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On some new sequence spaces

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ABSTRACT. In this paper we introduce and study some new sequence spaces.

1. Introduction

Let ℓ_∞ denote the Banach space of all real or complex bounded sequences $x = (x_k)$ normed as usual by $\|x\| = \sup_k |x_k|$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional Φ on ℓ_∞ is said to be an invariant mean or σ-limit if and only if

i) $\Phi(x) \geq 0$ whenever $x_n \geq 0$ for all n,

ii) $\Phi(e) = 1$, where $e = (1, 1, \dots)$

iii) $\Phi(x_{\sigma(n)}) = \Phi(x)$ for all $x \in \ell_\infty$.

Let V_σ denote the space of bounded sequences all of whose σ-means are equal, if $x = (x_k)$, we write $Tx = (x_{\sigma(n)})$. It can be shown [6] that

$$V_\sigma = \{x: \lim_m t_{mn}(x) = L \text{ exists uniformly in } n,\ L = \sigma\text{-}\lim x\}\,,$$

where

$$t_{mn}(x) = (x_n + Tx_n + \cdots + T^m x_n)/(m + 1) \qquad \text{and} \qquad t_{-1,n}(x) = 0\,. \tag{A}$$

In the case that σ is the translation mapping $n \to n + 1$, the σ-mean is often called a Banach limit and V_σ is the set of almost convergent sequences [1].

In accordance with Mursaleen [4], $x = (x_n) \in \ell_\infty$ is said to be strongly σ-convergent to a number L if

$$1/m \sum_{i=1}^{m} |x_{\sigma^i(n)} - L| = 0 \text{ as } m \to \infty \qquad \text{uniformly in } n\,.$$

Recently strongly σ-convergent sequences have been discussed and this concept of strong σ-convergence has been generalized by Savaş [5] in the following way:

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$$[V_\sigma](p_i) = \left\{x: \lim_m 1/m \sum_{i=1}^{m} |x_{\sigma^i(n)} - L|^{p_i} = 0 \text{ uniformly in } n\right\}$$

where (p_i) is a bounded sequence of positive real numbers.

If p_i is constant and $p_i = p > 0$ for all i, we write $[V_\sigma](p_i) = [V_\sigma](p)$ and if $p = 1$ then this coincides with the set of all strongly σ-convergent sequences introduced by Mursaleen [4].

Referring to [2] and [3], we introduce two spaces below:

$$w(p_i) = \left\{x: 1/n \sum_{i=1}^{n} |x_i - L|^{p_i} \to 0 \text{ as } n \to \infty\right\}$$

$$\text{ces}(p_i) = \left\{x: \sum_{n=1}^{\infty} 1/n \sum_{i=1}^{n} |x_i|^{p_i} < \infty\right\}.$$

The associate spaces of Cesaro sequence spaces have been discussed by several authors.

2. Main Result

In the present note we introduce a new sequence space. This is suggested by the notion of σ-convergence. We here denote this new space by $\text{ces}^\sigma(p)$. Topological properties and inclusion relations of $\text{ces}^\sigma(p)$ to known sequence spaces are discussed.

In order to define the sequence space, we put

$$z_{mn} = z_{mn}(x) = 1/m \sum_{i=1}^{m} |x_{\sigma^i(n)}|^{p_i}$$

where (p_i) is a bounded sequence of positive real numbers. Then

$$\text{ces}^\sigma(p_i) = \left\{x: \sum_m z_{mn} \text{ converges uniformly in } n\right\}$$

$$\text{ces}^{\sigma\sigma}(p_i) = \left\{x: \sup_n \sum_n z_{mn} < \infty\right\}$$

If p_i is constant and $p_i = p > 0$ for all i, we write $\text{ces}^\sigma(p)$ and $\text{ces}^{\sigma\sigma}(p)$ for $\text{ces}^\sigma(p_i)$ and $\text{ces}^{\sigma\sigma}(p_i)$ respectively. It is obvious that $\text{ces}^\sigma(p_i) \subset \text{ces}\,(p_i)$. It is seen that $\text{ces}\,(p) = \{0\}$ for $0 < p \leq 1$, hence we have $\text{ces}^\sigma(p) = \{0\}$ for $0 < p \leq 1$. We have the following result:

THEOREM 1.

i) *For* $p > 1$, $\ell_p \subset \text{ces}^\sigma(p)$.

ii) $\text{ces}^\sigma(p_i) \subset \text{ces}^{\sigma\sigma}(p_i)$

for any bounded sequence (p_i) *of positive real numbers.*

THEOREM 2. $\operatorname{ces}^{\sigma}(p_i)$ *is a complete linear metric space paranormed by* g, *where*

$$g(x) = \sup_n \left(\sum_m z_{mn} \right)^{1/M}$$

$M = \max\{1, \sup p_i\}$. *Moreover* $\operatorname{ces}^{\sigma\sigma}(p_i)$ *is parametrized by* g, *if* $\inf p_i > 0$.

PROOF. The proof is obtained through standard. However, it should be noted that there is an essential difference between $\operatorname{ces}^{\sigma}(p_i)$ and $\operatorname{ces}^{\sigma\sigma}(p_i)$.

If $x \in \operatorname{ces}^{\sigma}(p_i)$ then given $\varepsilon > 0$ there exists an integer k such that

$$\sum_{m \geq k} z_{mn}(x) < \varepsilon \qquad \text{for all } n. \tag{1}$$

So we conclude that for any $x \in \operatorname{ces}^{\sigma}(p_i)$, $\lambda x \to 0$ as $\lambda \to 0$. But if $x \in \operatorname{ces}^{\sigma\sigma}(p_i)$ we cannot assert (1). Now we assume that $\inf p_i = \theta > 0$. Then for $|\lambda| < 1$, $|\lambda|^{p_i} \leq |\lambda|^{\theta}$, so that $g(\lambda x) \leq |\lambda|^{\theta} g(x)$ and this proves that for any $x \in \operatorname{ces}^{\sigma\sigma}(p_i)$, $\lambda x \to 0$ as $\lambda \to 0$.

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