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Robustness of variance balanced designs against the unavailability of some observations

Dedicated to Professor Michihiko Kikkawa on the occasion of his 60th birthday

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ABSTRACT. Robustness of variance-balanced designs is investigated, when any number of observations in a block or any two blocks are lost in a design, in terms of efficiency of the residual design. The investigation shows that variance-balanced designs are fairly robust against the unavailability of observations in the set-up mentioned above.

1. Introduction

When some observations become unavailable in a designed experiment for some reason, it is of interest to examine the unavailability of information, defined suitably, that is incurred due to missing data. Designs for which this loss is "small" may be termed *robust.* The robustness of several kinds of block designs against the unavailability of data has been investigated in abundance, for example, see Hedayat and John (1974), Dey and Dhall (1988), Srivastava, Gupta and Dey (1990), Mukerjee and Kageyama (1990), Bhaumik and Whittinghill (1991), Ghosh, Kageyama and Mukerjee (1992), Das and Kageyama (1992) and Dey (1993). For variance-balanced (VB) block designs, Gupta and Srivastava (1992) investigated the robustness of the design against the unavailability of some disjoint blocks. As a special case, they also showed that resolvable balanced incomplete block (BIB) designs are fairly robust against the unavailability of one resolution set consisting of disjoint blocks. On the other hand, Bhaumik and Whittinghill (1991) discussed the optimality of VB designs by showing that the optimal design is derived by removing blocks which have disjoint sets of treatments, and the worst design appears when identical blocks are removed.

We here pay our attention to the following two situations: (i) any number

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of observations in a block are lost; (ii) all the observations in any two blocks which are not necessarily disjoint are going to missing. The purpose of this paper is to investigate the robustness of VB designs against the unavailability of observations in the patterns mentioned above.

Let d^* be the design obtained by removing some observations in the original design d. Assume d^* to be connected [this assumption is given only for the convenience of general presentation of the eigenvalues of C^*, because the calculation of $\phi_1(*)$ below can be made also for a disconnected design d^*]. Let C and C^* be the C-matrix of d and d^*, respectively. In this case, the criterion of robustness against the unavailability of such observations is the efficiency of the residual design d^*, given by

$$e(*) = \frac{\text{sum of reciprocals of non-zero eigenvalues of } C}{\text{sum of reciprocals of non-zero eigenvalues of } C^*}\left(= \frac{\phi_2}{\phi_1(*)}, \text{ say}\right) \tag{1.1}$$

(see Das and Kageyama (1992)), which is equivalent to the ratio of the average variance of all elementary treatment contrasts in the original and the residual design.

For the evaluation of eigenvalues, the following lemma (Mukerjee and Kageyama (1990)) is useful. Here $J_{s\times t} = \mathbf{1}_s\mathbf{1}_t'$, in which $\mathbf{1}_s$ denotes an s-dimensional column vector of all unity. Especially $J_s = J_{s\times s}$ (hereafter, J denotes such matrix of appropriate size). I_s is the identity matrix of order s.

LEMMA 1.1. *Let u, $s_1, \cdots, s_u$ be positive integers, and consider the $s \times s$ matrix*

$$A = \begin{bmatrix} a_1 I_{s_1} + b_{11}J_{s_1s_1} & b_{12}J_{s_1s_2} & \cdots & b_{1u}J_{s_1s_u} \\ b_{21}J_{s_2s_1} & a_2 I_{s_2} + b_{22}J_{s_2s_2} & \cdots & b_{2u}J_{s_2s_u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{u1}J_{s_us_1} & b_{u2}J_{s_us_2} & \cdots & a_u I_{s_u} + b_{uu}J_{s_us_u} \end{bmatrix},$$

where $s = s_1 + s_2 + \cdots + s_u$ and the $u \times u$ matrix $B = (b_{ij})$ is symmetric. Then the eigenvalues of A are a_i with multiplicity $s_i - 1$ $(1 \le i \le u)$ and $\mu_1^*, \ldots, \mu_u^*$, where $\mu_1^*, \ldots, \mu_u^*$ are the eigenvalues of $\Delta = D_a + D_s^{1/2}BD_s^{1/2}$ with $D_a = \operatorname{diag}\{a_1, \ldots, a_u\}$, $D_s = \operatorname{diag}\{s_1, \ldots, s_u\}$ and $D_s^{1/2} = \operatorname{diag}\{s_1^{1/2}, \ldots, s_u^{1/2}\}$.

