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# Limit sets and square roots of homeomorphisms

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**ABSTRACT.** In this paper, we define the positive and negative limit sets and characterize their dynamical properties in terms of the non-Hausdorff sets. By using these sets, we also consider the condition that a given homeomorphism has a square root, and give an example of a wandering homeomorphism without square roots.

## 1. Introduction

For a given homeomorphism $f$, a homeomorphism $g$ satisfying $g \cdot g = f$ is called a *square root* of $f$. Although the square root is not unique in general, we always denote $g$ by $\sqrt{f}$. In this paper, we consider the condition that a given homeomorphism of a non-compact space has a square root. For this purpose, we use the positive and negative limit sets, which will be defined and characterized in §2.

For homeomorphisms of compact spaces, we can use the nonwandering sets to show the non-existence of square roots as follows: Let $f$ be a homeomorphism. For any homeomorphism $h$, the nonwandering set $\Omega(f)$ of $f$ and the nonwandering set $\Omega(hfh^{-1})$ of $hfh^{-1}$ satisfy the relation $h\Omega(f) = \Omega(hfh^{-1})$. Since the square root $\sqrt{f}$ commutes with $f$, $\Omega(f)$ is also invariant under $\sqrt{f}$. By using this fact, we can construct a homeomorphism which has no square roots: We choose a homeomorphism $f$ of $S^2$ such that $f$ preserves each curve of Figure 1 and exchanges two thorns on the equator. In particular, the equator with two thorns is invariant under $f$, and $f$ exchanges their branch points. If $f$ has a square root $\sqrt{f}$, then the equator with two thorns is also invariant under $\sqrt{f}$ because $\Omega(f)$ consists of two fixed points and the equator with two thorns. Thus $\sqrt{f}$ either exchanges or fixes the branch points. However this contradicts the assumption that its square $f$ exchanges them.

The above argument cannot be applied to study the square roots of homeomorphisms of non-compact spaces with empty nonwandering set. Thus

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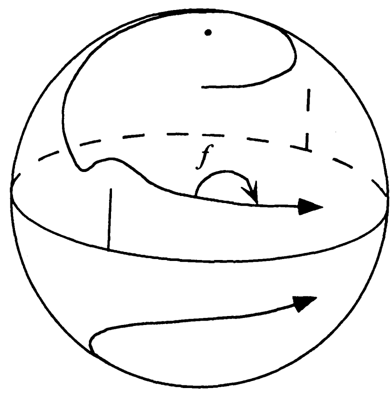


Figure 1: $f$ has no square roots.

we will define the positive and negative limit sets in §2 which replaces the role of the nonwandering sets in the above argument. In §2, we also characterize the positive and negative limit sets in terms of the non-Hausdorff sets, which were already studied in [3]. In §3, we show the relationship between the positive (negative) limit set of $f$ and that of $\sqrt{f}$. By using these limit sets, we give an example of a wandering homeomorphism without square roots and obtain the topological uniqueness of square roots in some special case.

## 2. Positive and negative limit sets

Let $X$ be a metric space and $f$ a homeomorphism of $X$. For a compact set $K$, we define its *ω-limit* (resp. *α-limit*) *set* by $\omega_f(K) = \bigcap_{n \in \mathbf{Z}} \overline{\bigcup_{m \geq n} f^m(K)}$ (resp. $\alpha_f(K) = \bigcap_{n \in \mathbf{Z}} \overline{\bigcup_{m \leq n} f^m(K)}$). By definition, $\omega_f(K)$ and $\alpha_f(K)$ are closed sets. Furthermore, we can prove the following Lemmas 1 and 2 immediately.

LEMMA 1. *Let $f$ and $h$ be homeomorphisms of $X$. For any compact set $K$, the following equalities hold*:

$$h\omega_f(K) = \omega_{hfh^{-1}}(h(K)),$$

$$h\alpha_f(K) = \alpha_{hfh^{-1}}(h(K)).$$

LEMMA 2. *Let $f$ be a homeomorphism of $X$. Then $\alpha_f(K)$ coincides with $\omega_{f^{-1}}(K)$ for any compact set $K$.*

As in the case of the $\omega$-limit ($\alpha$-limit) set of a point, we can also define the $\omega$-limit ($\alpha$-limit) set of a compact set by means of point sequences:

LEMMA 3. *A point* $p$ *is contained in* $\omega_f(K)$ *(resp.* $\alpha_f(K)$*) if and only if there are a converging point sequence* $\{q_n\}_{n=1,2,\dots}$ *of* $K$ *and integers* $m_n$ $(n = 1, 2, \dots)$ *satisfying that* $m_n \geqq n$ *(resp.* $m_n \leqq -n$*) and* $\lim_{n\to\infty} f^{m_n}(q_n) = p$.

