Hiroshima Math. J.
26 (1996), 245–251

A few results on block-colourings of designs

Dedicated to Professor Michihiko Kikkawa on the occasion of his 60th birthday

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(Received September 21, 1994)
(Revised January 17, 1995)

Abstract. A q-block colouring of a design is a partition of its blocks into q colour classes consisting of pairwise disjoint blocks. The chromatic index is the least q for which such a colouring exists. In this paper the bounds of chromatic indices of designs are given, and some greedy algorithms for colouring the blocks along with the worst-case colourings are discussed.

1. Introduction

A *design* is a pair $(\mathscr{V}, \mathscr{B})$ where $\mathscr{V}$ is a finite set of v elements and $\mathscr{B}$ is a collection of b subsets called blocks of $\mathscr{V}$, each of size k. A *balanced incomplete block design* $\mathrm{B}(k, \lambda; v)$ is a design $(\mathscr{V}, \mathscr{B})$ such that $|\mathscr{V}| = v$, $|B| = k$ for all $B \in \mathscr{B}$, and every pair of distinct elements of $\mathscr{V}$ is contained in precisely λ blocks of $\mathscr{B}$. The usual parameters of a $\mathrm{B}(k, \lambda; v)$ satisfy the relations $vr = bk$ and $\lambda(v-1) = r(k-1)$ where r is the number of blocks containing an element of $\mathscr{V}$. A *partial parallel class* in a design is a set of pairwise disjoint blocks.

Given a design $\mathscr{D} = (\mathscr{V}, \mathscr{B})$, a *block-colouring* of $\mathscr{D}$ is a mapping $\psi: \mathscr{B} \to \mathscr{C}$ such that if $\psi(B) = \psi(B')$ for $B, B' \in \mathscr{B}$, $B \neq B'$, then $B \cap B' = \phi$. If for the set of colours $\mathscr{C}$ we have $|\mathscr{C}| = q$, then ψ is called a *q-block-colouring*. For each $c \in \mathscr{C}$, the set $\psi^{-1}(c)$ is called a *block-colour class*. The *chromatic index* of $\mathscr{D}$, $\chi'(\mathscr{D})$, is defined by the smallest q such that there exists a q-block-colouring of $\mathscr{D}$. A design $\mathscr{D}$ with $\chi'(\mathscr{D}) \leq q$ is said to be *q-colourable*. Note that a block-colour class is a partial parallel class.

In the design of experiments where each block corresponds to a 'test' we can view disjoint blocks as tests which can be carried out simultaneously. The chromatic index is precisely the least time required for the entire experiment.

1991 *Mathematics Subject Classification.* 05B05, 62K10

Key words and phrases. Design; block-colouring; chromatic index; achromatic index; greedy algorithm; resolvable; near resolvable; projective plane.

Designs with small chromatic index have been studied under the guise of resolvable and near resolvable designs. A design is said to be *resolvable* (resp. *near resolvable*) if its collection of blocks can be partitioned into partial parallel classes $\{\mathscr{P}_i | i \in I\}$ such that $\cup_{B \in \mathscr{P}_i} B = \mathscr{V}$ (resp. $\cup_{B \in \mathscr{P}_i} B = \mathscr{V} - \{x\}$ for some $x \in \mathscr{V}$) for all $i \in I$. Clearly, a resolvable (resp. near resolvable) design, if it exists, has the minimum chromatic index among all designs with the same parameters. For the existence of resolvable and near resolvable designs, we refer the reader to a recent monograph [7].

The majority of research on block-colourings of designs has focussed on B(3, 1; v). For example, see [4, 5, 6, 9, 11]. Considerably less is known concerning block-colourings of designs with general k, λ and v. In this paper, the bounds of chromatic indices will be given, and some greedy algorithms for colouring the blocks along with the worst-case colourings will be discussed. These generalize some of the results in the literature mentioned above.

The results obtained here will be useful in constructions of resolvable designs, for example, the construction described in [13].

2. Upper and lower bounds

One of the primary questions on block-colourings is "What is the range of possible values of chromatic indices of designs?" It seems to be very difficult. We only know the upper and lower bounds of chromatic indices until now.

Let r_x denote the number of blocks containing an element x in a design $\mathscr{D}$, b the number of blocks in $\mathscr{D}$, p_s and p_l the smallest and largest cardinalities of the partial parallel classes in $\mathscr{D}$, respectively. Then clearly we can obtain the following.

