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On a geometric approach to distributions on a circle

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ABSTRACT. In this paper we aim to discuss a natural measure for means of the distributions on a circle, which plays a role similar to that of a usual mean in a Euclidean space. Because a circle is compact and not flat, it may be noted that we cannot define mean or expectation naturally by the same way as in a Euclidean space.

We introduce a measure of location without embedding the circle into a Euclidean plane. This measure is shown to be an extention of some other measures, the mean direction and the median direction. We also derive some properties of our measure by the use of the geometric nature of a circle.

1. Introduction

There are three basic approaches to directional statistics, (i) embedding, (ii) wrapping and (iii) intrinsic approaches. For a discussion of these approaches, see Jupp and Mardia [1989]. These are usually used in different areas, depending on their own merits. For examples, the embedding approach is mainly used for inferential problems, see Watson [1983]. This is because the embedding approach is comparatively easy to carry out various calculations. But this approach possesses an outer space in a sense that the dimension of the space considered is higher than that of the original space, and hence in some cases the results contradict our intuition. That is why we try to define a natural measure corresponding to the 'mean' intrinsically. For applications of distributions on $\mathscr{S}^1$, see Fisher [1993].

Throughout this paper we identify the unit circle $\mathscr{S}^1 = \{(x, y) \in \mathscr{R}^2 | x^2 + y^2 = 1\}$ with a quotient space $\mathscr{R}/2\pi\mathscr{Z}$, and consider the quotient map

$$q: \mathscr{R} \to \mathscr{S}^1 = \mathscr{R}/2\pi\mathscr{Z} ,$$

where $\mathscr{R}$ and $\mathscr{Z}$ denote the sets of real numbers and integers, respectively. For $\theta \in \mathscr{S}^1$ and $s \in \mathscr{R}$, we define a real number $x_\theta^{(s)}$ as a unique point such that $q(x_\theta^{(s)}) = \theta$, and $x_\theta^{(s)} \in (s - \pi, s + \pi] \subset \mathscr{R}$. Then we can define the function $r: \mathscr{S}^1 \to (0, 2\pi] \subset \mathscr{R}$ as

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$$r(\theta) = x_\theta^{(\pi)}.$$

It is easily seen that $q \circ r$ is the identity map of $\mathscr{S}^1$.

We note that for any $x, y \in \mathscr{R}$, the point $(x + y)/2$ is the barycenter. On the contrary, that is not true on $\mathscr{S}^1$. Moreover, for $\theta \in \mathscr{S}^1$, a usual product $a \times \theta$ for a real number a does not make a sense.

In §2 we consider backgrounds of statistics on $\mathscr{S}^1$ and some measures of location. Most of them are found in Mardia's book [1972] but the notations are slightly different. In §3 we give the definition of 'Mean point' with preparation of some notations. In §4 we derive some properties of the Mean point in the case, that the metric function is the geodesic, by the use of the geometric nature of $\mathscr{S}^1$.

2. Basic concepts and measures of location

Let Θ be a random variable which takes values on $\mathscr{S}^1$. The distribution function F of Θ is defined by the equation

$$F(\theta) = \Pr(0 < \Theta \leq \theta), \qquad 0 < \theta \leq 2\pi,$$

and $F(0) = 0$. If F is absolutely continuous, it has a pdf $f: \mathscr{S}^1 \to \mathscr{R}$ such that

$$\int_\alpha^\beta f(\theta)d\theta = F(\beta) - F(\alpha), \qquad 0 \leq \alpha < \beta \leq 2\pi.$$

As a periodic function introduced from f, we consider

$$\tilde{f} = f \circ q: \mathscr{R} \to \mathscr{R}.$$

For any function $g: \mathscr{S}^1 :\to \mathscr{R}$, let

$$E[g(\Theta)] = \int_0^{2\pi} g(\theta)f(\theta)d\theta.$$

As we mentioned before, a usual multiplication does not make a sense on $\mathscr{S}^1$, and hence, in general, $E[\Theta]$ is no longer appropriate as the expectation of Θ.