PROOF. Let $p$ be a point of $\omega_f(K)$. Then $p$ is contained in $\overline{\bigcup_{m\geqq n} f^m(K)}$ for any integer $n$. Hence there is a point $p_n$ $(n = 1, 2, \dots)$ of $\bigcup_{m\geqq n} f^m(K)$ such that $d(p, p_n) < 1/n$. We choose an integer $m_n$ greater than or equal to $n$ such that $p_n$ is contained in $f^{m_n}(K)$. Let $q_n = f^{-m_n}(p_n)$. Then $q_n$ is contained in $K$ and $\lim_{n\to\infty} f^{m_n}(q_n) = \lim_{n\to\infty} p_n = p$. Since $K$ is compact, we can choose a subsequence $\{n_i\}_{i=1,2,\dots}$ such that $\{q_{n_i}\}$ converges to some point of $K$ as $i \to \infty$. Since $m_{n_i} \geqq n_i \geqq i$ and $\lim_{i\to\infty} f^{m_{n_i}}(q_{n_i}) = p$, the point sequence $\{q_{n_i}\}_{i=1,2,\dots}$ and integers $m_{n_i}$ $(i = 1, 2, \dots)$ satisfy the condition of Lemma 3.

Conversely, we assume that there are a converging point sequence $\{q_n\}_{n=1,2,\dots}$ of $K$ and integers $m_n$ $(m_n \geqq n)$ satisfying $\lim_{n\to\infty} f^{m_n}(q_n) = p$. We fix an arbitrary integer $N$. If $n \geqq N$, then $m_n \geqq n \geqq N$, and hence $f^{m_n}(q_n)$ is an element of $\bigcup_{m\geqq N} f^m(K)$. Therefore $p$ is contained in $\overline{\bigcup_{m\geqq N} f^m(K)}$. This implies that $p$ is an element of $\omega_f(K)$.

The statement for $\alpha_f(K)$ follows from Lemma 2. □

By taking sufficiently many compact sets, we will define the positive and negative limit sets. Let $X$ be a metric space with a countable base $\mathcal{O} = \{U_i\}_{i=1,2,\dots}$ such that each $\overline{U_i}$ is a compact set. For a homeomorphism $f$ of $X$, we define the *positive* (resp. *negative*) *limit set* by $\omega^{\mathcal{O}}(f) = \bigcup_{i=1,2,\dots} \omega_f(\overline{U_i})$ (resp. $\alpha^{\mathcal{O}}(f) = \bigcup_{i=1,2,\dots} \alpha_f(\overline{U_i})$). Then these sets do not depend on the choice of countable bases by the following lemma:

LEMMA 4. *Let* $\mathcal{O}_1 = \{U_i\}_{i=1,2,\dots}$ *and* $\mathcal{O}_2 = \{V_j\}_{j=1,2,\dots}$ *be countable bases such that* $\overline{U_i}$ *and* $\overline{V_j}$ *are compact sets. Then* $\omega^{\mathcal{O}_1}(f)$ *(resp.* $\alpha^{\mathcal{O}_1}(f)$*) coincides with* $\omega^{\mathcal{O}_2}(f)$ *(resp.* $\alpha^{\mathcal{O}_2}(f)$*) for any homeomorphism* $f$.

PROOF. Let $p$ be an element of $\omega^{\mathcal{O}_1}(f)$. Then $p$ is contained in $\omega_f(\overline{U_i})$ for some $U_i$. By Lemma 3, there are a converging point sequence $\{q_n\}_{n=1,2,\dots}$ of $\overline{U_i}$ and integers $m_n$ $(n = 1, 2, \dots)$ satisfying that $m_n \geqq n$ and $\lim_{n\to\infty} f^{m_n}(q_n) = p$. Since $\overline{U_i}$ is compact, $\overline{U_i}$ is covered by a finitely many member of $\{V_j\}$. Therefore infinitely many $q_n$ are contained in some $\overline{V_j}$. By taking a subsequence, we may assume that the points $q_n$ of $\overline{V_j}$ converge to some point of $\overline{V_j}$ and $\lim_{n\to\infty} f^{m_n}(q_n) = p$. By Lemma 3 again, $p$ is contained in $\omega_f(\overline{V_j})$, and also in $\omega^{\mathcal{O}_2}(f)$.

