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On plane curves with several singular points with high multiplicity

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ABSTRACT. In this paper we study some properties of families of smooth curves with several pencils, which are families of normalizations of the integral plane curves with several ordinary singular points of high multiplicity at fixed points $P_i \in \boldsymbol{P}^2$ and with several nodes as other singularities. We first show that every reasonably good components of the family has the full symmetric group as monodromy group of nodes. We then proceed to prove the existence of a component of the family whose general element has nodes of maximal rank. We also prove the nonexistence of some linear series under certain numerical assumption. Finally we discuss what we can gain from the point of view of real algebraic curves from our construction.

In this note we consider some properties of certain families of smooth curves with several pencils. These families are just the families of normalizations of the integral plane curves, say of degree n, with s ordinary singular points of multiplicity m_i, $1 \leq i \leq s$, at fixed points $P_i \in \boldsymbol{P}^2$ and with c nodes as other singularities. We do not claim (and it should be false for most data) that, for fixed data $\{n, m_i, P_i, c\}$ the corresponding family of curves is irreducible (Severi-Enriques type of problem: see [8], [11] and [12]). Curves with two pencils were studied in [3] and [4].

In §0 we fix a few notations and we list some interesting properties (such as good postulation, uniform position, monodromy) of the set of nodes which a general curve of an integral subfamily may have. In §1 we discuss the case of the families which are deformations of n lines (hence we need here $n \geq m_1 + \cdots + m_s$) and variations on the same theme (lines plus conics). In Remark 1.1, we show that every reasonably good components (called principal

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components) of the family has the full symmetric group as monodromy group of nodes. In Propositions 1.2 and 1.3, we prove the existence of a component of the family whose general element has nodes of maximal rank.

In § 2 we consider the existence of linear series on the normalization of such curves. Specifically, in Theorem 2.1 we prove the nonexistence of some linear series under certain numerical assumption. On this topic we stress the Key Remark 2.3 which points out two general phenomena which should arise "in general" for the normalization of "general" curves with assigned singularities on every rational surface. In Remark 2.4 we show why the proof of Theorem 2.1 works in positive characteristic.

In § 3 we consider the byproducts of the constructions considered in this paper from the point of view of real algebraic curves. Also in § 3 (with a loose end) we point out what we can gain from the point of view of real algebraic curves from this construction.

0. Notations

We work over an algebraically closed field with characteristic 0.

Fix nonnegative integers n, s, m_i (with $1 \leq i \leq s$), c, and s distinct points $P_1, \ldots, P_s$ of $\boldsymbol{P}^2$. The set $W := \{P_1, \ldots, P_s\}$ will be called the exceptional set. We will write $m_i P_i$ for the $(m_i - 1)$-th infinitesimal neighborhood of the point P_i in $\boldsymbol{P}^2$ (a so-called fat point). Let $W(n; s; \{m_i\}, \{P_i\}, c)$ (or just $W(n, m_i, P_i, c)$) be the subscheme of $Hilb(\boldsymbol{P}^2)$ parametrizing integral curves of degree n with ordinary multiple points of multiplicity m_i at P_i (for all integers i with $1 \leq i \leq s$) and c ordinary nodes as only singularities. In many steps of the proof we will have large values of c (say $c > 24$; compare with Remark 0.1). Indeed we need $W(n, m_i, P_i, c)$ just with its reduced structure, essentially as a set of curves. We do not claim that $W(n, m_i, P_i, c)$ is irreducible or nonempty. Let $NW(n, m_i, P_i, c)$ be the family of normalizations of the elements of $W(n, m_i, P_i, c)$; just as a subset of the family of all genus g curves with $g := (n-1)(n-2)/2 - c - \sum_i (m_i - 1)m_i/2$. Let S be the surface obtained by blowing-up $\boldsymbol{P}^2$ along the exceptional set $\{P_1, \ldots, P_s\}$; let E_i be the exceptional divisor of S with image P_i; $\mathcal{O}_S(t)$ will denote the pull-back on S of the corresponding degree t line bundle on $\boldsymbol{P}^2$. For any sheaf F on $\boldsymbol{P}^2$, $H^i(F)$ (resp. $h^i(F)$) will denote $H^i(\boldsymbol{P}^2, F)$ (resp. $h^i(\boldsymbol{P}^2, F)$).

