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Solvability of degenerate elliptic problems of higher order via Leray–Lions theorem

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ABSTRACT. We present existence result for nonlinear degenerate elliptic boundary value problems of higher order. The weak solution is seeked in a suitable weighted Sobolev space using Leray–Lions theorem.

1. Introduction

We study a general existence theorem for *degenerate* elliptic boundary value problems for equations of higher order of the form

$$(1.1)\qquad \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,u,\dots,D^m u)=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha f_\alpha(x)\qquad \text{in } \Omega$$

on a closed subspace V satisfying

$$W_0^{m,p}(v,\Omega)\subseteq V\subseteq W^{m,p}(v,\Omega)\,,$$

where $W^{m,p}(v,\Omega)$ is a certain *weighted Sobolev space*. The degeneracy is determined by a vector function $v(x)=(v_\alpha(x))$, $|\alpha|\le m$, with positive components $v_\alpha(x)$ in Ω satisfying certain integrability assumptions.

When we deal with $V\ne W_0^{m,p}(v,\Omega)$, we always assume that Ω satisfies the *cone property* (see e.g. Adams [1]). In fact the subspace V is determined by the *homogeneous boundary conditions* appearing in the boundary value problem for the equation (1.1). The case of $V=W_0^{m,p}(v,\Omega)$ corresponds to the Dirichlet problem (where formally $D^\beta u=0$ on $\partial\Omega$ for $|\beta|\le m-1$) and $V=W^{m,p}(v,\Omega)$ corresponds to the Neumann problem (where formally $D^\beta u=0$ on $\partial\Omega$ for $m\le|\beta|\le 2m-1$). However, we can also deal with *nonhomogeneous boundary value problems* considering the equation

$$\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,u+u_0,\dots,D^\alpha(u+u_0))=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha f_\alpha(x)\qquad \text{in } \Omega$$

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on V, where the traces of the function u_0 and its derivatives on $\partial\Omega$ correspond to nonhomogeneous boundary conditions (see e.g. Fučík, Kufner [4]).

In Section 2 we define weighted Sobolev spaces and formulate some useful imbeddings of the weighted Sobolev spaces into "classical" Sobolev spaces. In Section 3 we formulate some growth assumptions on "coefficients" $A_\alpha(x, \xi)$. The main result is presented in Section 4. The proof of the main result which consists in the verification of the *Leray–Lions theorem* is given in Section 5. In Section 6 we present some concrete applications in order to illustrate our (somewhat complicated) assumptions.

This paper may be regarded as a continuation of preceding papers of Guglielmino, Nicolosi [6] and Drábek, Nicolosi [3] concerning second order degenerate quasilinear elliptic equations. Since for higher order equations we cannot apply the truncation method due to De Giorgi [5] (see also Stampacchia [10]), we are not able to deal with either the case of unbounded domain Ω or prove the boundedness in $L^\infty(\Omega)$ of the weak solution to (1.1) as in [3], [6]. The solvability of degenerate equations of the form (1.1) via the degree theory is studied in Drábek, Kufner, Nicolosi [2], where the degeneracy is included in the terms $A_\alpha(x, \xi)$ of order $|\alpha| = m$. In our paper the degeneration may be included in all terms $A_\alpha(x, \xi)$ for any $|\alpha| \leq m$ in a way similar to Murthy, Stampacchia [9], where the case $m = 1$ is considered. It should also be mentioned that in our paper more general assumptions than those in [2] and [9] are made concerning the degeneracy (see (2.5) and Example 6.1 below).

2. Imbedding Theorems

Let $\boldsymbol{R}^n$ $(n \geq 1)$ be the n-dimensional Euclidean space with elements $x = (x_1, \dots, x_n)$ and Ω be an open nonempty bounded set in $\boldsymbol{R}^n$ with the boundary $\partial\Omega$. Denote by $M(j)$ the number of distinct multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ such that the components α_i are nonnegative and the length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ does not exceed j. For a given differentiable function u defined in Ω we denote

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} , \qquad D^k u(x) = \{D^\alpha u(x); |\alpha| = k\} .$$

Let us denote by $W(\Omega)$ the set of all weight functions $v(x)$, i.e. $v(x)$ is measurable and positive a.e. in Ω. For $v_0(x) \in W(\Omega)$ and $1 \leq p < +\infty$ we define the *weighted Lebesque space* $L^p(v_0, \Omega)$ as the space of all real-valued functions $u(x)$ for which

$$\|u\|_{p, v_0} = \left(\int_\Omega v_0(x) |u(x)|^p dx \right)^{1/p} < +\infty .$$