In order to overcome these difficulties, some devices have been done, which are summarized in the following. The Population Median direction ξ_0 is defined as a point such that (a) ξ_0 is any solution of

$$\int_{r(\xi_0)}^{r(\xi_0)+\pi} \tilde{f}(x)dx = \int_{r(\xi_0)+\pi}^{r(\xi_0)+2\pi} \tilde{f}(x)dx = \frac{1}{2},$$

and (b) $\tilde{f}(r(\xi_0)) > \tilde{f}(r(\xi_0) + \pi)$. This is also a point at which δ_0 attains its minimum, where

$$\delta_0(\alpha) = \pi - \mathrm{E}[|\pi - |r(\Theta) - r(\alpha)||]$$
$$= \int_0^{\pi} r(\theta) dF(\theta + \alpha) + \int_{\pi}^{2\pi} (2\pi - r(\theta)) dF(\theta + \alpha) .$$

Let $\theta_1, \ldots, \theta_n$ be a random sample of Θ. The Sample Median direction P is any point with the following properties:

(a) half of the sample points are on each side of the diameter PQ through P,

(b) the majority of the sample points are nearer to P than Q.

If P exists, then it is a point at which d_0 attains its minimum, where

$$d_0(\alpha) = \sum y_i/n = \pi - \sum |\pi - |x_i||/n$$
$$= \pi - \sum |\pi - |r(\theta) - r(\alpha)||/n ,$$

and $x_i = (r(\theta_i) - r(\alpha)) \bmod 2\pi$, $y_i = \min(x_i, 2\pi - x_i)$.

If $\mathrm{E}[\cos(\Theta)] \neq 0$, the Population Mean direction μ_0 is the solution of two equations,

$$\cos(\mu_0) = \mathrm{E}[\cos(\Theta)]/R , \qquad \sin(\mu_0) = \mathrm{E}[\sin(\Theta)]/R ,$$

where R denotes the resultant length,

$$R^2 = \{\mathrm{E}[\cos(\Theta)]\}^2 + \{\mathrm{E}[\sin(\Theta)]\}^2 .$$

This is also a point at which

$$V(\nu) = 1 - \mathrm{E}[\cos(\Theta - \nu)]$$

attains its minimum.

Where $\sum \cos(\theta_i) \neq 0$, the Sample Mean direction $\bar{\theta}$ is defined as any solution of two equations:

$$\cos(\bar{\theta}) = \sum \cos(\theta_i)/R , \qquad \sin(\bar{\theta}) = \sum \sin(\theta_i)/R ,$$

where $R^2 = \{\sum \cos(\theta_i)\}^2 + \{\sum \sin(\theta_i)\}^2$. The Sample Mean direction $\bar{\theta}$ has the following properties:

(1) $\sum \sin(\theta_i - \bar{\theta}) = 0$.

(2) Let

$$D(\alpha) = 1 - \sum \cos(\theta_i - \alpha)/n ,$$

then $D(\alpha)$ attains its minimum at $\alpha = \bar{\theta}$ and the minimum is given as follows,

$$S_0 = 1 - \sum \cos(\theta_i - \bar{\theta})/n = 1 - R/n .$$

(3) $\bar{\theta}$ is the direction of the barycenter of $(\cos\theta_i, \sin\theta_i)$ in $\mathscr{R}^2$.

3. Definitions of the Mean point

Now we consider a general measure of location on $\mathscr{S}^1$, based on the intrinsic approach. Let $\rho(\theta, v): \mathscr{S}^1 \times \mathscr{S}^1 \to \mathscr{R}$ be any metric function on $\mathscr{S}^1$. Consider a Dispersion function, $V: \mathscr{S}^1 \to \mathscr{R}_{\geq 0}$, which is defined as

$$V(v) = \mathrm{E}[\rho(\Theta, v)^2] .$$

Then it is natural to define the Population Mean point η_0 as the point of satisfying

$$V(\eta_0) = \min_{v \in \mathscr{S}^1} V(v) .$$

Here we note that these include the Median and Mean directions. Because if

$$\rho(\theta, v) = \sqrt{\pi - |\pi - |r(\theta) - r(v)||}$$

or

$$\rho(\theta, v) = \sqrt{1 - \cos(\theta - v)} ,$$

then the Mean point is equal to the Median direction or the Mean direction, respectively.

