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The K_*-local type of the smash product of real projective spaces

Dedicated to Professor Yasutoshi Nomura on his sixtieth birthday

Zen-ichi YOSIMURA

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ABSTRACT. We have already determined the K_*-local types of the real projective spaces RP^n and the stunted real projective spaces RP^n/RP^m in [11] and [12]. The purpose of this note is to determine the K_*-local types of the smash products of these two projective spaces.

0. Introduction

Given a ring spectrum E with unit, a CW-spectrum X is said to be quasi E_*-equivalent to a CW-spectrum Y if there exists an equivalence $h: E \wedge Y \to E \wedge X$ of E-module spectra. A map $f: Z \to X$ is said to be quasi E_*-equivalent to a map $g: W \to Y$ if there exist equivalences $h: E \wedge Y \to E \wedge X$ and $k: E \wedge W \to E \wedge Z$ of E-module spectra such that the equality $(1 \wedge f)k = h(1 \wedge g): E \wedge W \to E \wedge X$ holds. In this case the cofiber $C(f)$ is quasi E_*-equivalent to the cofiber $C(g)$. In particular, a map $f: Z \to X$ is said to be E_*-trivial if it is quasi E_*-equivalent to the trivial map, thus $1 \wedge f: E \wedge Z \to E \wedge X$ is trivial. Let KO and KU be the real and complex K-spectrum, respectively, and S_K denote the K_*-localization of the sphere spectrum S. Recall that two CW-spectra X and Y have the same K_*-local type if and only if X is quasi S_{K_*}-equivalent to Y (see [3] or [6]). In [9] and [10] we determined the quasi KO_*-equivalent types of the real projective spaces RP^n and the stunted real projective spaces $RP^n_{m+1} = RP^n/RP^m$, and then in [11] and [12] we established to determine completely the K_*-local types of these projective spaces after investigating the behavior of their real Adams operations ψ^k_R. The purpose of this note is to determine the K_*-local types of the smash products of these two projective spaces, which allows us to compute implicitly their J-groups as well as their KO-groups (see [16] for the computation of their KO-groups with ψ^k_R).

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According to [12, Theorems 2.7, 2.9 and 3.8] we have

THEOREM. i) *The stunted real projective space* $\Sigma^1 RP^{2s+n}_{2s+1}$ *has the same* K_**-local type as the small spectrum* $X_{n,s}$ *tabled below when* $s = 4k - 1$ *or* $4k$ *and the smash product* $X_{n,s} \wedge C(\bar{\eta})$ *when* $s = 4k + 1$ *or* $4k + 2$:

$s \backslash n =$	0	1	2	3	4	5	6	7
odd	$SZ/2^m$	J^t_m	$SZ/2^m$	M^t_m	V_m	${}_VJ^t_m$	V_m	${}_VM^t_m$
even	$SZ/2^m$	M^t_m	V_m	${}_VJ^t_m$	V_m	${}_VM^t_m$	$SZ/2^m$	J^t_m

ii) *The stunted real projective space* $\Sigma^1 RP^{2s+n}_{2s}$ *has the same* K_**-local type as the small spectrum* $Y_{n,s}$ *tabled below when* $s = 4k$ *or* $4k + 1$ *and the smash product* $Y_{n,s} \wedge C(\bar{\eta})$ *when* $s = 4k + 2$ *or* $4k + 3$:

$s \backslash n =$	0	1	2	3	4	5	6	7
even	I^s_{m+1}	$MI^{t,s}_{m+1}$	${}_VI^s_{m+1}$	${}_VJI^{t,s}_{m+1}$	${}_VI^s_{m+1}$	${}_VMI^{t,s}_{m+1}$	I^s_{m+1}	$JI^{t,s}_{m+1}$
odd	${}_VP^s_{m+1}$	${}_VJP^{t,s}_{m+1}$	${}_VP^s_{m+1}$	${}_VMP^{t,s}_{m+1}$	P^s_{m+1}	$JP^{t,s}_{m+1}$	P^s_{m+1}	$MP^{t,s}_{m+1}$

Here we set $m = [n/2]$ *and* $t = s + m + 1$ *in both cases.*

See the beginning parts in **1.1**, **2.1**, **2.2** and **3.1** for the construction of the small spectra $C(\bar{\eta})$, $X_{n,s}$ and $Y_{n,s}$ appearing in our theorem. In the above table the small spectra V_m, ${}_VM^t_m$, ${}_VJ^t_m$, P^s_{m+1}, ${}_VP^s_{m+1}$, I^s_{m+1}, ${}_VI^s_{m+1}$, ${}_VMI^{t,s}_{m+1}$ and ${}_VJP^{t,s}_{m+1}$ may be replaced by $U_m \wedge C(\bar{\eta})$, M^t_m, ${}_UJ^t_m \wedge C(\bar{\eta})$, $\Sigma^{2s+1} C_s \wedge {}'M^{-s}_m \wedge C(\bar{\eta})$, $\Sigma^{2s+1} C_s \wedge {}'M^{-s}_m$, $\Sigma^{2s+1} C_s \wedge {}'J^{-s}_m$, $\Sigma^{2s+1} C_s \wedge {}_V'J^{-s}_m$, $MI^{t,s}_{m+1}$ and $JP^{t,s}_{m+1} \wedge C(\bar{\eta})$, respectively, where $C_{4r} = C_{4r+1} = \Sigma^0$ and $C_{4r+2} = C_{4r+3} = C(\bar{\eta})$. Moreover $MP^{t,s}_{m+1} \wedge C(\bar{\eta})$ and ${}_VMP^{t,s}_{m+1}$ may be also replaced by $MP^{t,s}_{m+1}$. For our purpose it is sufficient to study the K_*-local types of the smash products $X_m \wedge Y_n$ where X_m, $Y_m = SZ/2^m$, V_m, M^t_m, ${}'M^t_m$, J^t_m, ${}_UJ^t_m$, ${}'J^t_m$, ${}_V'J^t_m$, $MP^{t,s}_m$, $MI^{t,s}_m$, $JP^{t,s}_m$, $JI^{t,s}_m$ or ${}_VJI^{t,s}_m$. These small spectra X_m and Y_n are constructed as the cofibers of certain maps $f: Z_0 \to Z_1$ and $g: W_0 \to W_1$. If either of the maps $f \wedge 1: Z_0 \wedge Y_n \to Z_1 \wedge Y_n$ and $1 \wedge g: X_m \wedge W_0 \to X_m \wedge W_1$ is S_{K*}-trivial, then the smash product $X_m \wedge Y_n$ admits a K_*-local splitting. Even if it is not so, the smash products $Z_i \wedge Y_n$ $(i = 0, 1)$ or $X_m \wedge W_i$ $(i = 0, 1)$ admit suitable K_*-local splittings in most cases. According to our plan we use these splittings so that either of the maps $f \wedge 1$ and $1 \wedge g$ is replaced by a simpler map h, whose cofiber has the same K_*-local type as the smash product $X_m \wedge Y_n$.

In § 1 and § 2 we give K_*-local splittings of the smash products $SZ/2^m \wedge SZ/2^n$ ($m \leq n$ and $n \geq 2$), $SZ/2^m \wedge V_n$ ($m \neq n$), $V_m \wedge V_n$ ($2 \leq m \leq n$) and $SZ/2^m \wedge M^t_n$, $V_m \wedge M^t_n$ ($m \leq n$), $SZ/2^m \wedge MP^{q,t}_n$, $V_m \wedge MP^{q,t}_n$ ($m < n$ and $n \geq 2$). In § 2 and § 3 we construct several small spectra concerned with the smash products

$M_m^t \wedge SZ/2^n$, $M_m^t \wedge V_n$ $(m < n)$, $M_m^t \wedge M_n^q$, $'M_m^t \wedge M_n^q$ $(m \le n)$ as well as $J_m^t \wedge SZ/2^n$, ${}_UJ_m^t \wedge V_n$ $(m < n)$, ${}_UJ_m^t \wedge SZ/2^n$, $J_m^t \wedge V_n$ $(m \le n)$, $J_m^t \wedge M_n^q$, ${}_UJ_m^t \wedge M_n^q$, $J_m^t \wedge J_n^q$, ${}_UJ_m^t \wedge J_n^q$, ${}_UJ_m^t \wedge {}_UJ_n^q$ and so on. In §4 we establish to detetmine the K_*-local types of the smash products $X_m \wedge Y_n$ by using the small spectra constructed in §2 and §3 where X_m, $Y_m = SZ/2^m$, V_m, M_m^t, $'M_m^t$, J_m^t, ${}_UJ_m^t$, $'J_m^t$ or ${}_V'J_m^t$. For the small spectra appearing in our main results (Theorems 4.1–4.4) we can easily study the KU-homologies with ψ_C^k and the KO-homologies with ψ_R^k by routine computations (see [16] for details). Similarly this can be done for the remaining smash products involving $MP_m^{t,s}$, $MI_m^{t,s}$, $JP_m^{t,s}$, $JI_m^{t,s}$ and ${}_VJI_m^{t,s}$. But we shall omit to describe them explicitly in this note.

1. Splittings of the smash products $SZ/2^m \wedge V_n$ and $V_m \wedge V_n$

1.1. Let $SZ/2^m$ be the Moore spectrum of type $Z/2^m$ $(m \ge 1)$, and $i : \Sigma^0 \to SZ/2^m$ and $j : SZ/2^m \to \Sigma^1$ denote the bottom cell inclusion and the top cell projection. It is well known [2] that the identity map $1 : SZ/2^m \to SZ/2^m$ is of order 2^m when $m \ge 2$ and of order 4 when $m = 1$. This implies that

(1.1) $$SZ/2^m \wedge SZ/2^n = \Sigma^1 SZ/2^m \vee SZ/2^m \qquad \text{if } m \le n \text{ and } n \ge 2.$$

In fact there exist maps

(1.2) $$\varphi : SZ/2^m \wedge SZ/2^n \to SZ/2^m \quad \text{and} \quad \psi : \Sigma^1 SZ/2^m \to SZ/2^m \wedge SZ/2^n$$

for any $m \le n$ and $n \ge 2$ such that $\varphi(1 \wedge i) = (1 \wedge j)\psi = 1$, $\varphi(i \wedge 1) = \pi$, $(j \wedge 1)\psi = \pi$, $ij\varphi = j \wedge \pi : SZ/2^m \wedge SZ/2^n \to \Sigma^1 SZ/2^{n-m}$ and $\psi ij = i \wedge \pi : SZ/2^{n-m} \to SZ/2^m \wedge SZ/2^n$ where π's are the obvious maps. Moreover there hold the relations $ij\varphi = 1 \wedge j + j \wedge 1 : SZ/2^n \wedge SZ/2^n \to \Sigma^1 SZ/2^n$ and $\psi ij = 1 \wedge i + i \wedge 1 : SZ/2^n \to SZ/2^n \wedge SZ/2^n$ when $m = n \ge 2$ (see [2]).

For the stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$ there exists its extension $\bar{\eta} : \Sigma^1 SZ/2^m \to \Sigma^0$ and its coextension $\tilde{\eta} : \Sigma^2 \to SZ/2^m$. Set $\eta_{1,n} = (\bar{\eta} \wedge 1)\psi : \Sigma^2 SZ/2 \to SZ/2^n$ and $\eta_{n,1} = \varphi(\tilde{\eta} \wedge 1) : \Sigma^2 SZ/2^n \to SZ/2$ for any $n \ge 2$. Using these maps we consider the following cofiber sequences

$$\Sigma^1 SZ/2 \xrightarrow{\bar{\eta}} \Sigma^0 \xrightarrow{\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{j}} \Sigma^2 SZ/2, \qquad \Sigma^2 \xrightarrow{\tilde{\eta}} SZ/2 \xrightarrow{\tilde{i}} C(\tilde{\eta}) \xrightarrow{\tilde{j}} \Sigma^3,$$

$$\Sigma^1 SZ/2 \xrightarrow{i\bar{\eta}} SZ/2^{m-1} \xrightarrow{i_V} V_m \xrightarrow{j_V} \Sigma^2 SZ/2, \qquad \Sigma^1 SZ/2^{m-1} \xrightarrow{\tilde{\eta}j} SZ/2 \xrightarrow{i'_V} V'_m \xrightarrow{j'_V} \Sigma^2 SZ/2,$$

$$\Sigma^2 SZ/2 \xrightarrow{\eta_{1,m+1}} SZ/2^{m+1} \xrightarrow{i_U} U_m \xrightarrow{j_U} \Sigma^3 SZ/2,$$

$$\Sigma^2 SZ/2^{m+1} \xrightarrow{\eta_{m+1,1}} SZ/2 \xrightarrow{i'_U} U'_m \xrightarrow{j'_U} \Sigma^3 SZ/2.$$

Since $\eta_{1,m+1} = (\bar{\eta} \wedge 1)\psi$ and $\eta_{m+1,1} = \varphi(\tilde{\eta} \wedge 1)$ we can choose maps $\bar{\lambda} : C(\bar{\eta}) \to$

Σ^0 and $\tilde{\lambda}:\Sigma^3 \to C(\tilde{\eta})$ satisfying $\bar{i}\bar{\lambda} = 4$, $\tilde{\lambda}\tilde{j} = 4$ and $\bar{\lambda}\bar{i} = \tilde{j}\tilde{\lambda} = 4$ (see [13]). Then the small spectra V_m, V'_m, U_m and U'_m are exhibited by the following cofiber sequences

$$(1.3)\qquad \begin{array}{ll} \Sigma^0 \xrightarrow{2^{m-1}\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{i}_V} V_m \xrightarrow{\bar{j}_V} \Sigma^1, & \Sigma^{-1}C(\tilde{\eta}) \xrightarrow{2^{m-1}\tilde{j}} \Sigma^2 \xrightarrow{\tilde{i}_V} V'_m \xrightarrow{\tilde{j}_V} C(\tilde{\eta}), \\ C(\bar{\eta}) \xrightarrow{2^{m-1}\bar{\lambda}} \Sigma^0 \xrightarrow{\bar{i}_U} U_m \xrightarrow{\bar{j}_U} \Sigma^1 C(\bar{\eta}) & \Sigma^3 \xrightarrow{2^{m-1}\tilde{\lambda}} C(\tilde{\eta}) \xrightarrow{\tilde{i}_U} U'_m \xrightarrow{\tilde{j}_U} \Sigma^4. \end{array}$$

Since $\tilde{\eta}\bar{\lambda}:\Sigma^2 C(\bar{\eta}) \to SZ/2$, $\bar{\lambda} \wedge \bar{\eta}:\Sigma^1 C(\bar{\eta}) \wedge SZ/2 \to \Sigma^0$ and $\tilde{\lambda} \wedge \tilde{\eta}:\Sigma^5 \to C(\tilde{\eta}) \wedge SZ/2$ are trivial, there exist K_*-equivalences

(1.4) $e: C(\bar{\eta}) \to \Sigma^{-3}C(\tilde{\eta})$, $\bar{e}: C(\bar{\eta}) \wedge C(\bar{\eta}) \to \Sigma^0$ and $\tilde{e}: \Sigma^6 \to C(\tilde{\eta}) \wedge C(\tilde{\eta})$

satisfying $\tilde{j}e = \bar{\lambda}$, $e\bar{i} = \tilde{\lambda}$, $\bar{e}(1 \wedge \bar{i}) = \bar{e}(\bar{i} \wedge 1) = \bar{\lambda}$ and $(1 \wedge \tilde{j})\tilde{e} = (\tilde{j} \wedge 1)\tilde{e} = \tilde{\lambda}$. Hence we notice that $\Sigma^{-3}C(\tilde{\eta})$ has the same K_*-local type as $C(\bar{\eta})$, and all of $\Sigma^{-2}V'_m \wedge C(\bar{\eta})$, $U_m \wedge C(\bar{\eta})$ and $\Sigma^{-3}U'_m$ have the same K_*-local type as V_m (cf. [11]).

