\boldsymbol{C}}.$$

Then we have $\Lambda^* = \Lambda^{-1}$. We put

$$P_n = \{k \in \boldsymbol{Z};\, k > 0,\, k \not\equiv 0 \pmod n\}\,.$$

2. Irreducible unitary representations of the infinite dimensional Heisenberg group

We denote by $\mathscr{S}$ the set of all $\boldsymbol{R}$-valued rapidly decreasing sequences $\{a_k\} = \{a_k\}_{k \in P_n}$ $(a_k \in \boldsymbol{R})$ and by $\mathscr{S}^{\boldsymbol{C}}$ the set of all $\boldsymbol{C}$-valued rapidly decreasing sequences $\{c_k\} = \{c_k\}_{k \in P_n}$ $(c_k \in \boldsymbol{C})$ and we define a subalgebra $\widehat{L\mathfrak{h}}^{\boldsymbol{C}}$ of $\widehat{L\mathfrak{g}}^{\boldsymbol{C}}$ by

$$\widehat{L\mathfrak{h}}^{\boldsymbol{C}} = \left\{\sum_{k \in P_n} x_k \Lambda^k + \sum_{k \in P_n} y_k \Lambda^{-k} + \lambda\gamma;\, \{x_k\}, \{y_k\} \in \mathscr{S}^{\boldsymbol{C}}, \lambda \in \boldsymbol{C}\right\}.$$

Put

$$\widehat{L\mathfrak{h}} = \left\{\sum_{k \in P_n} x_k \Lambda^k + \sum_{k \in P_n} (-\bar{x}_k) \Lambda^{-k} + \lambda\gamma;\, \{x_k\} \in \mathscr{S}^{\boldsymbol{C}}, \lambda \in \sqrt{-1}\boldsymbol{R}\right\}.$$

The $\widehat{L\mathfrak{h}}$ is a real form of $\widehat{L\mathfrak{h}}^{\boldsymbol{C}}$. We denote by $\widehat{LH}$ and $\widehat{LH}^{\boldsymbol{C}}$ the Lie subgroups of $\widehat{LG}$ and $\widehat{LG}^{\boldsymbol{C}}$ corresponding to $\widehat{L\mathfrak{h}}$ and $\widehat{L\mathfrak{h}}^{\boldsymbol{C}}$, respectively.

Put

$$\mathfrak{H}_\infty = \left\{\begin{bmatrix} 0 & a_1 & a_2 & \cdots & c \\ & \ddots & & & b_1 \\ & & \ddots & & b_2 \\ & & & \ddots & \vdots \\ & & & \ddots & 0 \end{bmatrix};\, \{a_k\}, \{b_k\} \in \mathscr{S}, c \in \boldsymbol{R}\right\}.$$

Then $\mathfrak{H}_\infty$ is an infinite dimensional Lie algebra with the bracket product defined by: $[X, Y] = XY - YX$ $(X, Y \in \mathfrak{H}_\infty)$. $\mathfrak{H}_\infty$ is called the infinite dimensional Heisenberg algebra.

Since

$$\begin{aligned}\Bigg[\sum_{k \in P_n} (\sqrt{-1}a_k + b_k)\Lambda^k + \sum_{k \in P_n} (\sqrt{-1}a_k - b_k)\Lambda^{-k} + \lambda\gamma\,, \\ \sum_{k \in P_n} (\sqrt{-1}a'_k + b'_k)\Lambda^k + \sum_{k \in P_n} (\sqrt{-1}a'_k - b'_k)\Lambda^{-k} + \lambda'\gamma\Bigg] \\ = -2\sqrt{-1} \sum_{k \in P_n} k(a_k b'_k - a'_k b_k)\gamma\,,\end{aligned}$$

the mapping

$$\sum_{k\in P_n}(\sqrt{-1}a_k+b_k)\Lambda^k+\sum_{k\in P_n}(\sqrt{-1}a_k-b_k)\Lambda^{-k}+\lambda\gamma$$
$$\mapsto\begin{bmatrix} 0 & \cdots & \sqrt{k}a_k & \cdots & \frac{\sqrt{-1}}{2}\lambda \\ & \ddots & & & \vdots \\ & & \ddots & & \sqrt{k}b_k \\ & & & \ddots & \vdots \\ & & & & 0 \end{bmatrix}$$

is an isomorphism $\widehat{L\mathfrak{h}}\cong\mathfrak{H}_\infty$. It follows that $\widehat{LH}$ is isomorphic to $\exp(\mathfrak{H}_\infty)$. In the following, we call $\widehat{LH}$ the infinite dimensional Heisenberg group.