Fix an integral subvariety G of $W(n, m_i, P_i, c)$. We are interested in the properties of the set A of c nodes of a general element $C \in G$. For instance Proposition 1.2 will give a case in which A has maximal rank, i.e. its postulation is the same as the postulation of c general points of $\boldsymbol{P}^2$. Since the family G is integral, it makes sense to speak of the monodromy (or Galois) group of its set of nodes. In the next section (Remark 1.1) we will give a case in which the monodromy is the full symmetric group. One way to check that the monodromy group is the full symmetric group is to check that it contains a simple transposition and that it is doubly transitive. To find a simple transposition usually it is sufficient to find a degenerate configuration with $c - 2$ ordinary nodes and a tacnode. For the double transitivity we may use the following obvious remark, which will be useful in the next section.

REMARK 0.1. Fix an integer $k \geq 1$. To check that the monodromy group, H, of the set of nodes of an integral family G of curves is at least k-transitive (assuming that it is at least $(k-1)$-transitive) it is sufficient to check that, for a sufficiently general $C \in G$ and for a subset, A, of $k - 1$ nodes, we may move C in G keeping fixed A as a subset of the nodes and acting transitively on the complementary set of $c - k + 1$ nodes. By the classification of finite permutation groups ([5]) to check that the monodromy group contains the alternating group (and hence that the Uniform Position Property holds) it is sufficient to check that it is at least 6-transitive (and except for $c = 11$, 12, 23 or 24 just that it is at least 4-transitive).

1. Deformations of lines

Here we consider the case in which $n \geq m_1 + \cdots + m_s$. Let T be the union of n distinct lines such that exactly m_i of them contain the point P_i and general with this restriction; here we only require that no such line contains two of the exceptional points and that outside the exceptional points T has only double points (hence exactly $b := n(n-1)/2 - m_1(m_1 - 1)/2 - \cdots - m_s(m_s - 1)/2$ nodes). Let B be the strict transform of T in S. Fix a subset, E, of the set of nodes of B (i.e. of T); set $e := \text{card}(E)$; let E' be the the set of nodes of B not in E (and no P_i even if $m_i = 2$). Note that B is connected and hence the linear system $|B|$ contains an irreducible member by Bertini's theorem. Since the strict transform, A, in S of any component of T has $A^2 = 0$, i.e. $K_S \cdot A = -2 < 0$, by [15], §2, we may deform B equisingularly at E, while smoothing the other nodes, and obtain a family, $F(E)$, of nodal curves of S with $\dim F(E) = B \cdot (B - K)/2 - e = (n+2)(n+1)/2 - 1 - (m_1 + 1)m_1/2 - \cdots - (m_s + 1)m_s/2 - e$. By [15], Th. 2.13, if the set E is not a disconnecting set of nodes (in the sense of Tannenbaum [15]) then a general element of $F(E)$ will be integral. It is easy to check the existence of a non-disconnecting set E for every integer e with $0 \leq e \leq b - s + 1$. We will call "principal" the component of $W(n, m_i, P_i, c)$ containing the normalizations of the irreducible nodal curves of geometric genus g which arise as partial smoothing of B for a fixed choice of unassigned nodes; we do not claim that the component is independent from the choice of the nodes which are smoothed (the unassigned nodes). Moving the lines of T (but moving continuosly in the same way the sets of assigned and non-assigned nodes) will produce genus g curves in the same principal component. Now move also the points P_i (hence the lines of T), and move continuosly in the same way the sets of assigned and non-assigned nodes; "big principal component" will be the name of the corresponding integral family of genus g curves; again, we do not claim that it is independent from the rules used to give the assigned nodes. It is easy to construct curves in the same big principal component, but with very different Brill-Noether theory (e.g. gonality or Clifford index and so on; see Remark 2.6 and use points P_i with bad postulation).