It is easily computed that $[C(\bar{\eta}), C(\bar{\eta})] \cong Z \oplus Z/2$ with generators 1 and $\bar{i}\nu j\bar{j}$, $[\Sigma^1 C(\bar{\eta}), C(\bar{\eta})] \cong Z/2$ with generator $\eta \wedge 1$ and $[C(\bar{\eta}), \Sigma^1 C(\bar{\eta})] = 0$, and moreover that $[C(\bar{\eta}), V_n] \cong Z/2^{n+1} \oplus Z/2$ with generators $\bar{i}_V$ and $i_V i\nu j\bar{j}$ in the $n \geq 2$ case and $[U_n, \Sigma^1 C(\bar{\eta})] \cong Z/2^{n+1} \oplus Z/2$ with generators $\bar{j}_U$ and $\bar{i}\nu j j_U$ in any case where $\nu: \Sigma^3 \to \Sigma^0$ is the stable Hopf map. Let $\alpha: SZ/2 \wedge SZ/2^m \to \Sigma^1$ denote the adjoint map to the obvious map $\pi: SZ/2 \to SZ/2^m$ with $\alpha(1 \wedge i) = j$, and $\omega: V_m \to V_n$ and $\omega: U_m \to U_n$ the obvious maps. Then it follows immediately that

(1.5) i) $[SZ/2^l, C(\bar{\eta}) \wedge SZ/2^n] \cong Z/2^n * Z/2^l$ with generator $\bar{i} \wedge \pi$; $[C(\bar{\eta}) \wedge SZ/2^l, SZ/2^n] \cong (Z/2^n * Z/2^l) \oplus Z/2 \oplus Z/2$ with generators $\bar{\lambda} \wedge \pi$, $i\nu\alpha(\bar{j} \wedge 1)$ and $\pi\bar{j} \wedge \nu j$; and $[C(\bar{\eta}) \wedge SZ/2^m, C(\bar{\eta}) \wedge SZ/2^n] \cong Z/4 \oplus Z/2 \oplus Z/2$ or $(Z/2^n * Z/2^m) \oplus Z/2 \oplus Z/2 \oplus Z/2$ according as $m = n = 1$ or otherwise, which is generated by $1 \wedge \pi$, $1 \wedge i\eta j$, $\bar{i} \wedge \pi\bar{j} \wedge \nu j$ and $(\bar{i} \wedge i)\nu\alpha(\bar{j} \wedge 1)$;

ii) $[V_m, V_n] \cong (Z/2^{n+1} * Z/2^{m+1}) \oplus Z/2$ with generators ω and $i_V i\nu j j_V$ for any $n \geq 2$; and $[U_m, U_n] \cong (Z/2^{n+1} * Z/2^{m+1}) \oplus Z/2$ with generators ω and $(\nu \wedge 1)i_U \pi j_U$.

Here $A * B$ stands for the torsion product Tor (A, B). By means of (1.3) and (1.5) we observe that

(1.6) i) $V_n \wedge SZ/2^m = \Sigma^1 SZ/2^m \vee (C(\bar{\eta}) \wedge SZ/2^m)$ and $U_n \wedge SZ/2^m = (\Sigma^1 C(\bar{\eta}) \wedge SZ/2^m) \vee SZ/2^m$ whenever $m < n$; and

ii) $V_m \wedge SZ/2^n = \Sigma^1 V_m \vee V_m$ and $U_m \wedge SZ/2^n = \Sigma^1 U_m \vee U_m$ whenever $m < n$.

Consider the four cofiber sequences

$$
(1.7)\qquad
\begin{gathered}
SZ/2^m \xrightarrow{i_V\pi} V_n \xrightarrow{\omega_1} V_{n-m} \xrightarrow{i\bar{j}_V} \Sigma^1 SZ/2^m,\\
\Sigma^{-1}C(\bar{\eta}) \wedge SZ/2^m \xrightarrow{\bar{i}_V \wedge j} V_{n-m} \xrightarrow{\omega_2} V_n \xrightarrow{\bar{\pi}_V} C(\bar{\eta}) \wedge SZ/2^m,\\
C(\bar{\eta}) \wedge SZ/2^m \xrightarrow{\bar{\pi}'_U} U_n \xrightarrow{\omega_3} U_{n-m} \xrightarrow{\bar{j}_U \wedge i} \Sigma^1 C(\bar{\eta}) \wedge SZ/2^m,\\
\Sigma^{-1}SZ/2^m \xrightarrow{\bar{i}_U j} U_{n-m} \xrightarrow{\omega_4} U_n \xrightarrow{\pi_U} SZ/2^m
\end{gathered}
$$

for any $m < n$ where ω_i's are the obvious maps. Since $\omega_2\omega_1 = 2^m$ and $\omega_4\omega_3 = 2^m$, we get maps

$$
(1.8)\qquad
\begin{gathered}
\varphi_V : V_n \wedge SZ/2^m \to C(\bar{\eta}) \wedge SZ/2^m, \qquad \psi_V : \Sigma^1 SZ/2^m \to V_n \wedge SZ/2^m,\\
\varphi_U : U_n \wedge SZ/2^m \to SZ/2^m \quad \text{and} \quad \psi_U : \Sigma^1 C(\bar{\eta}) \wedge SZ/2^m \to U_n \wedge SZ/2^m
\end{gathered}
$$

satisfying $\varphi_V(1 \wedge i) = \bar{\pi}_V$, $(\bar{i}_V \wedge j)\varphi_V = \omega_1 \wedge j$, $(1 \wedge j)\psi_V = i_V\pi$, $\psi_V i\bar{j}_V = \omega_2 \wedge i$, $\varphi_U(1 \wedge i) = \pi_U$, $\bar{i}_U j\varphi_U = \omega_3 \wedge j$, $(1 \wedge j)\psi_U = \bar{\pi}'_U$ and $\psi_U(\bar{j}_U \wedge i) = \omega_4 \wedge i$. The maps ψ_V and φ_U may be chosen to satisfy $(\bar{j}_V \wedge 1)\psi_V = 1$ and $\varphi_U(\bar{i}_U \wedge 1) = 1$. Moreover we can verify by means of (1.5) that the maps φ_V and ψ_U may be chosen to satisfy $\varphi_V(\bar{i}_V \wedge 1) = 1$ and $(\bar{j}_U \wedge 1)\psi_U = 1$.

We next consider the two cofiber sequences

$$
(1.9)\qquad V_m \xrightarrow{\pi_V} SZ/2^n \xrightarrow{i_U\pi} U_{n-m} \xrightarrow{\bar{i}_V\bar{j}_U} \Sigma^1 V_m, \quad U_{n-m} \xrightarrow{\bar{\pi}_U} C(\bar{\eta}) \wedge SZ/2^n \xrightarrow{\bar{\pi}'_V} V_m \xrightarrow{\bar{i}_U\bar{j}_V} \Sigma^1 U_{n-m}
$$

for any $m < n$. It is easily checked that $\bar{\pi}_U i_U\pi = 2^{m-1}(\bar{i} \wedge 1)$ and $\pi_V\bar{\pi}'_V = 2^{n-m-1}(\bar{\lambda} \wedge 1)$. Hence we get maps

$$
(1.10)\qquad
\begin{gathered}
\varphi'_V : V_m \wedge SZ/2^n \to V_m, \qquad \psi'_V : \Sigma^1 V_m \to V_m \wedge SZ/2^n,\\
\varphi'_U : U_{n-m} \wedge SZ/2^n \to U_{n-m} \quad \text{and} \quad \psi'_U : \Sigma^1 U_{n-m} \to U_{n-m} \wedge SZ/2^n
\end{gathered}
$$

satisfying $\varphi'_V(\bar{i}_V \wedge 1) = \bar{\pi}'_V$, $\bar{i}_U\bar{j}_V\varphi'_V = \bar{j}_V \wedge i_U\pi$, $(\bar{j}_V \wedge 1)\psi'_V = \pi_V$, $\psi'_V\bar{i}_V\bar{j}_U = (\bar{i}_V \wedge 1)\bar{\pi}_U$, $\varphi'_U(i_U \wedge 1) = i_U\pi$, $\bar{i}_V\bar{j}_U\varphi'_U = \bar{\pi}'_V(\bar{j}_U \wedge 1)$, $(\bar{j}_U \wedge 1)\psi'_U = \bar{\pi}_U$ and $\psi'_U\bar{i}_U\bar{j}_V = \bar{i}_U \wedge \pi_V$. Since $[\Sigma^1, V_m] = [U_m, \Sigma^0] = 0$ it follows immediately that the equalities $\varphi'_V(1 \wedge i) = 1$ and $(1 \wedge j)\psi'_U = 1$ hold. Note that $[V_n, C(\bar{\eta})] \cong Z/2$ with generator $\bar{i}_V vj\bar{j}_V$ and $[\Sigma^1 C(\bar{\eta}), U_m] \cong Z/2$ with generator $(v \wedge 1)i_U\pi\bar{j}$. Then we can observe by means of (1.5) that the equalities $(1 \wedge j)\psi'_V = 1$ and $\varphi'_U(1 \wedge i) = 1$ hold, too.

1.2. Choose maps $\bar{v}_C : \Sigma^3 C(\bar{\eta}) \to \Sigma^0$ and $\gamma : \Sigma^2 SZ/2 \to C(\bar{\eta}) \wedge C(\bar{\eta})$ with $\bar{v}_C\bar{i} = v$ and $(1 \wedge \bar{j})\gamma = \bar{i} \wedge 1$. The map γ satisfies $\gamma i\eta = (\bar{i} \wedge \bar{i})v$ because of $\bar{e}\gamma i = \eta^2 : \Sigma^2 \to \Sigma^0$ for the K_*-equivalence $\bar{e} : C(\bar{\eta}) \wedge C(\bar{\eta}) \to \Sigma^0$ given in (1.4). Then it is easily shown that $[\Sigma^0, C(\bar{\eta}) \wedge C(\bar{\eta})] \cong Z$ with generator $\bar{i} \wedge \bar{i}$, $[\Sigma^2 SZ/2, C(\bar{\eta}) \wedge C(\bar{\eta})] \cong Z/4$ with generator γ, $[C(\bar{\eta}) \wedge C(\bar{\eta}), \Sigma^0] \cong Z \oplus Z/2 \oplus Z/2 \oplus Z/2$ with generators $\bar{e}$, $\bar{v}_C \wedge j\bar{j}$, $j\bar{j} \wedge \bar{v}_C$ and $v^2(j\bar{j} \wedge j\bar{j})$, and $[C(\bar{\eta}) \wedge C(\bar{\eta}), \Sigma^2 SZ/2] \cong Z/2 \oplus Z/2$ with generators $vj\bar{j} \wedge \bar{j}$ and $\bar{j} \wedge vj\bar{j}$. We moreover choose a map $\tilde{v}_C : \Sigma^5 SZ/2 \to C(\bar{\eta})$ with $\bar{j}\tilde{v}_C = v \wedge 1$, which is contained in the

Toda bracket $\langle \bar{i}, \bar{\eta}, \nu \wedge 1 \rangle$ (see [7]). Since $\langle \bar{\eta}, \nu \wedge 1, i\eta \rangle = \nu^2$ in $[\Sigma^6, \Sigma^0] \cong Z/2$, this map $\tilde{\nu}_C$ satisfies $\tilde{\nu}_C i\eta = \bar{i}\nu^2$. Hence we get immediately that

(1.11) $[C(\bar{\eta}), C(\bar{\eta}) \wedge C(\bar{\eta})] \cong Z \oplus Z/4$ with generators $1 \wedge \bar{i}$ and $\gamma\bar{j}$; and $[C(\bar{\eta}) \wedge C(\bar{\eta}), C(\bar{\eta})] \cong Z \oplus Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$ with generators $\bar{i}\bar{e}$, $\bar{i}\tilde{\nu}_C \wedge j\bar{j}$, $j\bar{j} \wedge \bar{i}\tilde{\nu}_C$, $\tilde{\nu}_C\bar{j} \wedge j\bar{j}$ and $j\bar{j} \wedge \tilde{\nu}_C\bar{j}$.

Since $[C(\bar{\eta}), \Sigma^2 C(\bar{\eta}) \wedge SZ/2] \cong Z/2$ with generator $\bar{i} \wedge \bar{j}$ we may assume that the equality $\bar{i} \wedge 1 = 1 \wedge \bar{i} + \gamma\bar{j} : C(\bar{\eta}) \to C(\bar{\eta}) \wedge C(\bar{\eta})$ holds. On the other hand, the map $\bar{\lambda} \wedge 1 : C(\bar{\eta}) \wedge C(\bar{\eta}) \to C(\bar{\eta})$ is written to be $\bar{i}\bar{e} + a\tilde{\nu}_C\bar{j} \wedge j\bar{j} + bj\bar{j} \wedge \tilde{\nu}_C\bar{j}$ for some $a, b \in Z/2$ because $\bar{\lambda}\bar{i} = 4$ and $\bar{i}\bar{\lambda} = 4$. In this case $\bar{\lambda} \wedge 1 : C(\bar{\eta}) \wedge SZ/2 \to SZ/2$ is also written to be $a\bar{j} \wedge \nu j + b\nu j\bar{j} \wedge 1 + c\bar{j} \wedge \eta\bar{\eta}$ for some $c \in Z/2$. Note that $\nu\bar{\lambda} = 4\tilde{\nu}_C : \Sigma^3 C(\bar{\eta}) \to \Sigma^0$ because of $4(\nu \wedge 1) = 4\bar{i}\tilde{\nu}_C \in [\Sigma^3 C(\bar{\eta}), C(\bar{\eta})]$. Using this equality we see that $\bar{\lambda} \wedge \nu j = 0 : \Sigma^2 C(\bar{\eta}) \wedge SZ/2 \to \Sigma^0$ and $\eta^2 \wedge \bar{j} = i\bar{\lambda} : C(\bar{\eta}) \to SZ/2$. Now it is easily verified that $a = b = 0$ and $c = 1$. Thus we get the equality $\bar{\lambda} \wedge 1 = \bar{i}\bar{e}$ and similarly $1 \wedge \bar{\lambda} = \bar{i}\bar{e}$ in $[C(\bar{\eta}) \wedge C(\bar{\eta}), C(\bar{\eta})]$.