Put

$$L\mathfrak{n}^{\boldsymbol{C}}=\left\{\sum_{k\in P_n}x_k\Lambda^k;\ \{x_k\}\in\mathscr{S}^{\boldsymbol{C}}\right\}.$$

Then it is easy to see that the mapping

$$\sum_{k\in P_n}x_k\Lambda^k\mapsto\sum_{k\in P_n}x_k\Lambda^k+\sum_{k\in P_n}(-\bar{x}_k)\Lambda^{-k}+\sqrt{-1}\boldsymbol{R}\gamma$$

is an isomorphism $L\mathfrak{n}^{\boldsymbol{C}}\cong\widehat{L\mathfrak{h}}/\sqrt{-1}\boldsymbol{R}\gamma$. We define an inner product on $L\mathfrak{n}^{\boldsymbol{C}}$ by: for any $X,\ Y\in L\mathfrak{n}^{\boldsymbol{C}}$

$$(X,Y)=\sum_{k\in P_n}kx_k\bar{y}_k\,,$$

where $X=\sum_{k\in P_n}x_k\Lambda^k$ and $Y=\sum_{k\in P_n}y_k\Lambda^k$.

Further, for $X,\ Y\in L\mathfrak{n}^{\boldsymbol{C}}$ we define

$$\langle X,Y\rangle=(X,\bar{Y})\,,$$

where $\bar{Y}=\sum_{k\in P_n}\bar{y}_k\Lambda^k$.

We put $\boldsymbol{E}_c=L\mathfrak{n}^{\boldsymbol{C}}$ and denote by $\boldsymbol{H}_c$ the completion of $\boldsymbol{E}_c$. Then we have the Gel'fand triple

$$\boldsymbol{E}_c\subset\boldsymbol{H}_c\subset\boldsymbol{E}_c^*\,,$$

where $\boldsymbol{E}_c^*$ denotes the vector space of all $\boldsymbol{C}$-linear continuous mappings of $\boldsymbol{E}_c$ into $\boldsymbol{C}$.

Then we have a complex Gaussian measure ν_σ on $\boldsymbol{E}_c^*$ such that for any $\zeta_1,\ \zeta_2\in\boldsymbol{E}_c$

$$\int_{\boldsymbol{E}_c^*}\exp(\sqrt{-1}\{\langle z,\zeta_1\rangle+\overline{\langle z,\zeta_2\rangle}\})d\nu_\sigma(z)=\exp\left(-\frac{2}{\sigma}(\zeta_1,\zeta_2)\right),$$

where $\langle z,\zeta\rangle=z(\zeta)$, $(z\in\boldsymbol{E}_c^*,\ \zeta\in\boldsymbol{E}_c)$. This measure is called the complex white noise (see [11], [13]).

Further, for any $z \in E_c^*$ and $\zeta \in E_c$ we define

$$(z, \zeta) = \langle z, \bar{\zeta} \rangle .$$

We denote by $\Gamma^2(E_c^*, \nu_\sigma)$ the Hilbert space of all square integrable holomorphic functions on E_c^* with respect to the measure ν_σ. For $F \in \Gamma^2(E_c^*, \nu_\sigma)$ and $g = \exp\left(\sum_{k \in P_n} x_k \Lambda^k + \sum_{k \in P_n} (-\bar{x}_k) \Lambda^{-k} + \lambda\gamma\right) \in \widehat{LH}$ we define

$$(\pi_\sigma(g)F)(x) = \exp\left(\frac{\sigma}{2}\left((z, \alpha) - \frac{1}{2}\|\alpha\|^2 + \lambda\right)\right) F(z - \alpha) \qquad (z \in E_c^*),$$

where $\alpha = \sum_{k \in P_n} x_k \Lambda^k$. Then one can show that π_σ is an irreducible unitary representation of $\widehat{LH}$ on $\Gamma^2(E_c^*, \nu_\sigma)$.