By choosing an appropriate metric function ρ, we shall see that such a η_0 has some nice properties. In studying these properties, the following notions are fundamental. For each $s \in \mathscr{R}$ we define a density function $f^{(s)}$ on $\mathscr{R}$, as follows:

$$f^{(s)}(t) = \begin{cases} (f \circ q)(t) & (s - \pi < t \leq s + \pi) \\ 0 & (\text{otherwise}) . \end{cases}$$

$f^{(s)}$ is the distribution function on the tangent line at $s \in \mathscr{R}$ and $f^{(s)} = \tilde{f}|_{(s-\pi, s+\pi]}$. Further, we use a periodic function defined from the dispersion function V,

$$\tilde{V} = V \circ q: \mathscr{R} \to \mathscr{R}_{\geq 0} .$$

Suppose that a random variable $\Theta: \Omega \to \mathscr{S}^1$ has pdf f. Corresponding to Θ and any real number s, we consider a random variable $X_\Theta^{(s)}: \Omega \to \mathscr{R}$ whose pdf is $f^{(s)}$, where $X_\Theta^{(s)}(\omega) = X_{\Theta(\omega)}^{(s)}$. Then we can apply the theory of distributions on $\mathscr{R}$ to $f^{(s)}$ and the theory of the Fourier transformation on the corresponding periodic function.

Similarly we can define the Sample Mean point θ_0. Let $\theta_1, \theta_2, \ldots, \theta_n \in \mathscr{S}^1$ be a sample of Θ. Consider a function $W = W_{\{\theta_i\}}: \mathscr{S}^1 \to \mathscr{R}_{\geq 0}$ as

$$W(v) = \sum \rho(\theta_i, v)^2 ,$$

and the corresponding periodic function $\tilde{W} = W \circ q: \mathscr{R} \to \mathscr{R}_{\geq 0}$. Then we define the Sample Mean point as the point at which W attains its minimum, i.e.,

$$S(\{\theta_i\}) = \min_{v \in \mathscr{S}^1} W(v) = W(\theta_0).$$

4. Properties of the Mean point when ρ is the geodesic

In this section we consider the case that ρ is the geodesic, so

$$\rho(\theta, v) = \text{the length of minimum arc between } \theta \text{ and } v.$$

Then we have

$$\rho(\theta, v) = \pi - |\pi - |r(\theta) - r(v)||.$$

First we consider the population case.

4.1. Population space

We assume that the pdf of Θ is continuous and that its density is always positive. Let

$$\sigma^2(f) = \min_{v \in \mathscr{S}^1} V(v).$$

Then it is easily seen that

$$\tilde{V}(t) = \int_{t-\pi}^{t+\pi} (t - x)^2 \tilde{f}(x) dx.$$

Let

$$V_s(t) = \int_{\mathscr{R}} (t - x)^2 f^{(s)}(x) dx = \int_{s-\pi}^{s+\pi} \tilde{f}(x) dx.$$

We can regard $V_s(t)$ as a function

$$V_s: \mathscr{R} \to \mathscr{R}_{\geq 0}, \qquad s \in \mathscr{R}.$$

By using fundamental properties of the ordinal mean in $\mathscr{R}$, it is shown that

$$\sigma^2(f^{(s)}) = \int_{s-\pi}^{s+\pi} x^2 \tilde{f}(x) dx - (\bar{\mu}(s))^2$$

$$= \min_{t \in \mathscr{R}} V_s(t),$$

where

$$\bar{\mu}(s) = \int_{s-\pi}^{s+\pi} x\tilde{f}(x)dx\,.$$

LEMMA 4.1.1. *The followings hold for any s.*

(i) $\bar{\mu}(s + 2\pi) = \bar{\mu}(s) + 2\pi$,
(ii) $\sigma^2(f^{(s+2\pi)}) = \sigma^2(f^{(s)})$.

PROOF. (i) It is easy seen that

$$\bar{\mu}(s + 2\pi) = \int_{s+\pi}^{s+3\pi} x\tilde{f}(x)dx = \int_{s-\pi}^{s+\pi} (x + 2\pi)\tilde{f}(x)dx\,.$$

(ii) Similarly we have

$$\begin{aligned}\sigma^2(f^{(s+2\pi)}) &= \int_{s+\pi}^{s+3\pi} x^2\tilde{f}(x)dx - (\bar{\mu}(s + 2\pi))^2 \\ &= \int_{s-\pi}^{s+\pi} x^2\tilde{f}(x)dx + 4\pi\bar{\mu}(s) + (2\pi)^2 - ((\bar{\mu}(s) + 2\pi)^2\,.\end{aligned}$$

From Lemma 4.1.1, two function $\mu: \mathscr{S}^1 \to \mathscr{S}^1$ and $v \mapsto \sigma_v^2(f) = \sigma^2(f^{(s)})$ ($v = q(s)$) are induced, where $\mu(q(s)) = q(\bar{\mu}(s))$. That is, the following diagrams are commutative.

$$\begin{array}{ccc} \mathscr{R} & \xrightarrow{\bar{\mu}} & \mathscr{R} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathscr{S}^1 & \xrightarrow{\mu} & \mathscr{S}^1 \end{array} \qquad \begin{array}{ccc} s & \longmapsto & \sigma^2(f^{(s)}) \\ \downarrow & & \| \\ v & \longmapsto & \sigma_v^2(f) \end{array}$$

Note that these two functions are continuously differentiable.