From (1.11) it follows immediately that $[C(\bar{\eta}), C(\bar{\eta}) \wedge V_n] \cong Z/2^{n-1} \oplus Z/4$ with generators $1 \wedge \bar{i}_V\bar{i}$ and $(1 \wedge \bar{i}_V)\gamma\bar{j}$ in the $n \geq 2$ case, and $[C(\bar{\eta}) \wedge U_n, \Sigma^1 C(\bar{\eta})] \cong Z/2^{n-1} \oplus Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$ with generators $1 \wedge \bar{\lambda}\bar{j}_U$, $\bar{i}\tilde{\nu}_C \wedge j\bar{j}_U$, $j\bar{j} \wedge \bar{i}\tilde{\nu}_C\bar{j}_U$, $\tilde{\nu}_C\bar{j} \wedge j\bar{j}_U$ and $j\bar{j} \wedge \tilde{\nu}_C\bar{j}_U$ in any case. On the other hand, it is easily computed that $[\Sigma^1, C(\bar{\eta}) \wedge V_n] = 0$ and $[C(\bar{\eta}) \wedge U_n, \Sigma^0] \cong Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$ because $[\Sigma^1 C(\bar{\eta}) \wedge C(\bar{\eta}), \Sigma^0] \cong [\Sigma^3 SZ/2 \wedge C(\bar{\eta}), \Sigma^0] \cong Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$. Further it is shown that $[C(\bar{\eta}) \wedge U_n, C(\bar{\eta})] \cong Z/2 \oplus Z/2$ with generators $\bar{i}\bar{j}\bar{\nu}_U(\bar{j} \wedge 1)$ and $\bar{i}\sigma(j\bar{j} \wedge j\bar{j}_U)$ because $[\Sigma^1 C(\bar{\eta}) \wedge C(\bar{\eta}), C(\bar{\eta})] \cong Z/2$ with generator $\bar{i}\sigma(j\bar{j} \wedge j\bar{j})$. Here $\sigma : \Sigma^7 \to \Sigma^0$ is the stable Hopf map and $\bar{\nu}_U : \Sigma^3 SZ/2 \wedge U_n \to SZ/2$ is a map satisfying $\bar{\nu}_U(1 \wedge \bar{i}_U) = \nu \wedge 1$. Using the equality $\bar{\nu}\lambda = 4\tilde{\nu}_C$ we notice that the composite map $\bar{\lambda}\tilde{\nu}_C : \Sigma^5 SZ/2 \to \Sigma^0$ is trivial. By these computations we immediately get that

(1.12) i) $[V_m, C(\bar{\eta}) \wedge V_n] \cong (Z/2^{n-1} * Z/2^{m-1}) \oplus Z/4$ for any $n \geq 2$, which is generated by $(1 \wedge i_V\pi)\bar{\pi}_V$ and $(1 \wedge \bar{i}_V)\gamma j_V$; and

ii) $[C(\bar{\eta}) \wedge U_m, U_n] \cong (Z/2^{n-1} * Z/2^{m-1}) \oplus \left(\bigoplus_9 Z/2\right)$, which is generated by $\bar{\pi}'_U(1 \wedge \pi\pi_U)$ and nine elements of order 2.

Here the maps $\bar{\pi}_V : V_m \to C(\bar{\eta}) \wedge SZ/2^{m-1}$, $\pi_U : U_m \to SZ/2^{m-1}$ and $\bar{\pi}'_U : C(\bar{\eta}) \wedge SZ/2^{n-1} \to U_n$ are given in (1.7).

Set ${}_V\eta_{1,n} = (1 \wedge \bar{\eta})\psi_V : \Sigma^2 SZ/2 \to V_n$ for any $n \geq 2$, and then write ${}_V\eta_{1,n} = \omega + a_n i_V i\nu j$ for some $a_n \in Z/2$. Since $\omega j_V = 2^{n-1} : V_n \to V_n$, we get the equality $2^{n-1}\bar{i}_V\tilde{\nu}_C = a_n i_V i\nu^2 j$, which asserts that $a_2 = 1$ and $a_n = 0$ if $n \geq 3$. Moreover this implies that $\bar{i} \wedge 1 : V_n \to C(\bar{\eta}) \wedge V_n$ has order 2^{n-1} whenever $n \geq 3$, but $2(\bar{i} \wedge 1) = \bar{i} \wedge i_V i\nu j j_V : V_2 \to C(\bar{\eta}) \wedge V_2$. Notice that the composite map $\bar{i}\nu j j_V : V_n \to C(\bar{\eta})$ is always S_{K_*}-trivial because $[C(\bar{\eta}), S_K] \cong Z$ and $[\Sigma^1, S_K \wedge C(\bar{\eta})] = 0$.

Hence it is observed that

(1.13) i) $V_m \wedge V_n = \Sigma^1 V_m \vee (C(\bar{\eta}) \wedge V_m)$ if $m \leq n$ and $n \geq 3$, and the smash product on the left side has the same K_*-local type as the wedge sum on the right side even if $m = n = 2$; and

ii) $U_m \wedge U_n = (\Sigma^1 C(\bar{\eta}) \wedge U_m) \vee U_m$ if $m \leq n$ and $n \geq 2$.

For the maps $\bar{\pi}_V : V_n \to C(\bar{\eta}) \wedge SZ/2^{n-m}$, $\pi_U : U_n \to SZ/2^{n-m}$ and $\bar{\pi}'_U : C(\bar{\eta}) \wedge SZ/2^{n-m} \to U_n$ there holds the following equality $2^{m-1}(\bar{i} \wedge 1) = (1 \wedge i_V \pi)\bar{\pi}_V : V_n \to C(\bar{\eta}) \wedge V_n$ when $m \geqq 3$ and $2^{m-1}(\bar{\lambda} \wedge 1) = \bar{\pi}'_U(1 \wedge \pi_U) : C(\bar{\eta}) \wedge U_n \to U_n$ when $m \geq 2$. Hence we get maps

$$(1.14)\qquad \begin{aligned} &\psi''_V : \Sigma^1 V_m \to V_m \wedge V_n \qquad \text{for } 3 \leq m \leq n, \qquad \text{and} \\ &\varphi''_U : U_m \wedge U_n \to U_m \qquad \text{for } 2 \leq m \leq n \end{aligned}$$

satisfying $(\bar{j}_V \wedge 1)\psi''_V = \omega$, $\psi''_V(\bar{i}_V \wedge j) = \bar{i}_V \wedge i_V \pi$, $\varphi''_U(\bar{i}_U \wedge 1) = \omega$ and $(\bar{j}_U \wedge i)\varphi''_U = \bar{j}_U \wedge \pi_U$. If $m < n$ we can verify that the equalities $(1 \wedge \bar{j}_V)\psi''_V = 1$ and $\varphi''_U(1 \wedge \bar{i}_U) = 1$ hold. Even if $m = n$ the maps ψ''_V and φ''_U can be taken to satisfy the same equalities because they may be replaced by $\psi''_V + i_V i v \wedge i_V \pi j_V$ and $\varphi''_U + i_U \pi \bar{v}_U (j_U \wedge 1)$. On the other hand, it is evident that there exist maps

$$\psi''_V : \Sigma^1 V_2 \to V_2 \wedge V_n \quad \text{for } n \geq 3 \qquad \text{and} \qquad \varphi''_U : U_1 \wedge U_n \to U_1 \quad \text{for } n \geq 2$$

with $(1 \wedge \bar{j}_V)\psi''_V = 1$ and $\varphi''_U(1 \wedge \bar{i}_U) = 1$. These maps are also taken to satisfy $(\bar{j}_V \wedge 1)\psi''_V = \omega$ and $\varphi''_U(\bar{i}_U \wedge 1) = \omega$ because they may be replaced by $\psi''_V + i_V j_V \wedge i_V i v$ and $\varphi''_U + i_U \pi \bar{v}_U (j_U \wedge 1)T$ where T denotes the twisted map.

Denote by X_m and $X_{n,m}$ the cofibers of the maps $(\bar{i} \wedge \bar{i})vjj_V : V_m \to C(\bar{\eta}) \wedge C(\bar{\eta})$ and $(\bar{i} \wedge \bar{i}_V \bar{i})vjj_V : V_m \to C(\bar{\eta}) \wedge V_n$. These spectra are related by the following cofiber sequence

$$C(\bar{\eta}) \xrightarrow{2^{n-1} i_X (1 \wedge \bar{i})} X_m \xrightarrow{\omega_X} X_{n,m} \xrightarrow{\rho_X} \Sigma^1 C(\bar{\eta})$$

in which $i_X : C(\bar{\eta}) \wedge C(\bar{\eta}) \to X_m$ denotes the canonical inclusion. Since the map $\bar{i}vjj_V$ is S_{K_*}-trivial, there exists a map $\psi_X : \Sigma^1 V_m \to S_K \wedge X_m$ with $(1 \wedge j_X)\psi_X = \iota_K \wedge 1$ for the K_*-localization map $\iota_K : \Sigma^0 \to S_K$ in which $j_X : X_m \to \Sigma^1 V_m$ denotes the canonical projection. Recall that $2(\bar{i} \wedge 1) = (\bar{i} \wedge \bar{i}_V \bar{i})vjj_V \in [V_2, C(\bar{\eta}) \wedge V_2]$. This implies that $X_{2,2} = V_2 \wedge V_2$ and the map $\rho_X : V_2 \wedge V_2 \to \Sigma^1 C(\bar{\eta})$ satisfies $\rho_X(\bar{i}_V \wedge 1) = 1 \wedge \bar{j}_V$ and $2(1 \wedge \bar{i})\rho_X = \bar{j}_V \wedge (\bar{i} \wedge \bar{i})vjj_V$. In this case we can assume that the equality $1 \wedge \bar{j}_V = \bar{j}_V \wedge 1 + \bar{i}_V \rho_X$ holds since ρ_X may be replaced by $\rho_X + \bar{j}_V \wedge \bar{i}vjj_V$. Setting

$$\psi''_V = (1 \wedge \omega_X)\psi_X : \Sigma^1 V_2 \to S_K \wedge V_2 \wedge V_2,$$

it satisfies $(1 \wedge \bar{j}_V \wedge 1)\psi''_V = (1 \wedge 1 \wedge \bar{j}_V)\psi''_V = \iota_K \wedge 1$ because of $(\bar{j}_V \wedge 1)\omega_X = j_X$.

2. Spectra derived from M_m^t and $'M_m^t$

2.1. Let us fix an Adams' K_*-equivalence $A_s: \Sigma^{8s}SZ/m(4s) \to SZ/m(4s)$ for $s \geq 1$ such that the composite map $jA_s i: \Sigma^{8s-1} \to \Sigma^0$ is exactly the generator ρ_s of order $m(4s)$ in the J-image. Set $\bar{\rho}_S = jA_s: \Sigma^{8s-1}SZ/m(4s) \to \Sigma^0$ and $\tilde{\rho}_s = A_s i: \Sigma^{8s} \to SZ/m(4s)$, whose cofibers $C(\bar{\rho}_s)$ and $C(\tilde{\rho}_s)$ have the same K_*-local type as Σ^0 and Σ^{8s+1}, respectively. Consider the map $k: \Sigma^2 C(\bar{\eta}) \to \Sigma^0$ of order 2 with $k\tilde{i} = \eta\bar{\eta}$, which admits an extension $\bar{k}: \Sigma^2 C(\bar{\eta}) \wedge SZ/2^m \to \Sigma^0$ satisfying $\bar{k}(\tilde{i} \wedge 1) = \bar{\eta} \wedge \bar{\eta}$ and $\bar{i}\bar{k} = 0$. As in [14] (or [11]) we now introduce the following maps of order 2 (cf. [1]):

$$\mu_s = \bar{\eta}A_s i: \Sigma^{8s+1} \to \Sigma^0, \qquad \mu_{-s} = \bar{\eta}i_s: \Sigma^{-8s+1}C(\bar{\rho}_s) \to \Sigma^0,$$

$$k_s = \bar{k}(1 \wedge A_s i): \Sigma^{8s+2}C(\bar{\eta}) \to \Sigma^0, \qquad k_{-s} = \bar{k}(1 \wedge i_s): \Sigma^{-8s+2}C(\bar{\eta}) \wedge C(\bar{\rho}_s) \to \Sigma^0$$

in which $i_s: C(\bar{\rho}_s) \to \Sigma^{8s}SZ/m(4s)$ is the bottom cell collapsing. For convenience' sake we put $\mu_0 = \eta: \Sigma^1 \to \Sigma^0$ and $k_0 = k: \Sigma^2 C(\bar{\eta}) \to \Sigma^0$. The cofibers of the maps μ_r and k_r are denoted by P^{4r+1} and P^{4r+3}. Since $2(1 \wedge \bar{\eta})$: $\Sigma^1 P^t \wedge SZ/2 \to P^t$ is S_{K_*}-trivial, there exists a K_*-equivalence $e_P: P^t \wedge C(\bar{\eta}) \to S_K \wedge P^t$ with $e_p(1 \wedge \bar{i}) = 2(\iota_K \wedge 1)$. This gives rise to a K_*-equivalence $e_{P,m}: P^t \wedge V_m \to S_K \wedge P^t \wedge SZ/2^m$. Thus we observe that

(2.1) $P^t \wedge C(\bar{\eta})$ has the same K_*-local type as P^t, and $P^t \wedge V_m$ has the same K_*-local type as $P^t \wedge SZ/2^m$ for any $m \geq 1$.

