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A class of vector fields on manifolds containing second order ODEs

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ABSTRACT. This paper presents a new class of ordinary differential equations on manifolds containing second order ordinary differential equations as a special subclass. The main result of the paper is the genericity of hyperbolicity of equilibrium points in this class. A subclass containing the classical mechanical systems is also discussed there.

1. Introduction

One of the important special classes of vector fields which can be defined in a coordinate-free manner are second order ordinary differential equations (ODEs) on manifolds (see e.g. [1, 3, 11]). We introduce a new special class of ODEs on manifolds containing second order ODEs as its special subclass. Locally they are represented by systems of the form

$$\dot{x} = P(x)y\,, \qquad \dot{y} = g(x, y)\,, \tag{1.1}$$

where $g \in C^r(U, R^n)$, $P \in C^r(U, M_n)$, $U \subset R^n$ is an open set and M_n is the set of all $n \times n$ matrices. The mapping $\Phi_P: TR \to TR^n$, $\Phi_P(x, y) = (x, P(x)y)$ is a fiber preserving bundle endomorphism of TR^n, linear on each fiber. Therefore we call such special vector fields E-vector fields. If $\Phi_P = id_{TR^n}$ then $P(x)$ is the unit matrix and the system (1.1) is a second order ODE on R^n.

A special subclass of the class of E-vector fields are mechanical systems of the form

$$\begin{aligned} \dot{x} &= P(x)y\,, \\ \dot{y} &= -\frac{1}{2}\frac{\partial}{\partial x}(y^T P(x) y) - \text{grad}\, V(x) + h(x, y)\,, \end{aligned} \tag{1.2}$$

where $P \in C^r(U, S_n)$, S_n is the set of all $n \times n$ symmetric matrices, $V \in C^r(U, R)$, $h \in C^r(U, R^n)$, y^T is the transpose of y. This system can be written as the

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sum of the Hamiltonian vector field with the hamiltonian $H(x, y) = \frac{1}{2}y^T P(x)y + V(x)$ and the E-vector field $\dot{x} = 0$, $\dot{y} = h(x, y)$. The notion of the E-vector field, which will be precisely defined in the next section, enables us to define similarly as above also mechanical systems on manifolds. A definition of a mechanical system on manifolds is given for instance in [1, 2, 7, 18, 21], where it is defined as a class of second order ODEs.

In this paper the main results deal with generic properties of singular points of E-vector fields and mechanical systems defined as E-vector fields.

2. E-vector fields

Throughout the paper M will be an n-dimensional C^∞-manifold without boundary and with C^∞-Riemannian metric $\langle .,.\rangle$. We use the notation $X^k(M)$ for the space of all C^k-vector fields on M. Let TM be the tangent bundle of M. The papers [16] and [17] inspired us to define E-vector fields on manifolds. Let us recall the following definitions given in these papers (see also e.g. [1, Chapter 3, Section 3.4]).

DEFINITION 2.1. A bundle endomorphism A of the tangent bundle TM is a C^k-mapping $A\colon TM \to TM$ whose restriction on each fiber T_xM is the linear endomorphism $A(x)$ of T_xM. The set of all C^k-bundle endomorphism of TM we denote by $\Gamma^k(\mathrm{End}\,(TM))$.

DEFINITION 2.2. A bundle endomorphism $A \in \Gamma^k(\mathrm{End}\,(TM))$ is called of constant corank if, for any $x \in M$, the rank of $A(x)$ is independent of x. In this case, if the constant rank equals $n - r$, we call A a bundle endomorphism of corank r. We write corank $A = r$. The set of all $A \in \Gamma^k(\mathrm{End}\,(TM))$ of corank r we denote by $\Gamma^k(\mathrm{End}\,(TM))$.

In Section 6 we shall give a criterion (Lemma 6.1) for the local constantness of rank $A(x)$ for $A \in \Gamma^k(\mathrm{End}\,(TM))$. Now let us define E-vector fields on M.

DEFINITION 2.3. Let $A \in \Gamma^k(\mathrm{End}\,(TM))$ and $\pi\colon TM \to M$ be the natural projection. A vector field $F \in X^k(TM)$ is called an E-vector field on M if

$$D\pi \circ F = A\,, \tag{2.1}$$

where $D\pi\colon T^2M \to TM$ is the derivative of π and $T^2M = T(TM)$ is the double tangent of M. The set of all such vector fields with A fixed we denote by $E^k_A(TM)$ and we define $E^k(TM) = \bigcup_{A \in \Gamma^k(\mathrm{End}(TM))} E^k_A(TM)$.

The set $E^k_{id_{TM}}$ is exactly the set of all second order ODEs on M of the class C^k, where id_{TM} is the identity map on TM. One can easily check that any $F \in E^k_A(TM)$ is locally represented by a system of ODEs of the form (1.1).

Now we globalize the definition of mechanical systems mentioned in the introduction. By [9, Example 6.10] for any manifold Y there exists a regular 2-form ν on the cotangent bundle T^*Y of Y which define a symplectic structure on Y. The 2-form induces a map η^ν: $TY \to T^*Y$, $v_x \mapsto \eta(v_x)w_x$, $v_x \in T_xY$, where $\eta(v_x)w_x := \nu(v_x, w_x)$, $w_x \in T_xY$. By [1, Proposition 13.7] the mapping η is a vector bundle isomorphism (in particular, a diffeomorphism of TM). This implies that there exists a regular 2-form ω on the cotangent bundle $T^*(TM)$ of the tangent bundle TM which define a symplectic structure on the double tangent T^2M. The form ω defines the isomorphism

$$b\colon C^k(T^2M) \to C^k(T^*(TM)), b(X) := i_X\omega ,$$

where $i_X\omega$ is the inner product of X and ω (see [1, Definition 14.13]), $C^k(T^*(TM))$ is the set of all C^k-sections of $T^*(TM)$ and $C^k(T^2M))$ is the set of all C^k-sections of T^2M. If $H \in C^{k+1}(TM, R)$ then $dH \in C^k(T^*(TM))$. Since the map b is an isomorphism, there exists a vector field X^H on TM such that $b(X_H) = i_{X_H}\omega = dH$ and this means that X_H is a Hamiltonian vector field with the Hamiltonian function H.

Consider the function $H \in C^{k+1}(TM, R)$, $H(v_x) = \frac{1}{2} < B(x)v_x - v_x,\ v_x >$, $v_x \in T_xM$, $x \in M$, where $B \in \Gamma^{k+1}(\mathrm{End}\,(TM))$ is such that the mapping $B(x)$: $T_xM \to T_xM$ is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_x$ on T_xM for any $x \in M$. The set of all such $B \in \Gamma^{k+1}(\mathrm{End}\,(TM))$ we denote by $\Gamma_S^{k+1}(\mathrm{End}\,(TM))$. By the above construction there is a Hamiltonian vector field F_1 with the Hamiltonian function H. Now we define another C^k-vector field F_2 on TM as a second order ODE on M with the property that each of its integral curve $t \mapsto \gamma(t)$ satisfies the condition

$$\nabla_{\dot\gamma}\dot\gamma = -\mathrm{grad}\ W(\gamma) + G(\dot\gamma)\,, \tag{2.2}$$

where $\dot\gamma = \dfrac{d\gamma}{dt}$, $\nabla_{\dot\gamma}\dot\gamma$ is the covariant derivative of $\dot\gamma$, grad W is the gradient vector field on TM defined by $dW(v_x) = \langle \mathrm{grad}\ W(x), v_x\rangle_x$ for all $v_x \in TM$, where $W \in C^{k+1}(M, R)$ and $G \in \Gamma^k(\mathrm{End}\,(TM))$. We call the vector field $F = F_1 + F_2$ a generalized mechanical system or shortly a mechanical system on M and we identify it with $\mathscr{S} = (B, W, G) \in \Gamma_S^{k+1}(\mathrm{End}\,(TM)) \times C^{k+1}(M, R) \times \Gamma^k(\mathrm{End}\,(TM))$. This vector field on TM is locally represented by a system of the form (1.2). If B is an isomorphism on each fiber then we have a classical mechanical system and the function K: $TM \to R$, $K(v_x) = \frac{1}{2}\langle B(x)v_x, v_x\rangle$ represents the kinetic energy, the function W: $M \to R$ is the potential energy and E: $TM \to R$, $E(v_x) = K(v_x) + W(\pi(v_x))$, $v_x \in TM$, is the total energy of the systm $\mathscr{S}$. The function $L = K - W \circ \pi$ is called the Langrangian of the mechanical system. The mapping $G \in X^k(TM)$ represents the external force of $\mathscr{S}$. We use these notions also in the case when B is not an isomorphism on each fiber. The space of all the above defined mechanical systems we denote by $\mathcal{M}^k = \mathcal{M}^k(TM)$. In Section 5 we shall study some basic generic properties of mechanical systems in the space $\mathcal{M}^k(TM)$.

