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On the Julia sets of rational functions of degree two with two real parameters

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ABSTRACT. In this paper, we firstly define an effective standard from of rational functions of degree two and determine the Julia sets when the two parameters of the above standardization are real. We especially consider the conditions under which the Julia sets are real closed intervals or Cantor sets. Finally we study the continuity and the analyticity of the Hausdorff dimension as a function of parameters.

1. Introduction

Quadratic polynomial maps on the Riemann sphere $\overline{\mathbf{C}}$ have been studied as typical examples of nonlinear complex dynamics which show chaotic behaviors. Especially the Julia sets of these polynomial maps are intensively researched (see [2] and [4]). In treating these quadratic polynomial maps, one may consider one parameter families such as $\{z^2 + c : c \in \mathbf{C}\}$ or $\{az(1-z) : a \in \mathbf{C}\}$ because any quadratic polynomial map can be transformed into these forms by suitable affine maps and the geometrical or the metrical properties do not change in this procedure. In other words, the family $\{z^2 + c : c \in \mathbf{C}\}$ or $\{az(1-z) : a \in \mathbf{C}\}$ is regarded as a standard form of quadratic polynomial maps.

In this paper we wish to investigate the Julia set $J(f)$ of a rational function f of degree two on the Riemann sphere. For this purpose we introduce a standard form of rational functions of degree two. Several forms of degree two rational functions have been studied. For example, Rees [7] used the family of a standard form $\left\{z\dfrac{z+\alpha}{1+\beta z} : \alpha, \beta \in \mathbf{C}, \alpha\beta \neq 1\right\}$. Recently, Saito-Saitô-Shimizu [9] studied the family $\left\{\dfrac{\mu z(1-\gamma z)}{1-\mu(1-\gamma)z} : 0 < \gamma \leq 1, 1 < \mu\right\}$, in connection with the logistic maps. We propose another family

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$$\mathscr{B} = \left\{ f_{a,b}(z) = \frac{az(1-z)}{1-bz} : (a, b) \in \mathbf{C}^2 \backslash E \right\}, \qquad E = \{(a, b) : a = 0 \quad \text{or} \quad b = 1\}.$$

We will see that any rational function f of degree two (excepting a very special case) is conjugate to one of $f_{a,b} \in \mathscr{B}$ (see Proposition 3.2). This form is very convenient to study the dependence of feature on parameter (a, b), because zeros of $f_{a,b}$ are fixed (i.e., 0 and 1) and the eigenvalue of the fixed point 0 of $f_{a,b}$ is a. Therefore for $f_{a,b}$, the research of the preimages of 0, which is a repelling or a rationally indifferent fixed point in our cases, seems easier than the previous two forms above. Moreover ours is more suitable for applying the method for studying the Julia sets of one dimensional maps. This will be used in the research of the structures of the Julia sets and their estimates of the Hausdorff dimensions.

The contents of the present paper are as follows. In §2 we review basic facts and tools which will be used in what follows. In §3 we show that most of the degree two rational functions are conjugate to $f_{a,b}$ for some a, $b \in \mathbf{C}$. We can choose a fundamental domain $D^{\mathbf{C}}$ of the parameter space; that is, any $f \in \mathscr{B}$ is conjugate to a unique $f_{a,b}$, $(a, b) \in D^{\mathbf{C}} \subset \mathbf{C}^2 \backslash E$. Since we are mainly interested in the case of $(a, b) \in \mathbf{R}^2$, we will discuss the fundamental domain D for $\mathbf{R}^2 \backslash E$ (see Theorem 3.8 and Figure 3.2). Our choice is very effective, since the dependence of feature of $f_{a,b}$ on (a, b) in the domain D is very clear, but it is very complicated on the outside of the domain. Having obtained the domain D, we will focus on the analysis of the Julia sets of $f_{a,b}$ when $(a, b) \in D$.

In §4 the conditions under which $J(f_{a,b}) \subset \mathbf{R} \cup \{\infty\}$ are discussed as in [9]. We also consider more precise structures by using the criterion for the hyperbolicity on the Julia sets and the fact that if a rational function f has an attracting fixed point and $J(f)$ is a proper subset of $\mathbf{R} \cup \{\infty\}$, there are no other attracting fixed points and the Fatou set of f coincides with the immediate basin of the attracting fixed points. Using the above, we see that the domain D is separated to some parts, on each side of which the Julia set of $f_{a,b}$ has different properties (see Theorem 4.3).

In §5 we analyze the Hausdorff dimensions of the Julia sets of functions in the above family. Especially we investigate the asymptotics of H-dim$(J(f_{a,b}))$ as $b \to 1$. In other words, we study how H-dim$(J(f_{a,b}))$ varies as a function of (a, b) on the boundary where the degree degenerates to one. When b is near 1, $J(f_{a,b})$ is self-similar so that the Moran-Hutchinson method in [3] can be applied to the estimate of H-dim$(J(f_{a,b}))$ (see Lemma 5.3). Here we obtain the result which says that H-dim$(J(f_{a,b}))$ is continuous on the domain up to the boundary, while the real analyticity does not hold on there (see Theorem 5.4).

2. Preliminaries

Throughout this paper we use the following terminologies and notations. $\overline{\mathbf{C}}$ denotes the Riemann sphere equipped with the spherical metric. We call f a rational function if f can be written in the form $f(z) = P(z)/Q(z)$, where $P(z)$, $Q(z)$ ($\not\equiv 0$) are polynomials with complex coefficients and no common factors. The *degree*, $\deg(f)$, of a rational function f is defined by $\deg(f) = \max\{\deg(P), \deg(Q)\}$. In what follows f always stands for a rational function unless otherwise stated especially.

A point $z \in \overline{\mathbf{C}}$ is an element of $F(f)$, the *Fatou set* of f, if there exists a neighborhood U of z in $\overline{\mathbf{C}}$ such that the family of iterates $\{f^n|_U\}_{n=1,2,\dots}$ is a normal family. The *Julia set* $J(f)$ is the complement of the Fatou set.

Let $\alpha \in \overline{\mathbf{C}}$ be a *p-periodic point*, i.e., $f^p(\alpha) = \alpha$ and $f^q(\alpha) \neq \alpha$ for $1 \leq q \leq p-1$. In particular, if $p = 1$, we call α a *fixed point*. Put $\mathrm{Fix}(f) = \{z \in \overline{\mathbf{C}}: f(z) = z\}$. $\alpha \in \mathrm{Fix}(f)$ is called a *fixed point of the order n* if α is an n zero of $f(z) - z$ when $\alpha \in \mathbf{C}$ and if 0 is n zero of $\varphi \circ f \circ \varphi^{-1}(z) - z$ ($\varphi(z) = 1/z$) when $\alpha = \infty$. Especially a fixed point of the order one is called *simple* and that of higher order, *multiple*.

In order to classify periodic points, we define $\lambda_f(\alpha)$ at $\alpha \in \overline{\mathbf{C}}$ as follows.

$$\lambda_f(\alpha) = \begin{cases} \lim\limits_{z \to \alpha} \dfrac{f(z) - f(\alpha)}{z - \alpha} & \text{if } \alpha \in \mathbf{C} \text{ and } f(\alpha) \in \mathbf{C}\,, \\[2ex] \lim\limits_{z \to \alpha} \dfrac{1}{f(z)(z - \alpha)} & \text{if } \alpha \in \mathbf{C} \text{ and } f(\alpha) = \infty\,, \\[2ex] \lim\limits_{z \to 0} \dfrac{f(1/z) - f(\infty)}{z} = \lim\limits_{z \to \infty} z(f(z) - f(\infty)) & \text{if } \alpha = \infty \text{ and } f(\infty) \in \mathbf{C}\,, \\[2ex] \lim\limits_{z \to 0} \dfrac{1}{zf(1/z)} = \lim\limits_{z \to \infty} z/f(z) & \text{if } \alpha = \infty \text{ and } f(\infty) = \infty\,. \end{cases}$$

For any rational function f and any $\alpha \in \overline{\mathbf{C}}$, it can be easily shown that $\lambda_f(\alpha) \in \overline{\mathbf{C}}$ indeed exists.

If α is a fixed point or a periodic point of f, $\lambda_f(\alpha)$ is called *eigenvalue*. $\alpha \in \mathrm{Fix}(f)$ is called *attracting* (*superattracting*), *rationally indifferent*, *irrationally indifferent* and *repelling* if $|\lambda_f(\alpha)| < 1$ ($\lambda_f(\alpha) = 0$), $\lambda_f(\alpha)$ is a root of unity, $|\lambda_f(\alpha)| = 1$ but $\lambda_f(\alpha)$ is not a root of unity and $|\lambda_f(\alpha)| > 1$, respectively.

$\Omega_f(\alpha)$ denotes the *basin* of an attracting p-periodic point α, i.e.

$$\Omega_f(\alpha) = \bigcup_{i=1}^{p} \left\{ z \in \overline{\mathbf{C}}: \lim_{n \to \infty} f^{pn+i}(z) = \alpha \right\}.$$

Moreover $\Omega_f^*(\alpha)$ denotes the connected component of $\Omega_f(\alpha)$ containing α and is called *immediate basin* of f at α.

A point α is said to be a *critical point* of f, if f fails to be injective in any neighborhood of α. It is not hard to see from the definition of $\lambda_f(\alpha)$ that $\alpha \in \mathbf{C}$ is a critical of point f if and only if $\lambda_f(\alpha) = 0$. The set of all the critical points of f is denoted by $\mathrm{Crit}\,(f)$, i.e., $\mathrm{Crit}\,(f) = \{z \in \overline{\mathbf{C}} \colon \lambda_f(z) = 0\}$.

