HIROSHIMA MATH. J.
26 (1996), 341–349

3-primary β-family in stable homotopy of a finite spectrum

Katsumi SHIMOMURA

(Received October 17, 1994)
(Revised March 13, 1995)

ABSTRACT. Different from the case for a prime $p > 3$, the β-element $\beta_t \in \pi_{16t-6}(S^0)$ is not defined for each positive integer t at the prime 3. Consider the 8-skeleton X of the Brown-Peterson spectrum BP. Then we will show here that the β-element $\bar{\beta}_t \in \pi_{16t-6}(X)$ is defined for any positive integer t even at the prime 3, and that they are all essential. These β-elements are obtained from v_2-maps on type 2 spectra. We use here $V(1) \wedge X$ as a type 2 spectrum instead of the Toda-Smith spectrum $V(1)$ that is used in the case $p > 3$.

1. Introduction

For each prime number p, a p-local finite spectrum X is said to have type n if $K(n)_*(X) \neq 0$ and $K(n-1)_*(X) = 0$ for the Morava K-theories $K(i)_*(-)$ with coefficient ring $K(i)_*(S^0) = Z/p[v_i, v_i^{-1}]$. A self-map $\varphi: \Sigma^k X \to X$ of a p-local finite spectrum X for some $k > 0$ is called v_n-map if $K(n)_*(\varphi) \neq 0$ and $K(m)_*(\varphi) = 0$ for $m \neq n$. M. Hopkins and J. Smith [1] show the existence of a v_n-map for every spectrum of type n, say, $\varphi: \Sigma^k X \to X$. Note that a v_n-map determines an integer $l > 0$ such that $K(n)_*(\varphi) = v_n^l$, and we cannot tell anything about l from Hopkins-Smith's theorem. For $n = 1$ and $p > 2$, Adams gives a v_1-map $\alpha: \Sigma^{2p-2} V(0) \to V(0)$ with $l = 1$, where $V(0)$ denotes the mod p Moore spectrum. Let $V(1)$ denote the cofiber of α. Then it is a spectrum of type 2. For $n = 2$ and $p > 3$, L. Smith [9] show the existence of the v_2-map $\beta: \Sigma^{2p^2-2} V(1) \to V(1)$ with $l = 1$. These maps α and β are used to define homotopy elements known as α- and β-families in the stable homotopy groups of spheres.

Now we restrict our attention to the prime 3. Then Toda shows the non-existence of v_2-map on $V(1)$ with $l = 1$. So defining the β-family is a different story from the case $p > 3$, while β-family is given even at the prime 3. For example, β_4 does not exist at the prime 3. Recently, S. Pemmaraju [5] shows the existence of a v_2-map on $V(1)$ with $l = 9$. In this paper, we shall

1991 *Mathematics Subject Classification.* 55Q10, 55Q45.
Key words and phrases. β-family, v_n-map, the Toda-Smith spectra, the Adams-Novikov spectral sequence, the Brown-Peterson spectrum.

show the existence of a v_2-map with $l = 1$ on a spectrum VX of type 2. Here, the spectrum VX is the smash product of $V(1)$ and $X = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$.

THEOREM 1.1. *There exists a v_2-map $\bar{\beta}: \Sigma^{2p^2-2} VX \to VX$ with $l = 1$.*

A similar result is shown by S. Oka and H. Toda [4], in which they use $S^0 \cup_{\beta_1} e^{11}$ instead of X here.

Let $BP_*(-)$ denotes the Brown-Peterson homology theory with coefficient ring $Z_{(p)}[v_1, v_2, \cdots]$ with $|v_i| = 2p^i - 2$, for each prime number p. Then the cofiber $V(2)$ of β at $p > 3$, which is known as the Toda-Smith spectrum as well as $V(1)$, is characterized by $BP_*(V(2)) = BP_*/I_3$ for the ideal $I_3 = (p, v_1, v_2)$.