REMARK. The wide-spread notion of a mechanical system on a manifold M (see e.g. [1, 3, 7, 18, 21]) is defined as a second order ODE on M defined by two functions $K: TM \to R$, $K(v_x) = \frac{1}{2}\langle v_x, v_x \rangle$, $v_x \in TM$, $W: M \to R$ and some external force. The main result concerning mechanical systems which we shall prove in Section 5 says that singular points of generalized mechanical systems on TM lying in the zero section $(TM)_0$ of TM consists generically of isolated points and the corresponding Hessian at these points is an isomorphism. This result is an extension of results by S. Shashahani [20] concerning generic properties of mechanical systems. We remark that G. Fusco and M. Oliva [6] recently obtained some results concerning dissipative mechanical systems with constraints. We remark that in our definition of generalized mechanical systems the vector bundle endomorphism defining the function K which in classical mechanical systems represents the kinetic energy, need not to be an isomorphism on each fiber. Of course, in the case of classical mechanical systems with non-degenerate Lagrangians this is not possible. We give an example of an E-vector field on R^{2m} which in a special case has the form of a generalized mechanical system. It is true that the above abstract approach is not necessary in this case, however Theorem 4.2 can be applied to this case.

Let us give examples of equations which have the form of E-vector fields.

EXAMPLE 1. Consider the following class of models coming from reaction kinetics. Let us consider a chemical reaction of reactants $a_1, a_2, \ldots, a_n$ consisting of m reactions expressed graphically as

$$\sum_{k=1}^{n} \alpha_{ik} a_k \mapsto \sum_{k=1}^{n} \beta_{ik} a_k \qquad (i = 1, 2, \ldots, m),$$

where α_{ik}, β_{ik} are non-negative integers. If u_k is the concentration of a_k then the dynamics of the reaction can be described by the system of differential equations

$$(*) \qquad \dot{u}_k = \sum_{i=1}^{m} \gamma_{ik} w_i \qquad (k = 1, 2, \ldots, n),$$

where $\gamma_{ik} = \beta_{ik} - \alpha_{ik}$ and w_i is the velocity of the i-th reaction. In many applications the velocity w_i is supported to be of the form $w_i = k_i \prod_{k=1}^{n} u_k^{\alpha_{ik}}$, where k_i is a constant. If there are some constraints on the variables $u_1, u_2, \ldots, u_n$ given by smooth functions which define a smooth manifolds, we have to deal with a vector field on a manifold. The structure of a chemical reaction determines the structure of the corresponding vector field. We give an example of such a structure where the dynamics of the corresponding reactions is described by an E-vector field.

Let $a_1, a_2, \dots, a_n$, $A_1, A_2, \dots, A_n$ be reactions for which the system of reactions has the form

$$\text{(I)} \qquad \sum_{k=1}^{n} \alpha_{ik} a_k + \sum_{k=1}^{n} \tilde{\alpha}_{ik} A_k \mapsto \sum_{k=1}^{n} \beta_{ik} a_k + \sum_{k=1}^{n} \tilde{\beta}_{ik} A_k \qquad (i = 1, 2, \dots, n)$$

$$\text{(II)} \qquad \sum_{k=1}^{n} \delta_{jk} a_k + \sum_{k=1}^{n} \tilde{\delta}_{jk} A_k \mapsto \sum_{k=1}^{n} \eta_{jk} a_k + \sum_{k=1}^{n} \tilde{\eta}_{jk} A_k \qquad (j = 1, 2, \dots, n).$$

Let $\mathscr{A} = (\alpha_{ik})$, $\tilde{\mathscr{A}} = (\tilde{\alpha}_{ik})$, $\mathscr{B} = (\beta_{ik})$, $\tilde{\mathscr{B}} = (\tilde{\beta}_{ik})$, $\mathscr{C} = (\delta_{ik})$, $\tilde{\mathscr{C}} = (\tilde{\delta}_{ik})$, $\mathscr{D} = (\eta_{ik})$, $\tilde{\mathscr{D}} = (\tilde{\eta}_{ik})$, $\Gamma_{11} = \mathscr{C} - \mathscr{A}$, $\Gamma_{12} = \mathscr{D} - \mathscr{C}$, $\Gamma_{21} = \tilde{\mathscr{C}} - \tilde{\mathscr{A}}$, $\Gamma_{22} = \tilde{\mathscr{D}} - \tilde{\mathscr{C}}$, u_k, v_k be concentrations of a_k and A_k, respectively, $u = (u_1, u_2, \dots, u_n)$, $v = (v_1, v_2, \dots, v_n)$, w_i is the velocity of the i-th reaction of the system (I) and W_j is the velocity of the j-th reaction of the system (II). Then the dynamics of the system of reactions (I) and (II) is described by the system

$$\dot{u} = \Gamma_{11} w + \Gamma_{12} W,$$

$$\dot{v} = \Gamma_{12} w + \Gamma_{22} W,$$

where $w = (w_1, w_2, \dots, w_n)^T$, $W = (W_1, W_2, \dots, W_n)^T$. If $\Gamma_{12} = 0$—the zero matrix and $w = w(u, v) = (\phi_1(v)v_1, \dots, \phi_n(u)v_n)^T$, where $\phi_1(u), \dots, \phi_n(u)$ are smooth functions, then the above system has the form

$$(**) \qquad \dot{u} = B(u)v, \qquad \dot{v} = g(u, v),$$

where $B(u) = \Gamma_{11}(\phi_1(u)v_1, \dots, \phi_n(u)v_n)^T$ and $g(u, v) = \Gamma_{21} w + \Gamma_{22} W$. In applications the functions $\phi_i(u)$ may have zero points. They are often supposed to have the form $\phi_i(u) = k_i u_1^{\gamma_1} \dots u_n^{\gamma_n}$.

More generally, we can define differential equations on graphs as follows. Let Γ be an oriented graph with two sets of vertices $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ and edges (a_k, b_i), (b_i, a_k) connecting the vertex a_k with b_i, with the orientation from a_k to b_i and b_i with a_k with the orientation from b_i to a_k, respectively. Let α_{ki} be the number of edges of the type (a_k, b_i) and β_{ik} the number of edges of the type (b_i, a_k). Let a function $u_k(t)$ correspond to the vertex a_k and a function $w_i(u)$ correspond to the vertex b_i, where $u = (u_1, u_2, \dots, u_n)$. The functions $w_i(u)$ are given and the functions u_k are unknown. The system (*), where $\gamma_{ik} = \beta_{ik} - \alpha_{ik}$, is called a system of differential equations on the graph Γ. The structure of the graph Γ determines the structure of the system (*) and as a special case we have a system of the form (**). If for instance, $\dim u = \dim v = 1$, $\Gamma_{12} = \Gamma_{21} = \Gamma_{22} = 1$, $w(u, v) = \phi(u)v$, $W(u, v) = -\phi(u)v - \frac{1}{2}\phi'(u)v^2 - \psi'(u) + h(u, v)$, where ψ, h are smooth functions then the system (**) has the form

$$\dot{u} = \frac{\partial}{\partial v} H(u, v), \qquad \dot{v} = -\frac{\partial}{\partial u} H(u, v) + h(u, v),$$

where $H(u, v) = \frac{1}{2}\phi(u)v^2 + \psi(u)$. The reactants producing the term h play the role of an "external force" during the reaction. Such systems in higher dimensions are studied in Section 5.

EXAMPLE 2. Consider an integrodifferential equation of the form

$$\dot{x} = A(x)\left[\{\exp(Bt)\}c + \int_0^t \{\exp B(t-s)\}g(x(s))ds\right], \tag{2.3}$$

where $A \in C^k(R^n, M_n)$, $g \in C^k(R, R^n)$, $k \geqq 1$, $B \in M_n$, $c \in R^n$. This equation can be written as the system

$$\dot{x} = A(x)v, \qquad \dot{v} = Bv + g(x) \tag{2.4}$$

with the condition $v(0) = c$. The system (2.4) is an E-vector field on R^n. If $A(x) = \operatorname{diag}\{x_1, x_2, \dots, x_n\}$, where $x = (x_1, x_2, \dots, x_n\}$ and $G_i(x, t)$ is the i-th coordinate of the vector $\{\exp(Bt)\}c + \int_0^t \{\exp B(t-s)\}g(x(s))ds$, then (2.3) represents a model of population dynamics, where the relative velocity $\frac{\dot{x}_i}{x_i}$ of grows of the i-th species is $G_i(x, t)$. Integrodifferential population models are recently intensively studied. One can study also more general equations than equation (2.3), e.g. equations of the form (2.3), however with integrals containing finite or infinite delays. Population models for such equations are studied e.g. in [4]. Such equations with delays can also be defined in an analogous way as E-vector fields (also on manifolds) and they can be studied from our generic point of view. However we do not pursue this problem.

Even though the right hand side of the second equation of (2.4) is linear in v, using the same procedure as in the proof of Theorem 4.2 one can prove that there is a residual subset $\mathscr{E}_{10}^k$ of the space $\mathscr{E}_0^k(TR^n)$ of systems of the form (2.4) such that if $F \in \mathscr{E}_{10}^k$, then the set $K(F)$ of all singular points of F consists of isolated points which are all hyperbolic. If $(x, v) \in K_0(F)$, then $c \in R^n$ in (2.3) must be equal 0 and therefore $K_0(F) = K(F) \cap (TR^n) = \emptyset$, $((TR^n)_0 = \{(x, v) \in TR^n: v = 0\})$.