Let $\mathscr{M}$ be the set of all Möbius maps, i.e.,

$$\mathscr{M} = \left\{\varphi \colon \overline{\mathbf{C}} \to \overline{\mathbf{C}} \colon \varphi(z) = \frac{az+b}{cz+d}, ad - bc \neq 0\right\}$$

We say two rational functions f and g are *conjugate* if there is $\varphi \in \mathscr{M}$ such that $f(z) = \varphi_* g(z) \equiv \varphi \circ g \circ \varphi^{-1}(z)$. In this case we write $f \sim g$, or more precisely $f \overset{\varphi}{\to} g$ if there is a necessity for making φ clear. It is easy to see that $f \sim g$ implies that $\deg(f) = \deg(g)$ (see [1] for example).

We check here the following chain rule.

LEMMA 2.1. *Let f and g be rational functions. Then we have*

$$\lambda_{f \circ g}(\alpha) = \lambda_f(g(\alpha)) \cdot \lambda_g(\alpha), \qquad \alpha \in \overline{\mathbf{C}}. \tag{2.1}$$

PROOF. For $\zeta_1 \in \mathbf{C}$, $\zeta_2 \in \overline{\mathbf{C}}$, we define

$$D(\zeta_1, \zeta_2) = \begin{cases} \zeta_1 - \zeta_2 & \text{for } \zeta_2 \in \mathbf{C}, \\ 1/\zeta_1 & \text{for } \zeta_2 = \infty. \end{cases}$$

Then we have

$$\lambda_f(\alpha) = \lim_{z \to \alpha} \frac{D(f(z), f(\alpha))}{D(z, \alpha)} \qquad \text{for } \alpha \in \overline{\mathbf{C}}.$$

Therefore we obtain

$$\begin{aligned} \lambda_{f \circ g}(\alpha) &= \lim_{z \to \alpha} \frac{D(f \circ g(z), f \circ g(\alpha))}{D(z, \alpha)} \\ &= \lim_{z \to \alpha} \frac{D(f \circ g(z), f \circ g(\alpha))}{D(g(z), g(\alpha))} \cdot \frac{D(g(z), g(\alpha))}{D(z, \alpha)} = \lambda_f(g(\alpha)) \cdot \lambda_g(\alpha). \end{aligned}$$ ■

In view of Lemma 2.1, the following lemma is immediately obtained. The proof is easy and omitted.

LEMMA 2.2. *If $f \overset{\varphi}{\to} g$, then*

$$\mathrm{Fix}(f) = \varphi\, \mathrm{Fix}\,(g), \tag{2.2}$$

$$F(f) = \varphi F(g),\ J(f) = \varphi J(g), \tag{2.3}$$

$$\lambda_f(\alpha) = \lambda_g(\varphi^{-1}(\alpha)) \qquad \textit{if } \alpha \in \mathrm{Fix}\,(f), \tag{2.4}$$

$$\mathrm{Crit}(f) = \varphi(\mathrm{Crit}(g)), \tag{2.5}$$

$$\Omega_f(\alpha) = \varphi\Omega_g(\varphi^{-1}(\alpha)), \Omega_f^*(\alpha) = \varphi\Omega_g^*(\varphi^{-1}(\alpha)) \qquad \text{if } \alpha \in \mathrm{Fix}(f). \tag{2.6}$$

By this lemma we have the next fact which will be used in the standardizations of the degree two rational functions with two fixed points.

LEMMA 2.3. *$\alpha \in \mathrm{Fix}(f)$ has the order not less than two if and only if $\lambda_f(\alpha) = 1$.*

PROOF. The lemma is clear for $\alpha \in \mathbf{C}$. When $\alpha = \infty$, it is sufficient to show that 0 is a fixed point of the order $n (n \geq 2)$ of $g(z) = 1/f(1/z)$ if and only if $\lambda_g(0) = 1$. This is derived from the fact that $\lambda_g(0) = \lambda_f(\infty)$ and the orders of the corresponding fixed points 0 and ∞ are the same. The former fact has already been stated in (2.4). The latter can be easily seen from the definition of the order of the fixed points. Therefore we may only consider the case $\alpha \in \mathbf{C}$. ■

REMARK 2.4. (i) If $\deg(f) \geq 1$ and f is not identity, then f has precisely $\deg(f) + 1$ fixed points (see [1, Theorem 2.6.3]).
(ii) $\sharp\mathrm{Crit}(f) \leq 2\deg(f) - 2$ (see [1, Corollary 2.7.2]).
(iii) Repelling fixed points of f and irrationally indifferent fixed points are elements of $J(f)$ (see [1, Theorem 4.2.6, Theorem 6.5.1]).
(iv) Generalizing the fact in Lemma 2.3, it is easily seen that $f \overset{\varphi}{\to} g$, $\varphi \in \mathscr{M}$ means $\mathrm{ord}_f(\alpha) = \mathrm{ord}_g(\varphi^{-1}\alpha)$, where $\mathrm{ord}_f(\alpha)$ stands for the order at the fixed point α and so on.

3. A standard form of rational functions of degree two

Set

$$\mathscr{A} = \{f\colon f \text{ is a rational function such that } \deg(f) = 2 \text{ and } \sharp\mathrm{Fix}(f) \geq 2\},$$

$$\mathscr{B} = \left\{f_{a,b}\colon f_{a,b}(z) = \frac{az(1-z)}{1-bz}, (a, b) \in \mathbf{C}^2 \backslash E\right\},$$

where $E = \{(a, b) \in \mathbf{C}^2\colon a = 0 \text{ or } b = 1\}$.

We put

$$\alpha_1(a, b) = 0, \qquad \alpha_2(a, b) = \infty, \qquad \alpha_3(a, b) = (a-1)/(a-b).$$

Then

$$\mathrm{Fix}(f_{a,b}) = \{\alpha_1(a, b), \alpha_2(a, b), \alpha_3(a, b)\}.$$

Note that $\alpha_1(a, b) \neq \alpha_2(a, b)$ always holds but it may happen that $\alpha_1(a, b) = \alpha_3(a, b)$ or $\alpha_2(a, b) = \alpha_3(a, b)$.

By elementary calculus, we obtain

$$\lambda_{f_{a,b}}(\alpha_1(a, b)) = a\,, \qquad \lambda_{f_{a,b}}(\alpha_2(a, b)) = \frac{b}{a}\,, \qquad \lambda_{f_{a,b}}(\alpha_3(a, b)) = \frac{-a^2 + 2a - b}{a(1 - b)}\,.$$

We will write $\lambda_k(a, b) = \lambda_{f_{a,b}}(\alpha_k(a, b))$ $(k = 1, 2, 3)$ and $\Lambda(a, b) = \{\lambda_k(a, b): k = 1, 2, 3\}$ for convenience.

Since clearly $\deg(f_{a,b}) = 2$ and $\sharp\mathrm{Fix}(f_{a,b}) \geq 2$, we first see that $\mathscr{A} \supset \mathscr{B}$. On the other hand, we can prove that each $f \in \mathscr{A}$ is conjugate to some element of $\mathscr{B}$. In order to prove the statement, we prepare the following lemma.

LEMMA 3.1. *Suppose $f \in \mathscr{A}$ and $\alpha, \beta \in \mathrm{Fix}(f)$ $\alpha \neq \beta$. Then the following* (i)–(iii) *hold.*

(i) *If $\lambda_f(\alpha) = 0$, then there exists no $\varphi \in \mathscr{M}$ such that $\varphi(\alpha) = 0$ and $\varphi_* f \in \mathscr{B}$.*

(ii) $\lambda_f(\alpha)\lambda_f(\beta) \neq 1$.

(iii) *If $\lambda_f(\alpha) \neq 0$, then there exists a unique $\varphi \in \mathscr{M}$ (which depends on α, β and f) such that $\varphi(\alpha) = 0$, $\varphi(\beta) = \infty$ and $\varphi_* f \in \mathscr{B}$. Moreover $\varphi_* f = f_{\lambda_f(\alpha), \lambda_f(\alpha)\lambda_f(\beta)}$ in this case, that is,*

$$\varphi_* f(z) = \frac{\lambda_f(\alpha) z(1 - z)}{1 - \lambda_f(\alpha)\lambda_f(\beta) z}\,.$$

PROOF. (i) Suppose that there exists $\varphi \in \mathscr{M}$ such that $\varphi(\alpha) = 0$ and $\varphi_* f = f_{a,b} \in \mathscr{B}$. By (2.4) and (3.1), we see $\lambda_f(\alpha) = \lambda_{f_{a,b}}(0) = a = 0$. This is impossible by the definition of $\mathscr{B}$.

(iii) We will construct a conjugate map φ as follows. From the assumption that $\lambda_f(\alpha) \neq 0$ and $\deg(f) = 2$, there exists $\zeta \in \overline{\mathbf{C}}$ such that $f(\zeta) = \alpha$, $\zeta \neq \alpha$. Here we see that $\zeta \neq \beta$, because $f(\zeta) = \alpha \neq \beta = f(\beta)$. Set

$$\psi_{\alpha,\beta}(z) = \begin{cases} \dfrac{z - \alpha}{z - \beta} & \text{if } \alpha,\ \beta \in \mathbf{C}\,, \\[2ex] \dfrac{1}{z - \beta} & \text{if } \alpha = \infty,\ \beta \in \mathbf{C}\,, \\[2ex] z - \alpha & \text{if } \alpha \in \mathbf{C},\ \beta = \infty\,. \end{cases}$$

Since $\zeta \neq \alpha, \beta$, we can easily see that $\psi_{\alpha,\beta}(\zeta) \neq 0, \infty$ and that $\varphi(z) = \dfrac{\psi_{\alpha,\beta}(z)}{\psi_{\alpha,\beta}(\zeta)} \in \mathscr{M}$. Moreover φ satisfies the following conditions:

$$\mathrm{Fix}(\varphi_* f) \ni 0,\ \infty \qquad \text{and} \quad \varphi_* f(1) = 0\,. \tag{3.2}$$

In fact,

$$\varphi_* f(0) = \varphi \circ f \circ \varphi^{-1}(0) = \varphi \circ f(\alpha) = \varphi(\alpha) = 0\,,$$

$$\varphi_* f(\infty) = \varphi \circ f \circ \varphi^{-1}(\infty) = \varphi \circ f(\beta) = \varphi(\beta) = \infty\,,$$

$$\varphi_* f(1) = \varphi \circ f \circ \varphi^{-1}(1) = \varphi \circ f(\zeta) = \varphi(\alpha) = 0\,.$$

Furthermore we see that

$$\deg(\varphi_* f) = \deg(f) = 2\,. \tag{3.3}$$

Therefore by (3.2) and (3.3), $\varphi_* f$ must be written in the form

$$\varphi_* f(z) = \frac{\alpha' z(z-1)}{b'z + c'} \qquad \text{with} \quad a'c' \neq 0\,. \tag{3.4}$$

Putting $a = a'/c' \in \mathbf{C}$, $b = -b'/c' \in \mathbf{C}$, we have

$$f_{a,b}(z) = \frac{az(1-z)}{1-bz} \in \mathscr{B}\,. \tag{3.5}$$

Here (3.1) and (2.4) mean $a = \lambda_{f_{a,b}}(0) = \lambda_f(\alpha)$ and $b/a = \lambda_{f_{a,b}}(\infty) = \lambda_f(\beta)$, which consequently imply $a = \lambda_f(\alpha)$ and $b = \lambda_f(\alpha) \cdot \lambda_f(\beta)$.

