

On absolutely continuous invariant measures with respect to Hausdorff measures on self-similar sets

Munetaka NAKAMURA
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ABSTRACT. We treat here measures which are invariant with respect to the renormalization map on a self-similar set. A criterion for their absolute continuity with respect to their associated Hausdorff measures is given in terms of symbolic dynamics. Using this criterion, we give a striking characterization of the equilibrium state for a certain potential function.

1. Introduction

In this paper we shall study the properties of ergodic invariant measures on self-similar sets. Especially we shall investigate the relations between invariant measures and the Hausdorff measures associated to them from the point of the absolute continuity. In the sense of [8], a self-similar set K is constructed from a system $\Phi = \{\varphi_i \mid i \in S\}$, $S = \{1, \dots, N\}$ of contractions and a bounded open set $V \subset \mathbf{R}^d$ by $K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in S} \varphi_{i_1} \circ \dots \circ \varphi_{i_n}(\bar{V})$. Here we assume that Φ and V satisfy $\bigcup_{i=1}^N \varphi_i(V) \subset V$ and $\varphi_i(V) \cap \varphi_j(V) = \emptyset$ if $i \neq j$, which is often referred to as the open set condition. See [8] for the details. It is easy to see that there exists a continuous surjection $\psi: S^{\mathbf{N}} \rightarrow K$ and this fact enables us to work on the symbolic dynamics $(S^{\mathbf{N}}, \sigma)$, $\sigma((x_n)_{n \in \mathbf{N}}) = (x_{n+1})_{n \in \mathbf{N}}$. In order to guarantee the existence of the renormalization transformation f on K corresponding to σ with a measure theoretically negligible exceptional set, we have to make a further assumption (A7) below. By virtue of this assumption we can immediately have a one to one correspondence between f -invariant measures and σ -invariant measures through the map ψ . In this paper we shall make a little extension of the above notion of self-similar sets. That is, as in [1], for each $N \times N$ aperiodic matrix A we construct a compact set K_A by $K_A = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in \Sigma_{A,n}} \varphi_{i_1} \circ \dots \circ \varphi_{i_n}(\bar{V})$ where $\Sigma_{A,n}$ is the set of all A -admissible words of length n , that is, those $(i_1, \dots, i_n) \in S^n$ such that $A_{i_j i_{j+1}} = 1$, $j = 1, \dots, n-1$. Deterministic Markov self-similar sets which are considered

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in [20] are contained in our cases so that the theory developed in this paper is applicable. Corresponding to the restriction of the renormalization map f on K_A , we consider the subshift of finite type (Σ_A, σ) , i.e. $\Sigma_A = \{(x_n)_{n \in \mathbf{N}} \in \{1, \dots, N\}^{\mathbf{N}} \mid A_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbf{N}\}$ and its left shift is denoted by σ as in the previous case. As an application of Ruelle-Perron-Frobenius theory, it is known that the Hausdorff dimension δ_A of K_A is equal to the unique zero point of the pressure, that is, the unique positive solution δ of the equation $P^A(-\delta L) = 0$ for some function $L: \Sigma_A \rightarrow \mathbf{R}$. Furthermore if $\mu_{-\delta_A L}$ is the equilibrium state for the function $-\delta_A L$, its image measure $\mu_{-\delta_A L} \circ \psi^{-1}$ and \mathcal{H}^{δ_A} are equivalent as Borel measures on K_A and the Radon-Nikodym derivative $\frac{d\mu_{-\delta_A L} \circ \psi^{-1}}{d\mathcal{H}^{\delta_A}}$ is bounded away both from 0 and $+\infty$. See [1].

The notion of the (Hausdorff) dimension of measures was introduced in [21] for invariant measures of certain diffeomorphisms and investigated elaborately in case of self-similar sets in [7]. Roughly speaking, for an ergodic σ -invariant measure μ , its dimension δ_μ can be characterized as the number δ for which $\mu([x_1, \dots, x_n])$ decays with the same speed as $|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta$ as $n \rightarrow \infty$. Since $\{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta\}$ generates the Borel measure $\mathcal{H}_\mu^{\delta_\mu}$ which differs from \mathcal{H}^{δ_μ} only by a constant factor, it will be not meaningless to compare $\mu \circ \psi^{-1}$ and \mathcal{H}^{δ_μ} as Borel measures on K_A . Actually in [15], though there is some difference in the contexts, some equilibrium states are compared with the (generalized) Hausdorff measures associated to them.

The contents of this paper are as follows. After reviewing some basic facts in §2, we examine in §3 the absolute continuity and singularity of the ψ -images of σ -invariant measures on Σ_A with respect to the Hausdorff measures. That is, for an ergodic σ -invariant probability measure μ , $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^δ on K^A if $\delta < \delta_\mu$ while these two measures turn out to be mutually singular if $\delta > \delta_\mu$. On the other hand when $\delta = \delta_\mu$ we see by virtue of the ergodicity that ergodic invariant probability measures are divided into two extreme classes; absolute continuous ones and singular ones. In other words, a striking feature of an ergodic measure is that it never has the absolute continuous part and the singular part simultaneously. These considerations lead to the Lebesgue decomposition of non-ergodic σ -invariant measures with respect to the Hausdorff measures. In §4 we deduce a cohomologous relation between the two functions $-\delta_\mu L$ and $-I_\mu$ below under the assumption that the measures considered are equivalent to their dimensional Hausdorff measures. In §5 we shall show that within a certain class $\mathcal{D}_\sigma(\Sigma_A)$ there exists no equivalent invariant ergodic measure supported by Σ_A except $\mu_{-\delta_A L}$. See §5 for the definition of $\mathcal{D}_\sigma(\Sigma_A)$. In §6, restricting the class of measures to smaller one than §5, we shall show that

there exists no absolutely continuous measures supported by Σ_A except for $\mu - \delta_{AL}$.

Though there are slight differences, our strategies are almost common in both §5 and §6. That is, we shall design to deduce some kind of cohomologous relations under the given assumptions and to apply these relations to Ruelle-Perron-Frobenius theory to determine the measures. We should be careful in these procedures because our cohomologous relations are only given in the measure theoretic sense. This is a main difference between our arguments and those in [15]. That is, we do not assume the Hölder continuity of I_μ . Under our assumptions the law of iterated logarithm does not necessarily hold so that it may be difficult to compare invariant measures with the Hausdorff measures in more precise form as in [15].

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2. Notations and preliminaries

For later use we shall begin this section with some definitions of the symbolic dynamics called subshifts of finite type. Suppose that S is a finite set with $\#S = N$.

Let \mathcal{M}_N be the set of all $N \times N$ type 0–1 aperiodic matrices, that is,

$$\mathcal{M}_N = \{A = (A_{ij})_{i,j \in S}, A_{ij} \in \{0, 1\}, A^k > 0 \text{ for some } k \in \mathbb{N}\}.$$

Let us fix $A \in \mathcal{M}_N$ and set

$$\begin{aligned} \Sigma_{A,n} &= \{(x_1, x_2, \dots, x_n) \in S^n \mid A_{x_i x_{i+1}} = 1, i = 1, 2, \dots, n-1\}, \\ \Sigma_A &= \{\underline{x} = (x_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}} \mid A_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{N}\}. \end{aligned}$$

Let us also define

$$\begin{aligned} [i_k, i_{k+1}, \dots, i_l]_k^l &= \{\underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma_A \mid x_j = i_j, j = k, k+1, \dots, l\}, \\ \mathcal{C}_{A,k}^l &= \{[i_k, i_{k+1}, \dots, i_l]_k^l \mid (i_k, i_{k+1}, \dots, i_l) \in \Sigma_{A, l-k+1}\}, \quad \mathcal{C}_{A,k}^\infty = \bigcup_{l=k}^\infty \mathcal{C}_{A,k}^l, \quad k \in \mathbb{N}. \end{aligned}$$

We shall simply write $[i_1, \dots, i_n]$ for $[i_1, \dots, i_n]_1^n$.

We introduce the topology into the space Σ_A with $\mathcal{C}_A = \mathcal{C}_{A,1}^\infty$ as its open basis. It is easy to see that this topology is nothing but the product topology induced by the discrete topology on S , which is also equal to the topology generated by the metric $d(\underline{x}, \underline{y}) = \sum_{n=1}^\infty \frac{\delta_{x_n, y_n}}{2^n}$ for $\underline{x} = (x_n)_{n \in \mathbb{N}}, \underline{y} = (y_n)_{n \in \mathbb{N}} \in \Sigma_A$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

The topological Borel σ -algebra on Σ_A is denoted by \mathcal{B}_A . In other words, $\mathcal{B}_A = \sigma[\mathcal{C}_A]$, i.e. \mathcal{B}_A is the σ -algebra generated by \mathcal{C}_A . Let I_N be the $N \times N$ matrix whose entries are all equal to 1. Σ_{I_N} , $\mathcal{C}_{I_N, n}^\infty$ and $\mathcal{C}_{I_N, 1}^\infty$ are denoted by simply Σ , \mathcal{C}_n and \mathcal{C} respectively. The left shift on Σ is denoted by σ , i.e. $\sigma((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$. The restriction of σ to Σ_A is denoted by the same symbol σ . But later we need to treat the inverse images of σ on Σ_A . In these cases in order to avoid the confusion we write in such a way as $\sigma_A^{-1}x$. More concretely $\sigma_A^{-1}x = \{j\bar{x} \mid A_{jx_1} = 1, \bar{x} = (x_n)_{n \in \mathbb{N}}\}$.

For a differentiable map φ on a certain domain in \mathbf{R}^d let us denote by $J(\varphi, x)$ the Jacobian matrix of φ at x and by $j(\varphi, x)$ the operator norm of the linear map induced by $J(\varphi, x)$.

Suppose that a system $\Phi = \{\varphi_i\}_{i \in S}$ of maps on \mathbf{R}^d and a bounded open subset V of \mathbf{R}^d satisfy the following conditions.

- (A1) $\varphi_i, i \in S$ are all C^1 in an open set which contains \bar{V} .
- (A2) There exists $0 < c_1 < c_2 < 1$ such that $c_1 \leq |j(\varphi_i, x)| \leq c_2$ for all $x \in \bar{V}, i \in S$.
- (A3) $j(\varphi_i, x)$ is α -Hölder continuous on \bar{V} where $0 < \alpha \leq 1$, that is, there exists $C > 0$ such that $|j(\varphi_i, x) - j(\varphi_i, y)| \leq C|x - y|^\alpha$ for all $x, y \in \bar{V}, i \in S$.
- (A4) $j(\varphi, x)^{-1}J(\varphi, x) \in O(d)$ where $O(d)$ stands for the d -dimensional orthogonal group.
- (A5) \bar{V} and $\varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\bar{V}), (i_1, \dots, i_n) \in S^n, n \in \mathbb{N}$ are all convex.
- (A6) $\bigcup_{i \in S} \varphi_i(V) \subset V, \varphi_i(V) \cap \varphi_j(V) = \emptyset (i \neq j)$.
- (A7) The set $Z_0 \equiv \bigcup_{i, j \in S, i \neq j} \varphi_i(\bar{V}) \cap \varphi_j(\bar{V})$ is at most countable and $A_n(x) \cap A_m(x) = \emptyset$ if $n \neq m$ for every $x \in Z_0$, where $A_0(x) = \{x\}, A_n(x) = \{\varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(x) \mid (i_1, \dots, i_n) \in S^n\}, n \in \mathbb{N}$.