Denote by M_m^t and $_VM_m^t$ for $t = 4r + 1$ the cofibers of the maps $i\mu_r$ and $\bar{i}_V(\mu_r \wedge 1)$ composed with $i: \Sigma^0 \to SZ/2^m$ and $\bar{i}_V: C(\bar{\eta}) \to V_m$, and dually by $'M_m^t$ and $_V'M_m^t$ for $t = 4r + 1$ those of the maps $\mu_r j$ and $\mu_r(1 \wedge \bar{j}_V)$ composed with $j: SZ/2^m \to \Sigma^1$ and $\bar{j}_V: V_m \to \Sigma^1$. Use the map k_r instead of the map μ_r to construct small spectra denoted by the same symbols for $t = 4r + 3$. By virtue of (2.1) it is easily seen that $_VM_m^t$ and $_V'M_m^t$ have the same K_*-local types as M_m^t and $'M_m^t \wedge C(\bar{\eta})$, respectively (see [15, Theorem 3.1]). The spectra M_m^t and $'M_m^t$ are related to P^t by the following cofiber sequences

(2.2) $\quad \Sigma^0 \xrightarrow{2^m i_P} P^t \xrightarrow{i_M} M_m^t \xrightarrow{h_M} \Sigma^1 \quad$ and $\quad \Sigma^{2t-1}C_t \xrightarrow{h'_M} {'M_m^t} \xrightarrow{i'_M} P^t \xrightarrow{2^m j_P} \Sigma^{2t}C_t$

in which $i_P: \Sigma^0 \to P^t$ and $j_P: P^t \to \Sigma^{2t}C_t$ denote the canonical inclusion and projection, respectively. Here $C_{4s+1} = \Sigma^0$, $C_{4s+3} = \Sigma^{-3}C(\bar{\eta})$, $C_{-4s-3} = C(\bar{\rho}_{s+1})$ and $C_{-4s-1} = \Sigma^{-3}C(\bar{\eta}) \wedge C(\bar{\rho}_{s+1})$ for $s \geq 0$.

Since $[\Sigma^1, S_K \wedge P^t] \cong Z$ or $Z/m(t-1)$ depending if $t = 1$ or not, the composite map $i_P\eta: \Sigma^1 \to P^t$ is at least divisible by 4 in $[\Sigma^1, S_K \wedge P^t]$. This implies that the map $i_P \wedge i\eta: \Sigma^1 \to P^t \wedge SZ/2$ is S_{K_*}-trivial. By virtue of (2.1), (2.2) and this fact it is immediately observed that

(2.3) i) $M_n^t \wedge SZ/2^m = \Sigma^1 SZ/2^m \vee (P^t \wedge SZ/2^m)$ and $'M_n^t \wedge SZ/2^m = (P^t \wedge SZ/2^m) \vee (\Sigma^{2t-1}C_t \wedge SZ/2^m)$ if $m \leq n$ and $n \geq 2$, and the smash products on the left sides have the same K_*-local types as the wedge sums on the right sides, respectively, even if $m = n = 1$; and

ii) $M_n^t \wedge V_m$ and $'M_n^t \wedge V_m$ have the same K_*-local types as the wedge sums $\Sigma^1 V_m \vee (P^t \wedge SZ/2^m)$ and $(P^t \wedge SZ/2^m) \vee (\Sigma^{2t-1}C_t \wedge V_m)$, respectively, whenever $2 \leq m \leq n$.

When $m \leq n$ we have the following cofiber sequences

$$\Sigma^{-1}SZ/2^m \wedge P^t \xrightarrow{j \wedge l_M} M_{n-m}^t \xrightarrow{\omega_M} M_n^t \xrightarrow{\lambda_M} SZ/2^m \wedge P^t, \tag{2.4}$$

$$\Sigma^{-1}SZ/2^m \wedge P^t \xrightarrow{\lambda'_M} {}'M_n^t \xrightarrow{\omega'_M} {}'M_{n-m}^t \xrightarrow{i \wedge l'_M} SZ/2^m \wedge P^t,$$

where M_0^t and $'M_0^t$ stand for $\Sigma^{2t}C_t$ and Σ^0, respectively. According to (2.3) there exist maps

$$\text{(2.5)} \quad \varphi_M : SZ/2^m \wedge M_n^t \to S_K \wedge SZ/2^m \wedge P^t, \quad \psi_M : \Sigma^1 SZ/2^m \to S_K \wedge SZ/2^m \wedge M_n^t,$$

$$_U\varphi_M : U_m \wedge M_n^t \to S_K \wedge SZ/2^m \wedge P^t \quad \text{and} \quad _U\psi_M : \Sigma^1 U_m \to S_K \wedge U_m \wedge M_n^t$$

for any $m \leq n$ satisfying $\varphi_M(1 \wedge l_M) = \iota_K \wedge 1 \wedge 1$, $(1 \wedge 1 \wedge h_M)\psi_M = \iota_K \wedge 1$, $_U\varphi_M(1 \wedge l_M) = e_{P,m}$ and $(1 \wedge 1 \wedge h_M)_U\psi_M = \iota_K \wedge 1$ where $e_{P,m} : U_m \wedge P^t \to S_K \wedge SZ/2^m \wedge P^t$ is a K_*-equivalence with $e_{P,m}(\bar{i}_U \wedge 1) = \iota_K \wedge i \wedge 1$. As is easily seen, we can find maps $f : \Sigma^1 SZ/2^m \to S_K \wedge SZ/2^m \wedge P^t$ and $f_U : \Sigma^1 U_m \to S_K \wedge SZ/2^m \wedge P^t$ such that $\varphi_M(i \wedge 1) = \iota_K \wedge \lambda_M + fih_M$ and $_U\varphi_M(\bar{i}_U \wedge 1) = \iota_K \wedge \lambda_M + f_U\bar{i}_U h_M$. Hence the maps φ_M and $_U\varphi_M$ are chosen to satisfy $\varphi_M(i \wedge 1) = {}_U\varphi_M(\bar{i}_U \wedge 1) = \iota_K \wedge \lambda_M$. Similarly the maps ψ_M and $_U\psi_M$ are chosen to satisfy $(1 \wedge j \wedge 1)\psi_M = \iota_K \wedge i_M \pi$ and $(1 \wedge \bar{j}_U \wedge 1)_U\psi_M = \iota_K \wedge (1 \wedge i_M)\bar{\pi}_U$ for the canonical inclusion $i_M : SZ/2^n \to M_n^t$. In fact we may take $\psi_M = (1 \wedge i_M)\psi$ if $m \leq n$ and $n \geq 2$, and $_U\psi_M = (1 \wedge i_M)\psi'_U$ if $m < n$ where ψ and ψ'_U are given in (1.2) and (1.10).

2.2. Note that $\bar{\lambda} \wedge \bar{\eta} = 0$ and hence $\bar{\lambda} \wedge \bar{k} = 0$ since $[\Sigma^1 C(\bar{\eta}) \wedge SZ/2, \Sigma^0] = 0$. Choose maps $\zeta_P : P^t \to \Sigma^0$, $_V\zeta_P : P^t \to C(\bar{\eta})$, $_U\zeta_P : P^t \wedge C(\bar{\eta}) \to \Sigma^0$, $\xi_P : \Sigma^{2t}C_t \to P^t$, $_V\xi_P : \Sigma^{2t}C_t \to P^t \wedge C(\bar{\eta})$ and $_U\xi_P : \Sigma^{2t}C_t \wedge C(\bar{\eta}) \to P^t$ satisfying $\zeta_P i_P = 2$, $_V\zeta_P i_P = \bar{i}$, $_U\zeta_P(i_P \wedge 1) = \bar{\lambda}$, $j_P\xi_P = 2$, $(j_P \wedge 1)_V\xi_P = 1 \wedge \bar{i}$ and $j_P{}_U\xi_P = 1 \wedge \bar{\lambda}$. The cofibers of the maps $2^{n-1}\zeta_P$, $2^{n-1}{}_V\zeta_P$, $2^{n-1}{}_U\zeta_P$, $2^{n-1}\xi_P$, $2^{n-1}{}_V\xi_P$ and $2^{n-1}{}_U\xi_P$ are denoted by P_n^t, $_VP_n^t$, $_UP_n^t$, $'P_n^t$, $_V'P_n^t$ and $_U'P_n^t$, respectively. For the map μ_r we can suitably choose its coextensions $\tilde{\mu}_r$, $_V\tilde{\mu}_r$, $_U\tilde{\mu}_r$ and its extensions $\bar{\mu}_r$, $_V\bar{\mu}_r$, $_U\bar{\mu}_r$ so that their cofibers coincide with P_n^t, $_VP_n^t$, $_UP_n^t$, $'P_n^t$, $_V'P_n^t$ and $_U'P_n^t$ $(t = 4r + 1)$, respectively Similarly this can be done for the map k_r $(t = 4r + 3)$. By means of (2.1) we observe that $_VP_n^t \wedge C(\bar{\eta})$ and $_UP_n^t$ have the same K_*-local type as P_n^t, and dually that ${}_V{}'P_n^t$ and ${}_U{}'P_n^t \wedge C(\bar{\eta})$ have the same K_*-local type as $'P_n^t$. Moreover we notice that $'P_1^t$ and P_1^t have the same K_*-local types as $C(\bar{\eta})$ and $\Sigma^{2t+1}C_t \wedge C(\bar{\eta})$, and more generally $'P_n^t$ and P_n^t have the same K_*-local types as $\Sigma^{2t}C_t \wedge M_{n-1}^{-t}$ and $\Sigma^{2t+1}C_t \wedge {}'M_{n-1}^{-t} \wedge C(\bar{\eta})$, respectively (see [15, Theorem 3.1]).

Using the maps l_M, h_M and λ_M in (2.2) and (2.4) we consider the following mixed maps

$$(2.6)\qquad (i\bar{\mu}_r, i\mu_s \wedge j): \Sigma^{8r+1}D_{r,s} \wedge SZ/2^n \to SZ/2^m \vee \Sigma^{8r-8s+1}SZ/2^l,$$

$$({}_V\bar{\mu}_r \wedge i, i\mu_s \wedge \bar{j}_V): \Sigma^{8r+1}D_{r,s} \wedge V_n \to (C(\bar{\eta}) \wedge SZ/2^m) \vee \Sigma^{8r-8s+1}SZ/2^l,$$

$$((i\bar{\mu}_r \wedge 1)(1 \wedge \lambda_M), i\mu_s \wedge h_M): \Sigma^{8r+1}D_{r,s} \wedge M_n^q \to (SZ/2^m \wedge P^q) \vee \Sigma^{8r-8s+1}SZ/2^l,$$

$$(\mu_r \wedge j \wedge l_M) \vee i_M(\tilde{\mu}_s \wedge j): (\Sigma^{8r}D_r \wedge SZ/2^m \wedge P^q) \vee (\Sigma^{8s+1}D_s \wedge SZ/2^l) \to M_n^q$$

whose cofibers are denoted by $'PM_{m,l,n}^{t,p}$, ${}_V{}'PM_{m,l,n}^{t,p}$, $'PMM_{m,l,n}^{t,p,q}$ and $MP'M_{n,l,m}^{q,p,t}$ for $(t, p) = (4r+1, 4s+1)$, respectively. Here we set $D_s = \Sigma^0$, $D_{-s-1} = C(\bar{\rho}_{s+1})$ for $s \geq 0$ and $D_{r,s} = \Sigma^0$, $C(\bar{\rho}_{-r})$, $C(\bar{\rho}_{-s})$ or $C(\bar{\rho}_{rs})$ depending if Min $\{r, s\} \geq 0$, $r < 0 \leq s$, $s < 0 \leq r$ or Max $\{r, s\} < 0$. In addition the maps $\bar{\mu}_r$, ${}_V\bar{\mu}_r$ and μ_s are the composed ones with a suitable K_*-equivalence $\varepsilon_r: D_{r,s} \to D_r$ or $\varepsilon_s: D_{r,s} \to D_s$ as given in [15, (1.3)]. When $i\bar{\mu}_r$ or $\tilde{\mu}_r \wedge j$ is replaced by $i\bar{\mu}_r + \tilde{\mu}_r \wedge j$, we substitute $''P$ for $'P$ or P in the above notations. Next we use the maps k_s, $\bar{k}_r$, ${}_V\bar{k}_r$ and $\tilde{k}_r$ as well as μ_s, $\bar{\mu}_r$, ${}_V\bar{\mu}_r$ and $\tilde{\mu}_r$ to construct small spectra denoted by the same symbols for the other pairs (t, p) of odd integers.

Denote by $MP_n^{q,t}$ and ${}_VMP_n^{q,t}$ for $t = 4r+1$ the small spectra constructed as the cofibers of the composite maps

$$i_M\tilde{\mu}_r: \Sigma^{8r+2}D_r \to M_n^q \qquad \text{and} \qquad i_M{}_V\tilde{\mu}_r: \Sigma^{8r+2}D_r \to {}_VM_n^q$$

in which i_M's are the canonical inclusions (see [8] or [12]). Use the maps $\tilde{k}_r$ and ${}_V\tilde{k}_r$ instead of $\tilde{\mu}_r$ and ${}_V\tilde{\mu}_r$ to construct small spectra denoted by the same symbols for $t = 4r+3$. Evidently these spectra are exhibited by the following cofiber sequences

$$(2.7)\qquad \begin{gathered} P^t \xrightarrow{2^{n-1}i_P\zeta_P} P^q \xrightarrow{i_{P,MP}} MP_n^{q,t} \xrightarrow{j_{MP,P}} \Sigma^1 P^t, \\ P^t \xrightarrow{2^{n-1}i_P \wedge {}_V\zeta_P} P^q \wedge C(\bar{\eta}) \xrightarrow{i_{P,MP}} {}_VMP_n^{q,t} \xrightarrow{j_{MP,P}} \Sigma^1 P^t. \end{gathered}$$

By means of (2.1) and (2.7) we observe that ${}_VMP_n^{q,t}$ has the same K_*-local type as $MP_n^{q,t}$. Moreover it is immediately shown that

(2.8) $MP_n^{q,t} \wedge SZ/2^m = (\Sigma^1 P^t \wedge SZ/2^m) \vee (P^q \wedge SZ/2^m)$ if $m < n$ and $n \geq 3$, and the smash product on the left side has the same K_*-local type as the wedge sum on the right side even if $m = 1$ and $n = 2$.