REMARK. The second order ODE on a manifold M can also be defined as a vector field on TM such that any solution $\beta: I \to TM(I \subset R$ is an interval) satisfies the condition $D(\pi \circ \beta) = \beta$, where $\pi: TM \to M$ is the natural projection (see e.g. [2, 3, 11]). Similarly one can define an E-vector field as a vector field F on TM such that any solution $\beta: I \to TM$ satisfies the condition

$$D(\pi \circ \beta) = A \circ \beta\,, \tag{2.5}$$

where $A \in \Gamma^k(\mathrm{End}\,(TM))$. This means that if $\tilde{\beta} = (\gamma, \delta)$ is a local representation of β and $\tilde{\pi}$ is a local representation of π then $\tilde{\pi} \circ \tilde{\beta} = \gamma$, $D(\tilde{\pi} \circ \tilde{\beta}) = (\gamma, \dot{\gamma})\left(\dot{\gamma} = \dfrac{d\gamma}{dt}\right)$ and if $\tilde{A}(x, y) = (x, P(x)y)$ is a local representation of A then the condition (2.5) yields $\dot{\gamma} = P(\gamma)\delta$, $\dot{\delta} = g(\gamma, \delta)$.

DEFINITION 2.4. If $A \in \Gamma^k(\mathrm{End}_r(TM))$, then a vector field $F \in E_A^k(TM)$ is called an E-vector field of corank r. The set of all such vector fields we denote by $E_{A,r}^k(TM)$ and $E_r^k(TM) := \{E_{A,r}^k(TM) \colon A \in \Gamma^k(\mathrm{End}_r(TM))\}$.

THEOREM 2.5. *If $F \in E_{A,0}^k(TM)$, $k \geqq 1$, the there exists a C^k-diffeomorphism $h \colon TM \to TM$ such that $F_* \in E^{k-1}(TM)$ defined via*

$$F_*(y) := Dh(h^{-1}(y))F(h^{-1}(y))\,, \qquad y \in TM\,,$$

is a second order ODE on M.

PROOF. Given any $x \in M$ define the mapping $h_x \colon T_xM \to T_xM$, $h_x(y) = A(x)y$, and the mapping $h \colon TM \to TM$, $h(v_x) = h_x(v_x)$, $v_x \in T_xM$. Since the bundle endomorphism A is of corank 0, the linear mapping $A(x) \colon T_xM \to T_xM$ is invertible for any $x \in M$. Define the mapping $h^{-1} \colon TM \to TM$, $h^{-1}(v_x) = (A(x))^{-1}(v_x)$, $v_x \in T_xM$, where $(A(x))^{-1}$ is the inverse of $A(x)$. The inverse mapping theorem implies that h^{-1} is of the class C^k and obviously it is the inverse of h. Let F_* be defined as in the theorem. The vector field F_* is locally represented by a system of the form (1.1), with $P(x) \equiv I$—the unit matrix. Since the C^k-diffeomorphism h is defined globally, the C^{k-1}-vector field F_* is a second order ODE on M.

3. Generic bundle endomorphisms

We shall study in the next section generic properties of E-vector fields. Therefore we need to describe basic generic properties of bundle endomorphisms. Basic generic properties of matrix functions defined on the n-dimensional disk are presented in the paper [12]. In this section we give a globalization of these results in a form useful for our prupose.

One can consider an $A \in \Gamma^k(\mathrm{End}\,(TM))$ as a C^k-section of the fiber bundle $L(TM) := (p, \mathrm{End}\,(TM), L(R^n))$, where $p \colon \mathrm{End}\,(TM) \to M$, $p(B) = x$ for $B \in \mathrm{End}\,(T_xM)$. The coordinates on the fiber bundle $L(TM)$ are defined as follows:

Let (U, α) be a chart on M and (T_α, α, U) be the natural chart on TM induced by (U, α) (see e.g. [3]). Then define the natural chart on $L(TM)$ as a triple (L_α, α, U), where $L_\alpha(x) = (\alpha(x), B_\alpha(x))$ for $B \in \text{End}(T_xM)$, $b_\alpha(x) = A_\alpha(y)$, $y = \alpha(x)$, $T_\alpha \circ B \circ T_\alpha^{-1}(y, v) = (y, A_\alpha(y)v)$. Thus for $A \in \Gamma^k(\text{End}(TM))$ we have the mapping

$$A_\alpha: \alpha(U) \to M_n\,, \tag{3.1}$$

defined via

$$\Phi_\alpha(y) := T_\alpha \circ A(x) \circ T_\alpha^{-1}(y, v) = (y, A_\alpha(y)v)\,. \tag{3.2}$$

DEFINITION 3.1. Let S_n be the set of all $n \times n$ symmetric matrices. Define the sets

$$R_r = \{B \in M_n: \text{rank } B = r\}\,,$$

$$R_r^s = \{B \in S_n: \text{rank } B = r\}\,.$$

Let us recall the following well-known results.

PROPOSITION 3.2.

(1) *The set* R_r *is a smooth submanifold of* M_n *of codimension* $(n - r)^2$

(2) *The set* R_r^s *is a smooth submanifold of* S_n *of codimension* $\frac{1}{2}(n - r + 1)(n - r)$

(*see e.g.* [8, *Proposition* 3.2]).

DEFINITION 3.3. A bundle endomorphism $A \in \Gamma^k(\text{End}(TM))$ has the property T (or the transversality property) if for any $x \in M$ there is a chart (U, α) on M such that $x \in U$ and

$$A_\alpha \pitchfork R_r\,, \qquad r = 0, 1, \ldots, n \tag{3.3}$$

(i.e. A_α transversally intersects R_r), where $A_\alpha: \alpha(U) \to L(R^n)$ is defined by (3.1) and we identify $L(R^n)$ with M_n. Analogously we define the property T for $A \in \Gamma_S^k(\text{End}(TM))$ by the condition

$$A_\alpha \pitchfork R_r^s\,, \qquad r = 0, 1, \ldots, n\,. \tag{3.4}$$

If (V, β) is another chart on M as in Definition 3.3, where $x \in V$, then

$$T_\beta \circ A(x) \circ T_\beta^{-1}(y, v) = T_\beta \circ T_\alpha^{-1} \circ A(x) \circ T_\alpha^{-1}((T_\beta \circ T_\alpha)^{-1}(y, v))$$

and therefore there is a regular linear map $C_{\alpha\beta} \in L(R^n)$ such that $\Phi_\beta(y) = (y, A_\beta(y))$ (see (3.2)), where $A_\beta(y) = C_{\alpha\beta} \circ A_\alpha(y) \circ C_{\alpha\beta}^{-1}$. This implies that $A_\alpha \pitchfork R_k$ *iff* $A_\beta \pitchfork R_k$. Thus Definition 3.3 is independent of coordinates.

Let M be a compact Riemannian manifold. The set $\Gamma^k(\text{End}(TM))$ has the natural linear structure. We can define the norm $|||A|||_k$ of $A \in$

$\Gamma^k(\mathrm{End}\,(TM))$ as

$$\|\!|A|\!\|_k = \sup_{x \in M} (\|A(x)\|_0, \|DA(x)\|_1, \ldots, \|D^kA(x)\|_k), \tag{3.5}$$

where $\|\cdot\|_i$ is the norm in $L^i(T_xM)$—the space of continuous i-linear mappings of T_xM into itself and A is considered to be a C^k-section of End (TM). One can check that $B^k = (\Gamma^k(\mathrm{End}\,(TM)), \|\!|\cdot|\!\|_k)$ is a Banach space. Analogously $B_S^k = (\Gamma_S^k(\mathrm{End}\,(TM)), \|\!|\cdot|\!\|_k)$ is a Banach space.

Define the mappings

$$\rho_\alpha: B^k \times \alpha(U) \to L(R^n), \qquad (A, y) \mapsto A_\alpha(y),$$

where (U, α) is a chart on M, A_α is defined by (3.1) and

$$\rho_{\alpha,A}: \alpha(U) \to L(R^n), \qquad \rho_{\alpha,A}(y) = \rho_\alpha(A, y),$$

where $A \in B^k$ is fixed.

Let $V \subset \alpha(U)$ be an open set with $\bar{V} \subset \alpha(U)$ and $k \geqq n$. Then by the Abraham's transversality theorems (see [3, Theorems 18.2, 19.1]) the set of all $A \in B^k$ for which $\rho_{\alpha,A} \pitchfork_{\bar{V}} R_r$ for $r = 0, 1, \ldots, n$ is open and dense in B^k. An analogous assertion is valid in the case of the Banach space B_S^k. We have the following theorem.

THEOREM 3.4. *Let M be a compact smooth Riemannian manifold of dimension n, $B^k = (\Gamma^k(\mathrm{End}\,(TM)), \|\!|\cdot|\!\|_k)$, $B_S^k = (\Gamma_S^k(\mathrm{End}\,(TM)), \|\!|\cdot|\!\|_k)$ be the Banach spaces with the norm defined by* (3.5) *and $k \geqq n$. Then there is an open dense subset B_T of B^k (of B_S^k) such that if $A \in B_T$, then A has the property T in the sense of Definition* 3.2.

DEFINITION 3.5. If $A \in B_T$ then we say that A is generic.