Let $p > 1$ be a real number, $m \geq 1$ an integer and let $v(x) = (v_\alpha(x))$, $|\alpha| \leq m$, be a vector function with components $v_\alpha(x) \in W(\Omega)$. Further, we suppose that every component $v_\alpha(x)$ satisfies

$$v_\alpha(x) \in L^1_{\text{loc}}(\Omega)\,, \tag{2.1}$$

$$\frac{1}{v_\alpha(x)} \in L^{1/(p-1)}_{\text{loc}}(\Omega) \tag{2.2}$$

for any $|\alpha| \leq m$.

Now we denote by $W^{m,p}(v, \Omega)$ the function space of all real-valued functions u such that the derivatives in the sense of distributions satisfy:

$$(v_\alpha)^{1/p} D^\alpha u \in L^p(\Omega) \qquad \text{for all } |\alpha| \leq m\,.$$

Condition (2.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{m,p}(v, \Omega)$ and, consequently, we can introduce the subspace $W_0^{m,p}(v, \Omega)$ of $W^{m,p}(v, \Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{m,p,v} = \left(\sum_{|\alpha| \leq m} \int_\Omega v_\alpha(x) |D^\alpha u(x)|^p dx \right)^{1/p}\,.$$

Moreover, condition (2.2) implies that $W^{m,p}(v, \Omega)$ as well as $W_0^{m,p}(v, \Omega)$ are *reflexive Banach spaces.*

LEMMA 2.1. *Suppose that*

$$\frac{1}{v_\alpha(x)} \in L^{g^*}(\Omega) \tag{2.3}$$

for any $|\alpha| \leq m$ with some $g^* > 0$. Then the space $W^{m,p}(v, \Omega)$ is continuously imbedded into the "classical" Sobolev space $W^{m,p_1}(v, \Omega)$, where

$$p_1 = \frac{pg^*}{g^* + 1}\,. \tag{2.4}$$

The proof follows from (2.3), (2.4) and Hölder inequality (see Leonardi [7]).

In the subsequent discussions we will assume that

$$g^* > \frac{n}{p + n(p-1)}\,, \qquad n \geq 2 \tag{2.5}$$

and write

$$\kappa_1 = m - \frac{n}{p_1} = \frac{mpg^* - n(g^* + 1)}{pg^*}\,. \tag{2.6}$$

LEMMA 2.2. *Let $k < m$ denote a nonnegative integer. Then the following continuous imbeddings are valid:*

(i) *for* $k < \kappa_1$,

$$W^{m,p}(v, \Omega) \hookrightarrow C^k(\overline{\Omega})\,;$$

(ii) *for* $k = \kappa_1$

$$W^{m,p}(v, \Omega) \hookrightarrow W^{k,r}(\Omega)$$

with arbitrary r, $1 < r < +\infty$;

(iii) *for* $k > \kappa_1$,

$$W^{m,p}(v, \Omega) \hookrightarrow W^{k,r_k}(\Omega)\,,$$

where r_k *satisfies*

$$1 \le r_k \le q_k = \frac{p_1 n}{n - p_1(m-k)} = \frac{pg^* n}{n(g^*+1) - pg^*(m-k)}\,.$$

Moreover, the imbeddings (i) *and* (ii) *are compact,* i.e. $W^{m,p}(v, \Omega) \hookrightarrow \hookrightarrow C^k(\overline{\Omega})$ *for* $k < \kappa_1$ *and* $W^{m,p}(v, \Omega) \hookrightarrow \hookrightarrow W^{m,r}(\Omega)$ *for* $k = \kappa_1$, *and the imbedding* (iii) *is compact,* i.e. $W^{m,p}(v, \Omega) \hookrightarrow \hookrightarrow W^{k,r_k}(\Omega)$, *if* $r_k < q_k$.

The proof of this lemma follows from the preceding Lemma 2.1 and Sobolev imbedding theorems (see e.g. Adams [1], Leonardi [7]).

REMARK 2.1. (i). The number κ_1 given by (2.6) may be negative (or nonpositive) for certain values g^*, m, n, p. In this case we have no imbedding theorems of the type (i) and (ii) (or (i)), respectively.