Now we modify the construction, taking as T the union of $n-4$ suitable lines and 2 suitable conics which are tangents (not on the exceptional set); this is always possible if $n \geq m_1 + \cdots + m_s + 2$, but it is usually possible for suitable $\{m_i\}$ even if n is lower; the two tangent conics give a tacnode of T. It is easy to check that this implies that the monodromy group of the nodes contains a simple transposition. By Remark 0.1 and the way in which the general theory allows T to be smoothed partially, we see that the monodromy group is the full symmetric group. We record here this fact as Remark 1.1.

REMARK 1.1. We have just checked that if $n \geq m_1 + \cdots + m_s + 2$ then every principal component has the full symmetric group as monodromy group of the nodes.

The construction with conics may be done for much lower values of n; indeed if the integers $\{m_i\}$ are nice enough, it is sufficient to have $n \geq (m_1 + \cdots + m_s)/4$. When there is a principal component, choosing the conics and the unassigned nodes carefully, we arrive also at the conic on the boundary of that principal component.

Now we consider the postulation of the set of nodes of a sufficiently general $C \in W(n, m_i, P_i, c)$.

PROPOSITION 1.2. *For every numerical data with* $n \geq m_1 + \cdots + m_s$ *(and with the corresponding geometric genus* $g \geq 0$*, as always) there is a principal component,* V*, such that the general* $C \in V$ *has nodes which are of maximal rank (as set of points in* $\boldsymbol{P}^2$*).*

PROOF. Fix a configuration, T, of lines defining, after partial smoothing, the principal components. Let c be the number of nodes of a general integral curve in V and let r be the integer such that $r(r+1)/2 \leq c < (r+1)(r+2)/2$. It is sufficient to prove that T contains a non-disconnecting set M of c nodes such that $h^0(\mathscr{I}_M(r-1)) = 0$ and $h^0(\mathscr{I}_M(r)) = (r+1)(r+2)/2 - c$. For any ordering, say $L_1, \ldots, L_n$, of the lines of T the two cohomological conditions are satisfied if we take as M the nodes $L_i \cap L_j$ for $i < j < r$ and the nodes $L_r \cap L_j$ with $1 \leq j \leq c - r(r+1)/2$ (e.g. use Horace method—first introduced and used in [10] and [9]—after adding to M the remaining points of $L_r \cap L_j$ with $j < r$). We need to check that, since $g \geq 0$, we may also satisfy the non-disconnectedness condition. If $n > m_1 + \cdots + m_s$, to obtain this condition it is sufficient to take as unassigned nodes the singular points on L_n. Assume $n = m_1 + \cdots + m_s$. Fix a line L' through P_1 and a line L'' through P_2 and take as unassigned nodes at least all the nodes on L and $L' \cap L''$. ■

PROPOSITION 1.3. *Fix a set of numerical data with $n \geq m_1 + \cdots + m_s$ and corresponding geometric genus $g \geq 0$. Define r by the relation $r(r+1)/2 \leq c < (r+1)(r+2)/2$ and set $t := \max\{r + m_1, m_1 + \cdots + m_s - 1\}$. Then there is a big principal component, V, such that the general $C \in V$ has a set, N, of nodes which are of maximal rank (as set of points in $\boldsymbol{P}^2$) and such that $h^1(\mathscr{I}_{W \cup N}(t)) = 0$.*

PROOF. We will take the exceptional set in a very particular way. We fix a line, D, and we assume that $P_i \in D$ for every i. Fix a configuration, T, of lines defining, after partial smoothing, a principal component with respect to this exceptional set. We apply Horace method m_1 times exploiting the line D (and loosing after the first step many conditions if $s > 1$). We reduce to prove that $h^1(\mathscr{I}_N(t - m_1)) = 0$. By the choice of t and r, the proof of Proposition 1.2 works and gives the thesis. ■

2. Linear systems

In this section we consider the existence of linear series on curves $Y \in NW(n, m_i, P_i, c)$.