By Lemma 2, $\alpha^{\mathcal{O}_1}(f) = \omega^{\mathcal{O}_1}(f^{-1})$ and $\alpha^{\mathcal{O}_2}(f) = \omega^{\mathcal{O}_2}(f^{-1})$. Therefore $\alpha^{\mathcal{O}_1}(f)$ coincides with $\alpha^{\mathcal{O}_2}(f)$. □

Since $\omega^{\mathcal{O}}(f)$ (resp. $\alpha^{\mathcal{O}}(f)$) does not depend on the choice of such a countable base $\mathcal{O}$, we denote it by $\omega(f)$ (resp. $\alpha(f)$) in the following.

THEOREM 1. *For any homeomorphism $f$, $\omega(f)$ and $\alpha(f)$ are invariant under $f$.*

PROOF. Let $\{U_i\}_{i=1,2,\dots}$ be a countable base such that $\overline{U_i}$ is compact. Then $\omega(f)$ is equal to $\bigcup_{i=1,2,\dots}\omega_f(\overline{U_i})$. By taking $f$ as a homeomorphism $h$ of Lemma 1, we obtain the equation $f(\omega_f(\overline{U_i})) = \omega_f(f(\overline{U_i}))$. Hence $f(\omega(f)) = \bigcup_{i=1,2,\dots}\omega_f(\overline{f(U_i)})$. Since $\{f(U_i)\}_{i=1,2,\dots}$ is also a countable base such that each $\overline{f(U_i)}$ is compact, $\bigcup_{i=1,2,\dots}\omega_f(\overline{f(U_i)})$ is also the positive limit set by Lemma 4. Thus $f(\omega(f))$ coincides with $\omega(f)$. The negative limit set $\alpha(f)$ is also invariant under $f$ by Lemma 2. □

REMARK. By definition, $\omega_f(K)$ and $\alpha_f(K)$ are closed sets. However $\omega(f)$ and $\alpha(f)$ are not always closed. Moreover, there exists a homeomorphism $f$ of $\mathbf{R}^2$ such that $\omega(f)$ and $\alpha(f)$ are not closed and dense while $\Omega(f)$ is empty (See [2] and the following Example 1).

If $X$ is compact, then both $\omega(f)$ and $\alpha(f)$ coincide with $X$ because we can take $X$ as one of the open sets $\{U_i\}$. Thus $\omega(f)$ and $\alpha(f)$ are interesting only when we consider homeomorphisms of non-compact spaces. In the rest of this section, we will consider homeomorphisms whose nonwandering sets are empty, which are called *wandering homeomorphisms*. For such homeomorphisms, the limit sets play an important role. First we give a typical example of such homeomorphisms.

EXAMPLE 1. Let $X = \mathbf{R}^2$. Denote by $\mathfrak{F}$ the foliation of $X$ containing the Reeb component (Figure 2). We choose a leaf-preserving and fixed point free homeomorphism $f$ of $X$ as illustrated in Figure 2. Then $f$ is a wandering homeomorphism. We take a compact set $K$ intersecting the upper boundary of the Reeb component as in Figure 2. Then the $\omega$-limit set $\omega_f(K)$ is the lower boundary of the Reeb component. As a result, $\omega(f)$ is the lower boundary of the Reeb component and $\alpha(f)$ is the upper one.

In the case of wandering homeomorphisms, we can give a geometric characterization of $\omega(f)$ and $\alpha(f)$ by using the non-Hausdorff sets defined as follows: Let $f$ be a homeomorphism of $X$. Denote by $\pi: X \to X/f$ the quotient map which maps each orbit of $f$ to a point. In general, the orbit space $X/f$ with the quotient topology is not Hausdorff. A point $p$ of $X$ is called *non-Hausdorff* if $\pi(p)$ is not "Hausdorff" in $X/f$, i.e. there is a point $q$ of $X$ which is not contained in the orbit of $p$ and, for any neighborhoods $U$ and $V$ of $p$ and $q$ respectively, $(\bigcup_{n\in\mathbf{Z}} f^n(U)) \cap (\bigcup_{n\in\mathbf{Z}} f^n(V))$ is not empty. We call the set of non-Hausdorff points the *non-Hausdorff set*, denoted by $NH(f)$.