On the other hand, suppose that $\theta \in \mathscr{M}$ satisfies $\theta(\alpha) = 0$, $\theta(\beta) = \infty$ and $\theta_* f \in \mathscr{B}$. Since $\theta_* f(1) = 0$, $\theta^{-1}(1)$ is a preimage of $\alpha = \theta^{-1}(0)$ with respect to f. Therefore $\theta^{-1}(1) \neq \alpha = \theta^{-1}(0)$ and $\theta^{-1}(1) = \zeta$, i.e., $\theta(\zeta) = 1$. Thus we can see that both θ and φ map the distinct three points α, β, ζ to the distinct three points 0, ∞, 1 respectively. Since the Möbius map is uniquely determined by the images of three distinct points, we have $\varphi = \theta$.

(ii) If $b = \lambda_f(\alpha) \cdot \lambda_f(\beta) = 1$ in the proof of (iii), then $f_{a,b}(z) = az$. This contradicts to $\deg(f_{a,b}) = 2$. ■

PROPOSITION 3.2. *Each $f \in \mathscr{A}$ is conjugate to some element of $\mathscr{B}$.*

PROOF. By the assumption $\deg(f) = 2$ and Remark 2.4 (ii), we see $\sharp\mathrm{Crit}(f) \leq 2$. Therefore, if $\sharp\mathrm{Fix}(f) = 3$, then there exists $\alpha \in \mathrm{Fix}(f)$ such that $\lambda_f(\alpha) \neq 0$. If $\sharp\mathrm{Fix}(f) = 2$, the multiple fixed point α satisfies $\lambda_f(\alpha) = 1 \neq 0$ by Lemma 2.3. Therefore the assertion is true in view of Lemma 3.1 (iii). ■

Since $f_{a,b} \in \mathscr{B}$ has two distinct fixed points 0 and ∞, there exist no $\varphi \in \mathscr{M}$ such that $\varphi_* f \in \mathscr{B}$. But we have the following standardization for functions with only one fixed point.

PROPOSITION 3.3. *Suppose that f is a degree two rational function such that $\sharp\mathrm{Fix}(f) = 1$. Then there exists $\varphi \in \mathscr{M}$ such that* $\varphi_* f(z) = \dfrac{z(z-2)}{(z-1)(z+2)}$.

PROOF. Let $p_0 \in \mathbf{C}$ be the unique fixed point of f. Then p_0 has the order three. Let α_0 be one of the non-critical two periodic points of f. Since α_0 is not critical and $\deg(f) = 2$, there exists $\beta_0 \in f^{-1}(\alpha_0)$ such that $\beta_0 \neq \alpha_0$, p_0. Now we put

$$\varphi(z) = \begin{cases} \dfrac{\alpha_0 - \beta_0}{\alpha_0 - p_0}\dfrac{z - p_0}{z - \beta_0} & \text{if } \alpha_0,\ \beta_0 \in \mathbf{C}\,, \\[2ex] \dfrac{z - p_0}{z - \beta_0} & \text{if } \alpha_0 = \infty,\ \beta_0 \in \mathbf{C}\,, \\[2ex] \dfrac{z - p_0}{\alpha_0 - \beta_0} & \text{if } \alpha_0 \in \mathbf{C},\ \beta_0 = \infty\,. \end{cases}$$

Then $\varphi_* f$ satisfies

$$\operatorname{Fix}(\varphi_* f) = \{0\}\,, \tag{3.6}$$

$$\varphi_* f(1) = \infty\,, \qquad \varphi_* f(\infty) = 1\,. \tag{3.7}$$

By (3.6) and $\deg(\varphi_* f) = 2$, $\varphi_* f(z)$ is written in the form $\varphi_* f(z) = \dfrac{cz^3}{p'z^2 + q'z + r'} + z$ for some $c, p', q', r' \in \mathbf{C}$, $(c, p', r' \neq 0)$. Since $\deg(\varphi_* f) = 2$, we see that $\varphi_* f(z) = \dfrac{z(qz+1)}{pz^2 + qz + 1}$ for some $p(\neq 0)$, $q \in \mathbf{C}$. Moreover by (3.7), we obtain $p = q = -1/2$. Thus $\varphi_* f(z) = \dfrac{z(z-2)}{(z-1)(z+2)}$. ∎

In what follows we will designate to reduce the parameter space using the conjugation by Möbius maps. Identifying (a, b) with $f_{a,b}$, we define $(a_1, b_1) \sim (a_2, b_2)$ if f_{a_1,b_2} and f_{a_2,b_2} are conjugate.

We shall write $G \prec F$ if for any $(a, b) \in F$ there exists $(a', b') \in G$ such that $(a, b) \sim (a', b')$ and $G \prec\!\prec F$ if $G \prec F$ and $(a_1, b_1) \sim (a_2, b_2)$ implies $(a_1, b_1) = (a_2, b_2)$ for $(a_1, b_1), (a_2, b_2) \in G$. We call G a *fundamental domain* of F if $G \prec\!\prec F$.

REMARK 3.4. If $\tilde{F}_k \prec\!\prec F_k$, $(k = 1, \cdots, n)$ and there are no pairs $(a_l, b_l) \in \tilde{F}_l$, $(a_m, b_m) \in \tilde{F}_m$, $(l, m = 1, \cdots, n, l \neq m)$ such that $(a_l, b_l) \sim (a_m, b_m)$, then $\bigcup_{k=1}^n \tilde{F}_k \prec\!\prec \bigcup_{k=1}^n F$.

Restricting our arguments to real cases, we will first find a fundamental domain of $\mathbf{R}^2 \backslash E$. Complex cases will be slightly referred to later.

For $f_{a,b} \in \mathscr{F}(\mathbf{R}^2 \backslash E)$, $f_{a,b}$ has at most three distinct fixed points, two super-attracting fixed points and three distinct eigenvalues. Therefore we have

$$\mathbf{R}^2 \backslash E = \left(\bigcup_{k=0}^{2} \bigcup_{l=0}^{3} D_{k,l}\right) \cup D^{(2)}\,, \tag{3.8}$$

where

$D_{k,l} = \{(a, b) \in \mathbf{R}^2 \backslash E \colon \sharp(\mathrm{Fix}\,(f_{a,b})) = 3, \sharp(\mathrm{Fix}\,(f_{a,b}) \cap \mathrm{Crit}\,(f_{a,b})) = k, \sharp\Lambda(a, b) = l\}$

and $D^{(2)} = \{(a, b) \in \mathbf{R}^2 \backslash E \colon \sharp(\mathrm{Fix}\,(f_{a,b})) = 2\}$. Clearly $D_{k,l}$ ($k = 0, 1, 2, 3$, $l = 0, 1, 2$) and $D^{(2)}$ are pairwise disjoint. A simple observation immediately implies

$$D_{1,1} = D_{1,2} = D_{2,1} = D_{2,3} = \varnothing,\ D_{0,1} = \{(-2, 4)\},\ D_{2,2} = \{(2, 0)\}\,.$$

By a further precise classification of eigenvalues we can obtain a fundamental domain of $\mathbf{R}^2 \backslash E$ as follows.

LEMMA 3.5. *Set* $\tilde{D} = \{(-2, 4), (2, 0)\} \cup \tilde{D}_{0,3} \cup \tilde{D}_{0,2} \cup \tilde{D}_{1,3} \cup \tilde{D}^{(2)}$, *where*

$$\tilde{D}_{0,3} = \{(a, b) \in D_{0,3} \colon \lambda_1(a, b) > \lambda_2(a, b) > \lambda_3(a, b)\}\,,$$

$$\tilde{D}_{1,3} = \{(a, b) \in D_{1,3} \colon \lambda_1(a, b) > \lambda_3(a, b) > \lambda_2(a, b) = 0\}\,,$$

$$\tilde{D}_{0,2} = \{(a, b) \in D_{0,2} \colon \lambda_1(a, b) \neq \lambda_2(a, b) = \lambda_3(a, b)\}\,,$$

$$\tilde{D}^{(2)} = \{(a, b) \in D^{(2)} \colon \alpha_1(a, b)(=\alpha_3(a, b)) \neq \alpha_2(a, b)\}\,.$$

Then $\tilde{D} \prec\!\prec \mathbf{R}^2 \backslash E$.

PROOF. We show the following (i)–(iv).

(i) $\tilde{D}_{0,3} \prec\!\prec D_{0,3}$.