COROLLARY 1.2. *There exists a spectrum $\widetilde{V(2)}$ of type 3 such that*

$$BP_*(\widetilde{V(2)}) = BP_*/I_3 \oplus \Sigma^4 BP_*/I_3 \oplus \Sigma^8 BP_*/I_3$$

for the ideal $I_3 = (3, v_1, v_2)$.

Note that Oka and Toda show the existence of a spectrum whose BP_*-homology is $BP_*/I_3 \oplus \Sigma^{11} BP_*/I_3$ at the prime 3 in [4].

In the same way as the β-elements of $\pi_*(S^0)$ at the prime $p > 3$, we can define β-elements of $\pi_*(X)$ at the prime 3 as follows: Let $i: S^0 \to V(1)$, $i_X: S^0 \to X$ and $\pi: V(1) \to S^6$ be the inclusions to the bottom cells and the projection to the top cell, respectively. Then the β-elements are defined to be

$$\bar{\beta}_t = (\pi \wedge X)\bar{\beta}^t(i \wedge X)i_X.$$

As for the β-elements of the spheres at the prime 3, $\beta_t \in \pi_*(S^0)$ is known to exist for $t \leq 3$ and $t = 5$, 6 (*cf.* [4]). Recently, S. Pemmaraju [5] shows the existence β_t for $t > 0$ with $t \equiv 0$, 1, 2, 3, 5, 6 (9). Otherwise, β_t does not exist by [8]. Furthermore, $\bar{\beta}_t = i_X \beta_t$ up to the Adams-Novikov filtration if β_t exists, since β_t and $\bar{\beta}_t$ are detected by $v_2 \in E_2^{0,16}(S^0)$ and $i_{X_*}(v_2) = v_2 \in E_2^{0,16}(X)$, respectively. In [4], S. Oka and H. Toda show the existence and non-triviality of β-elements in the homotopy groups $\pi_*(S^0 \cup_{\beta_1} e^{11})$. Our result is:

THEOREM 1.3. *For $t > 0$, $\bar{\beta}_t \neq 0 \in \pi_*(X)$, where $X = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$.*

By the relation $\bar{\beta}_t = i_X \beta_t$ and Pemmaraju's result, this theorem implies the following

COROLLARY 1.4. *$\beta_t \neq 0 \in \pi_*(S^0)$ for $t > 0$ with $t \not\equiv 4$, 7, 8 (9).*

The existence theorems Theorem 1.1 and Corollary 1.2 are proved in §2 by using Toda's computation [11]. The non-triviality theorems Theorem 1.3 and Corollary 1.4 are proved by computing the chromatic spectral sequence converging to the Adams-Novikov E_2-term for $\pi_*(X)$ in §3.

2. The homotopy groups of VX

In the following, the prime number is fixed to be 3. Let $V = V(1)$ and $T(1)$ denote the Toda-Smith spectrum and the Ravenel spectrum, respectively, characterized by the BP_*-homologies

$$BP_*(V(1)) = BP_*/(3, v_1) \quad \text{and} \quad BP_*(T(1)) = BP_*[t_1].$$

Here $(BP_*, BP_*(BP))$ is the Hopf algebroid with

$$(BP_*, BP_*(BP)) = (Z_{(3)}[v_1, v_2, \cdots], BP_*[t_1, t_2, \cdots]),$$

where the degrees of these generators are $|v_i| = 2\cdot 3^i - 2 = |t_i|$. Consider the 8-skeleton X of $T(1)$. Then,

$$BP_*(X) = BP_*\{1, t_1, t_1^2\} \subset BP_*(T(1))$$

as a $BP_*(BP)$-subcomodule.

Actually, the Toda-Smith spectra $V(0)$ and $V(1)$ are defined by the cofiber sequences:

$$\text{(2.1)} \quad \begin{array}{c} S^0 \xrightarrow{3} S^0 \xrightarrow{i} V(0) \xrightarrow{\pi} \Sigma S^0 \quad \text{and} \\ \Sigma^4 V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{\pi_1} \Sigma^5 V(0) \end{array}$$

for the Adams map $\alpha \in [V(0), V(0)]_4$. Moreover, X has the cell structure

$$X = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$$

for the generator $\alpha_1 \in \pi_3(S^0)$, which is defined as $\alpha_1 = \pi\alpha i$ by the Adams map α.