The following theorem shows the relationship between $\omega(f)$, $\alpha(f)$ and $NH(f)$ for wandering homeomorphisms.

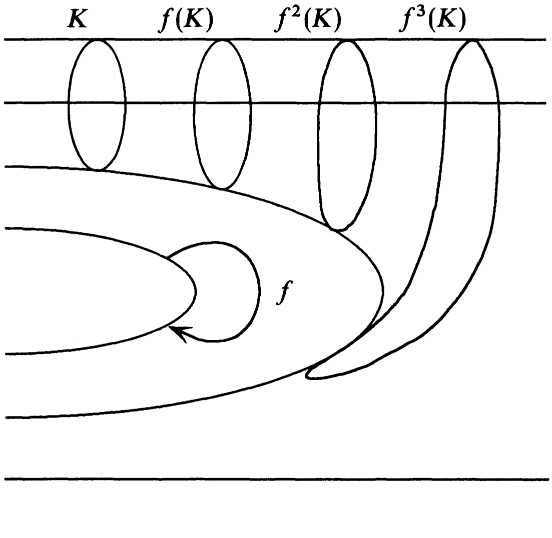


Figure 2: The Reeb component.

THEOREM 2. *Let $f$ be a wandering homeomorphism. Then the non-Hausdorff set is the union of $\omega(f)$ and $\alpha(f)$.*

PROOF. First we show that $NH(f)$ contains $\omega(f)$ and $\alpha(f)$. Let $p$ be a point of $\omega(f)$. By Lemma 3, there are a converging point sequence $\{q_n\}_{n=1,2,\dots}$ in some compact set $\overline{U_i}$ and integers $m_n$ $(n = 1, 2, \dots)$ such that $m_n \geqq n$ and $\lim_{n\to\infty} f^{m_n}(q_n) = p$. Denote by $q$ the limit point of $\{q_n\}$.

Suppose that $q$ is in the orbit $O(p)$ passing through $p$, i.e. $f^k(p) = q$ for some integer $k$. Since $\lim_{n\to\infty} f^{m_n+k}(q_n) = f^k(p) = q$, any neighborhood $V$ of $q$ contains $f^{m_n+k}(q_n)$ and $q_n$ for sufficiently large $n$, and hence $V$ intersects $\bigcup_{|m|\geqq N} f^m(V)$ for every $N \in \mathbf{Z}$. Therefore $q$ is nonwandering. Since this contradicts the assumption that $f$ is wandering, $q$ is not contained in the orbit $O(p)$.

For any neighborhood $U$ of $p$, $f^{m_n}(q_n)$ is contained in $U$ for sufficiently large $n$, and hence $q_n$ is an element of the saturation $\bigcup_{m\in\mathbf{Z}} f^m(U)$. Thus $q$ is contained in $\overline{\bigcup_{m\in\mathbf{Z}} f^m(U)}$. Therefore any neighborhood of $q$ intersects $\bigcup_{m\in\mathbf{Z}} f^m(U)$. By the above consideration, $p$ is non-Hausdorff. We can show $\alpha(f) \subset NH(f)$ in the same way.

Next we prove that $NH(f)$ is contained in $\omega(f)\cup\alpha(f)$. We take a countable base $\{U_i\}$ of $X$ such that each $\overline{U_i}$ is compact. Let $p$ be a point of $NH(f)$ and let $q$ be a pair of $p$, i.e. a point $q$ is disjoint from $O(p)$ and is contained in $\overline{\bigcup_{m\in\mathbf{Z}} f^m(U)}$ for any neighborhood $U$ of $p$. Since $\Omega(f)$ is empty, the orbit $O(p)$ is a closed set. Hence there is an open neighborhood $V$ of $q$ disjoint from $O(p)$. We choose an open set $U_i$ from the countable base such that $q \in U_i \subset \overline{U_i} \subset V$.