A binary connected VB design with parameters v, b, $\boldsymbol{r} = (r_1, \ldots, r_v)'$, $\boldsymbol{k} = (k_1, \ldots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ and $\theta = (n - b)/(v - 1)$ is only considered here to show the robustness against the unavailability of data. It is shown that the C-matrix of the VB design is given by $C = \theta(I_v - v^{-1}J_v)$, in which the non-zero eigenvalues of C are θ with multiplicity $v - 1$. Further assume that the present VB designs are not orthogonal and satisfy $k_j \ge 2$ for all $j = 1, 2, \ldots, b$. This is the usual assumption in this field to avoid trivial designs. On account of Corollary 6 of Kageyama and Tsuji (1980) and Proposition 3.1 of Kageyama

(1984), we can show that $\theta = (n - b)/(v - 1) > 1$. The property will be used later.

2. Unavailability of any number of observations in a block

The investigation, when one block is lost, has been done, in terms of efficiency, by Gupta and Srivastava (1992). Now suppose that s $(1 \leq s \leq k_j)$ observations in any one block of size k_j are lost in d. Let $k_{max} = \max\{k_1, \ldots, k_b\}$, $k_{min} = \min\{k_1, \ldots, k_b\}$. It follows that the C-matrix of the residual design d^* is given by C^*, where

$$C^* = \begin{bmatrix} C_{11} & c_{12}J & c_{13}J \\ c_{21}J & C_{22} & c_{23}J \\ c_{31}J & c_{32}J & C_{33} \end{bmatrix}$$

with

$$C_{11} = (\theta - 1)I_s - \left(\frac{\theta}{v} - \frac{1}{k_j}\right)J_s, \qquad C_{22} = \theta I_{k_j - s} - \left(\frac{\theta}{v} + \frac{1}{k_j - s} - \frac{1}{k_j}\right)J_{k_j - s},$$

$$C_{33} = \theta I_{v - k_j} - \frac{\theta}{v}J_{v - k_j}, \qquad c_{12} = c_{21} = -\frac{\theta}{v} + \frac{1}{k_j},$$

$$c_{13} = c_{31} = c_{23} = c_{32} = -\frac{\theta}{v}.$$

Hence we can obtain the following through Lemma 1.1.

LEMMA 2.1. *The $v - 1$ non-zero eigenvalues of C^*, $1 \leq s \leq k_j - 1$, are given by*

$$\begin{array}{ll} \theta - 1 & \textit{with multiplicity (w.m.) } s, \\ \theta & \textit{w.m. } v - s - 1. \end{array}$$

REMARK 2.1. When $s = k_j$ in the original design, the C-matrices for $s = k_j$ and $s = k_j - 1$ become identical. Hence the eigenvalues and their multiplicities of C^* for $s = k_j$ can be given by Lemma 2.1 with $s = k_j - 1$.

Recall that $\phi_1(s, k_j)$ and ϕ_2 are the sum of reciprocals of non-zero eigenvalues of C^* and C, respectively. Hence, in (1.1),

$$\phi_1(s, k_j) = \frac{s}{\theta - 1} + \frac{v - s - 1}{\theta},$$

$$\phi_2 = \frac{v - 1}{\theta},$$

which yield the efficiency of the residual design d^* as in

$$e(s, k_j) = \phi_2/\phi_1(s, k_j) = \frac{(v-1)(\theta-1)}{(v-1)(\theta-1)+s}. \tag{2.1}$$

Concerning behaviour of the values of efficiencies, we have the following.

THEOREM 2.1. *In a VB design with parameters* v, b, $\boldsymbol{r} = (r_1, \dots, r_v)'$, $\boldsymbol{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ *and* $\theta = (n-b)/(v-1)$, *for* $j = 1, 2, \dots, b$,

$$e(1, k_j) > e(2, k_j) > \cdots > e(k_j - 1, k_j) = e(k_j, k_j).$$

PROOF. For any integers s', s such that $s' < s \le k_j - 1$, it follows from (2.1) that

$$e(s', k_j) - e(s, k_j) = \frac{(v-1)(\theta-1)(s-s')}{\{(v-1)(\theta-1)+s'\}\{(v-1)(\theta-1)+s\}},$$

which is positive, since $\theta > 1$. It is obvious that $e(k_j - 1, k_j) = e(k_j, k_j)$. This completes the proof. □