Since $BP_*(X)$ is a free BP_*-module, the BP_*-homology of $VX = V(1) \wedge X$ is obtained by

$$BP_*(VX) = BP_*/(3, v_1)\{1, t_1, t_1^2\} \subset BP_*[t_1]/(3, v_1).$$

Thus the E_2-term of the Adams-Novikov spectral sequence for $\pi_*(VX)$ is $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(VX))$, which is isomorphic to

$$E_2^{s,t}(VX) = \mathrm{Ext}^{s,t}_{BP_*[t_1^3, t_2, \cdots]}(BP_*, BP_*/(3, v_1))$$

by a change of rings theorem using the comodule structure $BP_*(VX) = BP_*(BP)\square_{BP_*[t_1^3, t_2, \cdots]} BP_*/(3, v_1)$.

Note that the Ext-group $\mathrm{Ext}^{s,t}_\Gamma(A, M)$ is given as a homology of the reduced cobar complex (*cf.* [3, Note 1.15]) for a Hopf algebroid (A, Γ) and a Γ-comodule M. For $t - s < 20$, we have an isomorphism

$$\begin{aligned} E_2^{s,t}(VX) &= \mathrm{Ext}^{s,t}_{A(v_2, t_1^3, t_2)}(A(v_2), A(v_2)) \\ &= \mathrm{Ext}^{s,t}_{A(t_1^3, t_2)}(Z/3, Z/3) \otimes A(v_2). \end{aligned}$$

In fact, $\Lambda(v_2)$ is tensor out because v_2 is primitive. Besides, the internal degrees of v_2, t_1^3 and t_2 are 16, 12 and 16, respectively, and those of v_i and t_i for $i > 2$ are greater than 52. Therefore, if we restrict the total degree $t - s < 20$, then any product or any tensor product of such elements as

$$v_2^2, \quad (t_1^3)^2, \quad t_2^2, \quad v_i \quad \text{and} \quad t_i \quad \text{for } i > 2$$

leave our range, since these have total degrees at least 23. Thus the reduced cobar complex for computing $E_2^{s,t}(VX)$ equals to that for $\mathrm{Ext}^{s,t}_{\Lambda(v_2,t_1^3,t_2)}(\Lambda(v_2), \Lambda(v_2))$ if $t - s < 20$, and the first equality follows.

Consider the Cartan-Eilenberg spectral sequence

$$E_2^{s,t} = \Lambda(v_2) \otimes Z/3[h_{11}, h_{20}] \Rightarrow E_2^{s,t}(VX), \tag{2.2}$$

where h_{11} and h_{20} are the cohomology classes represented by t_1^3 and t_2, respectively. Then we see the following

LEMMA 2.3. *The homotopy groups $\pi_k(VX)$ with $k < 20$ are all trivial but for $k = 0$, 11, 15, and 16.*

In fact, for $t - s < 20$, the E_2-term $E_2^{s,t}$ in (2.2) is trivial unless $t - s = 0$, 11, 15, 16, since $t - s$ for 1, h_{11}, h_{20} and v_2 are 0, 11, 15 and 16, respectively.

COROLLARY 2.4. $[VX, VX]_4 = 0$.

PROOF. VX consists of cells of dimensions

$$0,\ 1,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 13,\ 14.$$

The homotopy group $[VX, VX]_4$ is computed from the homotopy groups $\pi_k(VX)$ for $k = 4$, 5, 8, 9, 10, 12, 13, 14, 17 and 18, which are all trivial by the above lemma. q.e.d.

Recall that the spectrum $V = V(1)$ is the cofiber of the Adams map $\alpha: \Sigma^4 V(0) \to V(0)$, where $V(0)$ denotes the mod p Moore spectrum. Since $V(1)$ is a $V(0)$-module spectrum (*cf.* [11]), we have the splitting

$$V(0) \wedge V = V \vee \Sigma V \tag{2.5}$$

which gives us the maps

$$\varphi: \Sigma V \to V(0) \wedge V \qquad \text{and} \qquad \mu: V(0) \wedge V \to V.$$

Here μ gives the module structure.