THEOREM 2.1. *In a design* $\mathscr{D}$, $\max\{b/p_l, \max\{r_x: x \in \mathscr{V}\}\} \leq \chi'(\mathscr{D}) \leq b/p_s$.

In particular, when $\mathscr{D}$ is a B(k, λ; v), we further have the following improvement. Let $\lfloor t \rfloor$ denote the largest integer not greater than a real number t.

THEOREM 2.2. *If* $\mathscr{D}$ *is a* B(k, λ; v), *then* $b/\lfloor v/k \rfloor \leq \chi'(\mathscr{D})$.

PROOF. In a B(k, λ; v), $r_x = r = \lambda(v-1)/(k-1)$ for every $x \in \mathscr{V}$. On the other hand, any block-colour class can include at most $\lfloor v/k \rfloor$ blocks, i.e., $p_l \leq \lfloor v/k \rfloor$. Since $b/\lfloor v/k \rfloor \geq b/(v/k) = \lambda(v-1)/(k-1)$, it holds that $\chi'(\mathscr{D}) \geq \max\{b/p_l, \max\{r_x: x \in \mathscr{V}\}\} \geq \max\{b/\lfloor v/k \rfloor, \lambda(v-1)/(k-1)\} = b/\lfloor v/k \rfloor$. This completes the proof. □

REMARK. Generally, the lower bound as in Theorem 2.2 cannot be improved further. For example, the chromatic indices of resolvable designs and near resolvable designs attain this bound.

Another upper bound is a generalization of [5]. We at first need Brooks' and Vizing's theorems [3, 12] in graph theory. The *chromatic number* of a graph G is defined to be the smallest number of colours needed to colour the vertices of the graph so that no two adjacent vertices have the same colour. The chromatic number of graph G is denoted by $\nu(G)$. A graph G with $\nu(G) \leq q$ is said to be *q-colourable*. For undefined terminologies on graph theory, the reader is referred to [1].

LEMMA 2.3 ([3, 12]). *Let G be a connected simple graph with maximum degree h. Then G is h-colourable, unless*

(1) $h \neq 2$, *and G is a K_{h+1}, the complete graph on $h+1$ vertices, which is $(h+1)$-colourable;*
(2) $h = 2$, *and G is an odd cycle, which is 3-colourable.*

This lemma can be used to show the following theorem.

THEOREM 2.4. *If $\mathscr{D}$ is a* $B(k, \lambda; v)$, *then* $\chi'(\mathscr{D}) \leq \lambda k(v-1)/(k-1) - \lambda(k-1)$.

PROOF. Construct the block-intersection graph of $\mathscr{D}$, whose vertices are the blocks of $\mathscr{D}$, and two vertices are adjacent if and only if the corresponding blocks intersect. Hence the chromatic index of $\mathscr{D}$ is the chromatic number of its block-intersection graph. For each vertex, there are at most

$$\left(\frac{\lambda(v-1)}{k-1} - 1\right) + (k-1)\left[\left(\frac{\lambda(v-1)}{k-1} - 1\right) - (\lambda - 1)\right]$$

other blocks which intersect the corresponding block. Thus the block-intersection graph has maximum degree less than $\lambda k(v-1)/(k-1) - \lambda(k-1)$. Therefore Lemma 2.3 shows that the chromatic number is at most $\lambda k(v-1)/(k-1) - \lambda(k-1)$. □

3. Computation

Another primary question on block-colourings is "Can we easily compute the chromatic index of a design?"

As far as algorithmic results are concerned, the complexity of computing the chromatic index is unknown. The current best method involves backtracking which could require an exponential amount of time. However, instead of employing the exhaustive algorithms, we may search for algorithms guaranteed to run in polynomial time but possibly giving only approximate answers. Two general classes of algorithmic greedy methods and hill-climbing methods, for approximating the chromatic index of a $B(3, 1; v)$, were studied in [6]. Note that the achromatic index defined later is always larger than or equal to the chromatic index.

Here the use of the greedy methods is generalized. We investigate two fairly simple greedy algorithms, and study the worst-case colouring for each of these two algorithms in the following two subsections.