REMARK 2.1. From conditions (A2) and the inverse function theorem it is easy to see that $\varphi_i (i \in S)$ are all injections in \bar{V} . The condition (A2) and (A5) also say that $\varphi_i (i \in S)$ are all contractions in V . Indeed $\|\varphi_i(x) - \varphi_i(y)\| \leq c_2 \|x - y\|$ for all $x, y \in \bar{V}$.

REMARK 2.2. When $d = 1$ some of the above conditions can be relaxed. Firstly (A4) clearly turns out to be a meaningless condition. So we can omit it. Next we may replace (A5) by

- (A5)' V is an interval.

This is equal to that V is convex. The continuity of $\varphi_i, i \in S$ automatically indicates that $\varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\bar{V}), (i_1, \dots, i_n) \in S^n, n \in \mathbb{N}$ are all convex (intervals). Moreover it can be easily checked that (A7) is automatically satisfied under the other conditions, so that we do not need it.

Fix $\Phi = \{\varphi_i\}_{i \in S}$ and V which satisfy the conditions (A1)–(A7). Let us set

$$K_A = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in \Sigma_{A,n}} \varphi_{i_1} \circ \varphi_{i_2} \circ \dots \circ \varphi_{i_n}(\bar{V})$$

for each $A \in \mathcal{M}_N$. K_{I_N} is simply denoted by K if there is no danger of confusion. It is easy to see that K_A is a compact subset of \bar{V} for all $A \in \mathcal{M}_N$.

A typical example of Φ and V satisfying the above conditions is $d = 2$,

$$\Phi = \{\varphi_1, \varphi_2, \varphi_3\}, \quad V = \{(x, y) \in \mathbf{R}^d \mid y > 0, y < \sqrt{3}x, y < -\sqrt{3}x + \sqrt{3}\},$$

$$\varphi_1(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right), \quad \varphi_2(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \varphi_3(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right).$$

In this case K is the Sierpiński gasket.

Let $\psi: \Sigma \rightarrow K$ be the map defined by

$$\{\psi((x_n)_{n \in \mathbf{N}})\} = \bigcap_{n=1}^{\infty} \varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}).$$

Note that the right hand side of the above definition is indeed a singleton because $\varphi_i, i \in S$ are all contractions as pointed out. Clearly ψ is a continuous surjection to K . For $A \in \mathcal{M}_N$ the restriction of ψ to Σ_A , which is also denoted by the same symbol ψ , gives the map to K_A .

Though the set K_A and the map ψ depends on the choice of the system Φ as well, we simply write in the above way.

For each $A \in \mathcal{M}_N$, K_A has the following self-similar structure:

$$K_A = \bigcup_{i \in S} K_i, \quad K_i = \bigcup_{j: (i,j) \in \Sigma_{A,2}} \varphi_i(K_j),$$

where $K_i = \bigcap_{n=2}^{\infty} \bigcup_{(i_1, i_2, \dots, i_n) \in \Sigma_{A,n}} \varphi_{i_1} \circ \varphi_{i_2} \circ \dots \circ \varphi_{i_n}(\bar{V})$.

REMARK 2.3. In [20] the notion of Markov-self-similar set was introduced and investigated elaborately. Markov-self-similar sets in deterministic cases can be regarded as self-similar sets in the above sense if we choose defining matrices appropriately. That is, a deterministic Markov self-similar set K is constructed from N^2 -tuple of contractions $\Phi = \{\varphi_{ij} \mid i, j \in \{1, \dots, N\}^2\}$ and a bounded open set V , which satisfy the open set condition, by $K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in \{1, \dots, N\}^n} \varphi_{i_1 i_2} \circ \varphi_{i_2 i_3} \circ \dots \circ \varphi_{i_{n-1} i_n}(\bar{V})$. This set can be viewed as the self-similar set constructed from the following situation in our case: $S = \{1, \dots, N\}^2$, Φ and V are the same and the matrix $A = (A_{\bar{i}\bar{j}})$ is given by

$$A_{\bar{i}\bar{j}} = \begin{cases} 1 & \text{if } \bar{i} = i_1 i_2, \bar{j} = j_1 j_2, i_2 = j_1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $A \in \mathcal{M}_{N^2}$. Indeed $A^2 > 0$.

Set $Z = \bigcup_{(i_1, \dots, i_n) \in S^n, n \in \mathbf{N}} \varphi_{i_1} \circ \dots \circ \varphi_{i_n}(Z_0)$, $W = \psi^{-1}(Z)$, $K^\# = K \setminus Z$ and $\Sigma^\# = \Sigma \setminus W (= \psi^{-1}(K^\#))$.

LEMMA 2.4. $\psi: \Sigma^\# \rightarrow K^\#$ is a bijection and $\sigma^{-1}(\Sigma^\#) = \Sigma^\#$.

PROOF. Since the original map $\psi: \Sigma \rightarrow K$ is surjective, it is clear from the definition $\Sigma^\# = \psi^{-1}(K^\#)$ that $\psi: \Sigma^\# \rightarrow K^\#$ is surjective.

Assume that $\psi(\underline{x}^{(i)}) = x \in K^\#$ for $\underline{x}^{(i)} = (x_n^{(i)})_{n \in \mathbb{N}}$ ($i = 1, 2$), $\underline{x}^{(1)} \neq \underline{x}^{(2)}$. Setting $m = \min \{n \in \mathbb{N} \mid x_n^{(1)} \neq x_n^{(2)}\}$ and recalling that $\varphi_i: \bar{V} \rightarrow \bar{V}$, $i \in S$ are injections, we have

$$\begin{aligned} x &\in \varphi_{x_1^{(1)}} \circ \cdots \circ \varphi_{x_m^{(1)}}(\bar{V}) \cap \varphi_{x_1^{(2)}} \circ \cdots \circ \varphi_{x_m^{(2)}}(\bar{V}) \\ &= \varphi_{x_1^{(1)}} \circ \cdots \circ \varphi_{x_{m-1}^{(1)}}(\varphi_{x_m^{(1)}}(\bar{V}) \cap \varphi_{x_m^{(2)}}(\bar{V})) \subset Z, \end{aligned}$$

which is impossible. Thus the injectivity of $\psi: \Sigma^\# \rightarrow K^\#$ was shown as well.

Next suppose that $\underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma^\#$ and $\sigma \underline{x} \notin \Sigma^\#$. By the definition of $\Sigma^\#$, we have $y \equiv \psi(\sigma \underline{x}) \notin K^\#$, that is, $y \in Z$. By the way $\underline{x} \in \Sigma^\#$ implies $\psi(\underline{x}) = \varphi_{x_1}(y) \notin Z$. This is clearly impossible. Therefore the relation $\Sigma^\# \subset \sigma^{-1}(\Sigma^\#)$ holds.

On the other hand assume that $\underline{x} = (x_n)_{n \in \mathbb{N}} \in \sigma^{-1}(\Sigma^\#)$ and $\underline{x} \notin \Sigma^\#$. Set $y = \psi(\sigma \underline{x})$ as before. Then $y \notin Z$ and $\psi(\underline{x}) = \varphi_{x_1}(y) \in Z$. The latter relation implies that there exists $(i_1, \dots, i_n) \in S^n$ ($n \in \mathbb{N}$) such that $\varphi_{x_1}(y) \in \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(Z_0)$ by the definition of Z . If $i_1 = x_1$, the injectivity of φ_{x_1} means $y \in \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(Z_0) \subset Z$, which is contrary to the condition $y \notin Z$. If $i_1 \neq x_1$, we have $y \in \varphi_{i_1}(\bar{V}) \cap \varphi_{x_1}(\bar{V}) \subset Z_0 \subset Z$, which is impossible as well. Thus we have shown that $\sigma^{-1} \Sigma^\# \subset \Sigma^\#$. ■

REMARK 2.5. Let $\Sigma_A^\# = \Sigma^\# \cap \Sigma_A$ and $K_A^\# = K^\# \cap K_A$ for $A \in \mathcal{M}_N$. Then clearly $\sigma_A^{-1}(\Sigma_A^\#) = \Sigma_A^\#$ and $\psi: \Sigma_A^\# \rightarrow K_A^\#$ is bijective.

In view of Lemma 2.4 the map $f: K^\# \rightarrow K^\#$, $f(x) = \varphi_i^{-1}(x)$ if $x \in \varphi_i(K)$ is well defined. This transformation is called the *renormalization transformation* on $K^\#$ and the following diagram is clearly commutative.

$$\begin{array}{ccc} \Sigma^\# & \xrightarrow{\sigma} & \Sigma^\# \\ \downarrow \psi & & \downarrow \psi \\ K^\# & \xrightarrow{f} & K^\# \end{array}$$

From now on we shall fix $A \in \mathcal{M}_N$. Let $\mathcal{E}_\sigma(\Sigma_A)$ be the set of all σ -invariant ergodic probability measures on Σ_A and set $\mathcal{E}_f(K_A) = \{\nu \in \mathcal{P}(K_A) \mid \nu(Z \cap K_A) = 0, \nu_A \circ f^{-1} = \nu_A\}$, where $\mathcal{P}(K_A)$ denotes the set of all the probability measures on K_A , ν_A the restriction of ν to $K_A^\#$. The following lemma shows that the correspondence between $\mathcal{E}_\sigma(\Sigma_A)$ and $\mathcal{E}_f(K_A)$ is easily given.

LEMMA 2.6. $\mu(\Sigma_A \cap W) = 0$ for every $\mu \in \mathcal{E}_\sigma(\Sigma_A)$.