(ii) It follows from our assumption (2.5) that $q_k > 1$ for any nonnegative integer $k < m$. Hence $W^{k,r_k}(\Omega)$ in Lemma 2.2 (iii) make sense as Sobolev spaces.

3. Growth conditions

Assume that $A_\alpha(x, \xi)$, $|\alpha| \le m$, are *Carathéodory* functions, i.e. $A_\alpha(\cdot, \xi)$ is measurable in Ω for all values of $\xi \in \boldsymbol{R}^{M(m)}$ and $A_\alpha(x, \cdot)$ is continuous in $\boldsymbol{R}^{M(m)}$ for a.e. $x \in \Omega$ and for all $|\alpha| \le m$. Let $g_1 \colon \boldsymbol{R} \to \boldsymbol{R}$ be a positive, continuous and nondecreasing function. Denote by p' the exponent conjugate to p, i.e. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

We then suppose that $A_\alpha(x, \xi)$, $|\alpha| \le m$, satisfy the following *growth conditions.*

(i) Let $|\alpha| = m$. Then we assume

$$A_\alpha(x, \xi) \le g_1(|\xi_0|) v_\alpha^{1/p} \left[a_\alpha(x) + \sum_{\kappa_1 \le |\beta| < m} |\xi_\beta|^{p_\beta} + \sum_{|\beta| = m} v_\beta^{1/p'} |\xi_\beta|^{p-1} \right],$$

where $\xi_0 = \{\xi_\beta : |\beta| < \kappa_1\}$, $a_\alpha(x) \in L^{p'}(\Omega)$,

$$1 < p_\beta = \frac{r_{|\beta|}}{p'} < \frac{(p-1)g^*n}{n(g^*+1) - pg^*(m-|\beta|)}$$

if $\kappa_1 < |\beta| < m$ and

$$1 < p_\beta < +\infty \text{ is arbitrary if } |\beta| = \kappa_1 .$$

(ii) Let $\kappa_1 < |\alpha| < m$. Then we assume

$$|A_\alpha(x, \xi)| \le g_1(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_1 \le |\alpha| < m} |\xi_\beta|^{p_{\alpha\beta}} + \sum_{|\beta|=m} v_\beta^{t_\alpha} |\xi_\beta|^{pt_\alpha}\right],$$

where $a_\alpha(x) \in L^{1/t_\alpha}(\Omega)$,

$$t_\alpha = \frac{1}{r'_{|\alpha|}} < \frac{pg^*n - n(g^*+1) + pg^*(m-|\alpha|)}{pg^*n},$$

$$1 < p_{\alpha\beta} = \frac{r_{|\beta|}}{r'_{|\alpha|}} < \frac{pg^*n - n(g^*+1) + pg^*(m-|\alpha|)}{n(g^*+1) - pg^*(m-|\beta|)}$$

if $\kappa_1 < |\beta| < m$ and

$$1 < p_{\alpha\beta} < +\infty \text{ is arbitrary if } |\beta| = \kappa_1 .$$

(iii) Let $|\alpha| = \kappa_1$. Then we assume

$$|A_\alpha(x, \xi)| \le g_1(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_1 \le |\alpha| < m} |\xi_\beta|^{r_\beta} + \sum_{|\beta|=m} v_\beta^{\sigma} |\xi_\beta|^{\tilde{p}}\right],$$

where $a_\alpha(x) \in L^{1/\sigma}(\Omega)$, $0 < \sigma < 1$, $1 < \tilde{p} < p$,

$$1 < r_\beta < q_{|\beta|} = \frac{pg^*n}{n(g^*+1) - pg^*(m-|\beta|)}$$

if $\kappa_1 < |\beta| < m$ and

$$1 < r_\beta < +\infty \text{ is arbitrary if } |\beta| = \kappa_1 .$$

(iv) Let $|\alpha| < \kappa_1$. Then we assume

$$|A_\alpha(x, \xi)| \le g_1(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_1 \le |\alpha| < m} |\xi_\beta|^{q_{|\beta|}} + \sum_{|\beta|=m} v_\beta |\xi_\beta|^{p}\right],$$

where $a_\alpha(x) \in L^1(\Omega)$.

REMARK 3.1. For $\kappa_1 \le 0$ it is obvious that $\xi_0 = \phi$. Hence set $g_1(t) \equiv 1$ in this case.