Suppose that $(a, b) \in D_{0,3}$. Then by the definition of $D_{0,3}$, there exists a unique $\sigma \in S_3$ (the symmetric group of degree three) such that

$$\lambda_{\sigma(1)}(a, b) > \lambda_{\sigma(2)}(a, b) > \lambda_{\sigma(3)}(a, b)\,. \tag{3.9}$$

By Lemma 3.1 (iii), there exists a unique $\varphi_\sigma \in \mathcal{M}$ such that

$$(\varphi_\sigma)_* f_{a,b} = f_{\xi_\sigma(a,b)}\,, \qquad \varphi_\sigma(\alpha_{\sigma(j)}(a, b)) = \alpha_j(\xi_\sigma(a, b)) \qquad \text{for } j = 1,\ 2,\ 3\,, \tag{3.10}$$

where

$$\xi_\sigma(a, b) = (\lambda_{\sigma(1)}(a, b), \lambda_{\sigma(1)}(a, b)\lambda_{\sigma(2)}(a, b)) \tag{3.11}$$

and $f_{(a',b')}$ stands for $f_{a',b'}$ and so on. Clearly $\xi_\sigma(a, b) \in \tilde{D}_{0,3}$, because

$$\begin{aligned}\lambda_1(\xi_\sigma(a, b)) &= \lambda_{\sigma(1)}(a, b) > \lambda_2(\xi_\sigma(a, b)) \\ &= \lambda_{\sigma(2)}(a, b) > \lambda_3(\xi_\sigma(a, b)) = \lambda_{\sigma(3)}(a, b)\,.\end{aligned}$$

Since $(a, b) \in D_{0,3}$ is arbitrary, we have $\tilde{D}_{0,3} \prec D_{0,3}$.

On the other hand, suppose that there exist $(a_1, b_1), (a_2, b_2) \in \tilde{D}_{0,3}$ such that $(a_1, b_1) \sim (a_2, b_2)$. Since $\Lambda(a_1, b_1) = \Lambda(a_2, b_2)$ and $\lambda_1(a_j, b_j) > \lambda_2(a_j, b_j) > \lambda_3(a_j, b_j)$, $(j = 1, 2)$, we must have

$$a_1 = \lambda_1(a_1, b_1) = \lambda_1(a_2, b_2) = a_2\,, \qquad \frac{b_1}{a_1} = \lambda_2(a_1, b_1) = \lambda_2(a_2, b_2) = \frac{b_2}{a_2}\,.$$

Therefore $(a_1, b_1) = (a_2, b_2)$ is easily obtained. Thus we can conclude $\tilde{D}_{0,3} \prec\prec D_{0,3}$.

(ii) $\tilde{D}_{1,3} \prec\prec D_{1,3}$.

Suppose that $(a, b) \in D_{1,3}$. Then there exists a unique $\sigma \in \{\tau \in S_3 \colon \tau(2) \neq 1\}$ such that

$$\lambda_{\sigma(2)}(a, b) = 0\,, \qquad \lambda_{\sigma(1)}(a, b) > \lambda_{\sigma(3)}(a, b)(\lambda_{\sigma(1)}(a, b)\lambda_{\sigma(3)}(a, b) \neq 0)\,.$$

Since $\lambda_{f_{a,b}}(\alpha_{\sigma(1)}(a, b)) = \lambda_{\sigma(1)}(a, b) \neq 0$, we can find $\varphi_\sigma \in \mathcal{M}$, which satisfies (3.10) by Lemma 3.1 (iii). Clearly $\xi_\sigma(a, b) \in \tilde{D}_{1,3}$, because

$$\lambda_2(\xi_\sigma(a, b)) = \lambda_{\sigma(2)}(a, b) = 0$$

and

$$\lambda_1(\xi_\sigma(a, b)) = \lambda_{\sigma(1)}(a, b) > \lambda_3(\xi_\sigma(a, b)) = \lambda_{\sigma(3)}(a, b)\,.$$

Therefore we have $\tilde{D}_{1,3} \prec D_{1,3}$ and it follows $\tilde{D}_{1,3} \prec\prec D_{1,3}$ by similar arguments as before.

(iii) $\tilde{D}_{0,2} \prec\prec D_{0,2}$.

Suppose that $(a, b) \in D_{0,2}$. Then there exists a (but not a unique) $\sigma \in S_3$ such that

$$\lambda_{\sigma(1)}(a, b) \neq \lambda_{\sigma(2)}(a, b) = \lambda_{\sigma(3)}(a, b)\,.$$

Therefore $\tilde{D}_{0,2} \prec D_{0,2}$ is deduced from this similarly.

Suppose that (a_1, b_1), $(a_2, b_2) \in \tilde{D}_{0,2}$ and $(a_1, b_1) \sim (a_2, b_2)$, or more precisely, $\varphi_* f_{a_1,b_1} = f_{a_2,b_2}$ for some $\varphi \in \mathcal{M}$. Then there exists a unique $\sigma \in S_3$ such that $\varphi(\alpha_j(a_1, b_1)) = \alpha_{\sigma(j)}(a_2, b_2)$, $(j = 1, 2, 3)$. If $\sigma(1) = k \neq 1$, i.e., $k = 2$ or 3, then

$$\lambda_1(a_1, b_1) = \lambda_{\sigma(1)}(a_2, b_2) = \lambda_{\sigma(k)}(a_2, b_2) = \lambda_k(a_1, b_1)\,.$$

This is impossible by the definition of $\tilde{D}_{0,2}$. Therefore $\sigma(1) = 1$ and consequently $\sigma(2) = 2$ or 3, which implies

$$\lambda_1(a_1, b_1) = \lambda_1(a_2, b_2)\,, \qquad \lambda_2(a_1, b_1) = \lambda_3(a_1, b_1) = \lambda_2(a_2, b_2) = \lambda_3(a_2, b_2)\,.$$

Thus we have $(a_1, b_1) = (a_2, b_2)$ from (3.1) as before and this implies $\tilde{D}_{0,2} \prec\prec D_{0,2}$.

(iv) $\tilde{D}^{(2)} \prec\prec D^{(2)}$.

Suppose that $(a, b) \in D^{(2)}$. Then there exists $\sigma \in S_3$ such that $\alpha_{\sigma(1)}(a, b) = \alpha_{\alpha(3)}(a, b) \neq \alpha_{\sigma(2)}(a, b)$. Since $\lambda_{\sigma(1)}(a, b) = 1 \neq 0$ in this case by Lemma 3.1, there exists $\varphi_\sigma \in \mathcal{M}$ which satisfies (3.10) by Lemma 3.1 (iii). Then $\alpha_1(\xi_\sigma(a, b)) = \alpha_3(\xi_\sigma(a, b)) \neq \alpha_2(\xi_\sigma(a, b))$, i.e., $\xi_\sigma(a, b) \in \tilde{D}^{(2)}$. Therefore $\tilde{D}^{(2)} \prec D^{(2)}$.

Next suppose that (a_1, b_1), $(a_2, b_2) \in \tilde{D}^{(2)}$ and $(a_1, b_1) \sim (a_2, b_2)$ or $\varphi_* f_{a_1,b_1} = f_{a_2,b_2}$ for some $\varphi \in \mathscr{M}$. Since the order of the fixed points are preserved as has been already stated, we must have

$$\varphi(\alpha_1(a_1, b_1)) = \alpha_1(a_2, b_2), \qquad \varphi(\alpha_2(a_1, b_1)) = \alpha_1(a_2, b_2).$$

Then

$$\lambda_1(a_1, b_1) = a_1 = a_2 = \lambda_1(a_2, b_2), \qquad \lambda_2(a_1, b_1) = \frac{b_1}{a_1} = \frac{b_2}{a_2} = \lambda_2(a_2, b_2).$$

Therefore $(a_1, b_1) = (a_2, b_2)$. Thus $\tilde{D}^{(2)} \prec\!\prec D^{(2)}$ is obtained.

In view of (i)–(iv), we can see that

$$\begin{aligned}\tilde{D} &= \{(-2, 4), (2, 0)\} \cup \tilde{D}_{0,3} \cup \tilde{D}_{0,2} \cup \tilde{D}_{1,3} \cup \tilde{D}^{(2)} \\ &\prec\!\prec \{(-2, 4), (2, 0)\} \cup D_{0,3} \cup D_{0,2} \cup D_{1,3} \cup D^{(2)} = \mathbf{R}^2 \backslash E\end{aligned}$$

by Remark 3.4. ■

REMARK 3.6. The explicit forms of $\{\varphi_\sigma\}$ and $\{\xi_\sigma\}$ which appeared in the above proposition are given as follows. See the construction of φ in Lemma 3.1 (iii) and recall the definition (3.11).

$$\varphi_{id}(z) = z, \qquad \xi_{id}(a, b) = (a, b),$$

$$\varphi_{(23)}(z) = \frac{(1-b)z}{(a-b)z - a + 1}, \qquad \xi_{(23)}(a, b) = \left(a, \frac{-a^2 + 2a - b}{1 - b}\right),$$

$$\varphi_{(13)}(z) = a \cdot \frac{(a-b)z - a + 1}{2a - b - a^2}, \qquad \xi_{(13)}(a, b) = \left(\frac{-a^2 + 2a - b}{a(1-b)}, \frac{b(-a^2 + 2a - b)}{a^2(1-b)}\right),$$

$$\varphi_{(12)}(z) = \frac{1}{bz}, \qquad \xi_{(12)}(a, b) = \left(\frac{b}{a}, b\right),$$

$$\varphi_{(123)}(z) = \frac{(b-a)z + a - 1}{(a^2 - 2a + b)z}, \qquad \xi_{(123)}(a, b) = \left(\frac{b}{a}, \frac{b(-a^2 + 2a - b)}{a^2(1-b)}\right),$$

$$\varphi_{(132)}(z) = \frac{1}{b} \cdot \frac{a(1-b)}{(a-b)z - a + 1} \qquad \xi_{(132)}(a, b) = \left(\frac{-a^2 + 2a - b}{a(1-b)}, \frac{-a^2 + 2a - b}{1 - b}\right).$$