PROOF. Let $x \in Z_0$ and set $\Gamma_n(x) = \Sigma_A \cap \psi^{-1}(A_n(x))$. Then from (A7) we have $\Gamma_n(x) \cap \Gamma_m(x) = \Sigma_A \cap \psi^{-1}(A_n(x) \cap A_m(x)) = \emptyset$ if $n \neq m$. Moreover it is easy to see that $\Gamma_n(x) = \sigma_A^{-n} \Gamma_0(x)$. Therefore we have $\mu(\bigcup_{n=0}^{\infty} \Gamma_n(x)) = \sum_{n=0}^{\infty} \mu(\Gamma_n(x))$ and $\mu(\Gamma_n(x)) = \mu(\sigma_A^{-n} \Gamma_0(x)) = \mu(\Gamma_0(x))$ for all $n \in \mathbb{N}$. This means that $\mu(\Gamma_n(x)) = \mu(\Gamma_0(x)) = 0$ for all $n \in \mathbb{N}$. Since Z_0 is countable, we have $\mu(\Sigma_A \cap \psi^{-1}(Z)) = \mu(\bigcup_{x \in Z_0} \bigcup_{n=0}^{\infty} \Gamma_n(x)) = 0$. ■

REMARK 2.7 From Lemma 2.6 it is clear that the map $\psi^*: \mathcal{E}_\sigma(\Sigma_A) \rightarrow \mathcal{E}_f(K_A)$, $\psi^*(\mu) = \mu \circ \psi^{-1}$ is a bijection. It is also easy to see that if $\mu \in \mathcal{E}_\sigma(\Sigma_A)$, then $\mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}) \cap K_A) = \mu([x_1, \dots, x_n])$ for all $(x_1, \dots, x_n) \in \Sigma_{A,n}$.

For $\mu \in \mathcal{E}_\sigma(\Sigma_A)$ let us set $\chi_\mu = \sum_{i \in S} 1_{[i]} \mu([i] | \sigma_A^{-1}(\mathcal{B}_A))$ and $I_\mu = -\log \chi_\mu$, where 1_B is the indicator function of the set B and $\mu([i] | \sigma_A^{-1}(\mathcal{B}_A))$ is the conditional probability measure. Note that $\chi_\mu > 0$ μ -a.e. so that I_μ is well-defined μ -a.e. It is not difficult to check from the definition of χ_μ that

$$\sum_{\mathcal{Y} \in \sigma_A^{-1} \sigma \mathcal{X}} \chi_\mu(\mathcal{Y}) f(\mathcal{Y}) = E_\mu(f | \sigma_A^{-1}(\mathcal{B}_A))(\mathcal{X}) \quad \mu\text{-a.e.}$$

for every μ -integrable function f , where $E_\mu(f | \sigma_A^{-1}(\mathcal{B}_A))$ stands for the conditional expectation. In other words, $\int [\sum_{\mathcal{Y} \in \sigma_A^{-1} \sigma \mathcal{X}} \chi_\mu(\mathcal{Y}) f(\mathcal{Y})] d\mu(\mathcal{X}) = \int f(\mathcal{X}) d\mu(\mathcal{X})$ for every μ -integrable function f . Since μ is σ -invariant we also have

$$\int [\sum_{\mathcal{Y} \in \sigma_A^{-1} \mathcal{X}} \chi_\mu(\mathcal{Y}) f(\mathcal{Y})] d\mu(\mathcal{X}) = \int f(\mathcal{X}) d\mu(\mathcal{X}). \quad (2.1)$$

Next we shall quote from [3] the thermodynamic formalism on (Σ_A, σ) in general. See also [17]. Let us denote the set of all the (real-valued) continuous functions on the space Σ_A by $C(\Sigma_A)$. For $\mathcal{G} \in C(\Sigma_A)$ we define an operator $\mathcal{L}_\mathcal{G}^A: C(\Sigma_A) \rightarrow C(\Sigma_A)$ by

$$\mathcal{L}_\mathcal{G}^A f(\mathcal{X}) = \sum_{\mathcal{Y} \in \sigma_A^{-1} \mathcal{X}} \exp[\mathcal{G}(\mathcal{Y})] f(\mathcal{Y}).$$

Its dual operator $(\mathcal{L}_\mathcal{G}^A)^*$ can be defined on the set of signed measures $\mathcal{M}(\Sigma_A)$, that is,

$$(\mathcal{L}_\mathcal{G}^A)^*: \mathcal{M}(\Sigma_A) \rightarrow \mathcal{M}(\Sigma_A), \quad \int f d(\mathcal{L}_\mathcal{G}^A)^* \mu = \int \mathcal{L}_\mathcal{G}^A f d\mu, \quad f \in C(\Sigma_A) \text{ for } \mu \in \mathcal{M}(\Sigma_A).$$

Here the usual topologies are given on $C(\Sigma_A)$ and $\mathcal{M}(\Sigma_A)$. That is, we give $C(\Sigma_A)$ the topology induced by the norm $\|f\|_\infty = \sup_{\mathcal{X} \in \Sigma_A} |f(\mathcal{X})|$, $f \in C(\Sigma_A)$ and $\mathcal{M}(\Sigma_A)$ the vague topology (weak-* topology).

For $f \in C(\Sigma_A)$, set

$\text{var}_A^{(m)}(f) = \sup \{|f(\underline{x}) - f(\underline{y})| \mid \underline{x} = (x_n)_{n \in \mathbf{N}}, \underline{y} = (y_n)_{n \in \mathbf{N}} \in \Sigma_A, x_n = y_n, n = 1, 2, \dots, m\}$

Then for $0 < \beta < 1$ the space

$$H_\beta = \{f \in C(\Sigma_A) \mid \text{there exists } C > 0 \text{ such that } \text{var}_A^{(n)}(f) \leq C\beta^n \text{ for all } n \in \mathbf{N}\}$$

is a Banach space with the norm $\|f\|_\beta = \|f\|_\infty + \sup_{n \in \mathbf{N}} \frac{\text{var}_A^{(n)}(f)}{\beta^n}$. Let us set $\mathcal{F}_A = \bigcup_{\beta \in (0,1)} H_\beta$.

Then the Ruelle-Perron-Frobenius operator theorem says that for every $\mathcal{G} \in \mathcal{F}_A$ there exist unique $\lambda_\mathcal{G} > 0$, $0 < h_\mathcal{G} \in C(\Sigma_A)$ and $\nu_\mathcal{G} \in \mathcal{P}(\Sigma_A)$ such that

$$\mathcal{L}_\mathcal{G}^A h_\mathcal{G} = \lambda_\mathcal{G} h_\mathcal{G}, (\mathcal{L}_\mathcal{G}^A)^* \nu_\mathcal{G} = \lambda_\mathcal{G} \nu_\mathcal{G}, \int h_\mathcal{G} d\nu_\mathcal{G} = 1$$

and

$$\lim_{n \rightarrow \infty} \|\lambda_\mathcal{G}^{-n} (\mathcal{L}_\mathcal{G}^A)^n f - \int f d\nu_\mathcal{G} h_\mathcal{G}\| = 0$$

for all $f \in C(\Sigma_A)$.

On the other hand there exist unique $\mu_\mathcal{G} \in \mathcal{E}_\sigma(\Sigma_A)$ (in fact weak Bernoulli; see [3]) and $P^A(\mathcal{G}) \in \mathbf{R}$ such that for some $\gamma_1, \gamma_2 > 0$

$$\gamma_1 \leq \frac{\mu_\mathcal{G}([x_1, x_2, \dots, x_n])}{\exp[-P^A(\mathcal{G})n + \sum_{i=0}^{n-1} \mathcal{G}(\sigma^i \underline{x})]} \leq \gamma_2$$

for all $n \in \mathbf{N}$ and for all $\underline{x} = (x_n)_{n \in \mathbf{N}} \in \Sigma_A$.

$\mu_\mathcal{G}$ and $P^A(\mathcal{G})$ are called the *equilibrium state* and the *pressure* respectively for $\mathcal{G} \in \mathcal{F}_A$.

Another characterization of $P^A(\mathcal{G})$ is given by $P^A(\mathcal{G}) = \sup_{\mu \in \mathcal{I}_\sigma(\Sigma_A)} \left\{ h_\mu(\sigma) + \int \mathcal{G} d\mu \right\}$, where $\mathcal{I}_\sigma(\Sigma_A)$ is the set of all the σ -invariant probability measure on Σ_A and $h_\mu(\sigma)$ is the entropy of σ with respect to μ . Furthermore $\mu_\mathcal{G}$ is characterized as the unique maximizer so that $P^A(\mathcal{G}) = h_{\mu_\mathcal{G}}(\sigma) + \int \mathcal{G} d\mu_\mathcal{G}$.

The relation between $\lambda_\mathcal{G}$, $\nu_\mathcal{G}$, $h_\mathcal{G}$ and $P^A(\mathcal{G})$, $\mu_\mathcal{G}$ above are given by $P^A(\mathcal{G}) = \log \lambda_\mathcal{G}$ and $\mu_\mathcal{G} = h_\mathcal{G} \nu_\mathcal{G}$ which means $\mu_\mathcal{G}(B) = \int_B h_\mathcal{G} d\nu_\mathcal{G}$ for all $B \in \mathcal{B}_A$.

A simple observation of the proof of Ruelle-Perron-Frobenius theorem tells the following fact: If $\mathcal{G} \in \mathcal{F}_A$ and there exist $\mu \in \mathcal{P}(\Sigma_A)$ and $\lambda > 0$ such that $(\mathcal{L}_\mathcal{G}^A)^* \mu = \lambda \mu$, then $\lambda = \lambda_\mathcal{G}$ and $\mu = \nu_\mathcal{G}$. Furthermore if the above μ belongs to $\mathcal{E}_\sigma(\Sigma_A)$ we can conclude that $\mu = \mu_\mathcal{G}$ and $h_\mathcal{G} = 1$, for $\mu_\mathcal{G}$ is absolutely continuous with respect to $\nu_\mathcal{G} = \mu$ and the ergodicity of $\mu_\mathcal{G}$ means $\mu_\mathcal{G} = \mu = \nu_\mathcal{G}$.

so that $h_g = 1$.

In what follows some definitions and notations on Hausdorff measures and dimensions are given. See [5], [6] or [16] for the details. For a bounded $B \subset \mathbf{R}^d$ and $\delta > 0$, a finite or countable family $\mathcal{U} = \{U_i \mid U_i \subset \mathbf{R}^d, i \in \mathbf{N}\}$ (\mathcal{U} is called finite if $U_i = \emptyset$ with finite exceptional $i \in \mathbf{N}$) is called δ -cover of B if $B \subset \bigcup_{i=1}^{\infty} U_i$ and $|U_i| \leq \delta$ ($i \in \mathbf{N}$), where $|U| = \sup_{x,y \in U} |x - y|$ for $U \subset \mathbf{R}^d$ ($|\emptyset| = 0$). For $\beta \geq 0$ we define $\mathcal{M}^\beta(\mathcal{U}) = \sum_{i=1}^{\infty} |U_i|^\beta$ if $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$.

For a family \mathcal{D} of subsets of \mathbf{R}^d we define

$$\mathcal{H}_{\mathcal{D}, \delta}^\beta(B) = \inf \{ \mathcal{M}^\beta(\mathcal{U}) \mid \mathcal{D} \supset \mathcal{U} \text{ is a finite or countable } \delta\text{-cover of } B \},$$

$$\mathcal{H}_{\mathcal{D}}^\beta(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_{\mathcal{D}, \delta}^\beta(B).$$

Here we conventionally put $\mathcal{H}_{\mathcal{D}}^\beta(A) = +\infty$ if there is no finite or countable δ -cover of B which is contained in \mathcal{D} .