Note that $[\Sigma^3 MP_n^{q,t}, KO \wedge MP_n^{q,t}] \cong Z \oplus Z/2^{n-1}$ and ψ_R^k behaves as $k^2\begin{pmatrix} k^{t-q} & 0 \\ k^{t-q}-1/2 & 1\end{pmatrix}$ on $(Z \oplus Z/2^{n-1}) \otimes Z[1/k]$, because there exists an isomorphism $j_{MP,P}^*: [\Sigma^4 P^t, KO \wedge MP_n^{q,t}] \cong [\Sigma^3 MP_n^{q,t}, KO \wedge MP_n^{q,t}]$. Since $\eta^2 \wedge 1: \Sigma^2 MP_n^{q,t} \to MP_n^{q,t}$ becomes KO_*-trivial, we can easily check that it is divisible by 2 in $[\Sigma^2 MP_n^{q,t}, S_K \wedge MP_n^{q,t}]$ whenever $n \geq 3$. On the other hand, we recall that $[\Sigma^2 P^q, KO \wedge P^q] \cong Z \oplus Z$ and ψ_R^k behaves as $k^{q+1}\begin{pmatrix} 1/k^{2q} & 0 \\ 1-k^{2q}/2k^{2q} & 1\end{pmatrix}$ on $(Z \oplus Z) \oplus Z[1/k]$. Then it is also checked that $\eta \wedge 1: \Sigma^1 P^q \to P^q$ is divisible by 2 in $[\Sigma^1 P^q, S_K \wedge P^q]$ and $\eta \wedge i_{P,MP}: \Sigma^1 P^q \to MP_n^{q,t}$ is divisible by 4 in $[\Sigma^1 P^q, S_K \wedge MP_n^{q,t}]$ under the assumption that $n = 1$ or 2. Hence it follows that $1 \wedge \eta^2 j: \Sigma^1 MP_n^{q,t} \wedge SZ/4 \to MP_n^{q,t}$ is divisible by 2 in $[\Sigma^1 MP_n^{q,t} \wedge SZ/4, S_K \wedge MP_n^{q,t}]$ if $n = 1$ or 2. Consequently we verify that $1 \wedge \eta^2 j: \Sigma^1 MP_n^{q,t} \wedge SZ/2 \to MP_n^{q,t}$ is always S_{K_*}-trivial. Therefore there exists a K_*-equivalence $e_{MP}: MP_n^{q,t} \wedge C(\bar{\eta}) \to S_K \wedge MP_n^{q,t}$ satisfying $e_{MP}(1 \wedge \bar{i}) = 2(\iota_K \wedge 1)$, which gives rise to a K_*-equivalence $e_{MP,m}: MP_n^{q,t} \wedge V_m \to S_K \wedge MP_n^{q,t} \wedge SZ/2^m$. Thus we observe that

(2.9) $MP_n^{q,t} \wedge C(\bar{\eta})$ has the same K_*-local type as $MP_n^{q,t}$, and $MP_n^{q,t} \wedge V_m$ and $MP_n^{q,t} \wedge U_m$ have the same K_*-local type as $MP_n^{q,t} \wedge SZ/2^m$ for any $m \geq 1$.

3. Spectra derived from $\mathbf{J_m^{t,a}}$, $\mathbf{{}_UJ_m^{t,a}}$, $\mathbf{'J_m^{t,a}}$ and $\mathbf{{}_V'J_m^{t,a}}$

3.1. We now use the following maps

$$\rho_r: \Sigma^{8r-1} D_r \to \Sigma^0 \qquad \text{and} \qquad n_r': \Sigma^{8r+3} C(\bar{\eta}) \to D'_{2r+1}$$

introduced in [14] where $D_s = D_s' = \Sigma^0$, $D_{-s-1} = C(\bar{\rho}_{s+1})$ and $D'_{-s-1} = \Sigma^{-8s-9} C(\tilde{\rho}_{s+1})$ for $s \geq 0$. These maps ρ_r and n_r' represent generators of $[\Sigma^{8r-1}, S_K] \cong Z/m(4r)$ and $[\Sigma^{8r+3} C(\bar{\eta}), S_K] \cong Z/m(4r+2)$, respectively. The cofibers of the maps $a\rho_r$ and an_r' $(a \geq 1)$ are denoted by $J^{4r,a}$ and $J^{4r+2,a}$. Consider the following maps

$$a(\rho_r \wedge i): \Sigma^{8r-1} D_r \to SZ/2^m, \qquad a(\rho_r \wedge j): \Sigma^{8r-2} D_r \wedge SZ/2^m \to \Sigma^0,$$

$$a(\rho_r \wedge \bar{i}_V): \Sigma^{8r-1} D_r \wedge C(\bar{\eta}) \to V_m, \qquad a(\rho_r \wedge \bar{j}_V): \Sigma^{8r-2} D_r \wedge V_m \to \Sigma^0,$$

$$a(\rho_r \wedge \bar{i}_U): \Sigma^{8r-1} D_r \to U_m, \qquad a(\rho_r \wedge \bar{j}_U): \Sigma^{8r-2} D_r \wedge U_m \to C(\bar{\eta})$$

whose cofibers are denoted by $J_m^{t,a}$, $'J_m^{t,a}$, ${}_VJ_m^{t,a}$, ${}_V'J_m^{t,a}$, ${}_UJ_m^{t,a}$ and ${}_U'J_m^{t,a}$ $(a \geq 1)$ for $t = 4r \neq 0$, respectively. Use the map n_r' instead of ρ_r to construct small spectra denoted by the same symbols for $t = 4r + 2$. Note that ${}_VJ_m^{t,a}$ and ${}_V'J_m^{t,a}$ have the same K_*-local types as ${}_UJ_m^{t,a} \wedge C(\bar{\eta})$ and ${}_U'J_m^{t,a} \wedge C(\bar{\eta})$, respectively.

The spectra $J_m^{t,a}$, ${}_VJ_m^{t,a}$ and ${}_UJ_m^{t,a}$ are exhibited by the following cofiber sequences

$$
\begin{aligned}
&C'_t \xrightarrow{2^m i_J} J^{t,a} \xrightarrow{l_J} J_m^{t,a} \xrightarrow{h_J} \Sigma^1 C'_t,\\
(3.1)\qquad &C'_t \xrightarrow{2^{m-1}(i_J \wedge \bar{i})} J^{t,a} \wedge C(\bar{\eta}) \xrightarrow{vl_J} {}_VJ_m^{t,a} \xrightarrow{vh_J} \Sigma^1 C'_t,\\
&C'_t \wedge C(\bar{\eta}) \xrightarrow{2^{m-1}(i_J \wedge \bar{\lambda})} J^{t,a} \xrightarrow{ul_J} {}_UJ_m^{t,a} \xrightarrow{uh_J} \Sigma^1 C'_t \wedge C(\bar{\eta})
\end{aligned}
$$

in which $C'_{4r} = \Sigma^0$, $C'_{4r+2} = D'_{2r+1}$ and $i_J : C'_t \to J^{t,a}$ denotes the canonical inclusion. By means of (1.5) and (1.12) it is evident that

(3.2) i) $J_n^{t,a} \wedge SZ/2^m = (\Sigma^1 C'_t \wedge SZ/2^m) \vee (J^{t,a} \wedge SZ/2^m)$ and ${}_UJ_n^{t,a} \wedge U_m = (\Sigma^1 C'_t \wedge C(\bar{\eta}) \wedge U_m) \vee (J^{t,a} \wedge U_m)$ if $m \leq n$ and $n \geq 2$; and

ii) $J_n^{t,a} \wedge U_m = (\Sigma^1 C'_t \wedge U_m) \vee (J^{t,a} \wedge U_m)$ and ${}_UJ_n^{t,a} \wedge SZ/2^m = (\Sigma^1 C'_t \wedge C(\bar{\eta}) \wedge SZ/2^m) \vee (J^{t,a} \wedge SZ/2^m)$ if $m < n$.

When $a = m(t)/2$ we shall drop the superscript "a" in $J^{t,a}$, $J_m^{t,a}$, $'J_m^{t,a}$, ${}_UJ_m^{t,a}$, ${}_V'J_m^{t,a}$ and so on for simplicity. We are only interested in the small spectra J_m^t, $'J_m^t$, ${}_UJ_m^t$ and ${}_V'J_m^t$ as treated in the introduction. Choose a map $\zeta_J : J^t \to C'_t$ with $\zeta_J i_J = 2$, whose cofiber I_1^t has the same K_*-local type as $\Sigma^{2t+1} C_t$ with $C_{4r} = D_r$ and $C_{4r+2} = C(\bar{\eta})$. Then there exists a map $\tilde{\alpha}_t : \Sigma^{2t} C_t \to C'_t \wedge SZ/2$ whose cofiber coincides with I_1^t where $\tilde{\alpha}_{4r}$ and $\tilde{\alpha}_{4r+2}$ are coextensions of $a\rho_r$ and an'_r with $a = m(t)/2$. Denote by I_n^t and ${}_VI_{n+1}^t$ $(n \geq 1)$ the cofibers of the composite maps $(1 \wedge \pi)\tilde{\alpha}_t : \Sigma^{2t} C_t \to C'_t \wedge SZ/2^n$ and $(1 \wedge i_V \pi)\tilde{\alpha}_t : \Sigma^{2t} C_t \to C'_t \wedge V_{n+1}$. By means of (1.7) we can show that I_n^t and ${}_VI_{n+1}^t$ have the same K_*-local types as $\Sigma^{2t+1} C_t \wedge {}'J_{n-1}^{-t}$ and $\Sigma^{2t+1} C_t \wedge {}_V'J_n^{-t}$, respectively (cf. [12, Lemma 1.4]).

The spectra I_n^t and ${}_VI_{n+1}^t$ may be regarded as the cofibers of the maps $2^{n-1}\zeta_J : J^t \to C'_t$ and $2^{n-1}(\zeta_J \wedge \bar{i}) : J^t \to C'_t \wedge C(\bar{\eta})$, respectively. Similarly to $MP_n^{q,t}$ and ${}_VMP_n^{q,t}$ in (2.7) we construct small spectra $MI_n^{q,t}$, ${}_VMI_{n+1}^{q,t}$, $JP_n^{q,t}$, ${}_VJP_n^{q,t}$, $JI_n^{q,t}$ and ${}_VJI_{n+1}^{q,t}$ as the cofibers of the maps $2^{n-1}(i_P \wedge \zeta_J) : J^t \to P^q \wedge C'_t$, $2^{n-1}(i_P \wedge \zeta_J \wedge \bar{i}) : J^t \to P^q \wedge C'_t \wedge C(\bar{\eta})$, $2^{n-1}(i_J \wedge \zeta_P) : C'_q \wedge P^t \to J^q$, $2^{n-1}(i_J \wedge {}_V\zeta_P) : C'_q \wedge P^t \to J^q \wedge C(\bar{\eta})$, $2^{n-1}(i_J \wedge \zeta_J) : C'_q \wedge J^t \to J^q \wedge C'_t$ and $2^{n-1}(i_J \wedge \zeta_J \wedge \bar{i}) : C'_q \wedge J^t \to J^q \wedge C'_t \wedge C(\bar{\eta})$, respectively. By means of (2.1) it is easily shown that ${}_VMI_{n+1}^{q,t}$ and ${}_VJP_n^{q,t}$ have the same K_*-local types as $MI_{n+1}^{q,t}$ and $JP_n^{q,t} \wedge C(\bar{\eta})$, respectively.

3.2. Consider the maps $\bar{\pi}_V$, $\bar{\pi}'_U$ and π_V given in (1.7) and (1.9) for $m < n$, and then set $\bar{\pi}_V = (1 \wedge \pi)\bar{\pi}_V : V_n \to C(\bar{\eta}) \wedge SZ/2^{n-1} \to C(\bar{\eta}) \wedge SZ/2^m$, $\bar{\pi}'_U = \bar{\pi}'_U(1 \wedge \pi) : C(\bar{\eta}) \wedge SZ/2^m \to C(\bar{\eta}) \wedge SZ/2^{n-1} \to U_n$ and $\pi_V = \pi\pi_V : V_m \to SZ/2^{m+1} \to SZ/2^n$ in case $m \geq n$. We denote by $SJ_{m,l,n}^{t,p,a,b}$, ${}_VSJ_{m,l,n}^{t,p,a,b}$, ${}_USJ_{m,l,n}^{t,p,a,b}$ and ${}_WSJ_{m,l,n}^{t,p,a,b}$ $(a, b \geq 1)$ for $(t, p) = (4r, 4s)$, respectively, the small spectra constructed as the cofibers of the following mixed maps

$$
(3.3)\quad
\begin{aligned}
&(a\rho_r \wedge \pi, bi\rho_s \wedge j): \Sigma^{8r-1}D_{r,s} \wedge SZ/2^n \to SZ/2^m \vee \Sigma^{8r-8s+1}SZ/2^l,\\
&(a\rho_r \wedge \bar{\pi}_V, bi\rho_s \wedge \bar{j}_V): \Sigma^{8r-1}D_{r,s} \wedge V_n \to (C(\bar{\eta}) \wedge SZ/2^m) \vee \Sigma^{8r-8s+1}SZ/2^l,\\
&(a\rho_r \wedge i_U\pi, b\bar{i}_U\rho_s \wedge j): \Sigma^{8r-1}D_{r,s} \wedge SZ/2^n \to U_m \vee \Sigma^{8r-8s+1}U_l,\\
&(a\rho_r \wedge \omega, b\bar{i}_U\rho_s \wedge \bar{j}_V): \Sigma^{8r-1}D_{r,s} \wedge V_n \to V_m \vee \Sigma^{8r-8s+1}U_l.
\end{aligned}
$$

Use the maps n_r' and n_s' as well as ρ_r and ρ_s to construct small spectra denoted by the same symbols for the other pairs (t, p) of non-zero even integers.