LEMMA 2.6. *The composition*

$$\lambda_V(\alpha): \Sigma^5 V \xrightarrow{\varphi} \Sigma^4 V(0) \wedge V \xrightarrow{\alpha \wedge V} V(0) \wedge V \xrightarrow{\mu} V$$

is trivial.

PROOF. Recall from [11] the operation $\lambda_X : [V(0), V(0)]_k \to [X, X]_{k+1}$ for a $V(0)$-module spectrum X defined by $\lambda_X(x) = \mu_X(x \wedge X)\varphi_X$. Since $\lambda_V(\alpha) \in [V, V]_5$ and $[V, V]_5 = \{\delta_1[\beta i_1]\delta\pi_1\}$ by [11, Th. 6.11], we put $\lambda_V(\alpha) = x\delta_1[\beta i_1]\delta\pi_1$ for some $x \in \boldsymbol{Z}/3$. Here $\delta_1 = i_1\pi_1$, $\delta = i\pi$ and $[\beta i_1]$ is the generator of $[V(0), V(1)]_{16}$ defined by Toda [11]. Note that Toda uses the notation $[\beta i_1]$, since it corresponds to βi_1 if $\beta \in [V(1), V(1)]_{16}$ exists. In fact, β does not exist at the prime 3, while it does at any prime > 3 (*cf.* [10]).

To show that $x = 0$, we further recall from [11] the operation θ such that

1) [11, Th. 6.1, Lemma 6.5]

$$\theta(\lambda_V(x)) = -\alpha''\lambda_V(\delta x) + (-1)^k\lambda_V(x\delta)\alpha''$$

for $x \in [V(0), V(0)]_k$.

2) [11, Lemma 6.6] $\lambda_V(\alpha\delta) = \beta'\delta_0$ and $\lambda_V(\delta\alpha) = -\beta'\delta_0$.

3) [11, (3.9)] $\beta' x = x\beta'$ for any $x \in [V, V]_*$.

4) [11, (4.2)] $\alpha''\delta_0 = 0 = \delta_0\alpha''$.

5) [11, Th. 2.2] θ is derivative.

6) [11, Cor. 2.7, Th. 4.1, (3.7)] $\theta(\delta) = -1$, $\theta(\delta_1) = 0$, $\theta(\pi_1) = 0$.

7) [11, Th. 6.4] $\theta([\beta i_1]) = \alpha''[\beta i_1]\delta$.

Here $\delta_0 = ii_1\pi_1\pi$ and the elements $\alpha'' \in [V(1), V(1)]_2$ and $\beta' \in [V(1), V(1)]_{10}$ are non-trivial elements defined in [11, p. 219 and p. 240]. Besides, note that the second equation of 2) is not stated in [11, Lemma 6.6], but can be shown in the same fashion as the first one. Now we compute

$$\begin{aligned} \theta(\lambda_V(\alpha)) &= \alpha''\beta'\delta_0 + \beta'\delta_0\alpha'' && \text{by 1) and 2)} \\ &= 0 && \text{by 3) and 4).} \end{aligned}$$

and

$$\begin{aligned} \theta(\delta_1[\beta i_1]\delta\pi_1) &= \delta_1\theta([\beta i_1]\delta)\pi_1 && \text{by 5) and 6)} \\ &= \delta_1(\alpha''[\beta i_1]\delta\delta - [\beta i_1])\pi_1 && \text{by 5, 6) and 7)} \\ &= -\delta_1[\beta i_1]\pi_1 && \text{since } \delta\delta = 0. \end{aligned}$$

Here by [11, Th. 6.11], we see that $\delta_1[\beta i_1]\pi_1$ is the generator of $[V, V]_6$. Thus we obtain $x = 0$ from the equation $0 = \theta(\lambda_V(\alpha)) = \theta(x\delta_1[\beta i_1]\delta\pi_1) = -x\delta_1[\beta i_1]\pi_1$. q.e.d.