REMARK 3.7. In complex cases, setting $D^{\mathbf{C}} = \{(-2, 4), (2, 0)\} \cup D^{\mathbf{C}}_{0,3} \cup D^{\mathbf{C}}_{1,3} \cup D^{\mathbf{C}}_{0,2} \cup D^{\mathbf{C},(2)}$, where

$$\begin{aligned}D^{\mathbf{C}}_{0,3} = &\{(a, b) \in \mathbf{C}^2 \backslash E\colon |\lambda_1(a, b)| > |\lambda_2(a, b)| > |\lambda_3(a, b)| > 0\} \\ &\cup \{(a, b) \in \mathbf{C}^2 \backslash E\colon |\lambda_1(a, b)| = |\lambda_2(a, b)| \neq |\lambda_3(a, b)| > 0, \\ &\quad \operatorname{Arg} \lambda_1(a, b) > \operatorname{Arg} \lambda_2(a, b)\}\end{aligned}$$

$$\cup \{(a, b) \in \mathbf{C}^2 \backslash E\colon |\lambda_1(a, b)| = |\lambda_2(a, b)| = |\lambda_3(a, b)| > 0,$$
$$\operatorname{Arg} \lambda_1(a, b) > \operatorname{Arg} \lambda_2(a, b) > \operatorname{Arg} \lambda_3(a, b)\},$$

$$D_{0,2}^{\mathbf{C}} = \{(a, b) \in \mathbf{C}^2 \backslash E\colon \lambda_1(a, b) = \lambda_2(a, b) \neq \lambda_3(a, b), \lambda_1(a, b)\lambda_2(a, b)\lambda_3(a, b) \neq 0\},$$

$$D_{1,3}^{\mathbf{C}} = \{(a, b) \in \mathbf{C}^2 \backslash E\colon \lambda_2(a, b) = 0, |\lambda_1(a, b)| > |\lambda_3(a, b)| > 0\}$$
$$\cup \{(a, b) \in \mathbf{C}^2 \backslash E\colon \lambda_2(a, b) = 0, |\lambda_1(a, b)| = |\lambda_3(a, b)| > 0,$$
$$\operatorname{Arg} \lambda_1(a, b) > \operatorname{Arg} \lambda_3(a, b)\},$$

$$D^{\mathbf{C},(2)} = \{(a, b) \in \mathbf{C}^2 \backslash E\colon \sharp\mathrm{Fix}\,(f_{a,b}) = 2\} = \{(a, b) \in \mathbf{C}^2 \backslash E\colon a = 1\},$$

we have $D^{\mathbf{C}} \prec\!\prec \mathbf{C} \backslash E$ in the same way as above.

Though we have already obtained a fundamental domain $\tilde{D}$ of $\mathbf{R}^2 \backslash E$, we will slightly transform it into another domain D below using maps $\{\xi_\sigma\colon \sigma \in S_3\}$. See Figure 3.2.

THEOREM 3.8. *Set* $D = D_0 \cup D_1 \cup D_2$, *where*

$$D_0 = \{(a, b) \in \mathbf{R}^2 \backslash E\colon a \geq 1, 2 - a \leq b \leq 2a/(a + 1)\},$$
$$D_1 = \{(a, b) \in \mathbf{R}^2 \backslash E\colon a \leq -2, 2a/(a + 1) \leq b \leq 2 - a\},$$
$$D_2 = \{(a, b) \in \mathbf{R}^2 \backslash E\colon a = 1\}.$$

Then $D \prec\!\prec \mathbf{R}^2 \backslash E$.

PROOF. Let $D_{0,3}$, $D_{0,2}$, $D_{1,3}$, $\tilde{D}_{0,3}$, $\tilde{D}_{0,2}$ and $\tilde{D}_{1,3}$ be the sets which appeared in Lemma 3.5. According to the definition

$$\begin{aligned}\tilde{D}_{0,3} &= \{(a, b) \in D_{0,3}\colon \lambda_1(a, b) > \lambda_2(a, b) > \lambda_3(a, b)\} \\ &= \left\{(a, b) \in D_{0,3}\colon a > \frac{a}{b} > \frac{-a^2 + 2a - b}{a(1 - b)}\right\}.\end{aligned}$$

We can obtain by elementary and easy calculations $\tilde{D}_{0,3} = \bigcup_{j=1}^{3} \tilde{D}_{0,3,j}$, where

$$\tilde{D}_{0,3,1} = \{(a, b) \in D_{0,3}\colon 2 - a < b < 1\},$$
$$\tilde{D}_{0,3,2} = \{(a, b) \in D_{0,3}\colon 1 < a < b < a^2\},$$
$$\tilde{D}_{0,3,3} = \{(a, b) \in D_{0,3}\colon b > 1, a < 0, a^2 < b < 2 - a\}.$$

Similarly we also obtain $\tilde{D}_{0,2} = \bigcup_{j=1}^{4} \tilde{D}_{0,2,j}$, where

$$\tilde{D}_{0,2,1} = \{(a, b) \in D_{0,2}\colon b = 2 - a, a < -2\},$$
$$\tilde{D}_{0,2,2} = \{(a, b) \in D_{0,2}\colon b = 2 - a, \ -2 < a < 0\},$$

$$\tilde{D}_{0,2,3} = \{(a, b) \in D_{0,2} : b = 2 - a, 0 < a < 1\},$$

$$\tilde{D}_{0,2,4} = \{(a, b) \in D_{0,2} : b = 2 - a, 1 < a\}$$

and $\tilde{D}_{1,3} = \tilde{D}_{1,3,1} \cup \tilde{D}_{1,3,2}$, where

$$\tilde{D}_{1,3,1} = \{(a, b) \in D_{1,3} : b = 0, 0 < a < 1\},$$

$$\tilde{D}_{1,3,2} = \{(a, b) \in D_{1,3} : b = 0, 2 < a\}.$$

See Figure 3.1. Put

$$D'_0 = \tilde{D}_{0,3,1} \cup \xi_{(123)}(\tilde{D}_{0,3,2}) \cup \tilde{D}_{0,2,4} \cup \xi_{(132)}(\tilde{D}_{0,2,3}) \cup \xi_{(13)}(\tilde{D}_{1,3,1}) \cup \tilde{D}_{1,3,2} \cup \{(2, 0)\}$$

and

$$D'_1 = \xi_{(132)}(\tilde{D}_{0,3,3}) \cup \tilde{D}_{0,2,1} \cup \xi_{(132)}(\tilde{D}_{0,2,2}) \cup \{(-2, 4)\}.$$

Since $\xi_{(123)}(\tilde{D}_{0,3,2}) \prec\prec \tilde{D}_{0,3,2}$ and so on, in view of Remark 3.4, we have

$$\begin{aligned} D'_0 \cup D'_1 \cup D^{(2)} \prec\prec{} & \tilde{D}_{0,3,1} \cup \tilde{D}_{0,3,2} \cup \tilde{D}_{0,2,4} \cup \tilde{D}_{0,2,3} \cup \tilde{D}_{1,3,1} \cup \tilde{D}_{1,3,2} \cup \{(2, 0)\} \\ & \cup \tilde{D}_{0,3,3} \cup \tilde{D}_{0,2,1} \cup \tilde{D}_{0,2,2} \cup \{(-2, 4)\} \cup D^{(2)} \\ ={} & (\cup_{j=1}^{3} \tilde{D}_{0,3,j}) \cup (\cup_{k=1}^{4} \tilde{D}_{0,2,k}) \cup (\cup_{l=1}^{2} \tilde{D}_{1,3,l}) \cup \{(2, 0), (-2, 4)\} \cup D^{(2)} \\ ={} & \tilde{D}_{0,3} \cup \tilde{D}_{0,2} \cup \tilde{D}_{1,3} \cup \{(2, 0), (-2, 4)\} \cup D^{(2)} = \tilde{D}. \end{aligned}$$

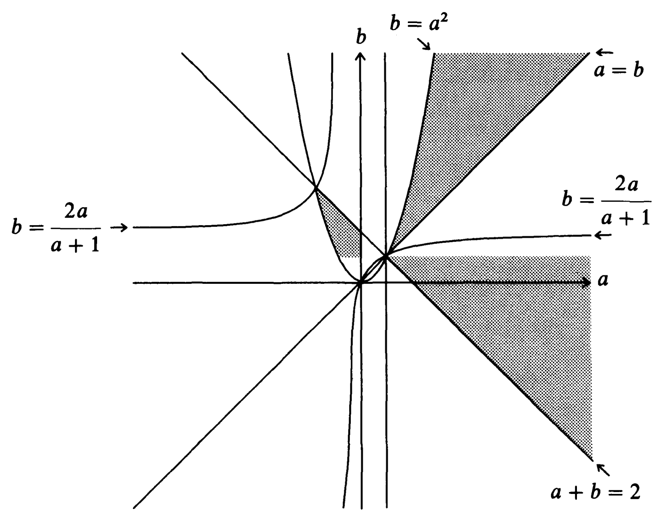


FIGURE 3.1

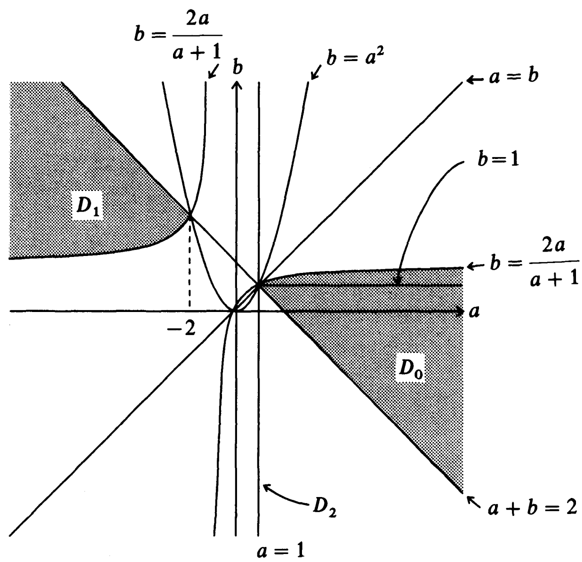


FIGURE 3.2

On the other hand easy calculations show that $D_0' = D_0$, $D_1' = D_1$ and $D^{(2)} = D_2$. Therefore $D = D_0 \cup D_1 \cup D_2$ is a fundamental domain of $\mathbf{R}^2 \backslash E$ by Lemma 3.5. ∎

REMARK 3.9. For $(a, b) \in D$, 0 always belongs to $J(f_{a,b})$. For, as can be seen from Figure 3.2, $|f'_{a,b}(0)| = |a| > 1$ for any $(a, b) \in D$ except when $a = 1$. Therefore 0 is either a repelling fixed point or a rationally indifferent one. In any case $0 \in J(f_{a,b})$ by Remark 2.4 (iii).