THEOREM 2.2. *In a VB design with parameters* v, b, $\boldsymbol{r} = (r_1, \dots, r_v)'$, $\boldsymbol{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ *and* $\theta = (n-b)/(v-1)$, *if* $k_{j'} < k_j$, *then*

$$e(k_{j'} - 1, k_{j'}) > e(k_j - 1, k_j).$$

PROOF. As in the proof of Theorem 2.1, let $s' = k_{j'} - 1$ and $s = k_j - 1$, then it follows from (2.1) that

$$e(k_{j'} - 1, k_{j'}) - e(k_j - 1, k_j) = \frac{(v-1)(\theta-1)(k_j - k_{j'})}{\{(v-1)(\theta-1)+k_{j'}-1\}\{(v-1)(\theta-1)+k_j-1\}},$$

which is positive, since $\theta > 1$. This completes the proof. □

THEOREM 2.3. *In a VB design with parameters* v, b, $\boldsymbol{r} = (r_1, \dots, r_v)'$, $\boldsymbol{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ *and* $\theta = (n-b)/(v-1)$,

$$e(s, k_{j'}) = e(s, k_j)$$

for any $1 \le s \le \min\{k_{j'}, k_j\} - 1$.

PROOF. It is clear from (2.1). □

If the loss of information in a design against the unavailability of some observations is not more than $1 - \alpha$, then it is assumed that the design is said to be robust, i.e. $e(s, k_j) \ge \alpha$, in general, $\alpha = 0.9$ or 0.8. In this sense, we have the following.

THEOREM 2.4. *In a VB design with parameters* v, b, $\boldsymbol{r} = (r_1, \dots, r_v)'$,

$k = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ and $\theta = (n - b)/(v - 1)$, the design is robust against the unavailability of any number of observations in a block, if

$$n \geq b + v + \frac{\alpha k_{max} - 1}{1 - \alpha}$$

for a given positive number α (< 1).

PROOF. Through Theorems 2.1 and 2.2, it is clear that the minimum of $e(s, k_j)$ is given by $e(k_{max} - 1, k_{max})$. If $n \geq b + v + (\alpha k_{max} - 1)/(1 - \alpha)$ for a given positive number α (< 1), then it can be shown that

$$e(k_{max} - 1, k_{max}) = \frac{(v - 1)(\theta - 1)}{(v - 1)(\theta - 1) + k_{max} - 1} \geq \alpha.$$

This completes the proof. □

Theorems 2.1 and 2.2 imply that the behaviour of $e(k_{max} - 1, k_{max})$ is important to judge whether the design is robust or not. Hence it is enough to evaluate the value of $e(k_{max} - 1, k_{max})$ to show the robustness of VB designs in the present case. For all VB designs listed in Kageyama (1976), Gupta and Jones (1983), Jones, Sinha and Kageyama (1987), and Gupta and Kageyama (1992), the evaluation of $e(k_{max} - 1, k_{max})$ has been worked out. It reveals that except for a few, all the VB designs have high values of $e(k_{max} - 1, k_{max})$. In fact, 315 designs satisfy $e \geq 0.90$, 6 designs get $0.90 > e \geq 0.80$, 2 designs satisfy $0.80 > e \geq 0.70$ and only one design of series number 1 in Kageyama (1976) has $e(3, 4) = 0.67$.

Thus, it appears that VB designs are fairly robust against the unavailability of any number of observations in a block.

3. Unavailability of any two blocks

Now suppose that all the observations in any two blocks, jth and j'th, say, of respective sizes k_j and $k_{j'}$ are lost in d. Let w be the number of treatments common to two such blocks. Then $0 \leq w \leq \min\{k_j, k_{j'}\}$. In this case the C-matrix of the residual design d^* can be given by C^*, where

$$C^* = \begin{bmatrix} C_{11} & c_{12}J & c_{13}J & c_{14}J \\ c_{21}J & C_{22} & c_{23}J & c_{24}J \\ c_{31}J & c_{32}J & C_{33} & c_{34}J \\ c_{41}J & c_{42}J & c_{43}J & C_{44} \end{bmatrix}$$

with

$$C_{11} = (\theta - 2)I_w - \left(\frac{\theta}{v} - \frac{1}{k_j} - \frac{1}{k_{j'}}\right)J_w, \qquad C_{22} = (\theta - 1)I_{k_j - w} - \left(\frac{\theta}{v} - \frac{1}{k_j}\right)J_{k_j - w},$$