When $\mathcal{D} = 2^{\mathbf{R}^d}$, we simply write \mathcal{H}_δ^β and \mathcal{H}^β respectively. Furthermore we define $\dim_{\mathbf{H}}(B) = \sup \{ \beta : \mathcal{H}^\beta(B) = +\infty \}$. It is well known that $\dim_{\mathbf{H}}(B)$ is also given by $\inf \{ \beta : \mathcal{H}^\beta(B) = 0 \}$, and it is easy to see from the definition that $\mathcal{H}_{\mathcal{D}}^\beta \leq \mathcal{H}^\beta$ for all \mathcal{D} and $\beta > 0$. $\mathcal{H}^\beta(B)$ and $\dim_{\mathbf{H}}(B)$ are called the β -dimensional Hausdorff measure of B and the Hausdorff dimension of B respectively.

Define the function $L: \Sigma_A \rightarrow \mathbf{R}$ by $L(\underline{x}) = -\log |j(\varphi_{x_1}, \psi(\delta \underline{x}))|$. Then it is standard to show the following facts. We mention here that the convexity condition (A5) is used in obtaining (B2) and (B3).

(B1) $L \in \mathcal{F}_A$.

(B2) There exist $0 < C_1 < C_2$ such that

$$C_1 |\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})| \leq \exp \left[\sum_{i=0}^{n-1} L(\sigma^i \underline{x}) \right] \leq C_2 |\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|$$

for all $n \in \mathbf{N}$ and all $\underline{x} \in \Sigma_A$.

(B3) For all $\beta \geq 0$ there exists a constant $0 < C_3 \leq 1$ such that

$$C_3 \mathcal{H}^\beta \leq \mathcal{H}_{\mathcal{C}}^\beta \leq \mathcal{H}^\beta$$

where $\mathcal{C} = \{ \varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}) \mid \underline{x} \in \Sigma, n \in \mathbf{N} \}$.

It immediately follows from these results that, if we put $\delta_A = \dim_{\mathbf{H}}(K_A)$, δ_A is equal to the unique positive solution δ of the equation $P^A(-\delta L) = 0$. Indeed by virtue of (B2) the equilibrium state $\mu_{-\delta_A L}$ satisfies

$$\gamma_1 \leq \frac{\mu_{-\delta_A L}([i_1, \dots, i_n])}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^{\delta_A}} \leq \gamma_2, \quad \text{for all } [i_1, \dots, i_n] \in \mathcal{C}_{A,n}, n \in \mathbf{N}$$

for some $0 < \gamma_1 \leq \gamma_2$. Therefore we can easily obtain $0 < \mathcal{H}_{\frac{\delta}{2}}^{\delta_A}(K_A) < +\infty$, or equivalently $0 < \mathcal{H}^{\delta_A}(K_A) < +\infty$ in view of (B3). Furthermore it can be also seen that \mathcal{H}^{δ_A} is equivalent to $\mu_{-\delta_A L}$ as Borel measures on K_A and the Radon-Nikodym derivative $\frac{d\mu_{-\delta_A L}}{d\mathcal{H}^{\delta_A}}$ has a version uniformly bounded away both from 0 and $+\infty$ on K_A . See [4], [7] and [18].

On the other hand the variational characterization of the pressure gives the equality $\delta_A = \sup_{\mu \in \mathcal{E}_\sigma(\Sigma_A)} \delta_\mu$ where $\delta_\mu = \frac{h_\mu(\sigma)}{\int L d\mu}$. Moreover $\mu_{-\delta_A L}$ is the unique maximizer, that is, $\dim_{\text{H}}(K_A) = \frac{h_{\mu_{-\delta_A L}}}{\int L d\mu_{-\delta_A L}}$.

3. Comparison with μ and \mathcal{H}^{δ_A}

In what follows the Hausdorff measures are always restricted to the Borel field on \mathbf{R}^d unless otherwise stated. Let μ and ν be (not necessarily σ -finite) Borel measures on a Borel measurable set $C \subset \mathbf{R}^d$. μ is said to be *absolutely continuous* with respect to ν if $\nu(B) = 0, B \in \mathcal{B}(C)$ (the Borel field on C) implies $\mu(B) = 0$. Note that the absolute continuity does not necessarily mean the existence of the derivative $\frac{d\mu}{d\nu}$ in Radon-Nikodym sense because the measures considered here are not necessarily σ -finite. μ and ν are said to be *equivalent* if each of them is absolutely continuous with respect to the other. μ and ν are said to be *singular* if there exists $B \in \mathcal{B}(C)$ such that $\mu(B) = 0$ and $\nu(B^c) = 0$.

Results similar to the following Lemma 3.1, Lemma 3.2, Theorem 3.6 and Lemma 4.1 in the next section can be found in [18] (Theorem 67) and [23] in general contexts. Specifying the situation, we can obtain the absolutely continuous or equivalent conditions in terms of the information function I_μ in a sharper and more useful form for the research below.

LEMMA 3.1. *Let ν be a Borel probability measure on K_A such that $\nu(K_A \cap Z) = 0$. For $\delta \geq 0$ and $0 \leq a < +\infty$, set*

$$B_a = \left\{ x \in K_A^\# \mid \sup_{n \in \mathbf{N}} \frac{\nu(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta} \leq a, (x_n)_{n \in \mathbf{N}} = \psi^{-1}(x) \right\}.$$

Then the restriction of ν to B_a which is denoted by ν_{B_a} is absolutely continuous with respect to \mathcal{H}^δ on B_a .

PROOF. From (B3) it is sufficient to show that ν_{B_a} is absolutely continuous with respect to $\mathcal{H}_{\tilde{\mathcal{C}}}^\delta$. Note that

$$\nu(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})) \leq a |\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta, \quad x \in B_a. \quad (3.1)$$

Suppose that $C \in \mathcal{B}(B_a)$ satisfies $\mathcal{H}_{\tilde{\mathcal{C}}}(C) = 0$. Then by the definition of $\mathcal{H}_{\tilde{\mathcal{C}}}$ for any given $\varepsilon > 0$ and $\varepsilon' > 0$ there exists $\mathcal{U} = \{U_i\}$ which is an ε' -cover of C such that $\mathcal{U} \subset \tilde{\mathcal{C}}$ and $\mathcal{M}^\delta(\mathcal{U}) < \mathcal{H}_{\varepsilon', \tilde{\mathcal{C}}}^\delta(C) + \varepsilon \leq \mathcal{H}_{\tilde{\mathcal{C}}}^\delta(C) + \varepsilon = \varepsilon$. Since we may assume that every $U \in \mathcal{U}$ contains at least one $x \in B_a$ (if otherwise we can exclude such U), (3.1) implies that $\nu(U) \leq a|U|^\delta$ for all $U \in \mathcal{U}$. Therefore

$$\mathcal{M}^\delta(\mathcal{U}) = \sum_{i=1}^{\infty} |U_i|^\delta \geq a^{-1} \sum_{i=1}^{\infty} \nu(U_i) \geq a^{-1} \nu(\bigcup_{i=1}^{\infty} U_i) \geq a^{-1} \nu(C).$$

Therefore $\nu(C) \leq a\varepsilon$. Since $\varepsilon > 0$ is arbitrary we have $\nu(C) = 0$. ■

LEMMA 3.2. Let $\delta \geq 0$ and ν be a Borel probability measure on K_A such that $\nu(K_A \cap Z) = 0$. If ν satisfies

$$\limsup_{n \rightarrow \infty} \frac{\nu(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta} = +\infty \text{ for } \nu\text{-a.e. } x, \quad \psi^{-1}(x) = (x_n)_{n \in \mathbb{N}}, \quad (3.2)$$

then \mathcal{H}^δ and ν are mutually singular on K_A .

PROOF. If we set

$$B = \left\{ x \in K_A^\# \mid \limsup_{n \rightarrow \infty} \frac{\nu(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta} = +\infty, \quad \psi^{-1}(x) = (x_n)_{n \in \mathbb{N}} \right\},$$

we have $\nu(K_A^\# \setminus B) = \nu(K_A \setminus B) = 0$ by the assumption $\nu(K_A \cap Z) = 0$ and (3.2). If we show $\mathcal{H}^\delta(A) = 0$, these equalities mean the singularities of ν and \mathcal{H}^δ . As in Lemma 3.1 we treat $\mathcal{H}_{\tilde{\mathcal{C}}}^\delta$ instead of \mathcal{H}^δ .

For a given $k \in \mathbb{N}$, $\varepsilon > 0$ and $x \in B$, there exists $n = n_{k, \varepsilon, x} \in \mathbb{N}$ such that

$$|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})| < \varepsilon, \quad \nu(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})) \geq k |\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta. \quad (3.3)$$

Therefore setting $C(x) = \varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})$ ($n = n_{k, \varepsilon, x}$), we obtain $\mathcal{C}_{k, \varepsilon} = \{C(x) \mid x \in B\}$ which is an ε -cover of B by (3.3). Recalling that $\mathcal{C}_{k, \varepsilon}$ is at most countable, we may choose a sequence $\{x^{(i)}\}_{i \in \mathbb{N}} \subset B$ such that $\mathcal{C}_{k, \varepsilon} = \{C(x^{(i)}) \mid i \in \mathbb{N}\}$. We may assume that $\nu(C(x^{(i)}) \cap C(x^{(j)})) = 0$ if $i \neq j$, for $\nu(C(x^{(i)}) \cap C(x^{(j)})) = 0$ if $i \neq j$, for $\nu(C(x^{(i)}) \cap C(x^{(j)})) \leq \nu(Z) = 0$. Therefore

$$\nu(\bigcup_{i=1}^{\infty} C(x^{(i)})) = \sum_{i=1}^{\infty} \nu(C(x^{(i)})). \quad (3.4)$$

Using (3.3) and (3.4), we have

$$\mathcal{M}^\delta(\mathcal{C}_{k,\varepsilon}) = \sum_{i=1}^{\infty} |C(x^{(i)})|^\delta \leq \frac{1}{k} \sum_{i=1}^{\infty} \nu(C(x^{(i)})) = \frac{1}{k} \nu(\bigcup_{i=1}^{\infty} C(x^{(i)})) = \frac{1}{k}.$$

Letting $k \rightarrow \infty$ and $\varepsilon \rightarrow 0$, we have $\mathcal{H}_{\mathcal{G}}^\delta(B) = 0$. Thus the lemma is proved. \blacksquare

The following Lemma 3.3 is easy to be assured. So we omit the proof.