THEOREM 2.1. *Fix the points $P_1, \ldots, P_s$ and the numerical data $\{n, m_i, c\}$. Set $W := \{P_1, \ldots, P_s\}$. Fix a subset A of $\boldsymbol{P}^2$ with card$(A) = c$ and $P_i \notin A$ for every i; assume that for every subset A' of A, the scheme $W \cup A'$ has maximal rank. Let j be the first integer with $j^2 + 3j \geq \sum_i (m_i + 1)m_i$. Fix an integer x with*

$$x \leq jn - 3(j^2 + j)/2 + \sum_i (m_i^2 - 1)/4 \tag{1}$$

Assume

$$n^2 \geq 9\left(\sum_i m_i\right) \qquad \text{and} \qquad c + x < n^2/18 - \sum_i (m_i + 1)^2/4 \tag{2}$$

Let Y be the normalization of an integral curve C in $W(n, m_i, P_i, c)$ with A as set of nodes. Then Y has no g_x^1.

PROOF. We will follow the outline of [2] (i.e. Reider-Lazarsfeld method). Assume the existence of a g_x^1. Shrinking x if necessary we may assume that this pencil is complete and base-point-free.

(i) Fix a general divisor B of the g_x^1 considered as a subset of the plane; B is reduced because we are in characteristic 0. By elementary adjunction theory we have $h^1(\mathscr{I}_{W\cup A\cup B}(n-3)) \neq 0$ and since $h^1(\mathscr{I}_W(n-3)) = 0$ (e.g. by the existence of C and the independence of the adjoint conditions ([1]; Chapt. I, Appendix A): we do not need here the maximal rank assumption) we may take a minimal subset A' of A and B' of B with $h^1(\mathscr{I}_{W\cup A'\cup B'}(n-3)) \neq 0$. Since the g_x^1 is complete and base-point-free we may assume $B' = B$. Set $c' := \operatorname{card}(A')$. Let k be the first integer with $h^0(\mathscr{I}_{W\cup A'\cup B}(k)) \neq 0$; by the maximal rank condition on W we have $k \geq j$ and

$$k(k+1)/2 \leq c' + x + \sum_i (m_i+1)m_i/2 < (k+1)(k+2)/2\,. \tag{3}$$

(ii) Note that by (2) we have $3k+1 \leq n$. We will check here the following inequality:

$$c' + x \geq k(n-k) - \sum_i (m_i+1)^2/4\,. \tag{4}$$

The nonvanishing of $h^1(\mathscr{I}_{W\cup A'\cup B}(n-3))$ gives (calling again A' and B the counterimages of A' and B in S) $h^1(S, \mathscr{I}_{A'\cup B}(n-3-\sum_i m_iE_i)) \neq 0$. By the minimality condition on A' and B we see that $A'\cup B$ satisfies the Cayley-Bacharach condition with respect to the line bundle $R := \mathscr{O}_S(n - \sum_i (m_i+1)E_i)$ and hence determines the following exact sequence

$$0 \to \mathscr{O}_S \to F \to \mathscr{I}_{A'\cup B} \otimes R \to 0 \tag{5}$$

with F rank 2 vector bundle with $c_2(F) = c' + x$ and $c_1(F)^2 = n^2 - \sum_i (m_i+1)^2$. Assume the contrary to (4), i.e. $c' + x < k(n-k) - \sum_i (m_i+1)^2/4$. Since $4c' + 4x < c_1(F)^2$, F is Bogomolov unstable and fits in an extension