Now apply $- \wedge X$ to the above splitting (2.5), and obtain

$$\varphi_X : \Sigma VX \to V(0) \wedge VX \quad \text{and} \quad \mu_X : V(0) \wedge VX \to VX.$$

Then Lemma 2.6 is applied to prove the following

THEOREM 2.7. *There exists a map $v : V \wedge VX \to VX$ that is an extension of the identity $VX \to VX$.*

PROOF. Consider the exact sequence

$$[V \wedge VX, VX]_0 \xrightarrow{i_1^*} [V(0) \wedge VX, VX]_0 \xrightarrow{\alpha^*} [V(0) \wedge VX, VX]_4$$

associated to the second cofiber sequence in (2.1). By the definition of the multiplication $\mu: V(0) \wedge V \to V$, it is an extension of the identity $V \to V$ and we obtain the map $\mu_X: V(0) \wedge VX \to VX$ such that $\mu_X i = id: VX \to VX$, for the inclusion $i: S^0 \subset V(0)$. By (2.5) and Corollary 2.4, we have the isomorphisms

$$[V(0) \wedge VX, VX]_4 = [VX, VX]_4 \bigoplus [VX, VX]_5 \overset{\varphi_X^*}{\cong} [VX, VX]_5.$$

Then by Lemma 2.6,

$$\varphi_X^* \alpha^*(\mu_X) = \mu_X(\alpha \wedge VX)\varphi_X = \lambda_V(\alpha) \wedge X = 0 \in [VX, VX]_5.$$

Therefore we obtain a map $\nu: V \wedge VX \to VX$ such that $\nu(i_1 \wedge VX) = \mu: V(0) \wedge VX \to VX$. q.e.d.

As a corollary of this theorem, we have a similar theorem to Oka-Toda's [4]:

COROLLARY 2.8. *There exists a map*

$$\bar{\beta}: \Sigma^{16} VX \to VX$$

such that $\bar{\beta}$ *induces the multiplication by* v_2 *on each factor of* $BP_*(VX)$, *and hence the* BP_**-module* $BP_*/(3, v_1, v_2) \otimes Z/3\{1, a, a^2\}$ *with* $|a| = 4$ *is realizable as the BP-homology of the mapping cone of* $\bar{\beta}$.

PROOF. Computing the E_2-term of the Adams-Novikov spectral sequence for $\pi_*(V)$, we obtain the permanent cycle $v_2 \in E_2^{0,16}(V)$, which is represented by a homotopy element $[\beta i_1]i \in \pi_{16}(V)$. In other words,

$$BP \wedge [\beta i_1]i = v_2 \in BP_*(V).$$

Here BP denotes the Brown-Peterson spectrum, which represents the BP_*-homology theory. The self-map $\bar{\beta}$ is now defined to be the composition

$$\Sigma^{16} VX = \Sigma^{16} S^0 \wedge VX \xrightarrow{[\beta i_1]i \wedge VX} V \wedge VX \xrightarrow{\nu} VX.$$

Next consider the map $BP_*(\bar{\beta}): BP_*(VX) \to BP_*(VX)$. Let $T: V \wedge BP \to BP \wedge V$ be the switching map, and $m: BP \wedge BP \to BP$ and $\iota: S^0 \to BP$, the structure maps of the ring spectrum BP. Then for $v_2 \in BP_*(V)$,

$$(m \wedge V)(BP \wedge T)(v_2 \wedge BP) = (m \wedge V)(BP \wedge T)(BP \wedge \iota)(v_2)$$
$$= v_2 = BP \wedge [\beta i_1]i.$$

This shows the commutativity of the center square in the following commutative diagram:

$$\begin{array}{ccccccc}
S^0 = S^0 \wedge S^0 & \xrightarrow{v_2 \wedge x} & BP \wedge V \wedge BP \wedge VX & \xrightarrow{BP \wedge T \wedge VX} & BP \wedge BP \wedge V \wedge VX & \xrightarrow{m \wedge v} & BP \wedge VX \\
& \searrow^{x} & \uparrow{\scriptstyle v_2 \wedge BP \wedge VX} & & \downarrow{\scriptstyle m \wedge V \wedge VX} & \nearrow_{BP \wedge v} & \\
& & BP \wedge VX & \xrightarrow{BP \wedge [\beta i_1] i \wedge VX} & BP \wedge V \wedge VX & &
\end{array}$$

for any element $x \in BP_*(VX)$. The upper and the lower sequences represent $v_2 x$ and $BP_*(\bar{\beta})(x)$, respectively, and so $BP_*(\bar{\beta}) = v_2$ as desired. q.e.d.