3. The construction of the representations of the infinite dimensional Heisenberg group by means of the path integral

In this section, generalizing the idea in [7] we construct irreducible unitary representations of the infinite dimensional Heisenberg group $\widehat{LH}$ by means of the Feynman path integral.

First we will show that what kind of difficulty we encounter if we try to compute the path integral in the exactly same way as in [7].

Fix a positive integer m such that $m \not\equiv 0 \pmod n$. Define

$$P_{n,m} = \{k \in P_n; 0 < k \leq m\} .$$

We put

$$\widehat{L\mathfrak{h}}_m = \left\{ \sum_{k \in P_{n,m}} x_k \Lambda^k + \sum_{k \in P_{n,m}} (-\bar{x}_k) \Lambda^{-k} + \lambda\gamma ; \begin{array}{c} x_k \in \boldsymbol{C}, \lambda \in \sqrt{-1}\boldsymbol{R} \\ (k \in P_{n,m}) \end{array} \right\} .$$

Then $\widehat{L\mathfrak{h}}_m$ is a finite dimensional subalgebra of $\widehat{L\mathfrak{h}}$. It is clear that $\widehat{L\mathfrak{h}}_m$ is isomorphic to the Heisenberg algebra

$$\mathfrak{H}_m = \left\{ \begin{bmatrix} 0 & a_1 & \cdots & a_m & c \\ & 0 & & & b_1 \\ & & \ddots & & \vdots \\ & & & 0 & b_m \\ & & & & 0 \end{bmatrix} ; \begin{array}{c} a_k, b_k, c \in \boldsymbol{R} \\ (k \in P_{n,m}) \end{array} \right\} .$$

The isomorphism is given by the mapping

$$\sum_{k\in P_{n,m}}(\sqrt{-1}a_k+b_k)\Lambda^k+\sum_{k\in P_{n,m}}(\sqrt{-1}a_k-b_k)\Lambda^{-k}+\lambda\gamma$$

$$\mapsto\begin{bmatrix} 0 & a_1 & \cdots & \sqrt{m}a_m & \frac{\sqrt{-1}}{2}\lambda \\ & 0 & & & b_1 \\ & & \ddots & & \vdots \\ & & & 0 & \sqrt{m}b_m \\ & & & & 0 \end{bmatrix}.$$

In [7], we proved that all irreducible unitary representations of $\mathfrak{H}_m$ are obtained by the path integrals. Using the above isomorphism we can construct all irreducible unitary representations of $\exp(\widehat{L\mathfrak{h}_m})$ by the path integrals.

Now let us investigate what happens if we take the limit $m\to\infty$. We review roughly the main part of the computation of the path integral. (Refer to [7] for the details.) For any $Y=\sum_{k\in P_{n,m}}x_k\Lambda^k+\sum_{k\in P_{n,m}}(-\bar{x}_k)\Lambda^{-k}+\lambda\gamma\in\widehat{L\mathfrak{h}_m}$, the corresponding Hamiltonian is given by

$$\frac{\sqrt{-1}\sigma}{2}((z,\alpha)-\overline{(z,\alpha)}+\lambda),$$

where we put $\alpha=\sum_{k\in P_{n,m}}x_k\Lambda^k$. The action integral is computed as follows.