LEMMA 3.3 *Suppose that $c_n, d_n > 0$ and $\lim_{n \rightarrow \infty} d_n = 0$ and $\lim_{n \rightarrow \infty} \frac{\log c_n}{\log d_n} = \varepsilon$. If $-\infty \leq \varepsilon < 1$, then $\lim_{n \rightarrow \infty} \frac{c_n}{d_n} = +\infty$ and if $1 < \varepsilon \leq +\infty$, then $\lim_{n \rightarrow \infty} \frac{c_n}{d_n} = 0$.*

PROPOSITION 3.4. *Suppose that $\mu \in \mathcal{E}_\sigma(\Sigma_A)$. If $0 < \delta < \delta_\mu$, then $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^δ on K_A and if $\delta_\mu < \delta$, then $\mu \circ \psi^{-1}$ and \mathcal{H}^δ are mutually singular on K_A .*

PROOF. The Shannon-McMillan-Breiman theorem and the Birkhoff ergodic theorem say that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu([x_1, \dots, x_n]) = -h_\mu(\sigma), \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} L(\sigma \underline{x}) = - \int L d\mu \quad \mu\text{-a.e.}$$

It follows from (B2) and Remark 2.7 that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}) \cap K_A) &= -h_\mu(\sigma), \\ \lim_{n \rightarrow \infty} \frac{1}{n} \log |\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta &= -\delta \int L d\mu \quad \mu\text{-a.e.} \end{aligned}$$

It is easy to see that for $\delta > 0$, $\lim_{n \rightarrow \infty} |\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta = 0$ μ -a.e. Moreover

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log \mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}) \cap K_A)}{\log |\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \log \mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}) \cap K_A)}{\frac{1}{n} \log |\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta} \\ &= \frac{h_\mu}{\delta \int L d\mu} = \frac{h_\mu}{\delta \int L d\mu} = \frac{\delta_\mu}{\delta} \quad \mu\text{-a.e.} \end{aligned}$$

Therefore if $0 < \delta < \delta_\mu$, then we have from Lemma 3.3 that

$$\lim_{n \rightarrow \infty} \frac{\mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}) \cap K_A)}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta} = 0 \quad \mu\text{-a.e. } \underline{x} = (x_n)_{n \in \mathbb{N}}.$$

In view of Lemma 3.1 we see that $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^δ on K_A .

Next suppose that $\delta_\mu < \delta$. Then we have again from Lemma 3.3 that

$$\lim_{n \rightarrow \infty} \frac{\mu \circ \psi^{-1}(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}) \cap K_A)}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta} = +\infty \quad \mu\text{-a.e. } \underline{x} = (x_n)_{n \in \mathbb{N}}.$$

Therefore from Lemma 3.2 we see that $\mu \circ \psi^{-1}$ and \mathcal{H}^δ are mutually singular on K_A . ■

REMARK 3.5. For $\mu \in \mathcal{E}_\sigma(\Sigma_A)$, δ_μ has another characterization. That is,

$$\delta_\mu = \dim_{\text{H}}(\mu) \equiv \inf \{ \dim_{\text{H}}(E) \mid E \in \mathcal{B}(K_A), \mu(E) = 1 \}.$$

The above equality can be shown as in Billingsley's book (see Section 14 in [2]) by using the set

$$M_\mu = \left\{ (x_n)_{n \in \mathbb{N}} \in \Sigma_A \mid \lim_{n \rightarrow \infty} \frac{\log \mu([x_1, \dots, x_n])}{\log |\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|} = \delta_\mu \right\}.$$

Indeed the set M_μ has the Hausdorff dimension equal to δ_μ and $\mu(M_\mu) = 1$. See also [21] and [7]. For general treatments of the computation of Hausdorff dimension in Billingsley's method, see [10].

THEOREM 3.6. For $\mu \in \mathcal{E}_\sigma(\Sigma_A)$ the only one of the following (1) or (2) holds.

- (1) $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} .
- (2) $\mu \circ \psi^{-1}$ and \mathcal{H}^{δ_μ} are mutually singular.

Furthermore (1) occurs if and only if

$$\limsup_{n \rightarrow \infty} \frac{\mu([x_1, \dots, x_n])}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^{\delta_\nu}} < +\infty \quad \mu\text{-a.e. } \underline{x} = (x_n)_{n \in \mathbb{N}}$$

and (2) occurs if and only if

$$\limsup_{n \rightarrow \infty} \frac{\mu([x_1, \dots, x_n])}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^{\delta_\nu}} = +\infty \quad \mu\text{-a.e. } \underline{x} = (x_n)_{n \in \mathbb{N}}.$$

PROOF. Suppose that $\mu \in \mathcal{E}_\sigma(\Sigma_A)$. Then $\mu \circ \psi^{-1}(Z \cap K_A) = 0$ by Lemma 2.4
Set

$$B = \left\{ \underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma_A^\# \mid \limsup_{n \rightarrow \infty} \frac{\mu \circ \psi^{-1}(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^{\delta_\nu}} < +\infty \right\}.$$

Then from (B2) and Remark 2.5 we have

$$B = \left\{ \underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma_A^\# \mid \limsup_{n \rightarrow \infty} \log \frac{\mu([x_1, \dots, x_n])}{\exp[-\delta_\mu \sum_{i=0}^{n-1} L(\sigma^i \underline{x})]} < +\infty \right\}.$$

Let us set for $\underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma_A^\#$ and $n \in \mathbb{N}$

$$g_n(\underline{x}) = \log \frac{\mu([x_1, \dots, x_n])}{\exp[-\delta_\mu \sum_{i=0}^{n-1} L(\sigma^i \underline{x})]} = \log \mu([x_1, \dots, x_n]) + \delta_\mu \sum_{i=0}^{n-1} L(\sigma^i \underline{x}).$$

Then the following equality is easy to be checked:

$$g_n(\sigma \underline{x}) = g_{n+1}(\underline{x}) + \log \frac{\mu([x_2, \dots, x_{n+1}])}{\mu([x_1, x_2, \dots, x_{n+1}])} - \delta_\mu L(x).$$

By Doob's theorem $\lim_{n \rightarrow \infty} \log \frac{\mu([x_2, \dots, x_n])}{\mu([x_1, x_2, \dots, x_n])} = I_\mu(x)$ μ -a.e. so that

$$\limsup_{n \rightarrow \infty} g_n(\sigma \underline{x}) = \limsup_{n \rightarrow \infty} g_n(\underline{x}) + I_\mu(x) - \delta_\mu L(x) \quad \mu\text{-a.e. } \underline{x}. \quad (3.5)$$

If we set $g = \limsup_{n \rightarrow \infty} g_n$, then $B = \{\underline{x} \in \Sigma_A^\# \mid g(\underline{x}) < \infty\}$ and (3.5) means that $\sigma_A^{-1} B = B \pmod{(\mu)}$ since I_μ is μ -a.e. finite. Therefore by the ergodicity of μ we have $\mu(B) = 0$ or 1 .

First assume that $\mu(B) = 0$. Then

$$\limsup_{n \rightarrow \infty} \frac{\mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^{\delta_\mu}} = +\infty \quad \mu\text{-a.e. } \underline{x} = (x_n)_{n \in \mathbb{N}}.$$

Therefore putting $\nu = \mu \circ \psi^{-1}$ and $\delta = \delta_\mu$ in Lemma 3.2 we see that $\mu \circ \psi^{-1}$ and \mathcal{H}^{δ_μ} are mutually singular on K_A .

Next suppose that $\mu(B) = 1$. Since

$$B = \left\{ \underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma_A^\# \mid \sup_{n \in \mathbb{N}} \frac{\mu(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^{\delta_\mu}} < +\infty \right\},$$

if we set for $k \in \mathbb{N}$

$$B_k = \left\{ \underline{x} = (x_n)_{n \in \mathbb{N}} \in \Sigma_A^\# \mid \sup_{n \in \mathbb{N}} \frac{\mu(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^{\delta_\mu}} < k \right\},$$

we have $B = \bigcup_{k=1}^{\infty} B_k$. Then we see that for each $k \in \mathbb{N}$ the restriction $\mu \circ \psi^{-1}$ to $\psi(B_k)$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} by Lemma 3.1. Therefore $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} on $\psi(B)$. Since $\mu \circ \psi^{-1}(\psi(B)) = \mu(B) = 1$, we can conclude that $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} on K_A itself.

The statement in the later part of the theorem is clear from the proof above. \blacksquare

Let $\mathcal{I}_\sigma(\Sigma_A)$ be the set of all the σ -invariant probability measures on

$(\Sigma_A, \mathcal{B}_A)$ and $\mu \in \mathcal{F}_\sigma(\Sigma_A)$. Let us denote by ζ the ergodic decomposition of $(\Sigma_A, \mathcal{B}_A, \mu, \sigma)$ and by π the projection from Σ_A to ζ . Let $\{\mu_C | C \in \zeta\}$ be the regular conditional probability measure with respect to ζ . Here we regard μ_C as a probability measure on Σ_A . Therefore $\mu_C \in \mathcal{E}_\sigma(\Sigma_A)$.

For $\delta \geq 0$ we put

$$(\zeta_1^\delta)' = \{C \in \zeta | \delta_{\mu_C} = \delta, \mu_C \circ \psi^{-1} \text{ is absolutely continuous with respect to } \mathcal{H}^\delta \text{ on } K_A\},$$

$$(\zeta_2^\delta)' = \{C \in \zeta | \delta_{\mu_C} = \delta, \mu_C \circ \psi^{-1} \text{ and } \mathcal{H}^\delta \text{ are mutually singular on } K_A\},$$

$$\zeta_1^\delta = \{C \in \zeta | \delta_{\mu_C} > \delta\} \cup (\zeta_1^\delta)' \text{ and } \zeta_2^\delta = \{C \in \zeta | \delta_{\mu_C} < \delta\} \cup (\zeta_2^\delta)'.$$

Under these notations we construct μ_0^δ and μ_1^δ by

$$\mu_1^\delta = \int_{\zeta_1^\delta} \mu_C d\mu \circ \pi^{-1}(C), \quad \text{and} \quad \mu_0^\delta = \int_{\zeta_2^\delta} \mu_C d\mu \circ \pi^{-1}(C).$$

The following Lebesgue decomposition of σ -invariant probability measures with respect to the Hausdorff measures is an easy consequence of Proposition 3.4 and Theorem 3.6.

COROLLARY 3.7. *Let $\mu \in \mathcal{F}_\sigma(\Sigma_A)$ and $\delta \geq 0$. Then $\mu = \mu_1^\delta + \mu_0^\delta$ and μ_1 is absolutely continuous with respect to \mathcal{H}^δ on K_A and μ_0 and \mathcal{H}^δ are mutually singular on K_A . Especially if $0 \leq \delta < \text{essinf}(\delta_{\mu_C})$ for $\mu \circ \pi^{-1}$ -a.e. C , then μ is absolutely continuous with respect to \mathcal{H}^δ on K_A and if $\delta > \text{essinf}(\delta_{\mu_C})$ for $\mu \circ \pi^{-1}$ -a.e. C , then μ and \mathcal{H}^δ are mutually singular on K_A .*

4. Equivalence of μ and \mathcal{H}^δ

We consider the conditions for invariant measures under which they are equivalent to their dimensional Hausdorff measures. In view of (3.5) a cohomologous relation can be easily obtained.