As a consequence of [3, Corollary 17.2] we have

PROPOSITION 3.6. *If $A \in B_T \subset B^k(\subset B_S^k)$, (U, α) is a chart on M and A_α is the mapping defined by* (3.1), *then the set $\Sigma_r(A_\alpha) = A_\alpha^{-1}(R_r)(\Sigma_r^s(A_\alpha) = A_\alpha^{-1}(R_r^s))$ is either empty or it is a C^k-submanifold of $\alpha(U)$ of codimension $(n-r)^2$, i.e. $\dim \Sigma_r = n - (n-r)^2 \frac{1}{2}(n-r+1)(n-r)$, i.e. $\dim \Sigma_r^s = n - \frac{1}{2}(n-r+1)(n-r)$.*

COROLLARY 3.7. *If $A \in \Gamma^k(\mathrm{End}\,(TM))(A \in \Gamma_S^k(\mathrm{End}\,(TM)))$, then $G_r(A) = \{x \in M: \mathrm{rank}\, A(x) = r\}$ is a C^k-submanifold of M of codimension $(n-r)^2$, i.e. $\dim G_r(A) = n - (n-r)^2$ dim, $G_r(A) = n - \frac{1}{2}(n-r)(n-r+1)$.*

4. Generic E-vector fields

Let F be an E-vector field from $E_A^k(TM)$, where M is a compact Riemannian manifold, $\{U_i, \alpha_i\}_{i=1}^m$ be an atlas on M and

(4.1) $$F_i: \begin{cases} \dot{x} = A_i(x)y, \\ \dot{y} = f_i(x, y), \qquad x \in \alpha_i(U_i), \quad y \in R^n \end{cases}$$

be the local representation of F with respect to the chart $(T_{\alpha_i}, \alpha_i, U_i)$ on TM. Let us define

$$\|F_i\|_k := \|\|A\|\|_k + \sup_{(x,y)\in\alpha_i(U_i)\times R^n} (\|f_i(x, y)\|, \ldots, \|D^k_{(x,y)}f_i\|_k),$$

where $\|\|A\|\|_k$ is defined by (3.5), $D^j_{(x,y)}f_i$ is the j-th derivative of f_i at (x, y) and let

(4.2) $$\|F\|_k = \max_{i\in\{1,2,\ldots,m\}} \|F_i\|_k .$$

Define the set

(4.3) $$\mathscr{E}^k(TM) = \{F \in E^k(TM): \|F\|_k < \infty\} .$$

The set $\mathscr{E}^k(TM)$ has a natural linear structure and one can check that $(\mathscr{E}^k(TM), \|\cdot\|_k)$ is a Banach space. The set of all $F \in \mathscr{E}^k(TM)$, where A is fixed, we denote by $\mathscr{E}^k_A(TM)$.

Generic properties of second order ODEs on a manifold M have been studied by S. Shahshahani [19] and generic properties of 1-parameter families of second order ODEs are described in [13, 14, 15]. Proposition 3.6 and its corollary show that generic bundle endomorphisms which are components of generic E-vector fields are not simple in general. Namely, the set of all $x \in M$ for which the linear map $A(x): T_xM \to T_xM$ is not an isomorphism, is locally an algebraic variety whose $(n-k)$-th stratum has the dimension $n - k^2$. We shall show that in spite of this fact we are able to prove some results concerning generic properties of singular points of E-vector fields.

Recall that $v_x \in T_xM$ is the singular point of $F \in \mathscr{E}^k(TM)$ if $F(v_x) = 0(v_x)$— the zero of $T_{v_x}(TM)$. We denote by $K(F)$ the set of all singular points of F.

DEFINITION 4.1. Let $F \in \mathscr{E}^k_A(TM)$. Define the sets

$$S^j(F) = \{v_x \in TM: v_x \in K(F),\ \operatorname{rank} A(x) = j\}, \qquad j = 0, 1, \ldots, n$$

and let $S(F) = \bigcup_{j=1}^n S^j(F)$.

If $A = \mathrm{id}_{TM}$ then $S^j(F) = \emptyset$ for $j = 1, 2, \ldots, n$ and $K(F) \subset (TM)_0 = \{v_x \in TM: v_x = 0_x$—the zero in $T_xM\}$—the zero section of TM. By S. Shahshahani [19] (see also [15]) the set $K(F)$ consists generically of isolated points which are all hyperbolic.

Let us write $K(F)$ as $K(F) = K_0(F) \cup K_1(F)$, where $K_0(F) = K(F) \cap (TM)_0$ and $K_1(F) = K(F) - K_0(F)$.

THEOREM 4.2. *Let M be a compact manifold. Then there exists a residual subset $\mathscr{E}^k_1$ of $\mathscr{E}^k(TM)$ such that if $F \in \mathscr{E}^k_1$ then*

(1) $F \in E_A^k(TM)$, *where* $A \in \Gamma^k(\mathrm{End}\,(TM))$ *is a generic bundle endomorphism in the sense of Definition* 3.5.
(2) *The set* $K(F)$ *of all singular points of* F *consists of isolated points which are all hyperbolic and* $K_0(F)$ *is finite.*

Before starting to prove this theorem we give an example which demonstrates the main idea of the proof and also a difference between generic properties of second order ODEs and E-vector fields.

EXAMPLE. Consider the plane E-vector field

$$\dot{u} = \phi(u)v\,,$$

$$\dot{v} = f(u, v)\,,$$

where $u,\ v \in R$, ϕ and f are smooth functions. If $\phi(u) \neq 0$ for all u then by Theorem 2.5 this system is conjugated to a second order ODE. If, for instance, $\phi(u) > 0$ for all u then the system has the same trajectories as the second order ODE

$$\dot{u} = v\,,$$

$$\dot{v} = [\phi(u)]^{-1} f(u, v)\,.$$

By the results of S. Shahshahani [19] the hyperbolicity of singular points is a generic property in the space of all smooth second-order ODEs. In this case all singular points lie on the submanifold $v = 0$. If $\phi(u)$ has some zeros then the system possesses singular points also outside the submanifold $v = 0$. Let, for instance, $\phi(u) = u^2$, $f(u, v) = au + bv$, $a \neq 0$, $b \neq 0$. Then $K = (0, 0)$ is the only singular point which is non-hyperbolic. If we perturbe ϕ into $\phi_\varepsilon(u) = u^2 - \varepsilon$, $\varepsilon > 0$, then the corresponding systems has three singular points $K = (0, 0)$, $K_1 = (-\sqrt{\varepsilon}, v_\varepsilon)$, $K_2 = (\sqrt{\varepsilon}, -v_\varepsilon)$, where $v_\varepsilon = \frac{a}{b}\sqrt{\varepsilon}$. One can easily check that all these singular points are hyperbolic. If $b = 0$ then it is necessary to perturb also f into the generic case with $b \neq 0$. In the following proof we shall find generic perturbations of the linear vector bundle endomorphism as well as of the vector field defining a given non-generic generalized vector field in such a way that the resulting generalized vector field has all singular points hyperbolic.

PROOF OF THEOREM 4.2. The assertion (1) is a direct consequence of Theorem 3.4. Let $TM = \bigcup_{i=1}^{\infty} T_i$, where T_i are compact sets and $T_{i+1} \supset T_i$ for all i. Let $F \in \mathscr{E}^k(TM)$, $v_{x_0} \in K(F) \cap T_i$ and (T_α, α, U) be a chart on TM such that $v_{x_0} \in \pi_M^{-1}(U)$. Then the local representation of F with respect to this chart has the form

$$(4.4) \qquad G: \begin{cases} \dot{u} = B(u)v\,, \\ \dot{v} = g(u, v)\,, \end{cases}$$

where $B \in C_b^k(\alpha(U), M_n)$, $g \in C_b^k(\alpha(U) \times R^n, R^n)$, $T_\alpha(v_{x_0}) = (u, v)$, C_b^k is the set of all bounded C^k-mappings with bounded derivatives up to order k. Let us define the mapping

$$\Phi_{g,B}\colon \alpha(U) \times R^n \to Y := R^n \times M_n \times R^n \times M_n \times M_n\,,$$

$$\Phi_{g,B}(u, v) = \left(v, B(u), g(u, v), \frac{\partial}{\partial u} g(u, v), \frac{\partial}{\partial v} g(u, v)\right)$$