$$\frac{\sigma}{2}\int_0^T((\dot{z}(t),z(t))+((z(t),\alpha)-\overline{(z(t),\alpha)}+\lambda))dt$$

$$=\frac{\sigma}{2}\sum_{k=1}^N\int_{(k-1/N)T}^{(k/N)T}((\dot{z}(t),z(t))+(z(t),\alpha)-\overline{(z(t),\alpha)}+\lambda)dt$$

$$=\frac{\sigma}{2}\sum_{k=1}^N\left((z_k-z_{k-1},z_{k-1})+\left(\frac{1}{2}(z_k+z_{k-1},\alpha)-\overline{(z_{k-1},\alpha)}+\lambda\right)\frac{T}{N}\right).$$

We denote by $p_{n,m}$ the number of elements in the set $P_{n,m}$. Then the path integral is computed as follows.

$$K_Y(z',z;T)$$

$$=\lim_{N\to\infty}\int_{L\mathfrak{n}_m^C}\cdots\int_{L\mathfrak{n}_m^C}\frac{\sigma^{p_{n,m}}}{(2\pi)^{p_{n,m}}}(dz_1,dz_1)\cdots\frac{\sigma^{p_{n,m}}}{(2\pi)^{p_{n,m}}}(dz_{N-1},dz_{N-1})$$

$$\times\exp\left(\frac{\sigma}{2}\sum_{k=1}^N\left((z_k-z_{k-1},z_{k-1})+\left(\frac{1}{2}(z_k+z_{k-1},\alpha)-\overline{(z_{k-1},\alpha)}+\lambda\right)\frac{T}{N}\right)\right)$$

$$= \lim_{N\to\infty} \int_{L\mathfrak{n}_m^C} \cdots \int_{L\mathfrak{n}_m^C} \frac{\sigma^{p_{n,m}}}{(2\pi)^{p_{n,m}}} \exp\left(-\frac{\sigma}{2}\|z_1\|^2\right)(dz_1, dz_1) \cdots$$

$$\times \frac{\sigma^{p_{n,m}}}{(2\pi)^{p_{n,m}}} \exp\left(-\frac{\sigma}{2}\|z_{N-1}\|^2\right)(dz_{N-1}, dz_{N-1})$$

$$\times \exp\left(\frac{\sigma}{2}\sum_{k=1}^{N}\left((z_k, z_{k-1}) + \left(\frac{1}{2}(z_k + z_{k-1}, \alpha) - \overline{(z_{k-1}, \alpha)} + \lambda\right)\frac{T}{N}\right)\right)$$

$$\times \exp\left(-\frac{\sigma}{2}\|z_0\|^2\right)$$

$$= \exp\left(\frac{\sigma}{2}\left((z', z) + ((z', \alpha) - \overline{(z, \alpha)})T - \frac{1}{2}\|\alpha\|^2T^2 + \lambda T\right)\right)\exp\left(-\frac{\sigma}{2}\|z_0\|^2\right).$$

We define

$$\boldsymbol{K}_Y(z', z; T) = K_Y(z', z; T)\exp\left(\frac{\sigma}{2}\|z\|^2\right).$$

Finally the unitary operator is obtained as follows.

$$\int_{L\mathfrak{n}_m^C} \frac{\sigma^{p_{n,m}}}{(2\pi)^{p_{n,m}}} \exp\left(-\frac{\sigma}{2}\|z\|^2\right)(dz, dz)\boldsymbol{K}_Y(z', z; T)F(z)$$

$$= \exp\left(\frac{\sigma}{2}\left((z', \alpha)T - \frac{1}{2}\|\alpha\|^2T^2 + \lambda T\right)\right)F(z' - \alpha T)$$

$$= (U_\sigma(\exp(TY))F)(z').$$

To generalize the above computation to the infinite dimensional Heisenberg group $\widehat{LH}$ it looks natural to replace $(\sigma^{p_{n,m}}/(2\pi)^{p_{n,m}})\exp(-(\sigma/2)\|z\|^2)(dz, dz)$ by the complex white noise $d\nu_\sigma(z)$. However there appears a crucial difficulty that (z_k, z_{k-1}) and $\|z_0\|^2$ are divergent on $\boldsymbol{E}_c^*$. Notice that $z_0 = z$. Then one finds that the factor $\exp(-(\sigma/2)\|z_0\|^2)$ should be used to form the complex white noise $d\nu_\sigma(z)$ when we compute the unitary operator.