LEMMA 4.1. *Suppose that $\delta \geq 0$ and ν is a finite measure on K_A such that $\nu(K_A \cap Z) = 0$ and*

$$\nu \left(\left\{ x \in K_A^\# \mid \lim_{n \rightarrow \infty} \frac{\nu(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta} = 0, \psi^{-1}(x) = (x_n)_{n \in \mathbb{N}} \right\} \right) > 0. \quad (4.1)$$

Then ν and \mathcal{H}^δ are not equivalent on K_A .

PROOF. As in §3, we show that ν is not equivalent to $\mathcal{H}_\mathbb{Q}^\delta$. Set

$$K_0 = \left\{ x \in K_A^\# \mid \lim_{n \rightarrow \infty} \frac{\nu(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^\delta} = 0, \psi^{-1}(x) = (x_n)_{n \in \mathbb{N}} \right\}$$

and

$$K_{m,\varepsilon} = \left\{ x \in K_0 \mid \sup_{n \geq m} \frac{v(\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V}))}{|\varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(\bar{V})|^\delta} \leq \varepsilon, \psi^{-1}(x) = (x_n)_{n \in \mathbb{N}} \right\}$$

for $m \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $m = m_\varepsilon \in \mathbb{N}$ such that

$$v(K_{m,\varepsilon}) > \frac{v(K_0)}{2} (> 0). \quad (4.2)$$

Suppose that $0 < \eta < \frac{c_1^m}{C_2}$, where c_1 and C_2 are the constants which appeared in (A2) and (B2). Then there exists an η -cover $\mathcal{U} = \{U_i\}$ of $K_{m,\varepsilon}$ such that

$$\mathcal{M}^\delta(\mathcal{U}) \leq \mathcal{H}_{\eta, \frac{\delta}{2}}^\delta(K_{m,\varepsilon}) + 1. \quad (4.3)$$

We may assume that for each $i \in \mathbb{N}$, U_i contains at least one point $x^{(i)} \in K_{m,\varepsilon}$ since if otherwise we can exclude such U_i from \mathcal{U} . Therefore U_i is written in the form $U_i = \varphi_{x_1^{(i)}} \circ \cdots \circ \varphi_{x_k^{(i)}}(\bar{V})$ for some $x^{(i)} \in K_{m,\varepsilon}$ and $k (= k_i) \in \mathbb{N}$. ($\psi^{-1}(x^{(i)}) = \underline{x}^{(i)} = (x_n^{(i)})_{n \in \mathbb{N}}$.) Since

$$\begin{aligned} \frac{c_1^{\delta m}}{C_2} &> \eta^\delta \geq |U_i|^\delta \geq |\varphi_{x_1^{(i)}} \circ \cdots \circ \varphi_{x_k^{(i)}}(\bar{V})|^\delta \\ &\geq \frac{\exp[-\delta \sum_{j=0}^{k-1} L(\sigma^j \underline{x}^{(i)})]}{C_2} \geq \frac{\exp[\delta k \log c_1]}{C_2} = \frac{c_1^{\delta k}}{C_2}, \end{aligned}$$

we must have $k \geq m$. Then it follows from the definition of $K_{m,\varepsilon}$ that $v(U_i) \leq \varepsilon |U_i|^\delta$ for all $i \in \mathbb{N}$. Therefore

$$\begin{aligned} \mathcal{H}_{\frac{\delta}{2}, \eta}^\delta(K_{m,\varepsilon}) + 1 &> \mathcal{M}^\delta(\mathcal{U}) = \sum_{i=1}^{\infty} |U_i|^\delta \geq \frac{1}{\varepsilon} \sum_{i=1}^{\infty} v(U_i) \\ &\geq \frac{1}{\varepsilon} v(\cup_{i=1}^{\infty} U_i) \geq \frac{1}{\varepsilon} v(K_{m,\varepsilon}), \end{aligned}$$

which yields

$$\begin{aligned} v(K_{m,\varepsilon}) &\leq \varepsilon (\mathcal{H}_{\frac{\delta}{2}, \eta}^\delta(K_{m,\varepsilon}) + 1) \leq \varepsilon (\mathcal{H}_{\frac{\delta}{2}}^\delta(K_{m,\varepsilon}) + 1) \\ &\leq \varepsilon (\mathcal{H}_{\frac{\delta}{2}}^\delta(K_0) + 1). \end{aligned}$$

Therefore together with (4.2) and (4.3) we have

$$v(K_0) \leq 2\varepsilon (\mathcal{H}_{\frac{\delta}{2}}^\delta(K_0) + 1). \quad (4.4)$$

If $\mathcal{H}_{\frac{\delta}{2}}^\delta(K_0) < +\infty$, then we have $v(K_0) = 0$ since $\varepsilon > 0$ in (4.4) is arbitrary. But

this is contrary to the assumption so that $\mathcal{H}_{\frac{\delta}{\varepsilon}}^{\delta}(K_0) = +\infty$. Then we see that there exists a compact $C \subset K_0$ such that $\mathcal{H}_{\frac{\delta}{\varepsilon}}^{\delta}(C) = 1$. (See [5].) Suppose that $\nu(C) > 0$. Then repeating the same argument as above for the set C instead of K_0 we have $\nu(C) \leq 2\varepsilon(\mathcal{H}_{\frac{\delta}{\varepsilon}}^{\delta}(C) + 1) = 4\varepsilon$. It follows that $\nu(C) = 0$ since $\varepsilon > 0$ is arbitrary. Thus $\mathcal{H}_{\frac{\delta}{\varepsilon}}^{\delta}$ and ν are not equivalent. ($\mathcal{H}_{\frac{\delta}{\varepsilon}}^{\delta}$ is not absolutely continuous with respect to ν on $K_A^{\#}$.) ■

PROPOSITION 4.2. *Suppose that $\mu \in \mathcal{E}_{\sigma}(\Sigma_A)$ and that μ is equivalent to $\mathcal{H}^{\delta\mu}$. Then there exists a μ -a.e. finite measurable function g such that*

$$I_{\mu} = \delta_{\mu}L + g \circ \sigma - g \quad \mu\text{-a.e.} \quad (4.5)$$

PROOF. Let $g((x_n)_{n \in \mathbb{N}}) = \limsup_{n \rightarrow \infty} \frac{\mu([x_1, \dots, x_n])}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^{\delta\mu}}$ be the function which appeared in the proof of Theorem 3.6. Then the absolute continuity of μ means that $g(\underline{x}) < +\infty$ μ -a.e. as was seen in the proof of Theorem 3.4. Next $g(\underline{x}) > -\infty$ μ -a.e. can be assured by putting $\nu = \mu$ and $\delta = \delta_{\mu}$ in Lemma 4.1. Therefore the relation (4.5) can be deduced from the equality (3.5). ■

5. Equivalent invariant measures

Let us write $\mathcal{D}_{\sigma}(\Sigma_A)$ for the set of all the measure $\mu \in \mathcal{E}_{\sigma}(\Sigma_A)$ such that the set $\left\{ \sum_{i=0}^{n-1} \{I_{\mu}(\sigma^i x) + \log \mu([x_1, \dots, x_n])\} \right\}_{n \in \mathbb{N}}$ is bounded μ -a.e. In this section we shall show with the help of Ruelle-Perron-Frobenius operator theorem that there is no measure in $\mathcal{D}_{\sigma}(\Sigma_A)$ which is equivalent to its dimensional Hausdorff measure except for $\mu_{-\delta_A L}$.

LEMMA 5.1. *Suppose that $\mu \in \mathcal{E}_{\sigma}(\Sigma_A)$, $\text{supp}(\mu) \equiv \{x \in \Sigma_A \mid \text{there exists an open set } U(x) \ni x \text{ such that } \mu(U(x)) = 0\}^c = \Sigma_A$ and that there exists μ -a.e. finite function g such that $-I_{\mu} + \delta_{\mu}L = g - g \circ \sigma$ μ -a.e. Then*

$$\limsup_{n \rightarrow \infty} \sum_{i=0}^{n-1} \{-I_{\mu}(\sigma^i \underline{x}) + \delta_{\mu}L(\sigma^i \underline{x})\} = g(\underline{x}) - \mu\text{-essinf}(g) \quad \mu\text{-a.e.}$$

PROOF. Since $\text{supp}(\mu) = \Sigma_A$, $\chi_{\mu} > 0$ μ -a.e. or equivalently I_{μ} is finite μ -a.e. Therefore we may assume that I_{μ} is everywhere defined and finite. Taking an everywhere finitely defined version g_1 of g and putting $g_2 = \delta_{\mu}L - I_{\mu} + g_1 \circ \sigma$, we have $g_1 = g_2 = g$ μ -a.e. By the way, it follows from the definition of g_1 and g_2 that $S_n(\underline{x}) \equiv \sum_{i=0}^{n-1} \{-I_{\mu}(\sigma^i \underline{x}) + \delta_{\mu}L(\sigma^i \underline{x})\} = \sum_{i=0}^{n-1} \{g_2(\sigma^i \underline{x}) - g_1(\sigma^{i+1} \underline{x})\}$. Let us set $G = \{\underline{x} \in \Sigma_A \mid g_1(\underline{x}) = g_2(\underline{x})\}$ and $G_{\infty} = \bigcap_{n=0}^{\infty} \sigma_A^{-n} G$. Then $\mu(G_{\infty}) = 1$ and for all $\underline{x} \in G_{\infty}$ we have $S_n(\underline{x}) = g_1(\underline{x}) - g_1(\sigma^n \underline{x})$. Therefore

Poincaré's recurrence theorem says that

$$\begin{aligned} \limsup_{n \rightarrow \infty} S_n(\underline{x}) &= \limsup_{n \rightarrow \infty} \{g_1(\underline{x}) - g_1(\sigma^n \underline{x})\} = g_1(\underline{x}) - \liminf_{n \rightarrow \infty} g_1(\sigma^n \underline{x}) \\ &= g_1(\underline{x}) - \mu\text{-essinf}(g) \quad \mu\text{-a.e.} \quad \blacksquare \end{aligned}$$

LEMMA 5.2 *Suppose that $\mu \in \mathcal{D}_\sigma(\Sigma_A)$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} with $\text{supp}(\mu) = \Sigma_A$ and that there exists a μ -a.e. finite function such that $-I_\mu + \delta_\mu L = g - g \circ \sigma$ μ -a.e. Then $-\infty < \mu\text{-essinf}(g) < +\infty$.*

PROOF. We may assume that I_μ is everywhere finitely defined. Let us set $d_n(\underline{x}) = \sum_{i=0}^{n-1} I_\mu(\sigma^i \underline{x}) + \log \mu([x_1, \dots, x_n])$. Then by the definition of $\mathcal{D}_\sigma(\Sigma_A)$

$$-\infty < \inf_{n \in \mathbf{N}} \{d_n(\underline{x})\} \leq \sup_{n \in \mathbf{N}} \{d_n(\underline{x})\} < +\infty \quad \mu\text{-a.e.}$$

First clearly $\mu\text{-essinf}(g) < +\infty$ since g is μ -a.e. finite.