and let $W_r = \{(y, A, z, C, D) \in Y\colon \operatorname{rank} A = r, Ay = 0, z = 0\}$. If $\operatorname{rank} A = r$ then there exist regular matrices $P, Q \in M_n$ such that $PAQ = \operatorname{diag}\{1, \dots, 1, 0, \dots, 0\}$. Therefore $Ay = 0$ iff $w_1 = 0, \dots, w_r = 0$, where $w = (w_1, \dots, w_n)^T = Q^{-1}y$. From this and Proposition 3.2 (1) it follows that W_r is a smooth submanifold of Y and $\operatorname{codim} W_r = n + r + (n - r)^2$. Let $N \subset \alpha(U) \times R^n$ be an open set with compact closure $\overline{N} \subset \alpha(U) \times R^n$ and $(p, q) \in N$. Abraham's transversality theorems ([3, Theorem 18.2, 19.1]) imply that the sets $\mathscr{F}_r^k(N) = \mathscr{F}_r^k = \{(g, B) \in X := C_b^k(\alpha(U) \times R^n, R^n) \times C_b^k(\alpha(U), M_n)\colon \Phi_{g,B} \pitchfork_{\overline{N}} W_r\}$, $r = 0, 1, \dots, n$ are open dense in X. If $(g, B) \in \mathscr{F}_r^k$ then $Z_{g,B}^r(N) := ((\Phi_{g,B})^{-1}(W_r)) \cap N$ is a C^{k-1}-submanifold of $(\alpha(U) \times R^n) \cap N$ of codimension $n + r + (n - r)^2$. This implies that $Z_{g,B}^r(N) = \emptyset$ for $r < n - 1$ and $\dim Z_{g,B}^{n-1}(N) = \dim Z_{g,B}^n = 0$. If $(p, q) \in Z_{g,B}^n(N)$ then $\det B(p) \neq 0$ (therefore $v = 0$) and generically also $C := \dfrac{\partial}{\partial u} g(p, q)$, $D := \dfrac{\partial}{\partial v} g(p, q)$ are regular matrices. One can easily show that the matrix $\begin{pmatrix} 0 & B(p) \\ C & D \end{pmatrix}$ is regular and therefore $\lambda = 0$ is not an eigenvalue of $DF(v_{x_0})$. Let $W(c) = \{(y, A, z, C, D) \in Y\colon A, C, D$ are regular matrices, $y = 0$, $z = 0$, $\begin{pmatrix} 0 & A \\ C & D \end{pmatrix}$ has a purely imaginary eigenvalue$\}$. This set is a semi-algebraic variety of codimension $\geqq 2n + 1$. Therefore if $(g, B) \in \mathscr{F}^k(N, c) := \{(g, B) \in X := \Phi_{g,B} \pitchfork_{\overline{N}} W(c)\}$ then $(\Phi_{g,B})^{-1}(W(c)) = \emptyset$. Therefore if $(p, q) \in Z_{g,B}^n(N)$ then $q = 0$ and the point $(p, 0)$ is the hyperbolic singular point of (4.4).

If $(p, q) \in Z_{g,B}^{n-1}(N)$ then $\operatorname{rank} B(p) = n - 1$. There exist regular matrices $P, Q \in M_n$ such that $PB(p)Q = \operatorname{diag}\{1, 1, \dots, 1, 0\}$. Let $PB(u)Q = (b_{ij}(u))$. Let us introduce new coordinates via the regular mapping $\Psi\colon x = Pu$, $y_i = b_{i1}(u)(Q^{-1}v)_1 + \cdots + b_{in}(u)(Q^{-1}v)_n$, $1 \leqq i \leqq n - 1$, $y_n = (Q^{-1}v)_n$ near (p, q), where $(Q^{-1}v)_i$ is the i-th coordinate of $(Q^{-1}v)$. In these coordinates the system (4.4) has the form

$$
(4.5) \qquad H: \begin{cases} \dot{x}_1 &= y_1 , \\ &\vdots \\ \dot{x}_{n-1} &= y_{n-1} \\ \dot{x}_n &= a_1(x)y_1 + \cdots + a_n(x)y_n , \\ y &= \tilde{g}(x, y) , \end{cases}
$$

where $a_j(p) = 0$, $1 \leqq j \leqq n$, $(p, g) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$. Obviously, $q_1 = 0, \ldots, q_{n-1} = 0$, $q_n \neq 0$ (generically). Thus we have

$$
DH(p, q) = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & & & & & & & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \\ \dfrac{\partial a_n(p)}{\partial x_1} q_1 & \ldots & \dfrac{\partial a_n(p)}{\partial x_n} q_n & 0 & 0 & \ldots & 0 & 0 \\ & & \tilde{C} & & & \tilde{D} & & \end{bmatrix},
$$

where $\tilde{C} = \dfrac{\partial \tilde{g}}{\partial x}(p, q)$, $\tilde{D} = \dfrac{\partial \tilde{g}}{\partial y}(p, q)$. If $\tilde{C} = (c_{ij})$ and $\tilde{D} = (d_{ij})$ then $\det DH(p, q) = 0$ iff $\det D(p, q) = 0$, where

$$
(4.6) \qquad D(x, y) = \begin{bmatrix} s_1(x)y_1 & \ldots & s_n(x)y_n & 0 \\ c_{11}(x, y) & \ldots & c_{1n}(x, y) & d_{1n}(x, y) \\ \ldots & \ldots & \ldots & \ldots \\ c_{n1}(x, y) & \ldots & c_{nn}(x, y) & d_{nn}(x, y) \end{bmatrix}
$$

$s_j(x) = \dfrac{\partial a_n(x)}{\partial x_j}$, $c_{ij}(x, y) = \dfrac{\partial g_i(x, y)}{\partial x_j}$, $d_{in}(x, y) = \dfrac{\partial g_i(x, y)}{\partial y_n}$, $1 \leqq i, j \leqq n$, $\tilde{g} = (g_1, g_2, \ldots, g_n)$. Let us define the mapping $\psi_{g,B}\colon V \times R^n \to R^{4n} \times M_{n+1}$ by

$$
\psi_{g,B}(x, y) = (y, \alpha_B(x), \operatorname{grad} \alpha_n(x), \tilde{g}(x, y), D(x, y)) ,
$$

where $V = \psi(\alpha(U)) \subset R^n$, $\alpha_B = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $D(x, y)$ is defined by (4.6). Let $D_1 = \{(y_1, \ldots, y_n, v, w, z, Q) \in R^{4n} \times M_{n+1}\colon y_1 = 0,\ y_2 = 0,\ \ldots,\ y_{n-1} = 0,\ w_n = 0,\ z = 0,\ \det Q = 0\}$, where w_n is the n-th coordinate of w. This set is an algebraic variety whose strata have codimension $\geqq 2n + 1$. The set $W_0 = \{(g, B) \in X\colon \psi_{g,B} \pitchfork_{\bar{V}_0} D_1\}$ is open dense in X, where X is defined as above and $V_0 \subset V$ is an open neighbourhood of (p, q), $\bar{V}_0 \subset V$. If $(g, B) \in W_0$ then $(\psi_{g,B})^{-1}(D_1) \cap V_0 = \emptyset$. This means that generically $\det DH(p, q) \neq 0$.

Now we show that $DH(p, q)$ has generically no pure imaginary eigenvalues. Since $q_1 = 0, \ldots, q_{n-1} = 0$ the characteristic polynomial of $DH(p, q)$ has the form $(\lambda - s_n q_n)P(\lambda)$, where $P(\lambda)$ is the determinant of the matrix

$$\begin{bmatrix} \lambda^2 - \lambda_{11}d_{11} - c_{11} & \dots & -c_{1n-1} - \lambda d_{1n-1} & -d_{1n} \\ -c_{21} - \lambda d_{21} & \dots & -c_{2n-1} - \lambda d_{2n-1} & -d_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots\ \lambda^2 - \lambda d_{n-1n-1} - c_{n-1n-1} & & -d_{n-1n} \\ -c_{n1} - \lambda d_{n1} & \dots & -c_{nn-1} - \lambda d_{nn-1} & \lambda - d_{nn} \end{bmatrix},$$

where $s_j = s_j(p)$, $c_{ij} = c_{ij}(p, q)$, $d_{ij} = d_{ij}(p, q)$. This polynomial is of degree $2n - 1$ and it has a purely imaginary eigenvalue ia iff it factors as $(\lambda^2 + a^2)P_{2n-3}(\lambda)$, where P_{2n-3} is a polynomial of degree $2n - 3$ with real coefficients which are polynomial functions of c_{ij}, d_{ij} and p_k. Therefore the coefficients of $P(\lambda)$ must satisfy some equalities which define an algebraic variety I in M_{2n} of codimension $\geqq 1$. Therefore the set $D_2 = \{(y_1, \dots, y_n, v, w, z, Q) \in R^{4n} \times M_{2n}: y_1 = 0, \dots, y_{n-1} = 0, w_n = 0, z = 0, Q \in I\}$ is an algebraic variety of codimension $\geqq 2n$. The set $W(c) = \{(g, B) \in \psi_{g,B} \pitchfork_{\bar{V}_0} D_2\}$ is open dense in X and if $(g, B) \in X$ then $(\psi_{g,B})^{-1}(D_2) \cap V_0 = \emptyset$. This means that $DH(p, q)$ has generically no purely imaginary eigenvalue. Since any T_i is a compact set, we can cover it by a finite number of charts $\{(T^i_{\alpha_i}, U^i_j, \alpha^i_j)\}_{j=1}^{m(i)}$ and using the above local results one can find open dense subsets $\mathscr{E}^k_i(j)$, $j = 1, 2, \dots, m$ of $\mathscr{E}^k(TM)$ such that if $F \in \mathscr{E}^k_i(j)$ then $K(F) \cap T^i_{\alpha_j}(U^i_j)$ consists of isolated points which are all hyperbolic. Obviously, the set $\mathscr{E}^k_1 := \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{m(i)} \mathscr{E}^k_i(j)$ is residual in $\mathscr{E}^k(TM)$ and it has the properties as in the theorem.