Compose the map $\lambda_M: M_n^q \to SZ/2^n \wedge P^q$ given in (2.4) before the obvious map $\pi \wedge 1: SZ/2^n \wedge P^q \to SZ/2^m \wedge P^q$ and denote it again by $\lambda_M: M_n^q \to SZ/2^m \wedge P^q$. Using the maps h_M and l_M' in (2.2) we consider the following mixed maps

$$
(3.4)\quad
\begin{aligned}
&(a\rho_r \wedge \lambda_M, bi\rho_s \wedge h_M): \Sigma^{8r-1}D_{r,s} \wedge M_n^q \to (SZ/2^m \wedge P^q) \vee \Sigma^{8r-8s+1}SZ/2^l,\\
&(a\rho_r \wedge \lambda_M, b\bar{i}_U\rho_s \wedge h_M): \Sigma^{8r-1}D_{r,s} \wedge M_n^q \to (SZ/2^m \wedge P^q) \vee \Sigma^{8r-8s+1}U_l,\\
&(ai\rho_r \wedge l_M', b\rho_s \wedge (1 \wedge \pi)j_M'): \Sigma^{8r-1}D_{r,s} \wedge {}'M_n^q\\
&\qquad\qquad \to (SZ/2^m \wedge P^q) \vee (\Sigma^{8r-8s+2q-1}C_q \wedge SZ/2^l),\\
&(ai\rho_r \wedge l_M', b\rho_s \wedge (1 \wedge i_U\pi)j_M'): \Sigma^{8r-1}D_{r,s} \wedge {}'M_n^q\\
&\qquad\qquad \to (SZ/2^m \wedge P^q) \vee (\Sigma^{8r-8s+2q-1}C_q \wedge U_l)
\end{aligned}
$$

whose cofibers are denoted by $SJM_{m,l,n}^{t,p,q,a,b}$, ${}_USJM_{m,l,n}^{t,p,q,a,b}$, $JS'M_{m,l,n}^{t,p,q,a,b}$ and ${}_UJS'M_{m,l,n}^{t,p,q,a,b}$ $(a, b \geq 1)$ for $(t, p) = (4r, 4s)$, respectively. Use the maps n_r' and n_s' as well as ρ_r and ρ_s to construct small spectra denoted by the same symbols for the other pairs (t, p) of non-zero even integers.

For any $m \leq n$ there exist the following cofiber sequences

$$
(3.5)\quad
\begin{aligned}
&\Sigma^{-1}SZ/2^m \wedge J^{q,a} \xrightarrow{j \wedge l_J} J_{n-m}^{q,a} \xrightarrow{\omega_J} J_n^{q,a} \xrightarrow{\lambda_J} SZ/2^m \wedge J^{q,a},\\
&\Sigma^{-1}SZ/2^m \wedge J^{q,a} \xrightarrow{j \wedge {}_Ul_J} {}_UJ_{n-m+1}^{q,a} \xrightarrow{{}_U\omega_J} {}_UJ_{n+1}^{q,a} \xrightarrow{{}_U\lambda_J} SZ/2^m \wedge J^{q,a},\\
&\Sigma^{-1}U_m \wedge J^{q,a} \xrightarrow{\bar{j}_U \wedge l_J} C(\bar{\eta}) \wedge J_{n-m}^{q,a} \xrightarrow{{}_U\bar{\lambda}_J'} {}_UJ_n^{q,a} \xrightarrow{{}_W\lambda_J} U_m \wedge J^{q,a},\\
&\Sigma^{-1}U_m \wedge J^{q,a} \xrightarrow{{}_Vl_J(\bar{j}_U \wedge 1)} {}_VJ_{n-m+1}^{q,a} \xrightarrow{{}_V\lambda_J} J_{n+1}^{q,a} \xrightarrow{{}_U\lambda_J'} U_m \wedge J^{q,a}
\end{aligned}
$$

in which ${}_U\lambda_J' = (i_U \wedge 1)\lambda_J$ and $J_0^{q,a}$ stands for $\Sigma^{2q}C_q$. Using the maps h_J, l_J, ${}_Uh_J$ and ${}_Ul_J$ in (3.1) and $\lambda_J: J_n^{q,a} \to SZ/2^n \wedge J^{q,a}$ and ${}_U\lambda_J: {}_UJ_n^{q,a} \to SZ/2^{n-1} \wedge J^{q,a}$ we consider the following mixed maps

$$((i\bar{\mu}_r \wedge 1)(1 \wedge \lambda_J), i\mu_s \wedge h_J): \Sigma^{8r+1}D_{r,s} \wedge J_n^{q,a} \to (SZ/2^m \wedge J^{q,a}) \vee (\Sigma^{8r-8s+1}SZ/2^l \wedge C'_q),$$

$$((i\bar{\mu}_r \wedge 1)(1 \wedge {}_U\lambda_J), i\mu_s \wedge {}_Uh_J): \Sigma^{8r+1}D_{r,s} \wedge {}_UJ_n^{q,a}$$

(3.6) $$\to (SZ/2^m \wedge J^{q,a}) \vee (\Sigma^{8r-8s+1}SZ/2^l \wedge C'_q \wedge C(\bar{\eta})),$$

$$i_J(1 \wedge \tilde{\mu}_r \wedge j) \vee (\mu_s \wedge j \wedge l_J):$$

$$(\Sigma^{8r+1}C'_q \wedge D_r \wedge SZ/2^m) \vee (\Sigma^{8s}D_s \wedge SZ/2^l \wedge J^{q,a}) \to J_n^{q,a},$$

$$i_J(1 \wedge {}_U\tilde{\mu}_r \wedge j) \vee (\mu_s \wedge j \wedge {}_Ul_J):$$

$$(\Sigma^{8r+1}C'_q \wedge D_r \wedge C(\bar{\eta}) \wedge SZ/2^m) \vee (\Sigma^{8s}D_s \wedge SZ/2^l \wedge J^{q,a}) \to {}_UJ_n^{q,a}$$

in which i_J's are the canonical inclusions. These cofibers are denoted by $'PMJ_{m,l,n}^{t,p,q,a}$, ${}_U'PMJ_{m,l,n}^{t,p,q,a}$, $J'MP_{n,l,m}^{q,p,t,a}$ and ${}_UJ'MP_{n,l,m}^{q,p,t,a}$ $(a \geq 1)$ for $(t, p) = (4r+1, 4s+1)$, respectively. Use the maps k_r, $\bar{k}_r$, $\tilde{k}_r$ and ${}_U\tilde{k}_r$ as well as μ_r, $\bar{\mu}_r$, $\tilde{\mu}_r$ and ${}_U\tilde{\mu}_r$ to construct small spectra denoted by the same symbols for the other pairs (t, p) of odd integers.

Next we take the maps $\lambda_J: J_n^{q,b} \to SZ/2^n \wedge J^{q,b}$, ${}_U\lambda_J: {}_UJ_{n+1}^{q,b} \to SZ/2^n \wedge J^{q,b}$, ${}_U\lambda'_J: J_{n+1}^{q,b} \to U_n \wedge J^{q,b}$ and ${}_W\lambda_J: {}_UJ_n^{q,b} \to U_n \wedge J^{q,b}$ given in (3.5) and then compose them before the obvious map $\pi \wedge 1: SZ/2^n \wedge J^{q,b} \to SZ/2^m \wedge J^{q,b}$ or $\omega \wedge 1: U_n \wedge J^{q,b} \to U_m \wedge J^{q,b}$. This compositions are again denoted by the same symbols λ_J, ${}_U\lambda_J$, ${}_U\lambda'_J$ and ${}_W\lambda_J$. Using the maps h_J, l_J, ${}_Uh_J$ and ${}_Ul_J$ in (3.1) we consider the following mixed maps

$$(a\rho_r \wedge \lambda_J, ci\rho_s \wedge h_J): \Sigma^{8r-1}D_{r,s} \wedge J_n^{q,b} \to (SZ/2^m \wedge J^{q,b}) \vee (\Sigma^{8r-8s+1}SZ/2^l \wedge C'_q),$$

$$(a\rho_r \wedge {}_U\lambda'_J, c\bar{i}_U\rho_s \wedge h_J): \Sigma^{8r-1}D_{r,s} \wedge J_n^{q,b} \to (U_m \wedge J^{q,b}) \vee (\Sigma^{8r-8s+1}U_l \wedge C'_q),$$

$$(a\rho_r \wedge {}_U\lambda_J, ci\rho_s \wedge {}_Uh_J): \Sigma^{8r-1}D_{r,s} \wedge {}_UJ_n^{q,b}$$

$$\to (SZ/2^m \wedge J^{q,b}) \vee (\Sigma^{8r-8s+1}SZ/2^l \wedge C'_q \wedge C(\bar{\eta})),$$

$$(a\rho_r \wedge {}_W\lambda_J, c\bar{i}_U\rho_s \wedge {}_Uh_J): \Sigma^{8r-1}D_{r,s} \wedge {}_UJ_n^{q,b}$$

$$\to (U_m \wedge J^{q,b}) \vee (\Sigma^{8r-8s+1}U_l \wedge C'_q \wedge C(\bar{\eta})),$$

(3.7)

$$(a\rho_r \wedge i_J(1 \wedge \pi)) \vee (c\rho_s \wedge j \wedge l_J):$$

$$(\Sigma^{8r-1}D_r \wedge C'_q \wedge SZ/2^m) \vee (\Sigma^{8s-2}D_s \wedge SZ/2^l \wedge J^{q,b}) \to J_n^{q,b},$$

$$(a\rho_r \wedge i_J(1 \wedge \pi_V)) \vee (c\rho_s \wedge \bar{j}_V \wedge l_J):$$

$$(\Sigma^{8r-1}D_r \wedge C'_q \wedge V_m) \vee (\Sigma^{8s-2}D_s \wedge V_l \wedge J^{q,b}) \to J_n^{q,b},$$

$(a\rho_r \wedge i_J(1 \wedge \bar{\pi}'_U)) \vee (c\rho_s \wedge j \wedge {}_Ul_J)$:

$$(\Sigma^{8r-1}D_r \wedge C'_q \wedge C(\bar{\eta}) \wedge SZ/2^m) \vee (\Sigma^{8s-2}D_s \wedge SZ/2^l \wedge J^{q,b}) \to {}_UJ_n^{q,b},$$

$(a\rho_r \wedge i_J(1 \wedge \omega)) \vee (c\rho_s \wedge \bar{j}_V \wedge {}_Ul_J)$:

$$(\Sigma^{8r-1}D_r \wedge C'_q \wedge U_m) \vee (\Sigma^{8s-2}D_s \wedge V_l \wedge J^{q,b}) \to {}_UJ_n^{q,b}$$

whose cofibers are denoted by $SJJ_{m,l,n}^{t,p,q,a,c,b}$, $S_UJJ_{m,l,n}^{t,p,q,a,c,b}$, $SJ_UJ_{m,l,n}^{t,p,q,a,c,b}$, $S_UJ_UJ_{m,l,n}^{t,p,q,a,c,b}$, $J'JS_{n,l,m}^{q,p,t,b,c,a}$, ${}_VJ'JS_{n,l,m}^{q,p,t,b,c,a}$, ${}_UJ'JS_{n,l,m}^{q,p,t,b,c,a}$ and ${}_WJ'JS_{n,l,m}^{q,p,t,b,c,a}$ $(a, b, c \geq 1)$ for $(t, p) = (4r, 4s)$, respectively. Use the maps n'_r and n'_s as well as ρ_r and ρ_s to construct small spectra denoted by the same symbols for the other pairs (t, p) of non-zero even integers.

3.3. Denote by $M_{n,l}^1$ and ${}_WM_{n,l}^1$ the cofibers of the maps $i\eta j: SZ/2^l \to SZ/2^n$ and $\bar{i}_U\eta\bar{j}_V: V_l \to U_n$, and by $MS_{n,l,m}^{1,t,a}$ and ${}_WMS_{n,l,m}^{1,t,a}$ $(a \geq 1)$ with $t = 4r$ those of the following mixed maps

$$(3.8)\qquad \begin{aligned} &(a\rho_r \wedge \pi) \vee i\eta j: (\Sigma^{8r-1}D_r \wedge SZ/2^m) \vee SZ/2^l \to SZ/2^n,\\ &(a\rho_r \wedge \omega) \vee \bar{i}_U\eta\bar{j}_V: (\Sigma^{8r-1}D_r \wedge U_m) \vee V_l \to U_n, \end{aligned}$$

respectively. Use the map n'_r instead of ρ_r to construct small spectra denoted by the same symbols for $t = 4r + 2$. By definition it is evident that

(3.9) $SZ/2 \wedge SZ/2 = M_{1,1}^1$, $J_1^{t,a} \wedge SZ/2 = MS_{1,1,1}^{1,t,a}$ and ${}_UJ_1^{t,a} \wedge U_1$ has the same K_*-local type as ${}_WMS_{1,1,1}^{1,t,a}$.

Choose maps $k_M: \Sigma^1 \to M_{n,l}^1$, $k'_M: M_{n,l}^1 \to \Sigma^1$, ${}_Wk_M: \Sigma^1C(\bar{\eta}) \to {}_WM_{n,l}^1$ and ${}_Wk'_M: {}_WM_{n,l}^1 \to \Sigma^1C(\bar{\eta})$ satisfying $j_Mk_M = i$, $2^lk_M = i_Mi\eta$, $k'_Mi_M = j$, $2^nk'_M = \eta jj_M$, $j_M\,{}_Wk_M = \bar{i}_V$, $2^{l-1}{}_Wk_M\bar{i} = i_M\bar{i}_U\eta$, ${}_Wk'_Mi_M = \bar{j}_U$ and $2^{n-1}\bar{\lambda}\,{}_Wk'_M = \eta\bar{j}_Vj_M$ in which i_M's and j_M's are the canonical inclusions and projections. Then the small spectra $M_{n,l}^1$ and ${}_WM_{n,l}^1$ are exhibited by the following cofiber sequences

$$SZ/2^l \xrightarrow{i\eta j} SZ/2^n \xrightarrow{i_M - k_Mj} M_{n,l}^1 \xrightarrow{j_M + ik'_M} \Sigma^1SZ/2^l,$$

$$V_l \xrightarrow{\bar{i}_U\eta\bar{j}_V} U_n \xrightarrow{i_M - {}_Wk_M\bar{j}_U} {}_WM_{n,l}^1 \xrightarrow{j_M + \bar{i}_V\,{}_Wk'_M} \Sigma^1V_l,$$

which give rise to the following cofiber sequences

$$(3.10)\qquad \begin{aligned} &J_l^{t,a'} \xrightarrow{\eta_J} J_n^{t,a''} \to MS_{n,l,m}^{1,t,a} \to \Sigma^1J_l^{t,a'},\\ &{}_VJ_l^{t,a'} \xrightarrow{{}_W\eta_J} {}_UJ_n^{t,a''} \to {}_WMS_{n,l,m}^{1,t,a} \to \Sigma^1\,{}_VJ_l^{t,a'}, \end{aligned}$$

respectively, where $a' = \operatorname{Max}\{a, 2^{m-n}a\}$ and $a'' = \operatorname{Max}\{a, 2^{n-m}a\}$.