DEFINITION 3.1. We will denote by **CAR** the set of all functions $A(x, \xi): \Omega \times \boldsymbol{R}^{M(m)} \to \boldsymbol{R}$ which satisfy the Carathéodory conditions and, moreover, satisfy the growth conditions (i)–(iv).

Let us denote by G_α the *Nemytskii operator* associated with the function $A_\alpha(x, \xi)$, i.e.

$$G_\alpha: u(x) \mapsto A_\alpha(x, u(x), \dots, u_{M(m)}(x)),$$

where $u(x) = (u_1(x), \dots, u_{M(m)}(x))$ is a vector function, $|\alpha| = m$. Let

$$X = \prod_{|\beta| \le m} X_\beta,$$

where $X_\beta = C^{|\beta|}(\overline{\Omega})$ for $|\beta| < \kappa_1$: $X_\beta = L^{r_\beta}(\Omega)$ for $\kappa_1 \le |\beta| < m$; $X_\beta = L^p(v_\beta, \Omega)$ for $|\beta| = m$.

LEMMA 3.1. *Let $A_\alpha(x, \xi) \in$ **CAR** for any $|\alpha| \le m$. Then the Nemytskii operator G_α is bounded and continuous from the space X into the dual space X_α^* for any $|\alpha| \le m$.*

PROOF. Let $|\alpha| \le m$, and fix any α. Using the growth conditions (i)–(iv) and the necessary and sufficient condition for the continuity of the Nemytskii operator between weighted Lebesgue spaces (see e.g. Drábek, Kufner, Nicolosi [2], Leonardi [7]), we see that G_α is a bounded and continuous operator

(i) from X into $[L^p(v_\alpha, \Omega)]^*$ if $|\alpha| = m$,

(ii) from X into $[L^{r_\alpha}(\Omega)]^*$ if $\kappa_1 \le |\alpha| < m$,

(iii) from X into $L^1(\Omega) \subset [L^\infty(\Omega)]^* \subset X_\alpha^*$ if $|\alpha| < \kappa_1$.

Let f_α be the functions appearing on the *right hand side* of (1.1). We will assume that functions f_α, $|\alpha| \le m$, satisfy the following *conditions*

(a) $f_\alpha(x) \in [L^p(v_\alpha, \Omega)]^* = L^{p'}(v_\alpha^{-(p'/p)}, \Omega)$ if $|\alpha| = m$,

(b) $f_\alpha(x) \in [L^{r_\alpha}(\Omega)]^* = L^{r'_\alpha}(\Omega)$ if $\kappa_1 \le |\alpha| < m$,

(c) $f_\alpha(x) \in L^1(\Omega)$ if $|\alpha| < \kappa_1$.

It follows from (a)–(c) that $f_\alpha \in X_\alpha^*$, $|\alpha| \le m$. We denote $f = (f_\alpha(x))$, $|\alpha| \le m$, i.e. $f \in X^*$.

Let V be a closed subspace of $W^{m,p}(v, \Omega)$ such that

$$W_0^{m,p}(v, \Omega) \subseteq V \subseteq W^{m,p}(v, \Omega).$$

Then V with norm $\|\cdot\|_{m,p,v}$ is a *reflexive Banach space*. Let us denote by $\langle \cdot, \cdot \rangle$ the pairing between the dual space V^* and V.

REMARK 3.2. Due to Lemma 2.2 we have $f \in V^*$ and for any $\varphi \in V$

$$D^\alpha \varphi \in X_\alpha$$

for $|\alpha| \le m$.

DEFINITION 3.2. A function $u \in V$ satisfying the identity

$$\sum_{|\alpha| \le m} \int_{\Omega} [A_\alpha(x, u, \dots, D^m u) - f_\alpha(x)] D^\alpha \varphi(x) dx = 0 \tag{3.1}$$

for any $\varphi \in V$ is called a *weak solution* of the boundary value problem for the equation (1.1) on the space V.

4. Main result

Let $g_2: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous, positive and nonincreasing function, $g_3: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous, positive and nondecreasing function. Let us denote $M_0 = M(m) - M(m-1)$. We will assume that the following *ellipticity condition* is satisfied:

$$\sum_{|\alpha| \le m} A_\alpha(x, \eta, \zeta)\zeta_\alpha \ge g_2(|\eta_0|) \sum_{|\beta| = m} v_\beta(x)|\zeta_\beta|^p - g_3(|\eta_0|) \sum_{\kappa_1 \le |\beta| < m} |\eta_\beta|^{r_\beta} \tag{4.1}$$

for a.e. $x \in \Omega$, for $\zeta = \{\zeta_\alpha; |\alpha| = m\}$ and $\eta = \{\eta_\beta: |\beta| < m\}$, $\zeta \in R^{M_0}$, $\eta \in \boldsymbol{R}^{M(m-1)}$, where $\eta_0 = \{\eta_\beta; |\beta| < \kappa_1\}$ and r_β is defined in Section 3.