$$C_{33} = (\theta - 1)I_{k_{j'} - w} - \left(\frac{\theta}{v} - \frac{1}{k_{j'}}\right)J_{k_{j'} - w}, \qquad C_{44} = \theta I_{v - k_j - k_{j'} + w} - \frac{\theta}{v}J_{v - k_j - k_{j'} + w},$$

$$c_{12} = c_{21} = -\frac{\theta}{v} + \frac{1}{k_j}, \qquad c_{13} = c_{31} = -\frac{\theta}{v} + \frac{1}{k_{j'}},$$

$$c_{14} = c_{41} = c_{23} = c_{32} = c_{24} = c_{42} = -\frac{\theta}{v}.$$

The situation can be treated by separating it into three cases of $w = 0$ (disjoint), $1 \le w \le \min\{k_j, k_{j'}\} - 1$ and $w = \min\{k_j, k_{j'}\}$. Hence by Lemma 1.1, we have the following.

LEMMA 3.1. *The $v - 1$ non-zero eigenvalues of $C^*(w)$ are given by*
(1) *for* $w = 0$

$\theta - 1$	*w.m.*	$k_j + k_{j'} - 2$,
θ	*w.m.*	$v - k_j - k_{j'} + 1$;

(2) *for* $w = k_j = k_{j'}$

θ	*w.m.*	$v - w$,
$\theta - 2$	*w.m.*	$w - 1$;

(3) *for* $w = k_j < k_{j'}$

θ	*w.m.*	$v - k_{j'}$,
$\theta - 1$	*w.m.*	$k_{j'} - k_j$,
$\theta - 2$	*w.m.*	$k_j - 1$; *and*

(4) *for* $1 \le w \le k - 1$ *and* $k_j = k_{j'} = k$

θ	*w.m.*	$v - 2k + w$,
$\theta - 1$	*w.m.*	$2k - 2w - 2$,
$\theta - 2$	*w.m.*	$w - 1$,
$\theta - w/k$	*w.m.*	1,
$\theta - 2 + w/k$	*w.m.*	1.

Let us consider the final case, i.e. $1 \le w \le \min\{k_j, k_{j'}\} - 1$. In this case, Lemma 1.1 shows that the eigenvalues of C^* are given by

$$\begin{aligned}
&\theta - 2 \quad && w.m. \quad w-1,\\
&\theta - 1 \quad && w.m. \quad k_j + k_{j'} - 2w - 2,\\
&\theta \quad && w.m. \quad v - k_j - k_{j'} + w - 1
\end{aligned}$$

and the remaining four eigenvalues of $\Delta = D_a + D_s^{1/2} B D_s^{1/2}$ with

$$D_a = \operatorname{diag}\{\theta - 2, \theta - 1, \theta - 1, \theta\},$$

$$D_s = \operatorname{diag}\{w, k_j - w, k_{j'} - w, v - k_j - k_{j'} + w\},$$

$$B = -\frac{\theta}{v} J_4 + \frac{1}{k_j} V + \frac{1}{k_{j'}} \xi\xi',$$

where $J_4 = \mathbf{1}_4 \mathbf{1}_4'$, $V = (1, 1, 0, 0)'(1, 1, 0, 0)$ and $\xi = (1, 0, 1, 0)'$.

Now, for $\Delta = D_a + D_s^{1/2} B D_s^{1/2}$, it follows that $\Delta \boldsymbol{s}^{1/2} = \mathbf{0}$, where $\boldsymbol{s}^{1/2} = (w^{1/2}, (k_j - w)^{1/2}, (k_{j'} - w)^{1/2}, (v - k_j - k_{j'} + w)^{1/2})$. Hence 0 is an eigenvalue of Δ with the corresponding orthonormal eigenvector being by $\xi_0 = v^{-1/2} \boldsymbol{s}^{1/2}$. Letting $\mu_0 = 0$, μ_1, μ_2 and μ_3 be the eigenvalues of Δ and defining $\Delta_0 = \Delta + (\theta/v) D_s^{1/2} J_4 D_s^{1/2}$, we can show that the eigenvalues of Δ_0 are θ, μ_1, μ_2 and μ_3. Thus, we have