Now we are in a position to mention our idea how to compute the path integral of the infinite dimensional Heisenberg group $\widehat{LH}$.

For any $\zeta_0, \zeta_1, \cdots, \zeta_N \in \boldsymbol{E}_c$ we define

$$S_0(\zeta_0, \zeta_1, \cdots, \zeta_N)$$
$$= \exp\left(\frac{\sigma}{2}\sum_{k=1}^{N}\left((\zeta_k, \zeta_{k-1}) + \left(\frac{1}{2}(\zeta_k + \zeta_{k-1}, \alpha) - \overline{(\zeta_{k-1}, \alpha)} + \lambda\right)\frac{T}{N}\right)\right).$$

We regard $S_0(\zeta_0, \zeta_1, \cdots, \zeta_N)$ as a function of the variable ζ_1 on $\boldsymbol{E}_c$ fixing other variables. Then it is clear that replacing ζ_1 by z_1 this function is uniquely extended to a continuous function on E_c^*. And we carry out the first integration to get a new function

$$S_1(\zeta_0, \zeta_2, \cdots, \zeta_N) = \int_{E_c^*} dv_\sigma(z_1) S_0(\zeta_0, z_1, \zeta_2, \cdots, \zeta_N)\,.$$

Carrying out the integration successively we obtain finally

$$\begin{aligned} S_{N-1}(\zeta_0, \zeta_N) &= \int_{E_c^*} dv_\sigma(z_{N-1}) S_{N-2}(\zeta_0, z_{N-1}, \zeta_N) \\ &= \exp\left(\frac{\sigma}{2}\left((\zeta_{N-1}, \zeta_0) + ((\zeta_{N-1}, \alpha) - \overline{(\zeta_0, \alpha)})T - \frac{1}{2}\|\alpha\|^2 T^2 + \lambda T\right)\right). \end{aligned}$$

We define

$$K_Y(\zeta', z; T) = S_{N-1}(z, \zeta')\,.$$

Computing step by step, we get

$$K_Y(\zeta', z; T) dv_\sigma(z) = \exp\left(\frac{\sigma}{2}\left((\zeta', z) + ((\zeta', \alpha) - \overline{(z, \alpha)})T - \frac{1}{2}\|\alpha\|^2 T^2 + \lambda T\right)\right) dv_\sigma(z)\,.$$

Finally we have

$$\begin{aligned} \int_{E_c^*} dv_\sigma(z) K_Y(\zeta', z; T) F(z) &= \exp\left(\frac{\sigma}{2}\left((\zeta', \alpha)T - \frac{1}{2}\|\alpha\|^2 T^2 + \lambda T\right)\right) F(\zeta' - \alpha T) \\ &= (\pi_\sigma(\exp(TY))F)(\zeta')\,. \end{aligned}$$

Thus we proved the following theorem.

THEOREM 1. *The unitary representation π_σ of the infinite dimensional Heisenberg group $\widehat{LH}$ can be obtained by means of the Feynman path integral.*

4. The construction of the fundamental representation of the affine Kac–Moody Lie algebra $A_{n-1}^{(1)}$ by means of the path integral

We denote by $d\pi_\sigma$ the representation of the infinite dimension Heisenberg algebra $\widehat{L\mathfrak{h}}$ on the space of differentiable vectors in $\Gamma^2(E_c^*, v_\sigma)$ which is obtained by differentiating the unitary representation π_σ of $\widehat{LH}$ defined in Section 2.

Then for any $X \in \widehat{L\mathfrak{h}}$ and any differentiable $F \in \Gamma^2(E_c^*, v_\sigma)$ we have

$$d\pi_\sigma(X)F = \frac{d}{dt}\bigg|_{t=0} \pi_\sigma(\exp tX)F\,.$$

Computing explicitly we get

$$d\pi_\sigma(\Lambda^k - \Lambda^{-k}) = -\frac{\partial}{\partial z_k} + \frac{\sigma}{2} k z_k \,,$$

$$d\pi_\sigma(\sqrt{-1}\Lambda^k + \sqrt{-1}\Lambda^{-k}) = -\sqrt{-1}\frac{\partial}{\partial z_k} - \sqrt{-1}\frac{\sigma}{2} k z_k \,,$$

$$d\pi_\sigma(\sqrt{-1}\gamma) = \sqrt{-1}\frac{\sigma}{2} I \,,$$

where I denotes the identity operator on $\Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma)$.