Moreover we assume that the following *monotonicity condition in the principal part* is fulfilled:

$$\sum_{|\alpha| = m} [A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \zeta')(\zeta_\alpha - \zeta'_\alpha) > 0 \tag{4.2}$$

for a.e. $x \in \Omega$, $\eta = \{\eta_\beta; |\beta| < m\}$ and ζ, $\zeta' \in \boldsymbol{R}^{M_0}$ such that $\zeta \ne \zeta'$.

Let $c > 0$ be a positive constant. Then we assume that the following *coercivity condition* is satisfied

$$\sum_{|\alpha| \le m} A_\alpha(x, \xi)\xi_\alpha \ge c \sum_{|\alpha| \le m} v_\alpha(x)|\xi_\alpha|^p \tag{4.3}$$

for a.e. $x \in \Omega$ and $\xi \in \boldsymbol{R}^{M(m)}$.

THEOREM 4.1. *Let* $A_\alpha \in \mathbf{CAR}$ *for any* $|\alpha| \le m$ *and assume that* (a)–(c), (4.1)–(4.3) *be fulfilled. Then the boundary value problem for the equation* (1.1) *on the space* V *has at least one weak solution.*

The proof of Theorem 4.1 consists in the verification of the assumptions of the Leray–Lions theorem (see e.g. Leray–Lions [8], Fučík, Kufner [4])

5. Proof of theorem 4.1.

We begin by defining a mapping

$$A(u, v): V \times V \to V^*$$

by

$$(5.1)\quad \langle A(u, v), w\rangle = \sum_{|\alpha|=m} \int_\Omega A_\alpha(x, u(x), \dots, D^{m-1}u(x), D^m v(x)) D^\alpha w(x) dx$$
$$+ \sum_{|\alpha|<m} \int_\Omega A_\alpha(x, u(x), \dots, D^{m-1}u(x), D^m u(x)) D^\alpha w(x) dx$$

for $u, v, w \in V$.

LEMMA 5.1. *Under the assumptions of Theorem* 4.1 *the mapping* $A(u, v)$ *defined in* (5.1) *is bounded and continuous.*

PROOF. Let $u, v \in V$. In view of Remark 3.2 we infer $D^\alpha u(x) \in X_\alpha$ for $|\alpha| \le m$ and $D^\alpha v(x) \in X_\alpha$ for $|\alpha| = m$. The assertion now follows directly from Lemma 3.1.

Let

$$(5.2)\qquad Au = A(u, u)$$

for any $u \in V$. By Lemma 5.1 the operator $A: V \to V^*$ defined by (5.2) is *bounded* and *continuous.* Applying the coercivity condition (4.3) we obtain

$$\langle Au, u\rangle = \sum_{|\alpha|\le m} \int_\Omega A_\alpha(x, u, \dots, D^m u) D^\alpha u dx$$
$$\ge c \sum_{|\alpha|\le m} \int_\Omega v_\alpha(x) |D^\alpha u(x)|^p dx = c\|u\|^p_{m,p,v}$$

for any $u \in V$. Hence A is *coercive.*

In virtue of the monotonicity condition (4.2) we have

$$\langle A(u, u) - A(u, v), u - v\rangle = \sum_{|\alpha|=m} \int_\Omega [A_\alpha(x, u, \dots, D^{m-1}u, D^m u)$$
$$- A_\alpha(x, u, \dots, D^{m-1}u, D^m v)](D^\alpha u - D^\alpha v) dx \ge 0$$

for any $u, v \in V$, i.e. $A(u, v)$ is *monotone in the principal part.*

LEMMA 5.2. *Assume that* u_n *converges to* $v \in V$ *weakly,* $v \in V$ *and that* $A(u_n, v)$ *converges weakly to* v^* *in* V^*.