THEOREM 4.1.2. *The followings hold.*
(i) $\tilde{V}(t) = \min_{s \in \mathscr{R}} V_s(t)$.
(ii) $\sigma^2(f) = \min_{v \in \mathscr{S}^1} \sigma_v^2(f)$.
(iii) If $q(s)$ is a mean point of f, then $\bar{\mu}(s) = s$ and $\sigma^2(f) = \sigma^2(f^{(s)})$.

PROOF. (i) We fix $t \in \mathscr{R}$. Since f is continuous, we have

$$\begin{aligned}\frac{\partial}{\partial s} V_s(t) &= \frac{\partial}{\partial s}\left\{\int_0^{s+\pi} (t - x)^2\tilde{f}(x)dx - \int_0^{s-\pi} (t - x)^2\tilde{f}(x)dx\right\} \\ &= (t - s - \pi)^2\tilde{f}(s + \pi) - (t - s + \pi)^2\tilde{f}(s - \pi) \\ &= 4\pi(s - t)\tilde{f}(s + \pi)\,.\end{aligned}$$

Note that $f > 0$, the sign of $\partial V_s(t)/\partial s$ is coincident with the sign of $s - t$. Hence the minimum of $V_s(t)$ with respect to s is attained at $s = t$, and the minimum is given by

$$V_t(t) = \int_{t-\pi}^{t+\pi} (t - x)^2 \tilde{f}(x)dx = \tilde{V}(t)\,.$$

(ii) The second result follows from that

$$\begin{aligned}\sigma^2(f) &= \min_{v \in \mathscr{S}^1} V(v) = \min_{t \in \mathscr{R}} \tilde{V}(t)\\ &= \min_{t \in \mathscr{R}} \min_{s \in \mathscr{R}} V_s(t) = \min_{s \in \mathscr{R}} \min_{t \in \mathscr{R}} V_s(t)\\ &= \min_{s \in \mathscr{R}} \sigma^2(f^{(s)}) = \min_{v \in \mathscr{S}^1} \sigma_v^2(f)\,.\end{aligned}$$

(iii) Note that

$$\sigma^2(f^{(s)}) = \int_{s-\pi}^{s+\pi} x^2 \tilde{f}(x)dx - \left(\int_{s-\pi}^{s+\pi} x\tilde{f}(x)dx\right)^2,$$

it is shown that

$$\begin{aligned}\frac{d}{ds}\sigma^2(f^{(s)}) &= (s + \pi)^2 \tilde{f}(s + \pi) - (s - \pi)^2 \tilde{f}(s - \pi)\\ &\quad - 2\left(\int_{s-\pi}^{s+\pi} x\tilde{f}(x)dx\right)\{(s + \pi)\tilde{f}(s + \pi) - (s - \pi)\tilde{f}(s - \pi)\}\\ &= 4\pi\left(s - \int_{s-\pi}^{s+\pi} x\tilde{f}(x)dx\right)\tilde{f}(s + \pi)\,.\end{aligned}$$

So $d\sigma^2(f^{(s)})/ds = 0$ if and only if $s = \bar{\mu}(s)$, since $\tilde{f} > 0$. Therefore,

$$\sigma^2(f) = \min_{s \in \mathscr{R}} \sigma^2(f^{(s)})\,, \qquad \text{and}$$

$$V(q(s)) = \tilde{V}(s) = \left(s - \int_{s-\pi}^{s+\pi} x\tilde{f}(x)dx\right)^2 + \sigma^2(f^{(s)})\,.$$

This proves the third result.