3.1. Achromatic index

A block-colouring of a design $\mathscr{D}$ is said to be *complete* if the union of any two distinct colour classes does not form a partial parallel class. The *achromatic index* of $\mathscr{D}$, $\psi'(\mathscr{D})$, is defined by the maximum possible number of colours in a complete block-colouring of $\mathscr{D}$. It appears that the achromatic index is the maximum number of colours needed in the greedy colouring technique operating as follows: Initially, each block is assigned with a different colour, then join together two disjoint colour classes (eliminating a colour), until there are no two disjoint colour classes. The achromatic index reflects the worst-case behaviour of a greedy block-colouring algorithm that proceeds by merging two disjoint block-colour classes while possible.

THEOREM 3.1. *For any* $B(k, \lambda; v)$ $\mathscr{D}$, $\psi'(\mathscr{D}) \leq c(k, \lambda)v^{3/2}$ *for some constant* $c(k, \lambda)$.

PROOF. Suppose we have a t-complete-block-colouring; the ith colour class contains $b(i)$ blocks. Then $\sum_{i=1}^{t} b(i) = \lambda v(v-1)/\{k(k-1)\}$. Now for each i, $(r-1)kb(i) \geq t-1$, since each pair of the t classes intersect. Thus $\sum_{i=1}^{t} b(i) \geq t(t-1)/\{(r-1)k\}$. Hence $t(t-1) \leq \lambda v(v-1)/\{k(k-1)\} \cdot k \cdot \{\lambda(v-1)/(k-1) - 1\} = \lambda v(v-1)\{\lambda(v-1)-(k-1)\}/(k-1)^2 \leq \lambda^2 v^3/(k-1)^2$ and $t \leq \{\lambda/(k-1)\}v^{3/2} + 1 \leq \{\lambda/(k-1) + 1\}v^{3/2}$ which completes the proof. □

This bound is the best in some sense, as described below.

THEOREM 3.2. *For any prime power* q, *there exist infinitely many* $B(q+1, 1; v)$ $\mathscr{D}$, *where* v *is of form* $q^{2 \cdot 3^n} + q^{3^n} + 1$, *such that* $\psi'(\mathscr{D}) \geq c_1(q+1, 1)v^{3/2}$ *for some constant* $c_1(q+1, 1)$.

PROOF. The existence of a resolvable $B(q+1, 1; q^3+1)$ is shown for any prime power q by Bose [2]. By reduction, there exists a resolvable $B(q+1, 1; q^{3^n}+1)$ for all $n \in \mathscr{N}$. Now take a projective plane of order q^{3^n}, and replace each line (block) of the plane by a resolvable $B(q+1, 1; q^{3^n}+1)$ which has q^{3^n-1} parallel classes. Assign each parallel class of each resolvable $B(q+1, 1; q^{3^n}+1)$ a unique colour. Moreover, the plane properties ensure that there are $q^{2 \cdot 3^n} + q^{3^n} + 1$ blocks in the plane and every two blocks of the plane intersect. Then $\psi'(\mathscr{D}) \geq (q^{2 \cdot 3^n} + q^{3^n} + 1)q^{3^n-1} \geq (2q)^{-1}v^{3/2}$. □

THEOREM 3.3. *For any prime power q, there exist infinitely many* $B(q+1, 1; v)$ $\mathscr{D}$, where v *is of form* $q^{2(2n+1)} + q^{2n+1} + 1$, *such that* $\psi'(\mathscr{D}) \geq c_2(q+1, 1)v^{3/2}$ *for some constant* $c_2(q+1, 1)$.

PROOF. The existence of a resolvable $B(q+1, 1; v)$, where v is of form $q^{2n+1} + 1$, is shown by Ray-Chaudhuri and Wilson [10]. Take a projective plane of order q^{2n+1}, and replace each line (block) of the plane by a resolvable $B(q+1, 1; q^{2n+1}+1)$. By assigning each parallel class of each resolvable $B(q+1, 1; q^{2n+1}+1)$ a unique colour, we can get the required result. □

THEOREM 3.4. *For any k, there exist infinitely many* $B(k, k-1; v)$, $\mathscr{D}$, *where* v *is of form* $t^2k^2 - tk + 1$ *and* $tk - 1$ *is a prime power, such that* $\psi'(\mathscr{D}) \geq c_3(k, k-1)v^{3/2}$ *for some constant* $c_3(k, k-1)$.