Then

$$(5.3)\qquad \langle A(u_n, v), u_n\rangle \to \langle v^*, u\rangle .$$

PROOF. By the definition of $A(u, v)$ we have

$$\begin{aligned}\langle A(u_n, v) - A(u, v), u_n - u\rangle = \sum_{|\alpha|=m} \int_\Omega & [A_\alpha(x, u_n, \dots, D^{m-1}u_n, D^m v) \\ & - A_\alpha(x, u, \dots, D^{m-1}u, D^m v)](D^\alpha u_n - D^\alpha v)dx \\ & + \sum_{|\alpha|<m} \int_\Omega [A_\alpha(x, u_n, \dots, D^{m-1}u_n, D^m u_n) \\ & - A_\alpha(x, u, \dots, D^{m-1}u, D^m u)](D^\alpha u_n - D^\alpha u)dx\,.\end{aligned}$$

Since $\{u_n\} \subset V$ is bounded in V and $D^\alpha u_n$ converge strongly to $D^\alpha u$ in X_α for $|\alpha| < m$ (see Lemma 2.2), we apply Lemma 3.1 to obtain

$$\int_\Omega [A_\alpha(x, u_n, \dots, D^{m-1}u_n, D^m v) - A_\alpha(x, u, \dots, D^{m-1}u, D^m v)](D^\alpha u_n - D^\alpha u)dx \to 0$$

for $|\alpha| = m$, and

$$\int_\Omega [A_\alpha(x, u_n, \dots, D^{m-1}u_n, D^m u_n) - A_\alpha(x, u, \dots, D^{m-1}u, D^m u)](D^\alpha u_n - D^\alpha u)dx \to 0$$

for $|\alpha| < m$.

Hence

$$\langle A(u_n, v) - A(u, v), u_n - u\rangle \to 0\,,$$

and as we obtain

$$\begin{aligned}\langle A(u_n, v), u_n\rangle = {} & \langle A(u_n, v) - A(u, v), u_n - u\rangle \\ & - \langle A(u, v), u_n - u\rangle + \langle A(u_n, v), u\rangle \to \langle v^*, u\rangle\,.\end{aligned}$$

LEMMA 5.3. *Let u_n converge to u weakly in V and*

$$\langle A(u_n, u_n) - A(u_n, u), u_n - u\rangle \to 0\,. \tag{5.4}$$

Then $v_\alpha^{1/p} D^\alpha u_n$ converge to $v_\alpha^{1/p} D^\alpha u$ strongly in $L^p(\Omega)$ for any $|\alpha| = m$.

Using the ellipticity condition (4.1) and monotonicity condition in the principal part (4.2) the proof of this lemma is obtained by following the lines of the proof of Lemma 4.4 in Drábek, Kufner, Nicolosi [2].

Let u_n converge to u weakly in V and (5.4) hold. Then by Lemma 2.2 we have $D^\alpha u_n \to D^\alpha u$ in X_α for $|\alpha| < m$ and in virtue of Lemma 5.3 we have $D^\alpha u_n \to D^\alpha u$ in X_α for $|\alpha| = m$. Applying Lemma 3.1 we conclude that $A(u_n, v)$ *converge to* $A(u, v)$ strongly in V^* for any $v \in V$.

Since all the assumptions of the *Leray–Lions theorem* are verified, there exists at least one $u \in V$ such that

$$A(u, u) = f\,,$$

for any $f \in V^*$, i.e. the boundary value problem for the equation (1.1) on the space V has at least one weak solution.

6. Applications

EXAMPLE. Let us consider the plane domain $\Omega \subset \boldsymbol{R}^2$, $\Omega =]-1, 1[\times]-1, 1[$ and put (for $x = (x_1, x_2) \in \Omega$)

$$v(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 \leq 0 \\ x_2^{\lambda}(1 - x_1)^{\gamma} & \text{for } x_1 > 0,\ x_2 > 0 \\ |x_2|^{\mu}(1 - x_1)^{\gamma} & \text{for } x_1 > 0,\ x_2 < 0 \end{cases} \tag{6.1}$$

with λ, μ and γ real parameters (cf. Drábek, Kufner, Nicolosi [2]). We have $v(x) \in W(\Omega)$. Consider the differential equation

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha}(v(x)|D^{\alpha}u(x)|^{p-2}D^{\alpha}u(x)) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} f_{\alpha}(x) \qquad \text{in } \Omega . \tag{6.2}$$

In our case, it is

$$A_{\alpha}(x, \xi) = v(x)|\xi_{\alpha}|^{p-2}\xi_{\alpha} \qquad \text{for } |\alpha| \leq m .$$