4.2. Sample space

In this case we can give a more precise representation of $\tilde{W}$,

$$\tilde{W}(t) = \sum (t - x_{\theta,i}^{(t)})^2\,.$$

For each $s \in \mathscr{R}$, a function $W_s(t)\colon \mathscr{R} \to \mathscr{R}_{\geq 0}$ is defined as follows:

$$W_s(t) = \sum (t - x_i^{(s)})^2 .$$

Then we obtain that $W_s(t)$ attains its minimum at $t = \sum x_i^{(s)}$, and that the minimum is given as

$$S(s; \{\theta_i\}) = \min_t W_s(t) = \sum (x_{\theta,i}^{(s)})^2 - (\sum x_{\theta,i}^{(s)})^2/n .$$

LEMMA 4.2.1. *The followings hold for any s.*
(i) $x_\theta^{(s+2\pi)} = x_\theta^{(s)} + 2\pi$,
(ii) $\sum x_{\theta,i}^{(s+2\pi)}/n = \sum x_{\theta,i}^{(s)}/n + 2\pi$,
(iii) $S(s + 2\pi; \{\theta_i\}) = S(s; \{\theta_i\})$.

From Lemma 4.2.1 the function $\varphi\colon \mathscr{S}^1 \to \mathscr{S}^1$ with $\varphi(q(s)) = q(\sum x_{\theta,i}^{(s)}/n)$ is induced and the correspondence $v \mapsto S_v(\{\theta_i\}) = S(s; \{\theta_i\})$ $(v = q(s))$ can be defined. That is, the following diagrams are commutative.

$$\begin{array}{ccc} s & \longmapsto & \frac{1}{n}\sum_i x_{\theta,i}^s \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ v & \overset{\varphi}{\longmapsto} & q\left(\frac{1}{n}\sum_i x_{\theta,i}^s\right) \end{array} \qquad \begin{array}{ccc} s & \longmapsto & S(s; \{\theta_i\}) \\ \downarrow & & \| \\ v & \longmapsto & S_v(\{\theta_i\}) \end{array}$$

THEOREM 4.2.2. *The followings hold.*
(i) $\tilde{W}(t) = \min_{s \in \mathscr{R}} W_s(t)$.
(ii) $S(\{\theta_i\}) = \min_{v \in \mathscr{S}^1} S_v(\{\theta_i\})$.
(iii) If $q(s)$ is the mean point of $\{\theta_i\}$ then $\varphi(v) = v$ i.e., $\sum x_{\theta,i}^{(s)}/n = s$.

PROOF. (i) It is easily seen that

$$\begin{aligned} W_s(t) - W_t(t) &= \sum_i (t - x_{\theta,i}^{(s)})^2 - \sum_i (t - x_{\theta,i}^{(t)})^2 \\ &= \sum_i (2t - x_{\theta,i}^{(s)} - x_{\theta,i}^{(t)})(x_{\theta,i}^{(s)} - x_{\theta,i}^{(s)}) . \end{aligned}$$

Since $t - \pi < x_{\theta,i}^{(t)} \leq t + \pi$ and $x_{\theta,i}^{(t)} - x_{\theta,i}^{(s)} = 2k\pi$ $(k \in \mathscr{Z})$, we have that

$$\text{if } x_{\theta,i}^{(t)} > x_{\theta,i}^{(s)} \text{ then } x_{\theta,i}^{(s)} \leq t - \pi \quad \text{so} \quad x_{\theta,i}^{(s)} + x_{\theta,i}^{(t)} \leq (t - \pi) + (t + \pi) = 2t ,$$

$$\text{if } x_{\theta,i}^{(t)} < x_{\theta,i}^{(s)} \text{ then } x_{\theta,i}^{(s)} > t - \pi \quad \text{so} \quad x_{\theta,i}^{(s)} + x_{\theta,i}^{(t)} > (t + \pi) + (t - \pi) = 2t .$$

Therefore,

$$W_s(t) \geq W_t(t) = \tilde{W}(t) .$$

(ii) The second result follows from

$$\begin{aligned} S(\{\theta_i\}) &= \min_{t\in\mathscr{R}} \tilde{W}(t) = \min_{t\in\mathscr{R}} \min_{s\in\mathscr{R}} W_s(t) \\ &= \min_{s\in\mathscr{R}} \min_{t\in\mathscr{R}} W_s(t) = \min_{s\in\mathscr{R}} S(s; \{\theta_i\}) \\ &= \min_{v\in\mathscr{S}^1} S_v(\{\theta_i\}) . \end{aligned}$$

(iii) Note that

$$\begin{aligned} \tilde{W}(t) &= n\left(t - \frac{1}{n}\sum_i x_{\theta,i}^{(t)}\right)^2 + S(t; \{\theta_i\}) \\ &\geq n\left(t - \frac{1}{n}\sum_i x_{\theta,i}^{(t)}\right)^2 + S(\{\theta_i\}) . \end{aligned}$$

Hence, $\tilde{W}(t)$ attains its minimum at $t = \sum_i x_{\theta,i}^{(t)}/n$. This proves the third result.