In [9] the rational function

$$F_{\mu,\gamma}(z) = \frac{\mu z(1 - \gamma z)}{1 + \mu(1 - \gamma)z} \quad \text{for } \mu > 1,\ 0 < \gamma \le 1\ , \tag{3.12}$$

was investigated. It is easy to see that the Möbius map

$$\varphi(z) = \gamma z \tag{3.13}$$

gives $\varphi^* F_{\mu,\gamma} \in \mathscr{M}$. In fact,

$$f_{a,b}(z) = \varphi \circ F_{\mu,\gamma} \circ \varphi^{-1}(z) = \frac{az(1 - z)}{1 - bz}\ ,$$

where $a = \mu$ and $b = -\mu(1 - \gamma)/\gamma$.

We will consider the family $\{f_{a,b}: (a, b) \in D\}$ in the following sections. We shall use the same notations as those in this section.

4. Real Julia sets

In this section we will investigate a necessary and sufficient condition under which $J(f_{a,b}) \subset \mathbf{R} \subset \{\infty\}$ holds and the structure of $J(f_{a,b})$. As was stated, the Julia set is equal to the closure of the preimages of an arbitrary element in it. Therefore, for $(a, b) \in D$, we only have to investigate $f_{a,b}^{-n}(\{0\})$ $(n = 0, 1, 2, \cdots)$ because we have already pointed out that $0 \in J(f_{a,b})$ in Remark 3.9.

Our goal in this section is the following Theorem 4.3. For the preparation of the theorem we consider the following lemma in a general situation.

LEMMA 4.1. *Let f be a rational function. Suppose that $J(f)$ is a proper subset of $\mathbf{R} \cup \{\infty\}$ and that f has an attracting fixed point α. Then f has no other attracting periodic points and $F(f) = \Omega_f(\alpha) = \Omega_f^*(\alpha)$.*

PROOF. Since $J(f)$ is a proper subset of $\mathbf{R} \cup \{\infty\}$, $F(f) = \overline{\mathbf{C}} \backslash J(f)$ is connected so that the number of the connected components of $F(f)$ is clearly one. If there exists an attracting periodic point $\beta \neq \alpha$, then $F(f)$ has at least two distinct components $\Omega_f^*(\alpha)$ and $\Omega_f^*(\beta)$. This is a contradiction. Therefore f has no attracting fixed points other than α and $F(f) = \Omega_f(\alpha) = \Omega_f^*(\alpha)$. ■

A subset B of the complex sphere $\overline{\mathbf{C}}$ is said to be a *Cantor set* if it is non-empty, closed, perfect (there are no isolated points), and totally disconnected (each component of B is a single point). We quote from [1] and [5] the following well known criterion for the Julia set to be a Cantor set.

THEOREM 4.2 ([1], [5]). *Let f be a rational function. Then the following conditions* (i) *and* (ii) *are equivalent.*
(i) *The orbits of all the critical points of f converge to stable cycles.*
(ii) *There exist $c > 0$ and $\lambda > 1$ such that*

$$|\lambda_{f^n}(z)| \geq c\lambda^n \qquad \text{for } z \in J(f),\ n \in \mathbf{N}.$$

Moreover if ζ be an attracting fixed point of f and all of the critical points of f lie in $\Omega_f^*(\zeta)$, then $J(f)$ is a Cantor set.

A rational function f which satisfies (i) or (ii) is called *expanding* or *hyperbolic*. Now let us go back to the concrete analysis of $J(f_{a,b})$.

THEOREM 4.3. (i) *If $(a, b) \in D_0$ and $b > 1$, then $J(f_{a,b})$ is a Cantor set contained in $[\alpha_1(a, b), \alpha_3(a, b)]$ and $\alpha_1(a, b), \alpha_3(a, b) \in J(f_{a,b})$.*
(ii) *If $(a, b) \in D_0$ and $2\sqrt{a} - a < b - 1$, then $J(f_{a,b})$ is a Cantor set contained in $[0, 1]$ and $0, 1 \in J(f_{a,b})$.*
(iii) *If $(a, b) \in D_0$ and $b = 2\sqrt{a} - a$, then $J(f_{a,b}) = [0, 1]$.*
(iv) *If $(a, b) \in D_0$ and $b < 2\sqrt{a} - a$, then $J(f_{a,b})$ is not a subset of $\mathbf{R} \cup \{\infty\}$.*

(v) *If* $(a, b) \in D_1$ *and* $b < -a$, *then* $J(f_{a,b})$ *is a Cantor set contained in* $[p_-, p_+]$ *and* p_-, $p_+ \in J(f_{a,b})$, *where* $p_\pm$ *is 2-periodic points of* $f_{a,b}$.

(vi) *If* $(a, b) \in D_1$ *and* $-a \leq b$, *then* $J(f_{a,b}) = \mathbf{R} \cup \{\infty\}$.

REMARK 4.4. In (i)–(v) it is easily assured that $|\lambda_2(a, b)| = |b/a| < 1$. Therefore $\alpha_2(a, b) = \infty$ is an attracting fixed point of $f_{a,b}$ in these cases.

PROOF. (i) Suppose that $(a, b) \in D_0$ and $b > 1$. Then an easy calculation shows that

$$\lambda_3(a, b) = \frac{-a^2 + 2a - b}{a(1 - b)} > 1$$

in this case. Therefore $\alpha_3(a, b) \in J(f_{a,b})$. $0 = \alpha_1(a, b) \in J(f_{a,b})$ has already been stated in Remark 3.9. Since $J(f_{a,b})$ coincides with the closure of $\{f_{a,b}^{-n}(\{0\}): n = 0, 1, 2, \cdots\}$, it is clear from the graph of $f_{a,b}$ as a function from $\mathbf{R}$ to $\mathbf{R}$ that $f_{a,b}^{-n}(\{0\}) \subset [\alpha_1(a, b), \alpha_3(a, b)]$ for any $n \in \mathbf{N}$ and consequently $J(f_{a,b}) \subset [\alpha_1(a, b), \alpha_3(a, b)]$. See Figure 4.1.

On the other hand, both of the critical points $(1 \pm \sqrt{b - 1}i)/b$ are not real (recall the condition $b > 1$) so that they do not belong to $J(f_{a,b})$. Therefore by Lemma 4.1 they lie in $\Omega^*(\alpha_2(a, b))$. By the criterion Theorem 4.2, we can conclude that $J(f_{a,b})$ is a Cantor set.

(ii) Suppose that $(a, b) \in D_0$ and $2\sqrt{a} - a < b < 1$. Then of course $0 \in J(f_{a,b})$ and $1 \in J(f_{a,b})$ because 1 is one of the preimages of 0. A simple observation of the graph of $f_{a,b}$ again shows that $f_{a,b}^{-n}(\{0\}) \subset [0, 1]$ for any $n \in \mathbf{N}$. Therefore $J(f_{a,b}) \subset [0, 1]$. See Figure 4.2. On the other hand $\mathrm{Crit}\,(f_{a,b}) = \{(1 \pm \sqrt{1 - b})/b\}$ and it is easy to show that $(1 + \sqrt{1 - b})/b > 1$ and $f_{a,b}((1 - \sqrt{1 - b})/b) > 1$. Therefore both of the critical points do not belong to $J(f_{a,b})$. Again recalling that $\alpha_2(a, b) = \infty$ is attracting as before, we see that they must lie in $\Omega^*(\alpha_2(a, b))$ and that $J(f_{a,b})$ is a Cantor set.

(iii) Suppose that $(a, b) \in D_0$ and $b = 2\sqrt{a} - a$. In this case $J(f_{a,b}) \subset [0, 1]$ can be shown in the same manner as in (ii).

On the other hand, since $\Omega(\alpha_2(a, b)) = \Omega^*(\alpha_2(a, b)) = F(f_{a,b})$ as before and $f_{a,b}([0, 1]) = [0, 1]$ as is easily seen, we have $[0, 1] \cap \Omega^*(\alpha_2(a, b)) = [0, 1] \cap F(f_{a,b}) = \varnothing$, which means $[0, 1] \subset J(f_{a,b})$. Therefore we have $J(f_{a,b}) = [0, 1]$.

(iv) Suppose that $(a, b) \in D_0$ and $2\sqrt{a} - a < b < 1$. Then we can immediately see that both of the preimages of 1 with respect to $f_{a,b}$ are not real, i.e., the solutions of the equation $f_{a,b}(z) = 1$ or $az^2 - (a + b)z + 1 = 0$ are both not real. Therefore $J(f_{a,b})$ is not real.

(v) Suppose that $(a, b) \in D_1$ and $b < -a$. The graph of $f_{a,b}$ is as shown in Figure 4.3. The assertion can be easily shown in the same way as before.

(vi) Suppose that $(a, b) \in D_1$ and $b > -a$. Then the fixed points $\alpha_j(a, b)$, $(j = 1, 2, 3)$ are all repelling so that they belong to $J(f_{a,b})$. The relation

$J(f_{a,b}) \subset \mathbf{R} \cup \{\infty\}$ can be shown in the same way as before. See the graph of $f_{a,b}$ in Figure 4.4.