Obviously, we have $A_{\alpha}(x, \xi) \in \mathbf{CAR}$ for any $|\alpha| \leq m$ and the ellipticity, monotonicity and coercivity conditions are satisfied. The appropriate weighted Sobolev space is

$$W^{m,p}(v, \Omega) = \left\{ u = u(x);\ \int_{\Omega} |D^{\alpha}u(x)|^p v(x) < +\infty,\ |\alpha| \leq m \right\} .$$

The *degeneracy* (or *singularity*) in (6.2) is determined by $v(x) = v(x_1, x_2)$ which is given by (6.1) and is included in all terms $A_{\alpha}(x, \xi)$, $|\alpha| \leq m$. Moreover the degeneracy is on a part of

$$\Gamma_1 = \{x = (x_1, x_2);\ x_1 = 1,\ x_2 \in]-1, 1[\} ,$$

as well as on a segment Γ_2 in Ω:

$$\Gamma_2 = \{x = (x_1, x_2);\ x_1 \in]0, 1[,\ x_2 = 0\} .$$

Conditions (2.1) and (2.2) indicate that we have to choose exponents λ and μ from the interval $]-1, p - 1[$ (while no conditions on γ are imposed). Condition (2.5) indicates that λ, μ and γ have to be less than $\frac{3p-2}{2}$. Thus, we have the following conditions on λ, μ and γ:

$$\lambda,\ \mu \in]-1, p - 1[, \qquad \gamma \in \left]-\infty, \frac{3p-2}{2}\right[. \tag{6.3}$$

If we compare the conditions in (6.3) with an analogous conditions in Example 5.7 in [2], we put the same limitation for singularity but more general restriction for degeneracy. Note that γ can be taken as $\gamma = 1/2$ for any $p > 1$.

We now check the number κ_1 in the formula (2.6). In view of (2.5) we can take

$$g^* = \frac{2}{3p-2} + \varepsilon,$$

with some $\varepsilon > 0$. We infer from (2.6) that

$$\kappa_1 = m - \frac{2}{p}\frac{3p + \varepsilon(3p-2)}{2 + \varepsilon(3p-2)},$$

and hence

$$\kappa_1 \in \left]m - 3, m - \frac{2}{p}\right[$$

(depending on the value of $\varepsilon > 0$).

EXAMPLE 6.2. Let $\Omega \subset \boldsymbol{R}^2$ and $v(x) = v(x_1, x_2)$ be as in Example 6.1. Assume that $p > 2$. We now suppose that the exponents λ, μ and γ satisfy

$$\text{(6.4)} \qquad \lambda,\ \mu \in \left]-1, \frac{p-2}{2}\right[, \qquad \gamma \in \left]-\infty, \frac{p-2}{2}\right[.$$

Then the weight function $v(x)$ satisfies (2.1), (2.2) and (2.3) with $g^* = \dfrac{2+\delta}{p-2}$ for some $\delta > 0$. It follows from (2.6) that $\kappa_1 > 0$.

Consider the differential equation

$$\text{(6.5)} \qquad -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(v(x_1, x_2)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + e^{u^2} = f \qquad \text{in } \Omega,$$

where $v(x_1, x_2)$ is defined for λ, μ and γ satisfying (6.4) by (6.1). In this case

$$A_{(1,0)}(x, \xi) = v(x)|\xi_{(1,0)}|^{p-2}\xi_{(1,0)},$$

$$A_{(0,1)}(x, \xi) = v(x)|\xi_{(0,1)}|^{p-2}\xi_{(0,1)},$$

$$A_{(0,0)}(x, \xi) = e^{\xi^2_{(0,0)}},$$

and the associated weighted Sobolev space is

$$W^{m,p}(v, \Omega) = \left\{u = u(x); \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p v(x_1, x_2)dx_1 dx_2 < +\infty,\right.$$

$$\left. i = 1, 2, \int_\Omega |u|^p dx_1 dx_2 < +\infty\right\}$$

(namely, we take $v_{(1,0)}(x) = v_{(0,1)}(x) = v(x)$, $v_{(0,0)}(x) \equiv 1$).

Obviously, all of the assumptions are fulfilled in this case. Hence the boundary value problem for the equation (6.5) on V has at least one weak solution for an arbitrary forcing term $f \in L^1(\Omega)$ on the right-hand side.

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