Now we consider conditions that the mean point of $\{\theta_i\}$ is equal to $q(t)$ for some $t \in \mathscr{R}$.

PROPOSITION 4.2.3. *Let $u_i = x_{\theta,i}^{(0)}$, and assume*

$$-\pi < u_1 \leq u_2 \leq \cdots \leq u_k \leq 0 \leq u_{k+1} \leq \cdots \leq u_n \leq \pi .$$

The mean point is $q(0)$ if and only if the following three conditions are satisfied.

(i) $\sum u_i = 0$,

(ii) $\displaystyle\sum_{i=j+1}^{n} u_i \leq \frac{(n-j)j}{n}\pi \qquad (k \leq \forall j \leq n-1)$,

(iii) $\displaystyle\sum_{i=1}^{j} u_i \geq -\frac{(n-j)j}{n}\pi \qquad (1 \leq \forall j \leq k)$.

PROOF. From Theorem 4.2.2, $q(0)$ is the mean point of $\{\theta_i\}$, if and only if

$$\frac{1}{n}\sum_{i=1}^{n} u_i = 0 , \qquad \text{and} \qquad S(0, \{\theta_i\}) \leq S(s, \{\theta_i\}) \qquad (\forall s \in \mathscr{R}) .$$

Hence,

$$\sum_{i=1}^{n} u_i^2 \leq \sum_{i=1}^{j} u_i^2 + \sum_{i=j+1}^{n} (u_i - 2\pi)^2 - \frac{1}{n}\left(\sum_{i=1}^{n} u_i - (n-j)2\pi\right)^2 \qquad (k \leq j \leq n-1) ,$$

$$\sum_{i=1}^{n} u_i^2 \leq \sum_{i=1}^{j} (u_i + 2\pi)^2 + \sum_{i=j+1}^{n} u_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} u_i + j\cdot 2\pi\right)^2 \qquad (1 \leq j \leq k) .$$

This proves the proposition.

From Proposition 4.2.3, we obtain the necessary and sufficient condition for $q(t)$ to be the mean point. Let

$$D_n = \left\{ (u_1, \ldots, u_{n-1}) \in \mathscr{R}^{n-1} \;\middle|\; \begin{array}{l} -\dfrac{(n-j)j}{n}\pi \leq u_{i(1)} + \cdots + u_{i(j)} \leq \dfrac{(n-j)j}{n}\pi \\ \forall\{i(1), \ldots, i(j)\} \subset \{1, 2, \ldots, n-1\} \\ 1 \leq \forall j \leq n-1 \end{array} \right\},$$

and for $t \in \mathscr{R}$

$$\begin{aligned} D_n(t) &= (t, \ldots, t) + D_n \\ &= \{(u_1 + t, \ldots, u_{n-1} + t) \in \mathscr{R}^{n-1} | (u_1, \ldots, u_{n-1}) \in D_n\}. \end{aligned}$$

Then we have the following theorem.

THEOREM 4.2.4. *For $t \in \mathscr{R}$ and $\{\theta_i\}$, $q(t)$ is the mean point of $\{\theta_i\}$, if and only if*

$$\frac{1}{n}\sum_{i=1}^{n} x_{\theta,i}^{(t)} = t \qquad \textit{and} \qquad (x_{\theta,1}^{(t)}, \ldots, x_{\theta,n-1}^{(t)}) \in D_n(t).$$

It is left as a future problem to study inferential problems based on our sample mean points. For a practical use, we also need to solve a numerical problem. On the other hand, in general, it is important to extend the central limit theorem in $\mathscr{R}$ to the one on $\mathscr{S}^1$. We can expect that our approach will be also useful in this problem. In the following paper, Kakimizu and Watamori [1994], we derive the laws of large numbers on this mean point. Further, our approach will be useful in introducing the mean point on a general Riemannian manifold with an appropriate metric function. We can regard this mean point as a kind of extention of the "mean" in the usual notion.

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