5. Generalized mechanical systems

In this section we study generalized mechanical systems (shortly mechanical systems) defined in Section 2 which are locally represented by systems of the form (1.2) with $f(x, y) = Q(x)y$, where $Q(x, y)$ is a C^k-matrix function. The set of all such systems we denote by $\mathscr{M}^k = \mathscr{M}^k(TM)$ and a mechanical system $\mathscr{S} \in \mathscr{M}^k$ is identified with a triple $(B, W, G) \in \Gamma^{k+1}_S(\text{End}\,(TM)) \times C^{k+1}(M, R) \times \Gamma^k(\text{End}\,(TM))$. We shall need the topology of a Banach space on the space of all mechanical systems and therefore we restrict ourselves to a subset $\mathscr{B}^k(TM)$ of $\mathscr{M}^k(TM)$ defined as $\mathscr{B}^k(TM) = B^k_s \times C^k_b(M, R) \times B^k$, where $B^k = (\Gamma^k(\text{End}\,(TM), \|\cdot\|_k)$, $B^k_S = (\Gamma^k_S(\text{End}\,(TM)), \|\cdot\|_k)$ are Banach spaces defined in Section 3 and $C^k_b(M, R)$ is the space of all bounded C^k-functions on M with bounded derivatives up to order k. The form of mechanical systems enables us to define the natural structure of a vector space on $\mathscr{B}^k(TM)$, i.e. $\lambda\mathscr{S} + \mu\mathscr{S}' = (\lambda B + \mu B', \lambda W + \mu W', \lambda G + \mu G')$ for λ, $\mu \in R$, $\mathscr{S} = (B, W, G)$, $\mathscr{S}' = (B', W', G') \in \mathscr{B}^k(TM)$. The space $\mathscr{B}^k(TM)$ is a Banach space.

We shall study generic properties of mechanical systems near singular points lying in the zero section $(TM)_o$. We define $K_0(\mathscr{S}) := K(\mathscr{S}) \cap (TM)_0$, where $K(\mathscr{S})$ is the set of all singular points of the mechanical system $\mathscr{S} \in \mathscr{B}^k(TM)$.

Let us recall the definition of the Hessian $\dot{G}(x)$ of a vector field G on a manifold N. This is the linear map $\tau \circ T_x G\colon T_x N \to T_x N$ (see [3, Section 5]), where $T_x G\colon T_x N \to T_{0_x}(TN) = T_{0_x}(TN)_0 \oplus T_{0_x}(T_x N)$, 0_x is the zero in $T_x N$, $\tau\colon T_{0_x}(TN) \to T_x N$ is the projection onto the second summand followed by the canonical identification of $T_{0_x}(T_x N)$ with $T_x N$. The Hessian locally represents the linearization of the vector field at this singular point.

THEOREM 5.1. *Let M be a compact manifold. Then there is an open dense subset $\mathscr{B}_1^k$ of $\mathscr{B}^k(TM)$ such that if $\mathscr{S} \in \mathscr{B}_1^k$, then the set $K_0(\mathscr{S})$ is finite and if $v_x \in K_0(\mathscr{S})$, then the Hessian $\dot{\mathscr{S}}(v_x)\colon T_{v_x}(TM) \to T_{v_x}(TM)$ is an isomorphism.*

PROOF. Let $TM = \bigcup_{i=1}^{\infty} T_i$, where all T_i are compact sets and $T_{i+1} \supset T_i$ for all i. Let $F \in \mathscr{B}_b^k(TM)$, $v_x \in K(\mathscr{S}) \cap T_i$ and (T_α, α, U) be a chart on TM such that $v_x \in \pi_M^{-1}(U)$. Then the local representation of F with respect to this chart has the form

$$\dot{u} = A(u)v\,, \tag{5.1}$$

$$\dot{v} = -\frac{1}{2}\frac{\partial}{\partial u}(v^T A(u)v) - \operatorname{grad} V(u) + Q(u)v\,,$$

where $(A, V, Q) \in \mathscr{D}_b^k := C_b^k(\alpha(U), S_n) \times C_b^k(\alpha(U), R) \times C_b^k(\alpha(U), M_n)$, $T_\alpha(v_x) = (u_0, 0)$. Let us define the mapping

$$\rho_{A,V,Q}\colon \alpha(U) \to Z := R^n \times S_n \times R^n \times S_n \times M_n\,,$$

$$\rho_{A,V,Q}(u, v) = (v, A(u), \operatorname{grad} V(u), D(\operatorname{grad} V)(u), Q(u))$$

and let

$$X_0 = \{(y, C, z, D, E) \in Z\colon y = 0, z = 0, C, D, E \in M_n \text{ are regular}\}\,,$$

$$Y_r^1 = \{(y, C, z, D, E) \in Z\colon y = 0, z = 0, C \in R_r^s, D, E \text{ are regular}\}\,,$$

$$Y_r^2 = \{(y, C, z, D, E) \in Z\colon y = 0, z = 0, D \in R_r^s, C, D \text{ are regular}\}\,,$$

$$Y_r^3 = \{(y, C, z, D, E) \in Z\colon y = 0, z = 0, E \in R_r, C, D \text{ are regular}\}\,,$$

where R_r^s, R_r are the sets from Definition 3.1.

Obviously, the set X_0 is a smooth submanifold of Z of codimension $2n$. From Proposition 3.2 it follows that the sets Y_r^j, $j = 1, 2, 3$ are smooth submanifolds of Z, codim $Y_r^j = 2n + \frac{1}{2}(n - r + 1)(n - r)$, $j = 1, 2$ and codim $Y_r^3 = 2n + (n - r)^2$. From the Abraham's transversality theorems it follows that the sets $C_0^k = \{(A, V, Q) \in Z\colon \rho_{A,V,Q} \pitchfork_{\bar{U}} X_0\}$, $C_{jr}^k = \{(A, V, Q) \in Z\colon \rho_{A,V,Q} \pitchfork_{\bar{U}_0} Y_r^j\}$,

$j = 1, 2, 3$; $r = 0, 1, \ldots, n-1$, are open dense in Z, where $U_0 \subset \alpha(U)$ is an open neighbourhood of $(u_0, 0)$ with $\overline{U}_0 \subset \alpha(U)$. Therefore $C_1^k = \bigcap_{r=0}^{n-1} (\bigcap_{j=1}^{3} C_{jr}^k) \cap C_0^k$ is open dense in Z and if $(B, V, Q) \in C_1^k$ then $(\rho_{B,V,Q})^{-1}(Y_r^j) = \emptyset$ for $j = 1, 2, 3$; $r = 0, 1, \ldots, n-1$, $(\rho_{B,V,Q})^{-1}(X_0)$ consists of isolated points. This means that the system (5.1) has generically at most a finite number of singular points lying in $U_0 \cap \{(u, v): v = 0\}$. Using the procedure sketched at the end of the proof of Theorem 4.2 one can first extend this generic property to the whole T_i and then to the TM. Since the regularity of the Hessian $\dot{\mathscr{S}}(v_x)$ is independent of coordinates the proof is complete.

We do not pursue the problem of generic properties of singular points of generalized mechanical systems lying outside the zero section and we do not deal with the case in which linearizations of generalized mechanical systems at singular points have purely imaginary eigenvalues, either. Instead of this we give a sufficient condition for the nonexistence of purely imaginary iegenvalues of the linearizations.

Let us note that $v_x \in K_0(\mathscr{S})$ for $\mathscr{S} = (B, V, Q) \in \mathscr{B}^k(TM)$ iff $x \in C(V)$— the set of all singular points of the function V. Then the Hessian $H_x V: T_x M \times T_x M \to R$ (a bilinear map) of the function V at x is defined.

THEOREM 5.2. *Let $\mathscr{S} = (B, W, G) \in \mathscr{B}_1^k$, $\mathscr{B}_1^k$ being as in Theorem 5.1, $w_x \in K_0(\mathscr{S})$ and let the following conditions be satisfied:*

(1) $H_x V(u_x, B(x)v_x) = H_x V(B(x)u_x, v_x)$ *for all* $u_x \in T_x M$,

(2) $H_x V(u_x, B(x)u_x) > 0$ *for all* $u_x, v_x \in T_x M$,

(3) $\langle G(x)u_x, u_x \rangle < 0$ *for all* $u_x \in T_x M$.

Then the Hessian $\dot{\mathscr{S}}(w_x)$ has n eigenvalues with negative real parts and n eigenvalues with positive real parts.

PROOF. We proceed similarly as G. Fusco and M. Oliva in the proof of [6, Lemma 4.2]. Let (5.1) be the local representation of (B, W, G), where $T_\alpha(v_x) = (u_0, 0)$. Let $A_0 = A(u_0)$, B_0 be the Hessian of the function V at u_0 and $C_0 = Q(u_0)$. The matrices A_0, B_0 are symmetric and assumption (1) implies that $A_0 B_0 = B_0 A_0$. Therefore the matrix $B_0 A_0$ is also symmetric. Assumption (3) implies that the symmetric part of C_0 is negative.