$$0 \to R' \to F \to R'' \otimes \mathscr{I}_Z \to 0$$

with $\dim(Z) = 0$ and $R' \otimes R''^*$ pseudoeffective; set $R' := \mathscr{O}_S(w - \sum_i a_iE_i)$ for some w and $a_1, \dots, a_s$; since $R' \otimes R''^*$ is pseudoeffective, we have $w \geq n - w$. The section of F coming from (5) shows in a standard way that $n - w \geq k$ and that we have:

$$c' + x \geq w(n-w) + \sum_i a_i(a_i + m_i + 1)\,. \tag{6}$$

Since $w + k \leq n$ and $k \leq n/2$, we have $w(n - w) \geq k(n - k)$. Hence the worst case a priori coming from (6) is when for all integers i we have either $2a_i = -m_i$ or $2a_i = -m_i - 1$, $k = n - w$. Hence $c' + x \geq k(n - k) - \sum_i (m_i + 1)m_i/2$, a contradiction.

(iii) Since $3k + 1 \leq n$, the inequality (4) contradicts (3). ■

REMARK 2.2. The maximal rank condition for $W \cup A'$ with A' subset of A is satisfied if it holds for W and we take as A the union of c general points of $\boldsymbol{P}^2$. Instead of maximal rank for W, we may just take as data the postulation of W and obtain a result corresponding to Theorem 2.1, but with worst bounds.

KEY REMARK 2.3. The essential point is not to have upper bounds for the degree of pencils for curves of $NW(n, m_i, P_i, c)$. The essential point is the following phenomenon: under much weaker assumptions, if there is a g_x^1 for every or some or the general curve in some $NW(n, m_i, P_i, c)$ with, say, $c > 1$, there is an integer $c'' < c$ such that for every c' with $c'' \leq c' < c$ there is a g_x^1 on every or some or the general curve in $NW(n, m_i, P_i, c')$. Furthermore the g_x^1 on the curves, Y, in $NW(n, m_i, P_i, c)$ is not unique at least if we take a general subset A of $\boldsymbol{P}^2$ with card $(A) = c$ as the set of nodes: by monodromy Y will have at least one such g_x^1 for every choice of a subset A'' of A with card $(A'') = c''$. Both phenomena are known for plane nodal curves, when the minimal degree g_x^1 is induced by the pencil of lines through a node ([2]). These phenomena hold (under suitable assumptions) on relatively minimal rational ruled surfaces and indeed (as shown by S) we strongly believe that the same is true on every rational surface; however, due to the big Picard groups and the messy numerology we were unable to prove in a reasonable number of cases the nonemptiness of the numerical interval in which the proof of Theorem 2.1 shows that these phenomena must occur.

REMARK 2.4. A weaker form of Theorem 2.1 holds in positive characteristic p (with the same proof), i.e. with the additional condition that x is the minimal degree of a pencil; indeed in this case B is reduced since there is no pencil of degree x/p, while Bogomolov-Reider technique works in positive characteristic on every rational surface ([6]).

Now we will consider the case $c = 0$ (although we allow $m_j = 2$ for some j).

PROPOSITION 2.5. *Fix an integer* $s > 0$. *Assume* $m_1 + m_2 + \cdots + m_s \leq n + s - 2$, $x + m_2 + \cdots + m_s \leq n - 1$ *and* $x + m_1 \leq n$. *Fix* $Y \in NW(n, m_i, P_i, 0)$ *and assume that* Y *has a* g_x^1. *Then* $x = n - m_1$, *the* g_x^1 *is unique and induced by the pencil of lines through* P_1.