Denote by $L_{n,l}^1$, ${}_VL_{n,l}^1$ and ${}_UL_{n,l}^1$ the cofibers of the maps $k_{-1} \wedge \bar{\eta}\eta$: $\Sigma^{-3}C(\bar{\eta}) \wedge SZ/2^l \to SZ/2^n$, $2^k\omega: V_l \to V_n$ and $2^k\omega: U_l \to U_n$, and by $LS_{n,l,m}^{1,t,a}$ and ${}_WLS_{n,l,m}^{1,t,a}$ $(a \geq 1)$ for $t = 4r$ those of the following mixed maps

$$(3.11)\quad \begin{aligned} &(a\rho_r \wedge \pi) \vee 2^{k-1}(\bar{\lambda} \wedge \pi): (\Sigma^{8r-1}D_r \wedge SZ/2^m) \vee (C(\bar{\eta}) \wedge SZ/2^l) \to SZ/2^n, \\ &(a\rho_r \wedge \omega) \wedge 2^k\omega : (\Sigma^{8r-1}D_r \wedge V_m) \vee V_l \to V_n, \end{aligned}$$

respectively, where $k = \text{Min}\,\{n, l\}$. Use the map n_r' instead of ρ_r to construct small spectra denoted by the same symbols for $t = 4r + 2$.

The small spectra ${}_VL^1_{n,l}$ and ${}_UL^1_{n,l}$ are also obtained as the cofibers of the maps $2^{k-1}(\bar{i} \wedge \pi): SZ/2^l \to C(\bar{\eta}) \wedge SZ/2^n$ and $2^{k-1}(\bar{\lambda} \wedge \pi): C(\bar{\eta}) \wedge SZ/2^l \to SZ/2^n$, respectively. Therefore we observe that ${}_VL^1_{n,l} \wedge C(\bar{\eta})$ and ${}_UL^1_{n,l}$ have the same K_*-local type as $L^1_{n,l}$. By definition it is now evident that

(3.12) the smash product $V_n \wedge SZ/2^n$ has the same K_*-local type as $L^1_{n,n} \wedge C(\bar{\eta})$, and ${}_UJ^{t,a}_n \wedge SZ/2^n = LS^{1,t,a}_{n,n,n}$ and $J^{t,a}_n \wedge V_n = {}_WLS^{1,t,a}_{n,n,n}$.

More generally there exist the following cofiber sequences

$$(3.13)\quad \begin{aligned} &{}_UJ^{t,a'}_l \xrightarrow{\pi_J} {}_UJ^{t,a''}_n \to LS^{1,t,a}_{n,l,m} \to \Sigma^1\,{}_UJ^{t,a'}_l, \\ &J^{t,a'}_l \xrightarrow{{}_W\pi_J} C(\bar{\eta}) \wedge J^{t,a''}_n \to {}_WLS^{1,t,a}_{n,l,m} \to \Sigma^1 J^{t,a'}_l \end{aligned}$$

in which $a' = \text{Max}\,\{a, 2^{m-n}a\}$ and $a'' = \text{Max}\,\{a, 2^{n-m}a\}$.

Using the maps η_J, ${}_W\eta_J$, π_J and ${}_W\pi_J$ given in (3.10) and (3.13) we consider the following maps

$$(3.14)\quad \begin{aligned} &\eta_J \wedge i_J : J^{t,a'}_l \wedge C'_q \to J^{t,a''}_n \wedge J^{q,b}, \qquad {}_W\eta_J \wedge i_J : {}_VJ^{t,a'}_l \wedge C'_q \to {}_UJ^{t,a''}_n \wedge J^{q,b}, \\ &\pi_J \wedge i_J : {}_UJ^{t,a'}_l \wedge C'_q \to {}_UJ^{t,a''}_n \wedge J^{q,b}, \qquad {}_W\pi_J \wedge i_J : J^{t,a'}_l \wedge C'_q \to C(\bar{\eta}) \wedge J^{t,a''}_n \wedge J^{q,b}, \\ &\eta_J \wedge j_J : \Sigma^{-1}J^{t,a'}_l \wedge J^{q,b} \to \Sigma^{2q-1}J^{t,a''}_n \wedge C_q, \\ &{}_W\eta_J \wedge j_J : \Sigma^{-1}\,{}_VJ^{t,a'}_l \wedge J^{q,b} \to \Sigma^{2q-1}\,{}_UJ^{t,a''}_n \wedge C_q, \\ &\pi_J \wedge j_J : \Sigma^{-1}\,{}_UJ^{t,a'}_l \wedge J^{q,b} \to \Sigma^{2q-1}\,{}_UJ^{t,a''}_n \wedge C_q, \\ &{}_W\pi_J \wedge j_J : \Sigma^{-1}J^{t,a'}_l \wedge J^{q,b} \to \Sigma^{2q-1}C(\bar{\eta}) \wedge J^{t,a''}_n \wedge C_q \end{aligned}$$

whose cofibers are denoted by $MSJ^{1,t,q,a,b}_{n,l,m}$, ${}_WMSJ^{1,t,q,a,b}_{n,l,m}$, $LSJ^{1,t,q,a,b}_{n,l,m}$, ${}_WLSJ^{1,t,q,a,b}_{n,l,m}$, $JMS^{q,1,t,b,a}_{n,l,m}$, ${}_WJMS^{q,1,t,b,a}_{n,l,m}$, $JLS^{q,1,t,b,a}_{n,l,m}$ and ${}_WJLS^{q,1,t,b,a}_{n,l,m}$ $(a, b \geq 1)$, respectively.

4. K_*-local types of some smash products

4.1. The K_*-local types of the smash products $SZ/2^m \wedge SZ/2^n$, $V_m \wedge SZ/2^n$ and $V_m \wedge V_n$ have been determined in (1.1), (1.6), (1.13), (3.9) and (3.12). On the other hand, the determination of K_*-local types of $M^t_m \wedge SZ/2^n$, $M^t_m \wedge V_n$, $'M^t_m \wedge SZ/2^n$ and $'M^t_m \wedge V_n$ is established by (2.3) and the following result and its dual.

THEOREM 4.1. *The smash products* $M_m^t \wedge SZ/2^n$ *and* $M_m^t \wedge V_n$ *have the same* K_**-local types as* $'PM_{m,m,n}^{t,t}$ *and* $_V'PM_{m,m,n}^{t,t}$*, respectively, if* $m < n$.

PROOF. Use the splitting maps $\varphi: SZ/2^m \wedge SZ/2^n \to SZ/2^m$ and $\varphi_V: V_n \wedge SZ/2^m \to C(\bar{\eta}) \wedge SZ/2^m$ given in (1.2) and (1.8) for $m < n$. Then the maps $i\mu_r \wedge 1: \Sigma^{8r+1}D_r \wedge SZ/2^n \to SZ/2^m \wedge SZ/2^n$ and $i\mu_r \wedge 1: \Sigma^{8r+1}D_r \wedge V_n \to SZ/2^m \wedge V_n$ are rewritten to be $(\mu_r \wedge \pi, i\mu_r \wedge j): \Sigma^{8r+1}D_r \wedge SZ/2^n \to SZ/2^m \vee \Sigma^1 SZ/2^m$ and $(\mu_r \wedge \bar{\pi}_V, i\mu_r \wedge \bar{j}_V): \Sigma^{8r+1}D_r \wedge V_n \to (C(\bar{\eta}) \wedge SZ/2^m) \vee \Sigma^1 SZ/2^m$, respectively, when $m < n$. In this case we may assume that the maps $\mu_r \wedge \pi$ and $\mu_r \wedge \bar{\pi}_V$ are quasi S_{K_*}-equivalent to the composite maps $i\bar{\mu}_r$ and $_V\bar{\mu}_r \wedge i$, respectively. Therefore our result for $t = 4r + 1$ is immediate from (2.6). Use the map $k_r: \Sigma^{8r+2}C(\bar{\eta}) \wedge D_r \to \Sigma^0$ instead of μ_r in case $t = 4r + 3$.

The determination of the K_*-local types of $M_m^t \wedge M_n^q$, $'M_m^t \wedge M_n^q$ and $'M_m^t \wedge 'M_n^q$ is established by the following result and its dual.

THEOREM 4.2. *The smash products* $M_m^t \wedge M_n^q$ *and* $'M_m^t \wedge M_n^q$ *have the same* K_**-local types as* $'PMM_{m,m,n}^{t,t,q}$ *and* $MP'M_{n,m,m}^{q,t,t}$*, respectively, if* $m < n$*; and they have the same* K_**-local types as* $''PMM_{m,m,m}^{t,t,q}$ *and* $M''P'M_{m,m,m}^{q,t,t}$*, respectively, if* $m = n$.

PROOF. Use the splitting maps $\varphi_M: SZ/2^m \wedge M_n^q \to S_K \wedge SZ/2^m \wedge P^q$ and $\psi_M: \Sigma^1 SZ/2^m \to S_K \wedge SZ/2^m \wedge M_n^q$ given in (2.5) for $m \le n$. Then the maps $i\mu_r \wedge 1: \Sigma^{8r+1}D_r \wedge M_n^q \to SZ/2^m \wedge M_n^q$ and $\mu_r \wedge j \wedge 1: \Sigma^{8r}D_r \wedge SZ/2^m \wedge M_n^q \to M_n^q$ may be, respectively, rewritten to be $((i\bar{\mu}_r \wedge 1)(1 \wedge \lambda_M), i\mu_r \wedge h_M): \Sigma^{8r+1}D_r \wedge M_n^q \to (SZ/2^m \wedge P^q) \vee \Sigma^1 SZ/2^m$ and $(\mu_r \wedge j \wedge l_M) \vee i_M(\tilde{\mu}_r \wedge j): (\Sigma^{8r}D_r \wedge SZ/2^m \wedge P^q) \vee (\Sigma^{8r+1}D_r \wedge SZ/2^m) \to M_n^q$ when $m < n$, and to be the ones we obtain by substituting $i\bar{\mu}_r + \tilde{\mu}_r \wedge j$ for $i\bar{\mu}_r$ or $\tilde{\mu}_r \wedge j$ when $m = n$. Combining these facts with (2.6) we get our result for $t = 4r + 1$. Use the map k_r instead of μ_r in case $t = 4r + 3$.

When $X_m = J_m^{t,a}$, $_UJ_m^{t,a}$, $'J_m^{t,a}$ or $_V'J_m^{t,a}$, the determination of the K_*-local types of the smash products $X_m \wedge SZ/2^n$ and $X_m \wedge V_n$ is established by (3.2), (3.9), (3.12) and the following result and their duals.

THEOREM 4.3. *The smash products* $J_m^{t,a} \wedge SZ/2^n$, $J_m^{t,a} \wedge V_n$, $_UJ_m^{t,a} \wedge SZ/2^n$ *and* $_UJ_m^{t,a} \wedge V_n$ *have the same* K_**-local types as* $SJ_{m,m,n}^{t,t,a,a}$, $_VSJ_{m,m,n}^{t,t,a,a}$, $_USJ_{m,m,n}^{t,t,a,a}$ *and* $_WSJ_{m,m,n}^{t,t,a,a}$*, respectively, if* $m < n$.

PROOF. Consider the maps $i\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge SZ/2^n \to SZ/2^m \wedge SZ/2^n$, $i\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge V_n \to SZ/2^m \wedge V_n$, $\bar{i}_U\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge SZ/2^n \to U_m \wedge SZ/2^n$ and $\bar{i}_U\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge U_n \to U_m \wedge U_n$ when $m < n$. By use of the splitting maps $\varphi: SZ/2^m \wedge SZ/2^n \to SZ/2^m$, $\varphi_V: V_n \wedge SZ/2^m \to C(\bar{\eta}) \wedge SZ/2^m$ and $\varphi_U': U_m \wedge SZ/2^n \to U_m$ given in (1.2), (1.8) and (1.10) the first three of them can be rewritten as the first three maps in (3.3) with $a=b=1$, $r=s$ and $m=l<n$, respectively. On the other hand, the fourth map is rewritten to be $(\rho_r \wedge \omega, \bar{i}_U\rho_r \wedge \bar{j}_U): \Sigma^{8r-1}D_r \wedge U_n \to U_m \vee (\Sigma^1 U_m \wedge C(\bar{\eta}))$ by use of the splitting map $\varphi''_U: U_m \wedge U_n \to U_m$ given in (1.14). Therefore our result for $t=4r$ is immediate. Use the map $n'_r: \Sigma^{8r+3}C(\bar{\eta}) \to D'_{2r+1}$ instead of ρ_r in case $t=4r+2$.

4.2. Choose maps

$$\varphi_J: SZ/2^m \wedge J_n^{t,a} \to SZ/2^m \wedge J^{t,a}, \qquad \psi_J: \Sigma^1 C'_t \wedge SZ/2^m \to J_n^{t,a} \wedge SZ/2^m,$$

(4.1)

$$_W\varphi_J: U_m \wedge {}_UJ_n^{t,a} \to U_m \wedge J^{t,a} \quad \text{and} \quad {}_W\psi_J: \Sigma^1 C'_t \wedge V_m \to S_K \wedge {}_VJ_n^{t,a} \wedge V_m$$

satisfying $\varphi_J(1 \wedge l_J)=1$, $(h_J \wedge 1)\psi_J = 1$, ${}_W\varphi_J(1 \wedge {}_Ul_J)=1$ and $(1 \wedge {}_Vh_J \wedge 1)\cdot {}_W\psi_J = \iota_K \wedge 1 \wedge 1$ when $m \le n$ and $n \ge 2$, and moreover

$${}_U\varphi_J: SZ/2^m \wedge {}_UJ_n^{t,a} \to SZ/2^m \wedge J^{t,a},$$

(4.2)

$${}_U\psi_J: \Sigma^1 C'_t \wedge C(\bar{\eta}) \wedge SZ/2^m \to {}_UJ_n^{t,a} \wedge SZ/2^m,$$

$${}_U\varphi'_J: U_m \wedge J_n^{t,a} \to U_m \wedge J^{t,a} \quad \text{and} \quad {}_U\psi'_J: \Sigma^1 C'_t \wedge U_m \to J_n^{t,a} \wedge U_m$$