We extend $d\pi_\sigma$ complex linearly to the complexification $\widehat{L\mathfrak{h}}^C$. Then by the explicit computation we can prove the equations in the following proposition.

PROPOSITION. *For any $k \in P_n$, we have*

$$d\pi_\sigma(\Lambda^k) = -\frac{\partial}{\partial z_k} \,,$$

$$d\pi_\sigma(\Lambda^{-k}) = -\frac{\sigma}{2} k z_k \,,$$

$$d\pi_\sigma(\gamma) = \frac{\sigma}{2} I \,,$$

where I denotes the identity operator.

We define the isomorphism

$$\varphi\colon \Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma) \ni F \mapsto \boldsymbol{F} \in \Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma)$$

by

$$\boldsymbol{F}(z) = F(-z) \qquad (z \in \boldsymbol{E}_c^*) \,.$$

Then the representation ρ_σ of $\widehat{LH}$ is obtained so that the following commutative diagram should hold.

$$\begin{array}{ccc} \Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma) & \xrightarrow{\varphi} & \Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma) \\ {\scriptstyle \pi_\sigma(g)} \downarrow & & \downarrow {\scriptstyle \rho_\sigma(g)} \\ \Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma) & \xrightarrow{\varphi} & \Gamma^2(\boldsymbol{E}_c^*, \nu_\sigma) \end{array}$$

for all $g \in \widehat{LH}$.

It is easy to show that for $F \in \Gamma^2(E_c^*, \nu_\sigma)$ and $g = \exp(\sum_{k \in P_n} x_k \Lambda^k + \sum_{k \in P_n}(-\bar{x}_k)\Lambda^{-k} + \lambda\gamma) \in \widehat{LH}$ we have

$$(\rho_\sigma(g)F)(z) = \exp\left(\frac{\sigma}{2}\left(-(z, \alpha) - \frac{1}{2}\|\alpha\|^2 + \lambda\right)\right)F(z + \alpha) \qquad (z \in E_c^*),$$

where $\alpha = \sum_{k \in P_n} x_k \Lambda^k$.

For $\sigma = 2$ we have the following corollary.

COROLLARY. *For any $k \in P_n$, we have*

$$d\rho_2(\Lambda^k) = \frac{\partial}{\partial z_k},$$

$$d\rho_2(\Lambda^{-k}) = k z_k,$$

$$d\rho_2(\gamma) = I,$$

where I denotes the identity operator.

Now we need some results in [12]. Let $l \in \{0, \cdots, n-1\}$ and let τ_l be the fundamental representation of $\widehat{L\mathfrak{h}}^C$ which is defined in [12]. Put $\varepsilon = \exp((2\pi\sqrt{-1})/n)$. For $k \in \boldsymbol{Z}$ and $s \in \{1, \cdots, n-1\}$ define

$$A_{k,s} = \sum_{\substack{i,j=1 \\ i-j=k+rn}}^{n} t^r \varepsilon^{-sj} E_{ij}.$$

Then these elements and Λ^k, Λ^{-k} $(k \in P_n)$ form a basis of $\widehat{L\mathfrak{g}}^C$. The fundamental representation τ_l is defined as follows (see Proposition 9.1 in [12]). (We corrected some misprints there.)