PROOF. Let D be a g_x^1. First assume that there is no line containing

P_1 and other 2 points P_j's. Fix a general group of points (i.e. a divisor), B, of D; we may assume B reduced; regard B as a subset of $\boldsymbol{P}^2$. By the generality of B we may assume that no point of B is contained in a line containing P_1 and another point in the exceptional set. Furthermore, by the integrality of Y we may assume that either there is a line containing $\{P_1\} \cup B$ or that no line through P_1 contains two points of B; until the end of the proof we will assume that the second case occurs. Set $E = m_1 P_1 \cup \cdots \cup m_s P_s$ (the exceptional set which is the union of fat points as a scheme) and $E' := E \backslash \{m_1 P_1\}$. Choose homogeneous coordinates $(x_0; x_1; x_2)$ such that $P_1 = (1; 0; 0)$. Consider the family of homolographies $h_t \in \operatorname{Aut}(\boldsymbol{P}^2)$ defined by $h_t(x_0; x_1; x_2) := (tx_0; x_1; x_2)$. By the noncollinearity assumption we see that $\{h_t(E \cup B)\}$ has as flat limit when t goes to 0 the scheme E'' union of $m_1 P_1$, $s-1$ fat points Q_j, $2 \leq j \leq s$, on the line $R = \{x_0 = 0\}$ with Q_j with multiplicity m_j and the x points obtained projecting B from P_1 into R. Hence the length of the intersection, A', of this scheme with R is $x + m_2 + \cdots + m_s \leq n - 2$. Since the g_x^1 is special, by semicontinuity we have $h^1(\mathscr{I}_{E''}(n-3)) \neq 0$. Now we will use Horace method exploiting R. The residual scheme, A, of E'' with respect to R is the union of $m_1 P_1$ and the fat points $(m_j - 1)Q_j$, $2 \leq j \leq s$. Exploiting the line R $n-4$ times, we see by Horace method that $h^1(\mathscr{I}_A(n-4)) = 0$. Hence (again by Horace method applied using R) $h^1(E''(n-3)) \leq h^1(R, \mathscr{O}_R(n-3-x-m_2-\cdots-m_s)) = 0$, contradiction.

Now assume that there are two (or more) exceptional points P_j, $j > 1$, on a line containing P_1. One checks easily that the flat limit, E'', of the family $\{h_t(E \cup B)\}$ has length $(E'' \cap R) \leq x + m_2 + \cdots + m_s$ and that the same contradiction comes applying the Horace method $n-3$ times with respect to R.

Hence the linear system D is induced by the pencil of lines through P_1. We only have to check that this linear system is complete (i.e. that $x = n - m_1$). Assume $x < n - m_1$. Consider again the flat limit of the family $\{h_t(E \cup B)\}$. In the same way Horace method gives the contradiction. ■

REMARK 2.6. For many numerical data with $s \geq 3$, a general $Y \in NW(n, m_i, P_i, c)$ has a pencil of degree smaller than $n - m_j$ for every j (and this may occur even if $c = 0$). Assume $m_i \geq m_j$ for $i \geq j$. Fix an integer $t \geq 2$ and assume that for some integer $a \leq s$ with $a \leq (t^2 + 3t - 2)/2$ we have $m_2 + \cdots + m_a > n(t-1)$. Then there is at least a pencil of degree t plane curves containing $\{P_1, \ldots, P_a\}$ and this pencil induces a g_x^1 with $x < n - m_1$.

3. Real curves

In this short section we make a few remarks on the smooth real algebraic curves which we are able to find with this construction. For the background, see e.g. [7], [13], [16], [17] or [14]. We fix any principal component; let T be the union of lines in the plane invariant over conjugation. We consider the case in which all points P_i are reals; in the other possible cases we have the same type of informations (with different numbers) in a similar way; in this case we may (and will) assume that all the lines are real and all the nodes are real. We may apply the standard (see e.g. the survey [7]) technique of smoothing the nodes in a particular way to obtain (in S) smooth curves (if we consider only the case $c = 0$) with circles as real part with various properties. The maximal number of connected components of the real locus of such curves is $1 + b - e$. Using a theorem of Brusotti on the independence of simplification of nodes (for plane curves and even for curves in S) ([7]), it is easy to obtain curves with real locus with lower number of connected components. We do not know how to construct curves in this way with interesting real scheme and complex scheme (in the sense of [13] and [16]) as curves in S or as singular curves in the plane. Of course, the usual Bezout trick to bound the depth/number of nested connected components works in S, too.

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