3. The non-triviality of the β-elements

Using the element $\bar{\beta}$ of Corollary 2.8, we will define the β-elements of $\pi_*(X)$.

The β-elements $\bar{\beta}_t$ for $t > 0$ is defined to be the composition

$$\Sigma^{16t-6} S^0 \xrightarrow{\iota} \Sigma^{16t-6} VX \xrightarrow{\bar{\beta}^t} \Sigma^{-6} VX \xrightarrow{\pi_1 \wedge X} \Sigma^{-1} V(0) \wedge X \xrightarrow{\pi \wedge X} X.$$

Here $\iota : S^0 \to VX$ is the inclusion to the bottom cell. The change of rings theorem implies

$$\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, N \otimes_{BP_*} BP_*(X)) \cong \mathrm{Ext}^{s,t}_{BP_*[t_1^3, t_2, \cdots]}(BP_*, N)$$

for a comodule N, since $BP_*(X) = BP_*\{1, t_1, t_1^2\} = BP_* \square_\Sigma BP_*(BP)$, where $\Sigma = BP_*[t_1^3, t_2, \cdots]$. We denote this shortly by

$$\mathrm{Ext}^{s,t} N.$$

We also use the abbreviation

$$H^{s,t} M = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, M).$$

Now recall from [3] the chromatic spectral sequence. First consider the short exact sequences

$$0 \to N_0^0 \to M_0^0 \to N_0^1 \to 0 \qquad \text{and} \qquad 0 \to N_0^1 \to M_0^1 \to N_0^2 \to 0.$$

Here $N_0^0 = BP_*$ and $M_0^i = v_i^{-1} N_0^i$. Then the short exact sequences give rise to the long exact sequences

$$\cdots \to \mathrm{Ext}^1 M_0^0 \to \mathrm{Ext}^1 N_0^1 \xrightarrow{\delta} \mathrm{Ext}^2 N_0^0 \to \cdots \qquad \text{and}$$

$$\cdots \to \mathrm{Ext}^0 M_0^1 \to \mathrm{Ext}^0 N_0^2 \xrightarrow{\delta'} \mathrm{Ext}^1 N_0^1 \to \cdots.$$

Recall [6] and [3] the structure of modules $H^* M_0^0$ and $H^* M_0^1$, and we have

$$H^sM_0^0 = 0 \qquad (s > 0),$$

$$H^sM_0^1 = 0 \qquad (s > 1),$$

$$H^{1,t}M_0^1 = 0 \qquad (t \neq 0) \qquad \text{and}$$

$$H^0M_0^1 = \boldsymbol{Q}/\boldsymbol{Z}_{(p)} \oplus \oplus_{i\geq 0, s\in \boldsymbol{Z}-3\boldsymbol{Z}} \boldsymbol{Z}/(3^{i+1})\{v_1^{sp^i}/3^{i+1}\}.$$

Furthermore, by definition, $\mathrm{Ext}^0N_0^2 = \{x \in N_0^2 | \eta_R(x) = \eta_L(x)\}$, and $\eta_R(v_2^s/3v_1) = \eta_L(v_2^s/3v_1)$. In fact, $\eta_R(v_2^s) \equiv v_2^s \bmod (3, v_1)$ by Landweber's formula. Thus,

$$v_2^s/3v_1 \in \mathrm{Ext}^0N_0^2.$$

To get our modules we consider the exact sequences

$$0 \to M \to M \otimes \Lambda(t_1) \to \Sigma^4M \to 0 \qquad \text{and}$$

$$0 \to M \otimes \Lambda(t_1) \to M \otimes \boldsymbol{Z}[t_1]/(t_1^3) \to \Sigma^8M \to 0$$

for $M = M_0^0$ or $= M_0^1$. Note that $H^*M \otimes \boldsymbol{Z}[t_1]/(t_1^3) = \mathrm{Ext}^*M$. Applying H^*-, we see that