The matrix $L = \begin{pmatrix} 0 & A_0 \\ B_0 & C_0 \end{pmatrix}$ is the matrix of the linearization of (5.1) at $(u_0, 0)$ and

$$\det\begin{pmatrix} -\lambda I & A_0 \\ B_0 & -\lambda I + C_0 \end{pmatrix} = 0$$

iff

$$\det\begin{pmatrix} 0 & -\lambda^2 B_0^{-1} + \lambda B_0^{-1} C_0 + A_0 \\ B_0 & * \end{pmatrix} = 0\,.$$

Thus we have the following characteristic equation for L:

(5.2) $$\det(\lambda^2 I - \lambda C_0 - B_0 A_0) = 0 .$$

Therefore λ is an eigenvalue of L iff there exists $w \in R^n$, $w \neq 0$ such that

(5.3) $$\lambda^2 w - \lambda C_0 w - B_0 A_0 w = 0 .$$

If w^* is the conjugate transpose of w then $w^* B_0 A_0 w$ is positive definite and symmetric and $\operatorname{Re} w^* C_0 w < 0$ since the symmetric part of C_0 is negative. Therefore (5.3) implies that there exist α, β, $\gamma \in R$, α, $\gamma > 0$ such that

(5.4) $$\lambda^2 + (\alpha + i\beta)\lambda - \gamma = 0 .$$

This implies that (5.4) has not root on the imaginary axis. The same is true for εC_0 instead of C_0, $0 \leqq \varepsilon \leqq 1$. Therefore it suffices to consider $C_0 = 0$ and for this case the assertion of theorem is obvious.

COROLLARY. *If $\mathcal{S} = (B, W, G) \in \mathcal{B}_1^k$ and for any $w_x \in K_o(\mathcal{S})$ the conditions* (1), (2), (3) *of Theorem* 5.2 *are satisfied, then all critical points lying in $K_0(\mathcal{S})$ are hyperbolic.*

EXAMPLE 1. Consider the integrodifferential equation

(5.5) $$\dot{x} = A(x)\left[\{\exp(Qt)\}c - \int_0^t \{\exp Q(t-s)\} \operatorname{grad} V(x(s))ds\right],$$

where $A \in C^k(R^n, S_n)$, $Q \in M_n$, $c \in R^n$, $V \in C^k(R^n, R)$, $k \geqq 1$. This equation can be written as the system

(5.6) $$\begin{aligned} \dot{x} &= A(x)v , \\ \dot{v} &= -\operatorname{grad} V(x) + Qv \end{aligned}$$

with the condition $v(0) = c$. This is obviously a generalized mechanical system on R^n, i.e. an E-vector field on R^n of a special form. Using the same procedure as in the proof of Theorem 5.1 one can prove that there is a residual subset B_{10}^k of the set $B_0^k(TR^n)$ of all systems of the form (5.6) such that if $B \in B_{10}^k$, then the set K_0 consists of isolated points which are all hyperbolic.

EXAMPLE 2. Let $R^3 - \{0\}$ be the space of positions with coordinates x. Then tangent space $T(R^3 - \{0\}) = (R^3 - \{0\}) \times R^3$ is the space of positions and velocities with coordinates (x, v). A classical mechanical system is defined by a smooth Hamiltonian function $H: T(R^3 - \{0\}) \to R$. The Hamiltonian function describing the motion of two bodies in R^3 in the gravity field is

$H(x, v) = \frac{1}{2}\|v\|^2 - \frac{\mu}{\|x\|}$, where one of the bodies is fixed at the origin, $\mu \in R$ and $\|x\|$ is the norm of x. The corresponding Hamiltonian vector field X_H is

$$\dot{x} = v\,, \qquad \dot{v} = -\mu\frac{x}{\|x\|^3} \tag{5.7}$$

(see e.g. [2]). Consider the following perturbation of (5.7):

$$\dot{x} = v\,, \qquad \dot{v} = -\mu\frac{x}{\|x\|^3} + \frac{1}{\|x\|^3}f(x, v)\,. \tag{5.8}$$

This system possesses the same topological structure of trajectories on $M = (R^3 - \{0\}) \times R^3$ as the system

$$\dot{x} = \|x\|^3 v\,, \qquad \dot{v} = -\mu x + f(x, v)\,,$$

which can be obtained from (5.8) by rescaling the time. This is an E-vector field on M. If we extend this system to the manifold $\tilde{M} = R^3 \times R^3$ we obtain an E-vector field of the form (1.1), where $P(x) = \operatorname{diag}\{\|x\|^3, \|x\|^3, \dots, \|x\|^3)$, which however is not in the form of a generalized mechanical system (1.2).

EXAMPLE 3. Consider the Kepler problem in one dimension given by the system

$$\dot{x} = y\,, \qquad \dot{y} = -\frac{1}{x^2}\,,$$

where $(x, y) \in R^+ \times R$—the phase space. This is a Hamiltonian system with the Hamiltonian function $H(x, y) = \frac{1}{2}y^2 - \frac{1}{x}$. After introducing so called McGehee transformation $u = x$, $v = \sqrt{xy}$ of the half-plane $x > 0$ with time rescaling defined by $\frac{dt}{d\tau} = x^{3/2}$ (see e.g. [5, 10]), we obtain the system

$$\dot{u} = uv\,, \qquad \dot{v} = \frac{1}{2}v^2 - 1$$

which is no longer Hamiltonian. However, if we extend this system to R^2 we obtain an E-vector field on R^n with singular points $K_1 = (0, -\sqrt{2})$, $K_2 = (0, \sqrt{2})$. The singularities K_1, K_2 correspond to the collision of the particle with the attracting center $x = 0$.

The E-vector fields described in the above examples 2, 3 do not represent generalized mechanical systems, however their perturbations can be studied by using the method presented in Section 4.

6. E-vector fields of corank r

We shall study the local structure of linear parts of E-vector fields of corank r (see Definition 2.4). First let us formulate a modification of [16, Lemma 3.1].

LEMMA 6.1. *Let $A \in C^k(U, M_n)$, $k \geqq 1$, $U \subset R^n$ be an open set, $x_0 \in R^n$ and $A(x_0)$ be of corank r. Then there exists a neighbourhood V of x_0 such that $A(x)$ is of corank r for $x \in V$ if and only if*

$$E(x) = B(x)D(x)^{-1}C(x) \tag{6.1}$$

holds, where

$$PA(x_0)Q = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \tag{6.2}$$

$P, Q \in M_n$ are regular matrices,

$$PA(x)Q = \begin{pmatrix} E(x) & B(x) \\ C(x) & D(x) \end{pmatrix}, \tag{6.3}$$

$E(x)$, $B(x)$, $C(x)$ are close to the zero matrix and $D(x)$ is close to the unit matrix I_{n-r}.

PROOF. It is well-known that matrices P, Q satisfying (6.2) do exist. Let the matrices $E(x)$, $B(x)$, $C(x)$, $D(x)$ be defined by (6.3). If V is a sufficiently small neighbourhood of x_0 then $D(x)$ is regular for all $x \in V$ and the assertion of lemma follows from [16, Lemma 3.1].

REMARK. A more general result concerning the existence of a C^k-vector bundle endomorphism of constant corank is proved in [1]. As a consequence of [1, Proposition 3.4, 18] we obtain that $A \in \Gamma^k(\mathrm{End}\,(TM))$ is locally of constant corank iff $\mathrm{Ker}\, A = \bigcup_{x \in M} \mathrm{Ker}\, A(x)$ and $\mathrm{Range}\, A = \bigcup_{x \in M} \mathrm{Range}\, A(x)$ are subbundles of the tangent bundle TM.

THEOREM 5.2. *If $F \in E^k_{A,r}(TM)$ (see Definition 2.4) then for any $v_x \in TM$ there is a chart (T_α, U, α) on TM such that $v_x \in \pi_M^{-1}(U)$, $T_\alpha(v_x) = (0, 0)$ and the local representation of F with respect to this chart has the form*

$$\begin{aligned} \dot{u}_1 &= P(u)v_2\,, \\ \dot{u}_2 &= v_2\,, \\ \dot{v} &= g(u, v)\,, \end{aligned} \tag{6.4}$$

where $u_1 \in R^r$, u_2, $v_2 \in R^{n-r}$, $u = (u_1, u_2)$, $v = (v_1, v_2)$, $P(u)$ is an $(n-r) \times r$ matrix function, $P(0) = 0$, P, $g \in C^k$.

PROOF. The vector field F has a local representation of the form

$$\dot{x} = A(x)y\,, \qquad \dot{y} = f(x, y)\,, \tag{6.5}$$

where A is a C^k-matrix function and $f \in C^k$. Let $x_0 \in R^n$. Since rank $A(x_0) = n - r$, by Lemma 6.1 there exist regular $n \times n$ matrices P, Q satisfying equality (6.1). By the definition of $E^k_{A,r}(TM)$ we have rank $A(x) = n - r$ for all x. By Lemma 6.1 we have the equalities (6.1), (6.3). If $x = P^{-1}u$, $y = Qv$ then the system (6.5) becomes of the same form with $PA(x)Q$ instead of $A(x)$. Therefore we may suppose without loss of generality that $A(x)$ has the form

$$A(x) = \begin{pmatrix} B(x)D(x)^{-1}C(x) & B(x) \\ C(x) & D(x) \end{pmatrix}, \tag{6.6}$$

where $B(0)$, $C(0)$ are zero matrices and $D(0) = I_{n-r}$. After introducing the coordinates $x = u$, $y = \Phi(u)v$, where

$$\Phi(u) = \begin{pmatrix} I_r & 0 \\ -D(u)^{-1}C(u) & D(u)^{-1} \end{pmatrix},$$

$v = (v_1, v_2)$, $v_1 \in R^r$, $v_2 \in R^{n-r}$, the system (6.5) with $A(x)$ as in (6.6) becomes of the form (6.4), where $P(u) = B(u)D(u)^{-1}$. Obviously, $P \in C^k$, $g \in C^k$, $P(0) = 0$, $g(u, v) = f(u, \Phi(u)v)$.