$$\begin{aligned}
\Delta &= D_a + D_s^{1/2} B D_s^{1/2}\\
&= -\frac{\theta}{v} \boldsymbol{s}^{1/2} \boldsymbol{s}^{1/2\prime} + D_a + \frac{1}{k_j} D_s^{1/2} V D_s^{1/2} + \frac{1}{k_{j'}} D_s^{1/2} \xi\xi' D_s^{1/2},
\end{aligned}$$

$$\Delta_0 = \Delta + \frac{\theta}{v} \boldsymbol{s}^{1/2} \boldsymbol{s}^{1/2\prime} = \Delta_1 + \boldsymbol{u}\boldsymbol{u}',$$

where

$$\begin{aligned}
\Delta_1 &= D_a + \frac{1}{k_j} D_s^{1/2} V D_s^{1/2}\\
&= \begin{bmatrix}
\theta - 2 + w/k_j & \sqrt{w(k_j - w)}/k_j & 0 & 0\\
\sqrt{w(k_j - w)}/k_j & \theta - w/k_j & 0 & 0\\
0 & 0 & \theta - 1 & 0\\
0 & 0 & 0 & \theta
\end{bmatrix},
\end{aligned}$$

$$\boldsymbol{u} = \frac{1}{\sqrt{k_{j'}}} D_s^{1/2} \xi = (w^{1/2}, 0, (k_{j'} - w)^{1/2}, 0)'.$$

Hence

$$\begin{aligned}\operatorname{tr}(\Delta_0^{-1}) &= \operatorname{tr}(\Delta_1^{-1}) - \frac{\boldsymbol{u}'\Delta_1^{-1}\Delta_1^{-1}\boldsymbol{u}}{1+\boldsymbol{u}'\Delta_1^{-1}\boldsymbol{u}}\\ &= \frac{2}{\theta} + \frac{2(\theta-1)}{\theta^2-2\theta+w/k_j+w/k_{j'}-w^2/(k_jk_{j'})}.\end{aligned}$$

Since Δ_0 has eigenvalues θ, μ_1, μ_2 and μ_3, it follows that

$$\begin{aligned}\sum_{i=1}^{3}\frac{1}{\mu_i} &= \operatorname{tr}(\Delta_0^{-1}) - \frac{1}{\theta}\\ &= \frac{1}{\theta} + \frac{2(\theta-1)}{\theta^2-2\theta+w/k_j+w/k_{j'}-w^2/(k_jk_{j'})}.\end{aligned}$$

Recall that $\phi_1(k_j, k_{j'}, w)$ and ϕ_2 are the sum of reciprocals of non-zero eigenvalues of C^* and C, respectively. Hence, in (1.1),

$$\begin{aligned}\phi_1(k_j, k_{j'}, w) = {}& \frac{w-1}{\theta-2} + \frac{k_j+k_{j'}-2w-2}{\theta-1} + \frac{v-k_j-k_{j'}+w}{\theta}\\ &+ \frac{2(\theta-1)}{\theta^2-2\theta+w/k_j+w/k_{j'}-w^2/(k_jk_{j'})}\end{aligned} \tag{3.1}$$

which yields the efficiency of the residual design d^* as

$$e(k_j, k_{j'}; w) = \phi_2/\phi_1(k_j, k_{j'}, w). \tag{3.2}$$

REMARK 3.1. In fact, for the calculation of (3.1), $\theta - 2 > 0$ is implicitly assumed. Note that the expression, $e(k_j, k_{j'}; w)$, of efficiency mentioned above also holds for $w = 0$ and $w = \min\{k_j, k_{j'}\}$ as a consequence.

Concerning behaviour of the values of efficiencies, we have the following.

THEOREM 3.1. *In a VB design with parameters* v, b, $\boldsymbol{r} = (r_1, \dots, r_v)'$, $\boldsymbol{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ *and* $\theta = (n-b)/(v-1)$, *for* $j, j' = 1, 2, \dots, b$,

$$e(k_{max}, k_{max}; w) \le e(k_j, k_{j'}; w) \le e(k_{min}, k_{min}; w)$$

for a fixed nonnegative integer w *such that* $0 \le w \le \min\{k_j, k_{j'}\}$.

PROOF. The situation can be treated by separating it into three cases of $0 \le w \le \min\{k_j, k_{j'}\} - 1$, $w = \min\{k_j, k_{j'}\}$ $(k_j > k_{j'})$ and $w = \min\{k_j, k_{j'}\}$ $(k_j = k_{j'})$.