LEMMA 3.1.

$$\mathrm{Ext}^1M_0^0 = 0 \qquad \text{and} \qquad v_2^s/3v_1 \notin \mathrm{Im}\,\{\mathrm{Ext}^0M_0^1 \to \mathrm{Ext}^0N_0^2\}.$$

COROLLARY 3.2. *In the E_2-term $\mathrm{Ext}^2(BP_*)$ of the Adams-Novikov spectral sequence for $\pi_*(X)$,*

$$\delta\delta'(v_2^s/3v_1) \neq 0.$$

Consider the diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^0N_0^2 & \xrightarrow{\delta'} & \mathrm{Ext}^1N_0^1 & \xrightarrow{\delta} & \mathrm{Ext}^2N_0^0 \\
{\scriptstyle f}\uparrow & & \uparrow & & \| \\
\mathrm{Ext}^0N_2^0 & \xrightarrow[\partial']{} & \mathrm{Ext}^1N_1^0 & \xrightarrow[\partial]{} & \mathrm{Ext}^2N_0^0,
\end{array}$$

in which $f(x) = x/3v_1$. Here $N_0^0 = BP_*$, $N_1^0 = BP_*/(3)$ and $N_2^0 = BP_*/(3, v_1)$, whose Ext groups are the Adams-Novikov E_2-terms for computing $\pi_*(X)$, $\pi_*(V(0) \wedge X)$ and $\pi_*(VX)$. Then, $v_2^s \in \mathrm{Ext}^0N_2^0$ converges to $\bar{\beta}^s \in \pi_*(VX)$ by Corollary 2.8, and so does $\partial\partial'(v_2^s)$ to $\bar{\beta}_s \in \pi_*(X)$ by the Geometric Boundary Theorem [2]. Hence the commutativity of the diagram shows that $\bar{\beta}_s \in \pi_*(X)$ is detected by $\delta\delta'(v_2^s/3v_1) \in \mathrm{Ext}^*N_0^0$. New Corollary 3.2 shows the non-triviality of the element in the E_2-term. Besides, nothing kills it in the Adams-Novikov spectral sequence, and we obtain Theorem 1.3 stated in the introduction.

References

[1] M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory II, to appear in Ann. of Math.

[2] D. C. Johnson, H. R. Miller, W. S. Wilson and R. S. Zahler, Boundary homomorphisms in the generalized Adams spectral sequence and the nontriviality of infinitely many γ_t in the stable homotopy, Notas, de Matematicas y Simposia, Numero 1: Reunion Sobre Teoria de Homotopia, ed. D. Davis 29-46. Sociedad Matematica Mexicana, Mexico, D. F., 1975.

[3] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.

[4] S. Oka and H. Toda, 3-primary β-family in stable homotopy, Hiroshima Math. J. **5** (1975), 447–460.

[5] S. Pemmaraju, private communication.

[6] D. C. Ravenel, The cohomology of the Morava stabilizer algebras, Math. Z., **152** (1977), 287–297.

[7] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres, Academic Press, 1986.

[8] K. Shimomura, The homotopy groups of L_2-localized Toda-Smith spectrum $V(1)$ at the prime 3, to appear in Trans. Amer. Math. Soc.

[9] L. Smith, On realizing complex cobordism modules, IV, Applications to the stable homotopy groups of spheres. Amer. J. of Math., **99** (1971), 418–436.

[10] H. Toda, On spectra realizing exterior parts of the Steenrod algebra, Topology, **10** (1971), 53–65.

[11] H. Toda, Algebra of stable homotopy of Z_p-spaces and applications, J. Math. Kyoto Univ. **11** (1971), 197–251.

Faculty of Education,
Tottori University,
Tottori, 680,
Japan