On the other hand the equality $\mu([x_1, \dots, x_n]) = \exp[d_n(\underline{x})] \times \exp[-\sum_{i=0}^{n-1} I_\mu(\sigma^i \underline{x})]$ for $\underline{x} = (x_n)_{n \in \mathbf{N}} \in \Sigma_A$ means that the absolute continuous condition in Theorem 3.6 is equivalent to

$$\limsup_{n \rightarrow \infty} S_n(\underline{x}) < +\infty \quad \mu\text{-a.e.}, \text{ where } S_n(\underline{x}) = \sum_{i=0}^{n-1} \{-I_\mu(\sigma^i \underline{x}) + \delta_\mu L(\sigma^i \underline{x})\}.$$

Therefore by Lemma 5.1 we have $\limsup_{n \rightarrow \infty} S_n(\underline{x}) = g(\underline{x}) - \mu\text{-essinf}(g) < +\infty$ μ -a.e. Thus we obtain $-\infty < \mu\text{-essinf}(g)$. \blacksquare

LEMMA 5.3. *Suppose that $\mu \in \mathcal{E}_\sigma(\Sigma_A)$, $\text{supp}(\mu) = \Sigma_A$ and that everywhere defined (on Σ_A) and finite function \mathfrak{g} and μ -a.e. finite function g with $-\infty < \mu\text{-essinf}(g)$ satisfy the relation*

$$\mathfrak{g}(\underline{x}) = -I_\mu(\underline{x}) + g(\sigma \underline{x}) - g(\underline{x}) \quad \mu\text{-a.e.} \quad (5.1)$$

Then for every $f \in C(\Sigma_A)$

$$\int \left[\sum_{\underline{y} \in \sigma_A^{-1} \underline{x}} \exp[\mathfrak{g}(\underline{y})] f(\underline{y}) \right] d\mu_g(\underline{x}) = \int f(\underline{x}) d\mu_g(\underline{x}), \quad (5.2)$$

where μ_g denotes the probability measure on Σ_A such that

$$\mu_g(B) = \frac{1}{D} \int_B \exp[-g(\underline{x})] d\mu(\underline{x}), \quad B \in \mathcal{B}_A, \quad D = \int \exp[-g(\underline{x})] d\mu(\underline{x})$$

REMARK 5.4. By the assumption of the lemma $0 < D \leq \exp[-\mu\text{-essinf}(g)]$

$< +\infty$.

PROOF. We may assume that I_μ is everywhere defined and finite as in previous lemmas. Let us take an everywhere finitely defined version g_1 of g and put $g_2(\underline{x}) = -\mathcal{G}(\underline{x}) - I_\mu(\underline{x}) + g_1(\sigma\underline{x})$. Then (5.1) implies that $g = g_1 = g_2$ μ -a.e. Therefore we have

$$\begin{aligned} & D \times \int \left[\sum_{\underline{y} \in \sigma_A^{-1}\underline{x}} \exp[\mathcal{G}(\underline{y})] f(\underline{y}) \right] d\mu_g(\underline{x}) \\ &= \int \left[\sum_{\underline{y} \in \sigma_A^{-1}\underline{x}} \exp[-I_\mu(\underline{y}) + g_1(\sigma\underline{y}) - g_2(\underline{y}) - g_1(\underline{x})] f(\underline{y}) \right] d\mu(\underline{x}) \\ &= \int \left[\sum_{\underline{y} \in \sigma_A^{-1}\underline{x}} \exp[-I_\mu(\underline{y}) - g_2(\underline{y})] f(\underline{y}) \right] d\mu(\underline{x}) \\ &= \int \left[\sum_{\underline{y} \in \sigma_A^{-1}\underline{x}} \chi_\mu(\underline{x}) \exp[-g_2(\underline{y})] f(\underline{y}) \right] d\mu(\underline{x}) \\ &= \int \exp[-g_2(\underline{x})] f(\underline{x}) d\mu(\underline{x}) = D \times \int f(\underline{x}) d\mu_g(\underline{x}), \end{aligned}$$

where we used the fact that for $\underline{y} \in \sigma_A^{-1}(\underline{x})$, $g_1(\sigma\underline{y}) = g(\underline{x})$ and the equality (2.1) for the μ -integrable function $\exp[-g]f$. (Note that $|\exp[-g]f| \leq \exp[-\mu\text{-essinf}(g)]|f|$.) ■

REMARK 5.5. As can be easily seen, if $\mathcal{G} = I_\mu$ and $g = 0$ in (5.1), (5.2) is nothing but the relation (2.1).

COROLLARY 5.6. Let μ, \mathcal{G} and g are the same as in Lemma 5.3. Furthermore we assume that $\mathcal{G} \in \mathcal{F}_A$. Then $P^A(\mathcal{G}) = 0$ and $\mu = \mu_g$.

PROOF. By Lemma 5.3 we have $\int \mathcal{L}_g^A f d\mu_g = \int f d\mu_g$ for all $f \in C(\Sigma_A)$. In other words, $(\mathcal{L}_g^A)^* \mu_g = \mu_g$. Therefore by remarks in §2 we have $P^A(\mathcal{G}) = 0$ and $\mu = \mu_g$. ■

COROLLARY 5.7. If $I_\mu \in \mathcal{F}_A$, then $P^A(-I_\mu) = 0$ and $\mu_{-I_\mu} = \mu$.

PROOF. We have only to put $\mathcal{G} = -I_\mu$ and $g = 0$ in Corollary 5.6. ■

REMARK 5.8. (1) Since $(\mathcal{L}_{-I_\mu}^A)^* \mu = \mu$ and $\mu_{-I_\mu} = \mu$, the uniqueness of the eigenfunction implies that $\mathcal{L}_{-I_\mu}^A 1 = 1$, that is $\sum_{\underline{y} \in \sigma_A^{-1}\underline{x}} \exp[-I_\mu(\underline{y})] = \sum_{\underline{y} \in \sigma_A^{-1}\underline{x}} \chi_\mu(\underline{y}) = 1$ for all $\underline{x} \in \Sigma_A$.

(2) The result $P(-I_\mu) = \sup_{\lambda \in \mathcal{E}_\sigma(\Sigma_A)} \left\{ h_\lambda(\sigma) - \int I_\nu d\lambda \right\} = 0$ implies that $h_\lambda(\sigma) =$

$\int I_\lambda d\lambda \leq \int I_\nu d\lambda$ for $\lambda \in \mathcal{E}_\sigma(\Sigma_A)$ with the equality if and only if $\lambda = \mu$.

(3) In order to deduce $\mathcal{L}_{-I_\mu}^* \mu = \mu$, we do not need to assume that $I_\mu \in \mathcal{F}_A$. For that purpose we have only to assume that $I_\mu \in C(\Sigma_A)$. But in this case the uniqueness of the equilibrium states for $-I_\mu$ is not assured.

These results are already obtained in [12] and [22] in somewhat different contexts.

THEOREM 5.9. *Suppose that $\mu \in \mathcal{D}_\sigma(\Sigma_A)$ and $\mu \neq \mu_{-\delta_\mu L}$. Then $\mu \circ \psi^{-1}$ is not equivalent to \mathcal{H}^{δ_μ} .*

PROOF. Suppose that $\mu \in \mathcal{D}_\sigma(\Sigma_A)$ and $\mu \circ \psi^{-1}$ is equivalent to \mathcal{H}^{δ_μ} on K_A . Then clearly $\text{supp}(\mu) = \Sigma_A$. Furthermore Theorem 4.2 and Lemma 5.2 indicate the existence of μ -a.e. finite function g such that $-I_\mu + \delta_\mu L = g - g \circ \sigma$ μ -a.e. with μ -essinf(g) $> -\infty$. It follows from Lemma 5.6 that $P^A(-\delta_\mu L) = 0$ so that $\delta_\mu = \delta_A$ and again from Lemma 5.6 that $\mu = \mu_{-\delta_\mu L}$. But this is a contradiction. ■

6. Absolutely continuous measures with respect to their dimensional Hausdorff measures

In this section we treat a class of measures which have a considerable mixing property and examine their absolute continuity with respect to the Hausdorff measures. The central limiting property of these measures enables us to work under a weaker condition than the previous section. That is, we shall develop our arguments only assuming the absolute continuity to the Hausdorff measures.

First let \mathcal{B}^n and \mathcal{B}_n be sub σ -fields of \mathcal{B}_A generated by the families $\mathcal{C}_{A,1}^n$ and $\mathcal{C}_{A,n}^\infty$ respectively. For $\mu \in \mathcal{P}(\Sigma_A)$, we define

$$\phi_\mu(n) = \sup_{n \in \mathbb{N}} \sup_{B \in \mathcal{B}_{n+m}} (\text{esssup}_{x \in \Sigma_A} |\mu(B|\mathcal{B}^n)(x) - \mu(B)|).$$

Under these notations we define

$$\mathcal{F}_\sigma(\Sigma_A) = \mathcal{D}_\sigma(\Sigma_A) \cap \{\mu \in \mathcal{P}(\Sigma_A) \mid I_\mu \in L^2(\Sigma_A, \mu), \sum_{n=1}^{\infty} \sqrt{\phi_\mu(n)} < \infty\}.$$

Now we quote a central limit theorem from [11]. See also [9] and [14]. For $f \in L^2(\Sigma_A, \mu)$ the following limit $\bar{\sigma}(f)$ exists:

$$\bar{\sigma}(f) = \left(\lim_{n \rightarrow \infty} \frac{1}{n} \int \left| \sum_{k=0}^{n-1} f \circ \sigma^k - n \int f d\mu \right|^2 d\mu \right)^{1/2}.$$

Furthermore if $\bar{\sigma}(f) > 0$ then the following central limit theorem holds:

$$\lim_{n \rightarrow \infty} \mu \left(\left\{ \underline{x} \in \Sigma_A \left| \frac{\sum_{k=0}^{n-1} f(\sigma^k \underline{x}) - n \int f d\mu}{\bar{\sigma}(f) \sqrt{n}} < a \right. \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{x^2}{2}} dx.$$

On the basis of the above theorem we have the following theorem.