Case 1: $0 \le w \le \min\{k_j, k_{j'}\} - 1$. First note that $\phi_1(k_j, k_{j'}, w)$ is a symmetric function of k_j and $k_{j'}$. The partial derivatives of $\phi_1(k_j, k_{j'}, w)$ with respect to k_j and $k_{j'}$ can be given by

$$\frac{\partial \phi_1(k_j, k_{j'}, w)}{\partial k_j} = \frac{1}{\theta(\theta-1)} + \frac{2(\theta-1)(w/k_j^2)(1-w/k_{j'})}{\{\theta^2 - 2\theta + w/k_j + w/k_{j'} - w^2/(k_j k_{j'})\}^2},$$

$$\frac{\partial \phi_1(k_j, k_{j'}, w)}{\partial k_{j'}} = \frac{1}{\theta(\theta-1)} + \frac{2(\theta-1)(w/k_{j'}^2)(1-w/k_j)}{\{\theta^2 - 2\theta + w/k_j + w/k_{j'} - w^2/(k_j k_{j'})\}^2},$$

respectively, which are positive, since $\theta > 2$ and $0 \le w \le \min\{k_j, k_{j'}\} - 1$.

Case 2: $w = \min\{k_j, k_{j'}\}$ $(k_j < k_{j'})$. It follows from Lemma 3.1 that $\theta - 2$ is a non-zero eigenvalue of C-matrix of the residual design d^*. Since C^* is positive semidefinite, $\theta - 2 > 0$. In this case, the partial derivatives of $\phi_1(k_j, k_{j'}, w)$ with respect to k_j and $k_{j'}$ can be given by

$$\frac{\partial \phi_1(k_j, k_{j'}, w)}{\partial k_j} = \frac{1}{(\theta-2)(\theta-1)}, \qquad \frac{\partial \phi_1(k_j, k_{j'}, w)}{\partial k_{j'}} = \frac{1}{\theta(\theta-1)},$$

respectively, which are positive, since $\theta > 2$.

Case 3: $w = \min\{k_j, k_{j'}\}$ $(k_j = k_{j'})$. In this case, the partial derivatives of $\phi_1(k_j, k_{j'}, w)$ with respect to k_j and $k_{j'}$ can be given by

$$\frac{\partial \phi_1(k_j, k_{j'}, w)}{\partial k_j} = \frac{\partial \phi_1(k_j, k_{j'}, w)}{\partial k_{j'}} = \frac{2}{\theta(\theta-2)},$$

which is positive, since $\theta - 2 > 0$. Thus, $\phi_1(k_j, k_{j'}, w)$ is an increasing function of k_j and $k_{j'}$. It follows that

$$\phi_1(k_{max}, k_{max}, w) \ge \phi_1(k_j, k_{j'}, w) \ge \phi_1(k_{min}, k_{min}, w),$$

which through (3.2) implies that

$$e(k_{max}, k_{max}; w) \le e(k_j, k_{j'}; w) \le e(k_{min}, k_{min}; w).$$

This completes the proof. □

Theorem 3.1 implies that the behaviour of $e(k_{max}, k_{max}; w)$ is important to judge whether the design is robust or not for a fixed w. On the other hand, Bhaumik and Whittinghill (1991) have proved that the optimal design in some class of VB designs is derived by removing some equal-sized blocks which have disjoint sets of treatments ($w = 0$), and the worst design appears when identical blocks ($w = k$) are removed. Hence we have the following.

THEOREM 3.2. *In a VB design with parameters* v, b, $\boldsymbol{r} = (r_1, \ldots, r_v)'$, $\boldsymbol{k} = (k_1, \ldots, k_b)'$, $n = \sum_{i=1}^{v} r_i$ *and* $\theta = (n-b)/(v-1)$, *the design is robust against the unavailability of any two blocks, if*

$$n \ge b + 2v + \frac{2(\alpha k_{max} - 1)}{1 - \alpha}$$

for a given positive number α (< 1).