On the other hand suppose that there exists $x \in \mathbf{R} \cap F(f_{a,b})$. Then $J(f_{a,b})$ is a proper subset of $\mathbf{R} \cup \{\infty\}$. Therefore by Lemma 4.1, the number of the connected components of $F(f_{a,b})$ is only one. By Sullivan's Theorem [1, Theorem 7.1.2.], $F(f_{a,b})$ is a Siegel disc or a Herman ring. Therefore $F(f_{a,b})$ is simply connected or doubly connected in $\overline{\mathbf{C}}$. Hence the number of the connected components of the Julia set is one or two. Since $\alpha_j(a, b) \in \mathbf{R} \cup \{\infty\} \cap J(f_{a,b})$ $(j = 1, 2, 3)$, at least one of the intervals in $\mathbf{R} \cup \{\infty\}$ given by $I_1 = [\alpha_1(a, b), \alpha_3(a, b)]$, $I_2 = [\alpha_3(a, b), \alpha_2(a, b)] (= [\alpha_3(a, b), +\infty])$ and $I_3 = [\alpha_2(a, b), \alpha_1(a, b)] (= [-\infty, \alpha_1(a, b)])$ must be contained in $J(f_{a,b})$. Using $f_{a,b}$-invariance of $J(f_{a,b})$, we have

$$J(f_{a,b}) \supset I_j \cup f_{a,b}(I_j) = \mathbf{R} \cup \{\infty\} \qquad \text{for } j = 1,\ 2,\ 3\,.$$

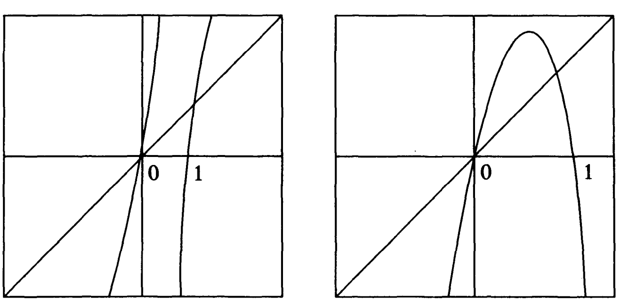
But this contradicts the assumption. Therefore $J(f_{a,b}) = \mathbf{R} \cup \{\infty\}$.

Next suppose that $(a, b) \in \{(a, b) \in D_2 \colon a + b = 0\}$. Then $\alpha_1(a, b)$ and $\alpha_3(a, b)$ are repelling fixed points and $\alpha_2(a, b)$ is a rationally indifferent fixed point which satisfies $\lambda_{f_{a,-a}}(\alpha_2(a, b)) = -1$.

Now put $g(z) = \varphi^{-1} \circ f_{a,-a} \circ \varphi(z)$ and $h(z) = g \circ g(z)$, where $\varphi(z) = \dfrac{1}{z}$. Since $J(f_{a,-a}) \subset \mathbf{R} \cup \{\infty\}$ can be shown similarly, $J(h) = J(g) = \varphi(J(f_{a,-a})) \subset \mathbf{R} \cup \{\infty\}$. On the other hand, an easy calculation shows that

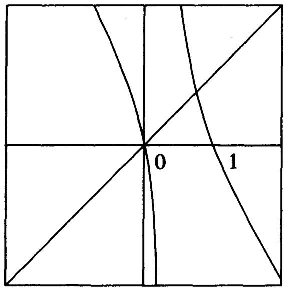
$$h(z) = z - \frac{(a+1)(a+2)}{a^2} z^3 + \cdots \qquad \text{for } z \text{ sufficiently near } 0\,,$$

and $\dfrac{(a+1)(a+2)}{a^2} \neq 0$. Applying Petal Theorem [1, Theorem 6.5.8 (b)], we



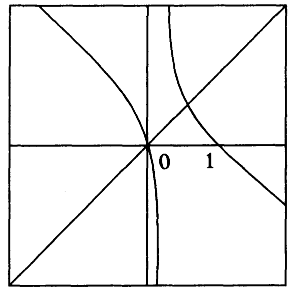
$a = 5,\ b = 1.5$ $\qquad$ $a = 4,\ b = 1.5$

FIGURE 4.1 $\qquad$ FIGURE 4.2



$a = -5,\ b = 3$

FIGURE 4.3



$a = -5,\ b = 5$

FIGURE 4.4

see that $F(h)$ has two distinct components. Therefore $J(h) = \mathbf{R} \cup \{\infty\}$. Thus $J(f_{a,-a}) = \varphi(J(h)) = \mathbf{R} \cup \{\infty\}$. ■

REMARK 4.5. If $(a, b) \in D_0$, $\alpha_1(a, b)$ is a repelling fixed point and $\alpha_2(a, b)$ is an attracting one as we have seen above. Suppose that $(a, b) \in D_0$ and $b < \dfrac{-a^2 + 3a}{a + 1}$. Then $\alpha_3(a, b)$ is an attracting fixed point. When (a, b) is in this region, we see by using computer simulations that the family $\{f_{a,b}\}$ causes period doubling phenomena similarly as logistic maps.

5. Asymptotics of Hausdorff dimension

We will treat here subsets of $\mathbf{R}$. Hence we give the definition of the Hausdorff dimension only for (bounded) subsets of $\mathbf{R}$. For a bounded set $A \subset X$ and $\beta \geq 0$ we firstly define

$$\mathscr{H}_\delta^\beta(A) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^\beta \colon A \subset \cup_{i=1}^{\infty} U_i,\ |U_i| \leq \delta\ (i = 1, 2, \dots)\right\}$$

$$\mathscr{H}^\beta(A) = \lim_{\delta \to 0} \mathscr{H}_\delta^\beta(A),$$

where $|U| = \sup_{x,y \in U} |x - y|$. The *Hausdorff dimension* of A is the number given by

$$\text{H-dim}\,(A) = \sup\{\beta \colon \mathscr{H}^\beta(A) = +\infty\}.$$

It is well known and can be easily verified that $\text{H-dim}\,(A) = \inf\{\beta \colon \mathscr{H}^{d,\beta}(A) = 0\}$. See for example [3].

According to [8], for a real analytically parameterized family of expanding rational functions of the same degree, the Hausdorff dimension of the Julia set of these functions is real analytic with respect to the parameter.

Set $D^* = \{(a, b) \in D_0 : b > 1\}$ and $D_* = \{(a, b) \in D_0 : 2\sqrt{a} - a < b < 1\}$. As has been seen in Theorem 4.3, the degree two functions $f_{a,b}$ are expanding on $J(f_{a,b})$ if $(a, b) \in D^* \cup D_*$ and the map $(a, b) \mapsto f_{a,b}(z)$ is clearly real analytic. Therefore the map $D^* \cup D_* \ni (a, b) \mapsto \text{H-dim}\,(J(f_{a,b}))$ is real analytic. In what follows we investigate the asymptotics near the boundary $\{(a, 1) : 1 < a\}$.

Assume that $(a, b) \in D_*$. Let $z_+(=z_+(a, b))$ and $z_-(=z_-(a, b))(z_- < z_+)$ be two real distinct solutions of the equation

$$f_{a,b}(z) = 1 \qquad \text{i.e.} \quad az^2 - (a + b)z + 1 = 0 \ .$$

Now we prepare the following convergences for the later arguments.

LEMMA 5.1

$$\lim_{(a,b)\in D_*, b\to 1-0} \lambda_{f_{a,b}}(z_+(a, b)) = -\infty \ , \qquad \lim_{(a,b)\in D_*, b\to 1-0} \lambda_{f_{a,b}}(z_-(a, b)) = a \ . \tag{5.1}$$

$$\lim_{(a,b)\in D^*, b\to 1+0} \lambda_{f_{a,b}}(\alpha_1(a, b)) = a \ , \qquad \lim_{(a,b)\in D^*, b\to 1+0} \lambda_{f_{a,b}}(\alpha_3(a, b)) = +\infty \ . \tag{5.2}$$

These convergences are locally uniform in a.

PROOF. First we see that

$$\lim_{(a,b)\in D_*, b\to 1-0} z_+(a, b)) = 1 \ , \qquad \lim_{(a,b)\in D_*, b\to 1-0} z_-(a, b) = 1/a \ .$$

These convergences are locally uniform in a. In fact, recalling $a > 1$, we have

$$\begin{aligned} z_\pm(a, b) &= \frac{(a + b) \pm \sqrt{(a + b)^2 - 4a}}{2a} \\ &= \frac{(a + 1) + (b - 1) \pm (a - 1)\sqrt{1 + (b - 1)(2a + b + 1)(a - 1)^{-2}}}{2a} \\ &\to \frac{a + 1 \pm (a - 1)}{2a} = \begin{cases} 1\,, & \text{for } z_+ \\ 1/a\,, & \text{for } z_- \end{cases} \end{aligned}$$

as $b \to 1 - 0$ for each a and these convergences are clearly locally uniform in a. Since $\lambda_{f_{a,b}}(z_\pm(a, b)) = f'_{a,b}(z_\pm(a, b)$ and

$$f'_{a,b}(z) = \frac{a(bz^2 - 2z + 1)}{(1 - bz)^2} = a + a(1 - b)\frac{bz - 2}{(1 - bz)^2} \ ,$$

the assertions are easily seen. ■

Let φ_1 and φ_2 be two branches of $f_{a,b}^{-1}|_{[0,1]}$ such that $\varphi_1([0, 1]) = [0, z_-]$

and $\varphi_2([0, 1]) = [z_+, 1]$. That is,

$$\varphi_1(x) = \frac{1}{2a}(a + bx - \sqrt{(bx + a)^2 - 4ax}),$$

$$\varphi_2(x) = \frac{1}{2a}(a + bx + \sqrt{(bx + a)^2 - 4ax}).$$

Since $J(f_{a,b})$ is the closure of $\cup_{n=1}^{\infty} f_{a,b}^{-n}(\{0\})$, if

$$\max_{x \in [0,1], j=1,2} |\varphi_j'(x)| < 1, \tag{5.3}$$

then $J(f_{a,b})$ is the self-similar set constructed by the contractions φ_1 and φ_2, i.e., $J(f_{a,b})$ is the unique compact set such that

$$J(f_{a,b}) = \varphi_1(J(f_{a,b})) \cup \varphi_2(J(f_{a,b})). \tag{5.4}$$

Furthermore $\varphi_1(J(f_{a,b})) \cap \varphi_2(J(f_{a,b})) = \varnothing$ holds and

$$J(f_{a,b}) = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \cdots, i_n) \in \{1,2\}^n} \varphi_i \circ \cdots \circ \varphi_{i_n}([0, 1]).$$