THEOREM 6.1 *Suppose that $\mu \in \mathcal{F}_\sigma(\Sigma_A)$ and $\mu \neq \mu_{\delta_\mu L}$. Then $\mu \circ \psi^{-1}$ and \mathcal{H}^{δ_μ} are mutually singular on K_A .*

PROOF. Set $\mathcal{G} = -I_\mu + \delta_\mu L$. By the assumption $\mathcal{G} \in L^2(\mu)$ and $\int \mathcal{G} d\mu = 0$.

Assume that $\mu \circ \psi^{-1}$ and \mathcal{H}^{δ_μ} are not mutually singular on K_A . Then by Theorem 3.6 $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} on K_A . Again by Theorem 3.6 we have

$$\limsup_{n \rightarrow \infty} \frac{\mu \circ \psi^{-1}(\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V}) \cap K_A)}{|\varphi_{x_1} \circ \dots \circ \varphi_{x_n}(\bar{V})|^{\delta_\mu}} < +\infty \quad \mu\text{-a.e.}$$

Since $\mu \in \mathcal{D}_\sigma(\Sigma_A)$, this is equivalent to the condition

$$\limsup_{n \rightarrow \infty} S_n(\underline{x}) < +\infty \quad \mu\text{-a.e. } \underline{x} \in \Sigma_A, \quad S_n(\underline{x}) = \sum_{i=0}^{n-1} \mathcal{G}(\sigma^i \underline{x}) \quad (6.1)$$

as we have seen in §5.

Set $\bar{\sigma} = \bar{\sigma}(\mathcal{G}) = \left(\lim_{n \rightarrow \infty} \frac{1}{n} \int |S_n|^2 d\mu \right)^{1/2}$. Suppose that $\bar{\sigma} > 0$. Then

$$\lim_{n \rightarrow \infty} \mu \left(\left\{ \underline{x} \in \Sigma_A \left| \frac{S_n(\underline{x})}{\bar{\sigma} \sqrt{n}} > a \right. \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} e^{-\frac{x^2}{2}} dx, \quad \text{for all } a \in \mathbf{R}.$$

But this is impossible in view of (6.1). For, if we put $h(\underline{x}) = \sup_{n \in \mathbf{N}} S_n(\underline{x})$, then for $n \in \mathbf{N}$ we have

$$\mu \left(\left\{ \underline{x} \in \Sigma_A \left| \frac{S_n(\underline{x})}{\bar{\sigma} \sqrt{n}} > 1 \right. \right\} \right) \leq \mu \left(\left\{ \underline{x} \in \Sigma_A \left| \frac{h(\underline{x})}{\bar{\sigma} \sqrt{n}} > 1 \right. \right\} \right) = \int 1_{A_n}(\underline{x}) d\mu(\underline{x}), \quad (6.2)$$

where $A_n = \left\{ \underline{x} \in \Sigma_A \left| \frac{h(\underline{x})}{\bar{\sigma}} > \sqrt{n} \right. \right\}$. Since $1_{A_n} \leq 1$ and $1_{A_n} \rightarrow 0$ μ -a.e. by (6.1),

letting $n \rightarrow \infty$, we have a contradiction that $\frac{1}{\sqrt{2\pi}} \int_1^{+\infty} e^{-\frac{x^2}{2}} dx \leq 0$.

Therefore $\bar{\sigma} = 0$. Then there exists $g \in L^2(\Sigma_A, \mu)$ such that $\mathcal{G} = -I_\mu + \delta_\mu L = g - g \circ \sigma$ μ -a.e. (see for example Lemma 2.1 of [15]) and Lemma 5.2

indicates that $\mu\text{-essinf}(g) > -\infty$. Therefore by Corollary 5.6 we easily have $\delta_\mu = \delta_A$ and $\mu = \mu_{-\delta_A L}$ as in Theorem 5.9. This does not match our assumption. Hence μ is not absolutely continuous with respect to \mathcal{H}^{δ_μ} and this means that μ and \mathcal{H}^{δ_μ} are mutually singular on K_A by Theorem 3.6. ■

REMARK 6.2. Suppose that $\int |I_\mu|^{2+\delta} d\mu < +\infty$ for some $\delta > 0$ and $\phi(1) < 1$. With these additional assumptions the following law of iterated logarithm

$$\limsup_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} \{-I_\mu(\sigma^i \underline{x}) + \delta_\mu L(\sigma^i \underline{x})\}}{\bar{\sigma} \sqrt{2n \log \log n}} = 1,$$

$$\liminf_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} \{-I_\mu(\sigma^i \underline{x}) + \delta_\mu L(\sigma^i \underline{x})\}}{\bar{\sigma} \sqrt{2n \log \log n}} = -1, \quad \mu\text{-a.e.}$$

holds if $\bar{\sigma} > 0$. Thus in this case the condition $\bar{\sigma} = 0$ can be obtained by the above equality as well. For general treatments of the law of iterated logarithm for weakly independent random variables, i.e. random variables with the condition corresponding to $\sum_{n=1}^{\infty} \sqrt{\phi(n)} < +\infty$, see [11] and [14].

We call a contraction map $\varphi: \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a similar map if there exists a constant $0 < r(\varphi) < 1$ such that $|\varphi(x) - \varphi(y)| = r(\varphi)|x - y|$ for all $x, y \in \mathbf{R}^d$. If φ is a similar map it is easy to see that $j_\varphi = r(\varphi)$ and that $L((x_n)_{n \in \mathbf{N}}) = -\log r(\varphi_{x_1})$ for the self-similar set K constructed from an N -tuples of contractions $\Phi = \{\varphi_1, \dots, \varphi_N\}$. In this case $P^A(-\delta L)$ is equal to $\log \lambda_\delta$, where λ_δ is the maximal eigenvalue of the matrix $L_\delta^A = ((L_\delta^A)_{ij})_{i,j \in S}$, $(L_\delta^A)_{ij} = A_{ij} r(\varphi_i)^{-\delta}$. Furthermore the equilibrium state for $-\delta L$ is the (simple) Markov measure $\nu_{\bar{P}}$, where the defining probability matrix $\bar{P} = (\bar{P}_{ij})_{i,j \in S}$ is given by $\bar{P}_{ij} = \frac{\tilde{A}_{ij} r_j}{\lambda_\delta r_i}$, $(r_i)_{i \in S}$ is the (simple positive) right eigenvector of \tilde{A} corresponding to the eigenvalue λ_δ . This is obtained just by an application of the classical Perron-Frobenius theorem. See [19] for the classical Perron-Froebnius theorem and [13] for its applications. The Hausdorff dimension of K_A is given by the unique positive solution of $P^A(-\delta L) = \log \lambda_\delta = 0$. Theorem 6.1 enables us to characterize $\mu_{-\delta_A L}$ as the unique absolutely continuous multiple Markov measure supported by Σ_A with respect to its dimensional Hausdorff measure. Moreover, since $I_{\mu_{-\delta_A L}} \in \mathcal{F}_A$ in this case, we see that there exists $g \in \mathcal{F}_A$ such that $-I_{\mu_{-\delta_A L}} = -\delta_A L + g - g \circ \sigma$. But Corollary 42 in [13] in fact says furthermore that g depends only on the first coordinate.

7. Remarks on cohomology

In §4 we obtained a cohomologous relation (4.5) for $\mu \in \mathcal{E}_\sigma(\Sigma_A)$ and used it in the arguments in §5 and §6. We mention here that the following stronger results holds.

LEMMA 7.1. *If $\mu \in \mathcal{D}_\sigma(\Sigma_A)$ and $\mu \circ \psi^{-1}$ is equivalent to \mathcal{H}^{δ_μ} , then there exists a measurable function g and $F \in \mathcal{B}_A$ which satisfy the following conditions.*

- (1) $g(\underline{x})$ is finite for all $\underline{x} \in F$.
- (2) $-I_\mu(\underline{x}) = -\delta_\mu L(\underline{x}) - g(\underline{x}) + g(\sigma \underline{x})$ for all $\underline{x} \in F$.
- (3) $\sigma_A^{-1} F = F$, $\mu(F) = 1$.

PROOF. As before we may assume that I_μ is everywhere finitely defined. As was seen in Lemma 5.5 we have $\limsup_{n \rightarrow \infty} S_n(\underline{x}) < +\infty$ μ -a.e. where $S_n(\underline{x}) = \sum_{i=0}^{n-1} \{-I_\mu(\sigma^i \underline{x}) + \delta_\mu L(\sigma^i \underline{x})\}$. Set $g(\underline{x}) = \limsup_{n \rightarrow \infty} S_n(\underline{x})$. Since $S_{n+1}(\underline{x}) = -I_\mu(\underline{x}) + \delta_\mu L(\underline{x}) + S_n(\sigma \underline{x})$ for all $n \in \mathbb{N}$, we clearly have

$$g(\underline{x}) = -I_\mu(\underline{x}) + \delta_\mu L(\underline{x}) + g(\sigma \underline{x}). \quad (7.1)$$

Set $F = \{\underline{x} \in \Sigma_A \mid -\infty < g(\underline{x}) < +\infty\}$. Then $\mu(F) = 1$. On the other hand since (7.1) indicates that $g(\underline{x})$ is finite if and only if $g(\sigma \underline{x})$ is finite, we have $\sigma_A^{-1} F = F$. Of course it follows from (7.1) that $-I_\mu(\underline{x}) = -\delta_\mu L(\underline{x}) + g(\underline{x}) - g(\sigma \underline{x})$ on F . ■

The lemma above enables us to relax some tediousness arising from the ambiguity of functions on null sets in §5. (But it seems of no direct use in §6.) In fact for an example we can carry on the computation in Lemma 5.3 with $g_1 = g_2 = g$ if we restrict the integrated domain to F .

We next consider the uniqueness of the function g in the cohomologous relation up to constant.

LEMMA 7.2. *Suppose that $\mu \in \mathcal{D}_\sigma(\Sigma_A)$ and $\mu \circ \psi^{-1}$ is absolutely continuous with respect to \mathcal{H}^{δ_μ} . If*

$$-I_\mu = -\delta_\mu L + g - g \circ \sigma, \quad -I_\mu = -\delta_\mu L + h - h \circ \sigma \quad \mu\text{-a.e.}$$

for μ -a.e. finite g and h , then $g = h + c$ μ -a.e. for some constant c .

PROOF. By Lemma 5.1 we have

$$\limsup_{n \rightarrow \infty} S_n(\underline{x}) = g(\underline{x}) - \mu\text{-essinf}(g) = h(\underline{x}) - \mu\text{-essinf}(h) \quad \mu\text{-a.e.}$$

Recalling that Lemma 5.2 says that $\mu\text{-essinf}(g)$ and $\mu\text{-essinf}(h)$ are both finite, we have $g = h + c$ μ -a.e. for $c = \mu\text{-essinf}(g) - \mu\text{-essinf}(h)$. ■

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*Department of Mathematics
Faculty of Science
Hiroshima University*