We will show that $p$ is contained in $\omega_f(\overline{U_i}) \cup \alpha_f(\overline{U_i})$. First remark that $p$ is not contained in $O(\overline{U_i}) = \bigcup_{m \in \mathbf{Z}} f^m(\overline{U_i})$ because $O(p) \cap \overline{U_i}$ is empty. Since $U_i$ is a neighborhood of $q$, $p$ is contained in $\overline{\bigcup_{m \in \mathbf{Z}} f^m(U_i)}$ by the choice of $p$ and $q$. Then we have

$$\begin{aligned} p &\in \overline{O(\overline{U_i})} - O(\overline{U_i}) \\ &= \left\{\left(\overline{\bigcup_{m \leqq -n} f^m(\overline{U_i})}\right) \cup \left(\overline{\bigcup_{|m| < n} f^m(\overline{U_i})}\right) \cup \left(\overline{\bigcup_{m \geqq n} f^m(\overline{U_i})}\right)\right\} - O(\overline{U_i}) \\ &= \left\{\left(\overline{\bigcup_{m \leqq -n} f^m(\overline{U_i})}\right) \cup \left(\bigcup_{|m| < n} f^m(\overline{U_i})\right) \cup \left(\overline{\bigcup_{m \geqq n} f^m(\overline{U_i})}\right)\right\} - O(\overline{U_i}) \\ &\subset \left(\overline{\bigcup_{m \leqq -n} f^m(\overline{U_i})}\right) \cup \left(\overline{\bigcup_{m \geqq n} f^m(\overline{U_i})}\right) \end{aligned}$$

for any $n \geqq 0$. If $p$ is contained in $\overline{\bigcup_{m \geqq n} f^m(\overline{U_i})}$ for infinitely many $n$ ($n \geqq 0$), then $p$ is contained in $\overline{\bigcup_{m \geqq n} f^m(\overline{U_i})}$ for every $n \in \mathbf{Z}$. Hence $p$ is a point of $\omega_f(\overline{U_i})$. Otherwise $p$ is a point of $\alpha_f(\overline{U_i})$.

Thus $NH(f)$ is cointained in $\omega(f) \cup \alpha(f)$. □

THEOREM 3. *Let $f$ be a wandering homeomorphism. If $\omega(f)$ (resp. $\alpha(f)$) is empty, then the non-Hausdorff set $NH(f)$ is also empty.*

PROOF. By Theorem 2 and Lemma 2, it is enough to show that $\omega(f)$ is not empty if $\alpha(f)$ is not empty. Let $U_i$ be an open set of the countable base $\{U_i\}_{i=1,2,\ldots}$ such that $\alpha_f(\overline{U_i})$ is not empty, where $\overline{U_i}$ is compact. Let $p$ be a point of $\alpha_f(\overline{U_i})$. By Lemma 3, there are a converging point sequence $\{q_n\}_{n=1,2,\ldots}$ of $\overline{U_i}$ and integers $m_n$ ($m_n \leqq -n$) such that $p = \lim_{n\to\infty} f^{m_n}(q_n)$. Let $q$ denote the limit point of $\{q_n\}$. We choose a neighborhood $U_j$ from the countable base $\{U_i\}_{i=1,2,\ldots}$ containing $p$. For sufficiently large $n$, $f^{m_n}(q_n)$ is contained in $\overline{U_j}$. Furthermore, $-m_n \geqq n$ and $\lim_{n\to\infty} f^{-m_n}(f^{m_n}(q_n)) = q$. Since the point sequence $\{f^{m_n}(q_n)\}_{n=1,2,\ldots}$ and integers $-m_n$ ($n = 1, 2, \ldots$) satisfy the condition of Lemma 3, $q$ is a point of $\omega_f(\overline{U_j})$. Therefore $\omega(f)$ is not empty. □

From the consideration of $NH(f)$ in [3], we obtain the following properties of $\omega(f)$ and $\alpha(f)$:

THEOREM 4. *Let $f$ be an orientation preserving and fixed point free homeomorphism of $\mathbf{R}^2$. If its positive (resp. negative) limit set is empty, then $f$ is topologically conjugate to the translation $T: \mathbf{R}^2 \to \mathbf{R}^2$; $T(x, y) = (x + 1, y)$.*

THEOREM 5. *Let $X$ be a metric space with a countable base $\{U_i\}$ such that each $\overline{U}_i$ is compact. If $X$ is, moreover, a Baire space, then the limit sets $\omega(f)$ and $\alpha(f)$ have no interior points for any wandering homeomorphism $f$ of $X$.*

PROOF. By Theorem 2 in [3], the non-Hausdorff set has no interior points. Hence $\omega_f(\overline{U}_i)$ $(i = 1, 2, \ldots)$ are closed sets without interior points. Since $X$ is a Baire space, their countable union has also no interior points. □

REMARK. In [3], only homeomorphisms of $\mathbf{R}^2$ are considered. However most results of this paper are valid for wandering homeomorphisms.