Next assume that $(a, b) \in D^*$. Let ϕ_1 and ϕ_2 be two branches of $f_{a,b}^{-1}|_{[0,\alpha_3(a,b)]}$ such that $\phi_1([0, \alpha_3(a, b)]) = [0, 1/a]$ and $\phi_2([0, \alpha_3(a, b)]) = [1, \alpha_3(a, b)]$. If

$$\max_{x \in [0,1], j=1,2} |\phi_j'(x)| < 1, \tag{5.5}$$

then $J(f_{a,b})$ is a self-similar set constructed from the contractions ϕ_1 and ϕ_2, i.e.,

$$J(f_{a,b}) = \phi_1(J(f_{a,b})) \cup \phi_2(J(f_{a,b})). \tag{5.6}$$

In this case as before $\phi_1(J(f_{a,b})) \cap \phi_2(J(f_{a,b})) = \varnothing$ holds and $J(f_{a,b})$ is given by the representation

$$J(f_{a,b}) = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \cdots, i_n) \in \{1,2\}^n} \phi_{i_1} \circ \cdots \circ \phi_{i_n}([0, \alpha_3(a, b)]).$$

Here we extend H-dim $(J(f_{a,b}))$ to the functions on the region $D = D^* \cup D_* \cup \{(a, 1): 1 < a\}$ by putting H-dim $(J(f_{a,b}))|_{b=1} = 0$. This is justified because $f_{a,1}(z) = az$ so that clearly H-dim $(J(f_{a,1})) = 0$. As stated, the asymptotics of this function will be researched when the degree of functions degenerates.

THEOREM 5.2.

$$\lim_{(a,b) \in D_*, b \to 1-0} \text{H-dim}\,(J(f_{a,b})) = 0, \qquad \lim_{(a,b) \in D^*, b \to 1+0} \text{H-dim}\,(J(f_{a,b})) = 0. \tag{5.7}$$

These convergences are locally uniform in a. In other words, H-dim $(J(f_{a,b}))$ *is continuous in* $D^* \cup D_* \cup \{(a, 1) \in \mathbf{R}^2 : a > 1\}$.

PROOF. Suppose that $(a, b) \in D_*$. In view of (5.1) we can see that the condition (5.3) is satisfied when b is sufficiently near 1. Therefore $J(f_{a,b})$ is the self-similar set defined by (5.4). Applying the Moran-Hutchinson's methods, we have the following estimate for H-dim $(J(f_{a,b}))$.

$$s(a, b) \leq \text{H-dim}\,(J(f_{a,b})) \leq t(a, b)\,,$$

where $s(a, b)$ and $t(a, b)$ are given by the following equations:

$$\left(\min_{x \in [0,1]} |\varphi_1'(x)|\right)^{s(a,b)} + \left(\min_{x \in [0,1]} |\varphi_2'(x)|\right)^{s(a,b)} = 1\,,$$

$$\left(\max_{x \in [0,1]} |\varphi_1'(x)|\right)^{t(a,b)} + \left(\max_{x \in [0,1]} |\varphi_2'(x)|\right)^{t(a,b)} = 1$$

respectively (see Falconer [3, Theorem 8.8]). Since

$$\max_{x \in [0,1]} |\varphi_1'(x)| = |\varphi_1'(1)| = |\lambda_{f_{a,b}}(z_-)|^{-1}\,, \ \max_{x \in [0,1]} |\varphi_2'(x)| = |\varphi_2'(1)| = |\lambda_{f_{a,b}}(z_+)|^{-1}\,,$$

Lemma 5.1 implies

$$\lim_{(a,b) \in D_*, b \to 1-0} \ \max_{x \in [0,1]} |\varphi_1'(x)| = \lim_{(a,b) \in D_*, b \to 1-0} |\lambda_{f_{a,b}}(z_-)|^{-1} = 1/a\,,$$

$$\lim_{(a,b) \in D_*, b \to 1-0} \ \max_{x \in [0,1]} |\varphi_2'(x)| = \lim_{(a,b) \in D_*, b \to 1-0} |\lambda_{f_{a,b}}(z_+)|^{-1} = 0\,,$$

where these convergences are locally uniform in a. Therefore it is clear that

$$\lim_{(a,b) \in D_*, b \to 1-0} t(a, b) = 0 \qquad \text{locally uniformly in } a\,.$$

Next suppose that $(a, b) \in D^*$. Then similarly we obtain

$$\lim_{(a,b) \in D^*, b \to 1+0} \ \max_{x \in [0, \alpha_3(a,b)]} |\phi_1'(x)| = \lim_{(a,b) \in D^*, b \to 1+0} |\lambda_{f_{a,b}}(0)|^{-1} = 1/a\,,$$

$$\lim_{(a,b) \in D^*, b \to 1+0} \ \max_{x \in [0, \alpha_3(a,b)]} |\phi_2'(x)| = \lim_{(a,b) \in D^*, b \to 1+0} |\lambda_{f_{a,b}}(\alpha_3(a, b))|^{-1} = 0\,.$$

Therefore defining $t(a, b)$ for ϕ_1 and ϕ_2 similarly, we obtain

$$\lim_{(a,b) \in D^*, b \to 1+0} t(a, b) = 0 \qquad \text{locally uniformly in } a\,.$$

Thus we have (5.7) in both cases. ■

The number $s(a, b)$ defined above is used to estimate H-dim $(J(f_{a,b}))$ from below and the estimates in the following lemma will be used to show the non-analyticity on the boundary.

LEMMA 5.3. *If* $(a, b) \in D_*$ *satisfies* (5.3), *then*

$$\text{H-dim}\,(J(f_{a,b})) \geq \frac{\log 2}{\log a - \log(1-b)}\,. \tag{5.8}$$

If $(a, b) \in D^*$ *satisfies* (5.5), *then*

$$\text{H-dim}\,(J(f_{a,b})) \geq \frac{\log 2}{\log a - \log(b-1)}\,. \tag{5.9}$$

PROOF. Note that

$$\min_{x\in[0,1]} |\varphi_1'(x)| = |\varphi_1'(0)| = |\lambda_{a,b}(0)|^{-1} = 1/a\,,$$

$$\min_{x\in[0,1]} |\varphi_2'(x)| = |\varphi_2'(0)| = |\lambda_{f_{a,b}}(1)|^{-1} = |-a/(1-b)|^{-1} = (1-b)/a$$

for $(a, b) \in D_*$ and

$$\min_{x\in[0,\alpha_3(a,b)]} |\phi_1'(x)| = |\lambda_{f_{a,b}}(1/a)|^{-1} = \frac{(a-b)^2}{a(b-2a+a^2)}\,,$$

$$\min_{x\in[0,\alpha_3(a,b)]} |\phi_2'(x)| = |\lambda_{f_{a,b}}(1)|^{-1} = |-a/(1-b)|^{-1} = (b-1)/a$$

for $(a, b) \in D^*$.

Suppose $(a, b) \in D_*$ and (5.3) holds. Then $|\lambda_{f_{a,b}}(0)| = a < a/(1-b) = |\lambda_{f_{a,b}}(1)|$ since $b > 1$, which implies

$$\begin{aligned} 1 &= (1/a)^{s(a,b)} + ((1-b)/a)^{s(a,b)} \\ &\geq 2((1-b)/a)^{s(a,b)}\,. \end{aligned}$$

Therefore it is clear that

$$\log 2/(\log a - \log(1-b)) \leq s(a, b) \leq \text{H-dim}\,(J(f_{a,b}))\,.$$

Similarly we can estimate (5.9) when $(a, b) \in D^*$ and (5.5) holds. In this case $|\lambda_{f_{a,b}}(1/a)| = \dfrac{a(b-2a+a)^2}{(a-b)^2} < a/(b-1) = |\lambda_{f_{a,b}}(1)|$. Therefore we have the estimate (5.9) similarly. ■

THEOREM 5.4. H-dim $(J(f_{a,b}))$ *is not real analytic in* b at 1 *for each* $a > 1$.

PROOF. Suppose that there exists some $a > 1$ such that H-dim $(J(f_{a,b}))$ is real analytic at $b = 1$. Considering continuity and analyticity, we have the following estimate for some $C > 0$ when $(a, b) \in D_*$, b is sufficiently near 1 and (5.3) holds.

$$(0<)\ \text{H-dim}\,(J(f_{a,b})) \leq C(1-b)^n \qquad \text{for some } n \in \mathbf{N}\,. \tag{5.10}$$

Comparing (5.8) with (5.10), we must have

$$\frac{\log 2}{\log a - \log(1-b)} \leq C(1-b)^n$$

for b sufficiently near 1, which is clearly impossible. Therefore the assertion must hold. ■

REMARK 5.5. We can show Theorem 5.4 by (5.9) as well.

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References

[1] A. Beardon, Iteration of Rational Functions, Springer-Verlag, GTM132, (1991).
[2] R. Devaney, An introduction to chaotic dynamical systems, second edition, Addison Wesley Publishing Company Inc, (1989).
[3] K. Falconer, The geometry of fractal sets, Cambridge Univ Press, (1985).
[4] K. Falconer, Fractal Geometry, Mathematical Foundations and Applications, John Willey & Sons, (1990).
[5] M. Lyubich, Dynamics of rational transformations: topological picture, Russ. Math. Surv., **41** (1986), 43–117.
[6] J. Milnor, Dynamics in One Complex Variable: Introductory Lectures (1990), SUNY Stony Brook Institute for Mathematical Sciences Preprint, #1990/5.
[7] M. Rees, Components of degree two hyperbolic rational maps, Invent. Math., **100** (1990), 357–382.
[8] D. Ruelle, Repellers for real analytic maps, Ergod. Th. & Dynam. Sys., (1982), 2:99–108.
[9] N. Saito, S. Saitô, & A. Shimizu, An Analysis of a Family of Rational Maps which Contains Integrable and Nonintegrable Difference Analogue of the Logistic Equation, Preprint, 1993.

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