satisfying ${}_U\varphi_J(1 \wedge {}_Ul_J)=1$, $({}_Uh_J \wedge 1){}_U\psi_J=1$, ${}_U\varphi'_J(1 \wedge l_J)=1$ and $(h_J \wedge 1){}_U\psi'_J = 1$ when $m<n$. For these maps φ_J, ${}_W\varphi_J$, ${}_U\varphi_J$ and ${}_U\varphi'_J$ we can find maps $f: \Sigma^1 SZ/2^m \wedge C'_t \to SZ/2^m \wedge J^{t,a}$, $f_W: \Sigma^1 U_m \wedge C'_t \wedge C(\bar{\eta}) \to U_m \wedge J^{t,a}$, $f_U: \Sigma^1 SZ/2^m \wedge C'_t \wedge C(\bar{\eta}) \to SZ/2^m \wedge J^{t,a}$ and $f'_U: \Sigma^1 U_m \wedge C'_t \to U_m \wedge J^{t,a}$ such that $\varphi_J(i \wedge 1) = \lambda_J + f(i \wedge h_J)$, ${}_W\varphi_J(\bar{i}_U \wedge 1) = {}_W\lambda_J + f_W(\bar{i}_U \wedge {}_Uh_J)$, ${}_U\varphi_J(i \wedge 1) = {}_U\lambda_J + f_U(i \wedge {}_Uh_J)$ and ${}_U\varphi'_J(\bar{i}_U \wedge 1) = {}_U\lambda'_J + f'_U(\bar{i}_U \wedge h_J)$ in which the maps $\lambda_J: J_n^{t,a} \to SZ/2^m \wedge J^{t,a}$, ${}_W\lambda_J: {}_UJ_n^{t,a} \to U_m \wedge J^{t,a}$, ${}_U\lambda_J: {}_UJ_n^{t,a} \to SZ/2^m \wedge J^{t,a}$ and ${}_U\lambda'_J: J_n^{t,a} \to U_m \wedge J^{t,a}$ are given in (3.5). When $m \ge 2$ our assertion is easily verified. Note that the map $\bar{\lambda} \wedge 1: C(\bar{\eta}) \wedge U_1 \to U_1$ is factorized as the composite map $\bar{i}_U\theta(1 \wedge \bar{j}_U)$ for some $\theta \in [\Sigma^1 C(\bar{\eta}) \wedge C(\bar{\eta}), \Sigma^0] \cong Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2$ because of $\bar{\lambda} \wedge 1 = 1 \wedge \bar{\lambda}: C(\bar{\eta}) \wedge C(\bar{\eta}) \to C(\bar{\eta})$. When $m=1$ it follows that $2\lambda_J = 2^{n-1}(i\eta \wedge i_Jh_J) = 0$, $2{}_U\lambda_J = 2^{n-2}(i\eta \wedge (i_J \wedge \bar{\lambda}){}_Uh_J) = 0$, $\bar{\lambda} \wedge {}_W\lambda_J = 2^{n-1}(\bar{i}_U\theta \wedge i_J)(1 \wedge T_Uh_J) = 0$ and $\bar{\lambda} \wedge {}_U\lambda'_J = 2^{n-2}(\bar{i}_U\theta \wedge i_J)(1 \wedge (\bar{i} \wedge 1)h_J) = 0$. By means of this result we can easily show that our assertion is also valid even if $m=1$. Consequently the maps φ_J, ${}_W\varphi_J$, ${}_U\varphi_J$ and ${}_U\varphi'_J$ are chosen to satisfy $\varphi_J(i \wedge 1) = \lambda_J$, ${}_W\varphi_J(\bar{i}_U \wedge 1) = {}_W\lambda_J$, ${}_U\varphi_J(i \wedge 1) = {}_U\lambda_J$ and ${}_U\varphi'_J(\bar{i}_U \wedge 1) = {}_U\lambda'_J$. On the other hand, the maps ψ_J, ${}_W\psi_J$, ${}_U\psi_J$ and ${}_U\psi'_J$ may be taken to be the composite maps $(i_J \wedge 1)(1 \wedge \psi)$, $(1 \wedge {}_Vi_J \wedge 1)(1 \wedge \psi''_V)$, $({}_Ui_J \wedge 1)(1 \wedge 1 \wedge \psi_U)$ and $(i_J \wedge 1)(1 \wedge \psi'_U)$, respectively, where $\psi: \Sigma^1 SZ/2^m \to SZ/2^n \wedge SZ/2^m$, $\psi_U: \Sigma^1 C(\bar{\eta}) \wedge SZ/2^m \to U_n \wedge SZ/2^m$, $\psi'_U: \Sigma^1 U_m \to SZ/2^n \wedge U_m$ and $\psi''_V: \Sigma^1 V_m \to S_K \wedge V_n \wedge V_m$ are given in (1.2), (1.8), (1.10) and (1.14). Therefore they satisfy $(1 \wedge j)\psi_J = i_J(1 \wedge \pi)$, $(1 \wedge 1 \wedge \bar{j}_V){}_W\psi_J = \iota_K \wedge {}_Vi_J(1 \wedge \omega'')$, $(1 \wedge j){}_U\psi_J = {}_Ui_J(1 \wedge \bar{\pi}'_U)$ and $(1 \wedge \bar{j}_U){}_U\psi'_J = i_J(1 \wedge T\bar{\pi}_U)$ where $\omega'' = \omega + i_Vivj$ or ω depending if $(m, n) = (1, 2)$ or not.

When $X_m = J_m^{t,a}$, ${}_UJ_m^{t,a}$, ${}'J_m^{t,a}$ or ${}_V'J_m^{t,a}$, the determination of the K_*-local types of the smash products $X_m \wedge M_n^q$ and $X_m \wedge {}'M_n^q$ is established by the following result and its dual.

THEOREM 4.4. i) *The smash products* $J_m^{t,a} \wedge M_n^q$, ${}_UJ_m^{t,a} \wedge M_n^q$, $J_m^{t,a} \wedge {}'M_n^q$ *and* ${}_UJ_m^{t,a} \wedge {}'M_n^q$ *have the same* K_**-local types as* $SJM_{m,m,n}^{t,t,q,a,a}$, ${}_USJM_{m,m,n}^{t,t,q,a,a}$, $JS'M_{m,m,n}^{t,t,q,a,a}$ and ${}_UJS'M_{m,m,n}^{t,t,q,a,a}$, *respectively, if* $m \le n$; *and*

ii) *the smash products* $M_m^t \wedge J_n^{q,a}$, $M_m^t \wedge {}_UJ_n^{q,a}$, ${}'M_m^t \wedge J_n^{q,a}$ *and* ${}'M_m^t \wedge {}_UJ_n^{q,a}$ *have the same* K_**-local types as* ${}'PMJ_{m,m,n}^{t,t,q,a}$, ${}_U'PMJ_{m,m,n}^{t,t,q,a}$, $J'MP_{n,m,m}^{q,t,t,a}$ *and* ${}_UJ'MP_{n,m,m}^{q,t,t,a}$, *respectively, if* $m < n$.

PROOF. i) Use the splitting maps $\varphi_M: SZ/2^m \wedge M_n^q \to S_K \wedge SZ/2^m \wedge P^q$ and ${}_U\varphi_M: U_m \wedge M_n^q \to S_K \wedge SZ/2^m \wedge P^q$ given in (2.5) and their dualized splitting maps $\varphi'_M: SZ/2^m \wedge {}'M_n^q \to S_K \wedge \Sigma^{2q-1}SZ/2^m \wedge C_q$ and ${}_U\varphi'_M: U_m \wedge {}'M_n^q \to S_K \wedge \Sigma^{2q-1}U_m \wedge C_q$ for $m \le n$. Then the maps $i\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge M_n^q \to SZ/2^m \wedge M_n^q$, $\bar{i}_U\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge M_n^q \to U_m \wedge M_n^q$, $i\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge {}'M_n^q \to SZ/2^m \wedge {}'M_n^q$ and $\bar{i}_U\rho_r \wedge 1: \Sigma^{8r-1}D_r \wedge {}'M_n^q \to U_m \wedge {}'M_n^q$ may be rewritten as in (3.4) with $a = b = 1$, $r = s$ and $m = l \le n$, respectively. Our result is now immediate.

ii) Use the splitting maps $\varphi_J: SZ/2^m \wedge J_n^{q,a} \to SZ/2^m \wedge J^{q,a}$, ${}_U\varphi_J: SZ/2^m \wedge {}_UJ_n^{q,a} \to SZ/2^m \wedge J^{q,a}$, $\psi_J: \Sigma^1C'_q \wedge SZ/2^m \to J_n^{q,a} \wedge SZ/2^m$ and ${}_U\psi_J: \Sigma^1C'_q \wedge C(\bar{\eta}) \wedge SZ/2^m \to {}_UJ_n^{q,a} \wedge SZ/2^m$ given in (4.1) and (4.2) for $m < n$. Then the maps $i\mu_r \wedge 1: \Sigma^{8r+1}D_r \wedge J_n^{q,a} \to SZ/2^m \wedge J_n^{q,a}$, $i\mu_r \wedge 1: \Sigma^{8r+1}D_r \wedge {}_UJ_n^{q,a} \to SZ/2^m \wedge {}_UJ_n^{q,a}$, $\mu_r \wedge j \wedge 1: \Sigma^{8r}D_r \wedge SZ/2^m \wedge J_n^{q,a} \to J_n^{q,a}$ and $\mu_r \wedge j \wedge 1: \Sigma^{8r}D_r \wedge SZ/2^m \wedge {}_UJ_n^{q,a} \to {}_UJ_n^{q,a}$ are rewritten as in (3.6) with $r = s$ and $m = l < n$. Our result is now immediate.

When X_m, $Y_m = J_m^{t,a}$, ${}_UJ_m^{t,a}$, ${}'J_m^{t,a}$ or ${}_V'J_m^{t,a}$, the determination of the K_*-local types of the smash products $X_m \wedge Y_n$ is established by the following result and its dual.

THEOREM 4.5. i) *The smash products* $J_m^{t,a} \wedge J_n^{q,b}$, ${}_UJ_m^{t,a} \wedge {}_UJ_n^{q,b}$, ${}'J_m^{t,a} \wedge J_n^{q,b}$ *and* ${}_V'J_m^{t,a} \wedge {}_UJ_n^{q,b}$ *have the same* K_**-local types as* $SJJ_{m,m,n}^{t,t,q,a,a,b}$, $S_UJ_UJ_{m,m,n}^{t,t,q,a,a,b}$, $J'JS_{n,m,m}^{q,t,t,b,a,a}$ *and* ${}_WJ'JS_{n,m,m}^{q,t,t,b,a,a}$, *respectively, if* $m \le n$ *and* $n \ge 2$, *and they have the same* K_**-local types as* $MSJ_{1,1,1}^{1,t,q,a,b}$, ${}_WMSJ_{1,1,1}^{1,t,q,a,b}$, $JMS_{1,1,1}^{t,1,q,a,b}$ *and* ${}_WJMS_{1,1,1}^{t,1,q,a,b} \wedge C(\bar{\eta})$, *respectively, if* $m = n = 1$; *and*

ii) *the smash products* ${}_UJ_m^{t,a} \wedge J_n^{q,b}$, $J_m^{t,a} \wedge {}_UJ_n^{q,b}$, ${}_V'J_m^{t,a} \wedge J_n^{q,b}$ *and* ${}'J_m^{t,a} \wedge {}_UJ_n^{q,b}$ *have the same* K_**-local types as* $S_UJJ_{m,m,n}^{t,t,q,a,a,b}$, $SJ_UJ_{m,m,n}^{t,t,q,a,a,b}$, ${}_VJ'JS_{n,m,m}^{q,t,t,b,a,a}$ *and* ${}_UJ'JS_{n,m,m}^{q,t,t,b,a,a}$, *respectively, if* $m < n$, *and they have the same* K_**-local types as* $LSJ_{m,m,m}^{1,t,q,a,b}$, ${}_WLSJ_{m,m,m}^{1,t,q,a,b} \wedge C(\bar{\eta})$, ${}_WJLS_{m,m,m}^{t,1,q,a,b}$ *and* $JLS_{m,m,m}^{t,1,q,a,b}$, *respectively, if* $m = n$.

PROOF. i) Use the splitting maps $\varphi_J : SZ/2^m \wedge J_n^{q,b} \to SZ/2^m \wedge J^{q,b}$, ${}_W\varphi_J$: $U_m \wedge {}_UJ_n^{q,b} \to U_m \wedge J^{q,b}$, $\psi_J : \Sigma^1 C'_q \wedge SZ/2^m \to J_n^{q,b} \wedge SZ/2^m$ and ${}_W\psi_J : \Sigma^1 C'_q \wedge V_m \to S_K \wedge {}_VJ_n^{q,b} \wedge V_m$ given in (4.1) for $m \leq n$ and $n \geq 2$. Then the maps $i\rho_r \wedge 1 : \Sigma^{8r-1} D_r \wedge J_n^{q,b} \to SZ/2^m \wedge J_n^{q,b}$, $\bar{i}_U \rho_r \wedge 1 : \Sigma^{8r-1} D_r \wedge {}_UJ_n^{q,b} \to U_m \wedge {}_UJ_n^{q,b}$ and $\rho_r \wedge j \wedge 1 : \Sigma^{8r-2} D_r \wedge SZ/2^m \wedge J_n^{q,b} \to J_n^{q,b}$ are rewritten as the first, forth and fifth maps in (3.7) with $a = c = 1$, $r = s$ and $m = l \leq n$, respectively. On the other hand, the map $\rho_r \wedge \bar{j}_V \wedge 1 : \Sigma^{8r-2} D_r \wedge V_m \wedge {}_VJ_n^{q,b} \to {}_VJ_n^{q,b}$ may be rewritten to be $(\rho_r \wedge {}_Vi_J(1 \wedge \omega)) \vee (\rho_r \wedge \bar{j}_V \wedge {}_Vl_J) : (\Sigma^{8r-1} D_r \wedge C'_q \wedge V_m) \vee (\Sigma^{8r-2} D_r \wedge V_m \wedge J^{q,a} \wedge C(\bar{\eta})) \to {}_VJ_n^{q,b}$. Hence the first half of our result is immediate. When $n = l = m = 1$ in (3.10) the map η_J may be taken to be $2 : J_1^{t,a} \to J_1^{t,a}$ and the map ${}_W\eta_J$ may be replaced by the map $\bar{\lambda} \wedge 1 : C(\bar{\eta}) \wedge {}_UJ_1^{t,a} \to {}_UJ_1^{t,a}$ if ${}_VJ_1^{t,a}$ is replaced by $C(\bar{\eta}) \wedge {}_UJ_1^{t,a}$. Therefore the latter half of our result is now obvious.

ii) The first half of our result is similarly shown as i) by use of the splitting maps ${}_U\varphi_J$, ${}_U\varphi'_J$, ${}_U\psi_J$ and ${}_U\psi'_J$ given in (4.2). When $n = l = m$ in (3.13) the maps π_J and ${}_W\pi_J$ may be taken to be $2^m : {}_UJ_m^{t,a} \to {}_UJ_m^{t,a}$ and $2^{m-1}(\bar{i} \wedge 1)$: $J_m^{t,a} \to C(\bar{\eta}) \wedge J_m^{t,a}$, respectively. Therefore the latter half of our result is now obvious.

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Department of Mathematics
Nagoya Institute of Technology
Nagoya 466, *Japan*