## 3. Limit sets for square roots

In this section, we use the positive (negative) limit sets for the square roots of homeomorphisms. In the following, we assume that $X$ is a metric space with a countable base $\{U_i\}_{i=1,2,\ldots}$ such that each $\overline{U}_i$ is compact and $f$ is a homeomorphism of $X$.

THEOREM 6. *If a homeomorphism $f$ of $X$ has a square root $\sqrt{f}$, then the positive (resp. negative) limit set $\omega(f)$ (resp. $\alpha(f)$) of $f$ is invariant under $\sqrt{f}$.*

PROOF. Since $f$ is the square of $\sqrt{f}$, $\sqrt{f}$ commutes with $f$. Hence we obtain the equality $\sqrt{f}(\omega_f(\overline{U}_i)) = \omega_f(\sqrt{f}(\overline{U}_i))$ by taking $\sqrt{f}$ as a homeomorphism $h$ of Lemma 1. Thus $\sqrt{f}(\omega(f))$ is equal to $\bigcup_{i=1,2,\ldots} \omega_f(\overline{\sqrt{f}(U_i)})$. Then $\{\sqrt{f}(U_i)\}_{i=1,2,\ldots}$ is also a countable base such that each $\overline{\sqrt{f}(U_i)}$ is compact. Since $\omega(f)$ does not depend on the choice of such countable bases (Lemma 4), $\sqrt{f}(\omega(f))$ coincides with $\omega(f)$. We can also prove $\sqrt{f}(\alpha(f)) = \alpha(f)$ by using Lemma 2. □

By using this theorem in the same way as in the case of the nonwandering set in the introduction, we can show the non-existence of a square root for the following wandering homeomorphism of $\mathbf{R}^2$:

EXAMPLE 2. Let $X = \mathbf{R}^2$. We take the singular foliation $\mathfrak{F}$ illustrated in Figure 3 (see also [1]). In the region between the two straight lines with thorns, the homeomorphism $f$ preserves the leaves of $\mathfrak{F}$. On the upper (resp. lower) straight line with thorns, $f$ maps each thorn to the next thorn on the right (resp. left) side. By modifying $f$ along the straight lines with thorns, we can construct a homeomorphism (diffeomorphism) of $X$ preserving the leaves of $\mathfrak{F}$. Then $\omega(f)$ is the lower straight line with thorns and $\alpha(f)$ is the upper one (See Example 1). If $f$ has a square root $\sqrt{f}$, then $\omega(f)$ must be invariant under $\sqrt{f}$ by Theorem 6. Furthermore, $\sqrt{f}$ maps the adjacent branch points of $\omega(f)$ on themselves because $\sqrt{f}$ is a homeomorphism.

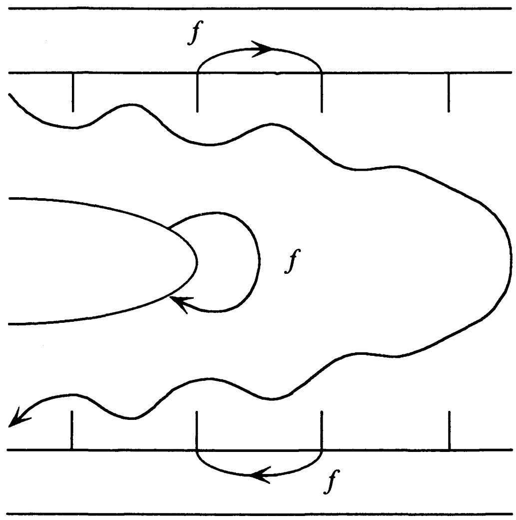


Figure 3: The Reeb component with thorns.

However this contradicts the assumption that $f$ maps each thorn on the lower straight line to the next thorn on the left side.

REMARK. The non-Hausdorff set is also useful to show the non-existence of square roots. However, for this example, the non-existence of square roots cannot be directly shown by the non-Hausdorff set.

Finally we give a more precise relationship between the positive (negative) limit set of $f$ and that of $\sqrt{f}$, and use it for the square roots of translations.