PROOF. Through Theorem 3.1 in this paper and Theorem 2 in Bhaumik and Whittinghill (1991), the possible minimum of $e(k_j, k_{j'}; w)$ is given by $e(k_{max}, k_{max}; k_{max})$. If $n \geq b + 2v + 2(\alpha k_{max} - 1)/(1 - \alpha)$ for a given positive number α (< 1), then it can be shown that

$$e(k_{max}, k_{max}; k_{max}) = \frac{n - b - 2v + 2}{n - b - 2v + 2k_{max}} \geq \alpha.$$

This completes the proof. □

All VB designs listed in Kageyama (1976), Gupta and Jones (1983), Jones, Sinha and Kageyama (1987), and Gupta and Kageyama (1992) were worked out. The evaluation reveals that except for a few, all the VB designs have high values of $e(k_{max}, k_{max}; k_{max})$. In fact, 282 designs satisfy $e \geq 0.90$, 29 designs get $0.90 > e \geq 0.80$, 7 designs satisfy $0.80 > e \geq 0.70$ and only 5 designs of series numbers 2, 4, 7, 9 in Kageyama (1976) and 2 in Gupta and Kageyama (1992) have $e(4, 4; 4) = 0.67$, $e(4, 4; 4) = 0.45$, $e(6, 6; 6) = 0.47$, $e(4, 4; 4) = 0.58$, and $e(4, 4; 4) = 0.60$, respectively.

REMARK 3.2. In the design of series number 1 in Kageyama (1976), if we remove two identical blocks of size 4, the residual design becomes disconnected. Thus, the efficiency of the design is not discussed for this case.

Thus, it appears that VB designs are fairly robust against the unavailability of any two blocks in a design.

REMARK 3.3. It is well known that a BIB design is a binary proper and equireplicate VB design. This case is noted below. For a BIB(v, b, r, k, λ) design, the number w of treatments common to any two blocks satisfies $-(r - \lambda - k) \leq w \leq 2\lambda k/r + (r - \lambda - k)$ ($= w_{max}$, say) (see Connor (1952)). Thus, Theorem 3.1 implies that the behaviour of $e(w_{max})$ ($= e(k, k; w_{max})$) is important to judge whether the design is robust or not. The values of $e(w_{max})$ for 168 existing BIB designs listed in Hall (1986) and Raghavarao (1971) were worked out. The evaluation reveals that, except for some cases, all the BIB designs have high values of $e(w_{max})$. In fact, 128 designs satisfy $e(w_{max}) \geq 0.90$, 23 designs get $0.90 > e(w_{max}) \geq 0.80$, 10 designs satisfy $0.80 > e(w_{max}) \geq 0.70$ and 7 designs of series numbers 1, 2, 3, 4, 8 and 11 in Raghavarao (1971) and of series number 1 in Hall (1986) have $e(1) = 0.47$, $e(2) = 0.41$, $e(1) = 0.69$, $e(3) = 0.57$, $e(4) = 0.65$, $e(2) = 0.69$ and $e(1) = 0.59$, respectively. Thus, it appears that BIB designs are fairly robust against the unavailability of any two blocks in a design.

4. Some remarks on numerical comparison

For a given VB design, we here consider how the efficiency of residual designs changes as a function of w, when any two blocks are lost.

An example is presented as an illustration. For a VB design with 6 treatments, 18 blocks of size $k_1 = 2$ and $k_2 = 4$, respectively, having 8 replicates, all values of efficiency for the residual design d^* are shown in Table 1. Here "—" means that such case does not exist.

Table 1. $e(w)$ for $0 \le w \le \min\{k_1, k_2\}$

Case i	$w = 0$	$w = 1$	$w = 2$	$w = 3$	$w = 4$
$e(k_1, k_1, w)$	0.9259	0.9218	0.9091	—	—
$e(k_2, k_2, w)$	0.8065	0.7993	0.7906	0.7805	0.7692
$e(k_1, k_2, w)$	0.8621	0.8567	0.8475	—	—

It is known through Bhaumik and Whittinghill (1991) that the optimal design is derived by removing some blocks of the same size which have disjoint sets of treatments ($w = 0$). In Table 1, the optimal design with $e(2, 2; 0) = 0.9259$ is derived by removing two disjoint blocks of size $k_{min} = 2$, the worst design with $e(4, 4; 4) = 0.7692$ is derived by removing two identical blocks of size $k_{max} = 4$. When any two unequal-sized blocks are lost, the best design is derived by removing two disjoint blocks ($w = 0$), which keeps the largest efficiency value as $e(2, 4; 0) = 0.8621$. This fact is also true for other VB designs. Hence, through Theorem 3.1 and our investigation for other examples, we may say more. When there is possibility that two arbitrary blocks may become unavailable, the design obtained by removing two repeated blocks of size k_{max} is the worst, and the best design is derived by removing two disjoint blocks of size k_{min}.

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