COROLLARY 6.3. *Let $F \in E^k_{A,r}(TR^n)_0$—the set of all vector fields in the set $E^k_{A,r}(TR^n)$ for which the matrix function $A(x)$ has the form*

$$A(x) = \begin{pmatrix} E(x) & 0 \\ C(x) & D(x) \end{pmatrix} \quad \text{for all} \quad x \in R^n\,, \tag{6.7}$$

where rank $D(x) = n - r$ *for all $x \in R^n$. Then $E(x) = 0$ for all $x \in R^n$ and the system* (6.5) *has the form*

$$\dot{x}_1 = 0\,,$$

$$\dot{x}_2 = y_2\,,$$

$$\dot{y}_2 = f(x, y)\,,$$

which is equivalent to an r-parameter family of vector fields on R^{2n-r} of the form

$$\dot{u} = z\,,$$

$$\dot{w} = f_1(\mu_1, \dots, \mu_r, u, w, z)\,,$$

$$\dot{z} = f_2(\mu^1, \dots, \mu^r, u, w, z)\,,$$

where $u = x_2$, $w = y_2$, $x = (\mu_1, \ldots, \mu_r, x_2)$, $y = (u, z)$, $\mu_1, \ldots, \mu_r$ *are parameters,* $f = (f_1, f_2)$.

Define the set $\mathscr{E}_r^k(TM) = \{F \in \mathscr{E}^k(TM): F \in \mathscr{E}_A^k(TM),\ A \in \Gamma^k(\mathrm{End}_r(TM))\}$, where $\mathscr{E}_A^k(M)$ is the set of all $F \in \mathscr{E}^k(TM)$ with common endomorphism A.

THEOREM 6.4. *Let M be a compact manifold. Then there is an open dense subset H^k of $\mathscr{E}_r^k(TM)$ such that if $F \in H^k$ then*

(1) $K_0(F) = \{v_x \in K(F): v_x \in (TM)_0\}$ *is either empty or it is a finite set, where $K(F)$ is the set of all singular points of F.*

(2) *If $z_0 \in K_0(F)$ then the linearization $L(z_0)$ of F at z_0 has the zero eigenvalue of multiplicity r and no purely imaginary eigenvalue.*

PROOF. The proof of assertion (1) is the same as the proof of Theorem 4.2. If $F \in \mathscr{E}_r^k(TM)$ and $z_0 \in K_0(F)$ then by Theorem 6.2 there are such coordinates on TM in which the vector field F has the local representation of the form (6.4), where $(x, y) = (0, 0)$ are coordinates of z_0. We have shown in the proof of Theorem 4.2 that the matrices $C = \frac{\partial}{\partial u} g(0, 0)$, $D = \frac{\partial}{\partial v} g(0, 0)$ are generically regular. The matrix L of the linearization of (6.4) at (0, 0) has the form

$$L = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

where $B = \mathrm{diag}\,\{0, \ldots, 0, 1, \ldots, 1\} \in M_n$ with r zeros on the diagonal. One can easily show that $P(\lambda) = \det(-\lambda I_{2n} + L) = 0$ iff $Q(\lambda) = \det R(\lambda) = 0$, where

$$R(\lambda) = \begin{pmatrix} R_1(\lambda) & R_2(\lambda) \\ R_3(\lambda) & R_4(\lambda) \end{pmatrix}.$$

$R_1(\lambda) = \mathrm{diag}\,\{-\lambda, \ldots, -\lambda, 0, \ldots, 0\} \in M_n$ with $n - r$ zeros on the diagonal, $R_2(\lambda) = \{0, \ldots, 0, 1, \ldots, 1\} \in M_n$ with $n - r$ units on the diagonal, $R_3(\lambda) = [c_1, \ldots, c_r, c_{r+1} + \lambda d_{r+1}, \ldots, c_n + \lambda d_n]$, where c_i, d_i are the i-th columns of the matrices C and D, respectively, and $R_4(\lambda) = -\lambda I_n + D$. Obviously, $\det R(\lambda) = (-1)^r \lambda^r . \det Q(\lambda)$, where $Q(\lambda) = [c_{r+1} + \lambda d_{r+1}, \ldots, c_n + \lambda d_n, d_1(\lambda), \ldots, d_r(\lambda)]$, $d_j(\lambda) = d_j + p_j(\lambda)$, d_j is the i-th column of the matrix D and $p_j(\lambda) = (0, \ldots, 0, -\lambda, 0, \ldots, 0)^T$, $-\lambda$ is the r-th element of $p_j(\lambda)$. It is obvious that there is an open dense subset N_1 of $M_n \times M_n$ such that if $(C, D) \in N_1$ then the matrix $Q(0)$ is regular. This implies that $\lambda = 0$ is generically the eigenvalue of L of multiplicity r. The other eigenvalues of L are zeros of the polynomial $q(\lambda) = \det Q(\lambda)$. Using the transversality argument (see e.g. [3]) one can show that if $(C, D) \in N_2$, then q has no purely imaginary root and the proof is complete.

REMARK. We have studied generic properties of E-vector fields and generalized mechanical systems near singular points only. The problems concerning closed orbits, bifurcations and some other global generic properties of these objects, including their equivariant version, remain open.

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References

[1] R. Abraham, J. E. Marsden and T. Ratiu, Manifolds, Tensor Analysis and Applications, Applied Math. Sciences 75, Second ed., Springer-Verlag, New York, Heidelberg 1988.

[2] R. Abraham and J. E. Marsden, Foundations of Mechanics, Benjamin, New York, Amsterdam 1967.

[3] R. Abraham and J. Robbin, Transversal Mappings and Flows, Benjamin, New York, 1967.

[4] J. M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics, Lecture Notes in Biomathematics 20, Springer-Verlag, Berlin, Heidelberg 1977.

[5] R. Devaney, Singularities in classical mechanical systems, in Ergodic Theory and Dynamical Systems I (A. Katok, Ed.), Birkhauser, Basel 1981, p. 211.

[6] G. Fusco and M. Oliva, Dissipative systems with constraints, J. Differential Equations **63** (1986), 362–388.

[7] G. Godbillon, Géométrie Differéntielle et Mécanique Analytique, Collection Méthodes Hermann, Paris, 1969.

[8] M. Golubitsky and V. Guillemin, Stable Mappings and Their Singularities, Springer-Verlag, New York, Heidelberg, 1973.

[9] S. Kobayashi and K. Nomizu, Foundation of Differential Geometry, Vol. II, Interscience Publ., New York, London and Sydney, 1969.

[10] E. A. Lacomba and G. Sienza, Blow up techniques in the Kepler problem, in: Holomorphic Dynamics (Eds. X. Gomez – Mont, J. Seade, A. Verjovski), LNM 1345, Springer-Verlag, Heidelberg, New York 1988.

[11] S. Lang, Introduction to Differentiable Manifolds, Interscience Publ., New York, 1962.

[12] J. N. Mather, Solutions of generic linear equations, Dynamical Systems (M. M. Peixoto, ed.), Academic Press, New York and London, 1973, pp. 185–193.

[13] M. Medveď, Generic bifurcations of second order ordinary differential equations near closed orbits, J. Differential Equations **36** (1980), 98–107.

[14] M. Medveď, On generic bifurcations of second order ordinary differential equations near closed orbits, Asterisque **50** (1977), 293–297.

[15] M. Medveď, Generic bifurcations of second order ordinary differential equations on differentiable manifolds, Mathematica Slovaca **27** (1977), 9–24.

[16] H. Oka, Constrained systems, characteristic surfaces and normal forms, Japan J. Math. **4** (1987), 393–431.

[17] H. Oka and H. Kokubu, An approach to constrained equations and strange attractors, Patterns and Waves—Qualitative Analysis of Nonlinear Differential Equations (T. Nishida, M. Mimura and H. Fujii, eds.), Kinokanija and North-Holland, 1986, pp. 607–630.

[18] J. Robbin, Relative equilibria in mechanical systems, Dynamical Systems (M. M. Peixoto, ed.), Academic Press, New York and London, 1973, pp. 425–441.

[19] S. Shahshahani, Second order ordinary differential equations on differentiable manifolds, Global analysis: Proc. Sym. Pure Math., vol. 14, AMS, 1970, pp. 265–272.
[20] S. Shahshahani, Dissipative systems on manifolds, Invent. Math. **16** (1972), 177–190.
[21] F. Takens, Mechanical and gradient systems, local perturbations and generic properties, Bol. Soc. Brasil **14** (1983), 147–162.

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