THEOREM 7. *If a homeomorphism $f$ of $X$ has a square root $\sqrt{f}$, then the positive* (*resp. negative*) *limit set of $\sqrt{f}$ coincides with that of $f$.*

PROOF. Let $\{U_i\}_{i=1,2,\ldots}$ be a countable base of $X$ such that each $\overline{U}_i$ is compact. Then we get

$$\begin{aligned}\omega(f) &= \bigcup_{i=1}^{\infty}\left\{\bigcap_{n\in\mathbf{Z}}\overline{\bigcup_{m\geq n} f^m(\overline{U}_i)}\right\}\\ &= \bigcup_{i=1}^{\infty}\left\{\bigcap_{n\geq 0}\overline{\bigcup_{m\geq n} f^m(\overline{U}_i)}\right\}\\ &\subset \bigcup_{i=1}^{\infty}\left\{\bigcap_{n\geq 0}\overline{\bigcup_{m\geq n} \sqrt{f}^m(\overline{U}_i)}\right\}\\ &= \bigcup_{i=1}^{\infty}\left\{\bigcap_{n\in\mathbf{Z}}\overline{\bigcup_{m\geq n} \sqrt{f}^m(\overline{U}_i)}\right\}\\ &= \omega(\sqrt{f}).\end{aligned}$$

Thus we conclude that $\omega(f)$ is contained in $\omega(\sqrt{f})$.

Now suppose that $p$ is an element of $\omega(\sqrt{f})$. Then there is an open set $U_i$ such that $\omega_{\sqrt{f}}(\overline{U_i})$ contains $p$. First observe that

$$\bigcup_{m\geqq n}\sqrt{f}^m(\overline{U_i})\subset\left(\bigcup_{m\geqq[n/2]}f^m(\overline{U_i})\right)\cup\left(\bigcup_{m\geqq[n/2]}f^m\sqrt{f}(\overline{U_i})\right)$$

where $[n/2]$ is the integer satisfying $[n/2]\leqq n/2<[n/2]+1$. Hence $p$ is contained in $\overline{\bigcup_{m\geqq[n/2]}f^m(\overline{U_i})}$ or $\overline{\bigcup_{m\geqq[n/2]}f^m\sqrt{f}(\overline{U_i})}$ for any $n\in\mathbf{Z}$. If $p$ is contained in $\overline{\bigcup_{m\geqq[n/2]}f^m(\overline{U_i})}$ for infinitely many $n$ ($n\geqq 0$), then $p$ is an element of $\bigcap_{n\in\mathbf{Z}}\overline{\bigcup_{m\geqq[n/2]}f^m(\overline{U_i})}$ because $\overline{\bigcup_{m\geqq[n/2]}f^m(\overline{U_i})}$ are decreasing subsets with respect to $n$. Thus $p$ is contained in $\omega_f(\overline{U_i})$. Otherwise $p$ is contained in $\overline{\bigcup_{m\geqq[n/2]}f^m\sqrt{f}(\overline{U_i})}$ for infinitely many $n$ ($n\geqq 0$). Then $p$ is contained in $\bigcap_{n\in\mathbf{Z}}\overline{\bigcup_{m\geqq[n/2]}f^m\sqrt{f}(\overline{U_i})}$, and is an element of $\omega_f(\sqrt{f}(\overline{U_i}))$. As a result, $p$ is contained in $\bigcup_{i=1}^{\infty}(\omega_f(\overline{U_i})\cup\omega_f(\sqrt{f}(\overline{U_i})))$. Then $\{U_i\}_{i=1,2,\dots}\cup\{\sqrt{f}(U_i)\}_{i=1,2,\dots}$ is also a countable base such that $\overline{U_i}$ and $\overline{\sqrt{f}(U_i)}$ are compact. Since $\omega(f)$ does not depend on the choice of such countable bases (Lemma 4), $\omega(\sqrt{f})$ is contained in $\omega(f)$.

Since $\sqrt{f}^{-1}$ is a square root of $f^{-1}$, $\omega(\sqrt{f}^{-1})=\omega(f^{-1})$. By Lemma 2, $\alpha(\sqrt{f})$ coincides with $\alpha(f)$. □

By Theorem 7, $\omega(f)=\varnothing$ implies that $\omega(\sqrt{f})=\varnothing$. Using Theorem 4, we obtain the uniqueness of square roots for the translation of $\mathbf{R}^2$.

COROLLARY. *For the translation of* $\mathbf{R}^2$, *its orientation preserving square root is topologically conjugate to the translation.*

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