For $k \in P_n$ one has

$$\tau_l(\Lambda^k) = \frac{\partial}{\partial z_k},$$

$$\tau_l(\Lambda^{-k}) = k z_k,$$

$$\tau_l(\gamma) = I.$$

And the operators $\tau_l(A_{k,s})$ are obtained by the coefficients of the parameter u of the following vertex operators:

$$\tau_l\left(\varepsilon^{sl}\left((1 - \varepsilon^s)\sum_{k \in Z} A_{k,s} u^k + \gamma\right)\right)$$

$$= \left(\exp \sum_{k \in P_n}(1 - \varepsilon^{sk}) u^k z_k\right)\left(\exp\left(-\sum_{k \in P_n} \frac{1 - \varepsilon^{-sk}}{k} u^{-k} \frac{\partial}{\partial z_k}\right)\right).$$

It follows from the above corollary that the representation $d\rho_2$ is equivalent with the restriction of the fundamental representation τ_l.

Assuming that $|u| = 1$, we put

$$\alpha = -\sum_{k \in P_n} \frac{1 - \varepsilon^{-sk}}{k} u^{-k} \Lambda^k .$$

Then as is easily seen, $\alpha \in \boldsymbol{E}_c^*$. Furthermore, for any $\zeta \in \boldsymbol{E}_c$ and any polynomial function $\boldsymbol{F}$ on $\boldsymbol{E}_c^*$, we have

$$\begin{aligned}
&\left(\left(\exp \sum_{k \in P_n} (1 - \varepsilon^{sk}) u^k \zeta_k\right)\left(\exp - \sum_{k \in P_n} \frac{1 - \varepsilon^{-sk}}{k} u^{-k} \frac{\partial}{\partial \zeta_k}\right) \boldsymbol{F}\right)(\zeta) \\
&= \exp\left(\sum_{k \in P_n} (1 - \varepsilon^{sk}) u^k \zeta_k\right) \boldsymbol{F}(\zeta + \alpha) \\
&= \exp(-(\zeta, \alpha)) \boldsymbol{F}(\zeta + \alpha) ,
\end{aligned}$$

where $\zeta = \sum_{k \in P_n} \zeta_k \Lambda^k$. For any $m \in P_n$ we put

$$Y_m = -\sum_{k \in P_{n,m}} \frac{1 - \varepsilon^{-sk}}{k} u^{-k} \Lambda^k + \sum_{k \in P_{n,m}} \frac{1 - \varepsilon^{sk}}{k} u^k \Lambda^{-k} ,$$

$$\alpha_m = -\sum_{k \in P_{n,m}} \frac{1 - \varepsilon^{-sk}}{k} u^{-k} \Lambda^k .$$

Then it is clear that $Y_m \in \widehat{L\mathfrak{h}}$. Using the kernel function obtained by means of the path integral in Section 3, we have

$$\begin{aligned}
\exp(-(\zeta, \alpha)) \boldsymbol{F}(\zeta + \alpha) &= \lim_{m \to \infty} \exp(-(\zeta, \alpha_m)) \boldsymbol{F}(\zeta + \alpha_m) \\
&= \lim_{m \to \infty} \exp\left(\frac{1}{2} \|\alpha_m\|^2\right) (\rho_2(\exp(Y_m)) \boldsymbol{F})(\zeta) \\
&= \lim_{m \to \infty} \exp\left(\frac{1}{2} \|\alpha_m\|^2\right) \int_{\boldsymbol{E}_c^*} d\nu_2(z) \boldsymbol{K}_{-Y_m}(\zeta, z; T) \boldsymbol{F}(z) .
\end{aligned}$$

We remark that $\lim_{m \to \infty} \exp(\|\alpha_m\|^2/2)$ is divergent to ∞ and that the restriction of τ_l to $\widehat{L\mathfrak{h}}$ coincides with $d\rho_2$.

In the above the assumption that $|u| = 1$ is essential. Thus, it follows that any vertex operator with the parameter u such that $|u| = 1$ can be obtained by the path integral if we modify the computation of the path integral by multiplying a divergent factor. Thus we proved the following theorem.

THEOREM 2. *Every fundamental representation τ_l of the affine Kac–Moody Lie algebra $\mathbf{A}_{n-1}^{(1)}$ can be given by means of the path integral with modification, if necessary, by multiplying a divergent factor.*

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