PROOF. Dirichlet's theorem guarantees the existence of infinitely many u such that $u \equiv 0 \bmod k$ and $u - 1$ be a prime power. Now, take a projective plane of order $u - 1$, and replace each line (block) of the plane by a resolvable $B(k, k-1; u)$ (see [8]). Then assign each parallel class of each resolvable $B(k, k-1; u)$ a unique colour. □

3.2. Block-by-block colourings

In this section, we consider a more sensible way of colouring blocks. The "block-by-block" greedy method proceeds as follows: We use integers to represent colours. Initially, the blocks are not coloured. We colour the blocks one at a time, by assigning a block the least integer so that the resulting colour class contains disjoint blocks.

THEOREM 3.5. *The block-by-block greedy method takes at most* $k\{\lambda(v-1)/(k-1) - 1\} + 1$ *colours for a* $B(k, \lambda; v)$.

PROOF. Suppose that the number of colours exceeds $k\{\lambda(v-1)/(k-1) - 1\} + 1$. Then, at some stage, a block $\{x_1, x_2, \ldots, x_k\}$ could not be assigned to any of the first $k\{\lambda(v-1)/(k-1) - 1\} + 1$ colours. Then the block $\{x_1, x_2, \ldots, x_k\}$ intersects each of these colour classes. But each of x_i, $1 \leq i \leq k$, appears in only $\lambda(v-1)/(k-1) - 1$ blocks other than $\{x_1, x_2, \ldots, x_k\}$. This is a contradiction which completes the proof. □

This bound cannot be improved further, as the following shows.

THEOREM 3.6. *If* $k - 1$ *is a prime power, then the block-by-block greedy method may require* $\lambda(v-k) + \lambda(v-1)/(k-1)$ *colours for a* $B(k, \lambda; v)$, *when* $v \equiv 1 \bmod k(k-1)$.

PROOF. Let a resolvable $B(k, \lambda; u)$ be based on $\{1, 2, \ldots, u\}$. We construct a $B(k, \lambda; (k-1)u+1)$ on $(\{1, 2, \ldots, u\} \times \{1, 2, \ldots, k-1\}) \cup \{\infty\}$ as follows: For each block B of the resolvable $B(k, \lambda; u)$, we have $(k-1)^2$ blocks B_i, $1 \le i \le (k-1)^2$, of the transversal design $TD(k, k-1)$ on $B \times \{1, 2, \ldots, k-1\}$. We also have blocks $\{(i, 1), (i, 2), \ldots, (i, k-1), \infty\}$ for all $i \in \{1, 2, \ldots, u\}$, each repeated λ times. (For a definition of the transversal design see [13].)

Now we colour the $B(k, \lambda; (k-1)u+1)$ by the block-by-block method. Let $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_r$, $r = \lambda(u-1)/(k-1)$, be the parallel classes of the resolvable $B(k, \lambda; u)$. For each block B of $\mathscr{P}_h$, assign B_i colour $(i-1)r+h$, $1 \le i \le (k-1)^2$. Finally, assign each $\{\infty, (i, 1), (i, 2), \ldots, (i, k-1)\}$ a different colour. This is a $(\lambda u + r(k-1)^2)$-block colouring. That is, a $B(k, \lambda; v)$ takes $\lambda(v-k) + \lambda(v-1)/(k-1)$ colours to colour its blocks by the block-by-block method. □

THEOREM 3.7. *If $k-1$ is a prime power, then the block-by-block greedy method may require $(k-1)(v-1)$ colours for a $B(k, k-1; v)$, when $v \equiv k$ mod $k(k-1)$.*

PROOF. Let a near resolvable $B(k, k-1; u)$ be based on $\{1, 2, \ldots, u\}$. We construct a $B(k, k-1; (k-1)u+1)$ similarly, and colour the block B_i corresponding to $\mathscr{P}_h$, $1 \le h \le u$, the near parallel classes missing $\{h\}$ of the near resolvable $B(k, k-1; u)$, by colour $(i-1)u+h$, $1 \le i \le (k-1)^2$. Finally, assign $\{\infty, (j, 1), \ldots, (j, k-1)\}$ colour $(i-1)u+j$ for some i, $1 \le i \le (k-1)^2$. This is a $(k-1)(v-1)$-block-colouring, which the block-by-block method proceeds. □

In subsections 3.1 and 3.2, we discussed two algorithms. Obviously, both of them can be improved by introducing certain heuristic improvement techniques.

Acknowledgements

The authors are grateful to the referee for his helpful suggestions. The research was supported in part by Grant-in-Aid for Scientific Research, The Ministry of Education, Science and Culture under Contract Numbers (A)06302015 and (C)06640311.

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