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Uniqueness of positive solutions for singular nonlinear boundary value problems

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ABSTRACT. We show that the singular nonlinear second-order differential equation

$$(E) \qquad u''(r) + \frac{m}{r}u'(r) + f(r, u(r)) = 0 \qquad in\ (\theta_1, \theta_2)$$

has at most one positive solution in $C^1[\theta_1, \theta_2]$ with boundary conditions

$$(BC_1) \qquad u(\theta_1) = \xi_1 \geq 0,\ u(\theta_2) = \xi_2 \geq 0$$

$$(BC_2) \qquad u'(\theta_1) = u(\theta_2) = 0$$

$$(BC_3) \qquad \begin{cases} \alpha_1 u(\theta_1) + \beta_1 u'(\theta_1) = 0\,, \\ \alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0\,. \end{cases}$$

where α_i, $\beta_i \in (-\infty, \infty)$ satisfy $\alpha_i^2 + \beta_i^2 \neq 0$, $i = 1, 2$.

1. Introduction

Singular, nonlinear boundary value problems appear in a variety of applications and often only positive solutions are of interest. This problem arises in the study of pseudoplastic fluids, boundary layer theory, the theory of tubular chemical reactors and reaction-diffusion theory. There are many authors considering the uniqueness and existence of positive solutions of

$$(I) \qquad u''(r) + \frac{k}{r}u'(r) + f(r, u(r)) = 0 \qquad in\ (0, 1)$$

with boundary conditions

$$(C1) \qquad u(0) = u(1) = 0\,,$$

$$(C2) \qquad u'(0) = u(1) = 0\,,$$

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(C3) $$c_1u(0) + c_2u'(0) = c_3u(1) + c_4u'(1) = 0\,,$$

where $k \in \{0, 1, 2, 3, \dots\}$ and $c_1c_3 + c_1c_4 - c_2c_3 > 0$.

See, for example, Brown and Hess [1], Baxley [2], Brezis and Oswald [3], Cohen and Laetsch [5], Callegari and Nachman [6], Fink, Gatica, Herandez and Waltman [7], Gatica, Oliker and Waltman [8], Kwong [10], O'Regan [17, 18].

Recently, Gatica, Oliker and Waltman [8], Kwong [10], Brezis and Oswald [3] showed the following important results:

THEOREM A. ([8]: Theorem 3.1) *Let $k = 0$ and $f\colon (0, 1) \times (0, \infty) \to (0, \infty)$ be continuous such that $f(r, u)$ is a (strictly) decreasing function of u for each fixed r and such that for any given $r_0 \in (0, 1)$, $\xi_1 > 0$, $\xi_2 \in R$. Equation* (I) *with initial condition $u(r_0) = \xi_1$ and $u'(r_0) = \xi_2$ has at most one solution in a neighborhood of r_0. Then* (I) *has at most one positive solution satisfying* (C3) *in $C^1[0, 1] \cap C^2(0, 1)$.*

THEOREM B. ([8]: Theorem 4.1) *Let $k \in \{1, 2, 3, \dots\}$, $p \in (0, 1)$ and $h \in C([0, 1]\colon [0, \infty))$ such that*

$$0 < \int_0^1 (1 - r)^{-p} h(r)dr < \infty\,.$$

Then

(*II*) $$u''(r) + \frac{k}{r}u'(r) + h(r)u^{-p}(r) = 0 \qquad \text{in } (0, 1)$$

has at least one positive solution satisfying (C2) *in $C^1[0, 1] \cap C^2(0, 1)$.*

THEOREM C. ([10]: Theorem 2) *Assume that $q(t) > 0$ in $(0, 1)$ and $f(u)/u$ is decreasing in $(0, \infty)$ and not constant in any neighborhood of $u = 0$. Then*

(*III*) $$u''(r) + q(r)f(u(r)) = 0 \qquad \text{in } (0, 1)$$

has a unique solution satisfying (C2) *in $C^1[0, 1] \cap C^2(0, 1)$.*

THEOREM D. ([3]: Theorem 1) *Consider the problem*

(*IV*) $$\begin{cases} \Delta u + f(x, u) = 0 & \text{in } \Omega\,, \\ u \geq 0,\ u \neq 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and make the following assumptions:

(1°) *For* a.e. *$x \in \Omega$, the function $u \to f(x, u)$ is continuous on $[0, \infty)$ and the function $u \to f(x, u)/u$ is strictly decreasing in $(0, \infty)$,*

(2°) *For each $u \geq 0$, the function $u \to f(x, u)$ belongs to $L^\infty(\Omega)$.*

Then, there exists at most one solution of (IV) *in* $H_0^1 \cap L^\infty(\Omega)$.

REMARK 1. The assumption "$f(u)/u$ is decreasing in $(0, \infty)$ and is not a constant in any neighborhood of $u = 0$" in Theorem C, is imposed to exclude the situation in which $f(u)$ behaves like a linear function in a neighborhood of $u = 0$, that is, $f(u)/u$ behaves like a strictly decreasing function near $u = 0$.

REMARK 2. The proof of Theorem D is based on the following results:

1° $$\int_{\partial\Omega} u_1 \nabla u_1 = \int_{\partial\Omega} u_2 \nabla u_2 = 0\,,$$

2° $$\int_{\partial\Omega} (u_2^2/u_1)\nabla u_1 = \int_{\partial\Omega} (u_1^2/u_2)\nabla u_2 = 0\,.$$

These results imply that $u_j'(\theta_i)$ exist and $u_2^2(\theta_i)/u_1(\theta_i) = u_1^2(\theta_i)/u_2(\theta_i) = 0$ $(i, j = 1, 2)$, if u_1, u_2, are radial and $\Omega = \{x \mid \theta_1 \le |x| \le \theta_2\}$.

It is clear that if $f(r, u)$ is (strictly) decreasing in $u \in (0, \infty)$,

$$f(r, u) \equiv h(r)u^{-p},\ h(r)u^q,\ u - h(r)u^\lambda,\ u^\alpha + u^{-\alpha}$$

for any given $p \in [0, \infty)$, $q \in [0, 1)$, $\lambda \in (1, \infty)$, $\alpha \in [0, 1]$ and $h \in C((0, 1); [0, \infty))$, then $f(r, u)$ satisfies "$f(r, u)/u$ is strictly decreasing in u". But, Theorems A, B, C and D can not be applied to most of these functions, for example,

$$f(r, u) \equiv u^{1/2},\ u - u^2,\ u^{1/2} - h(r)u \quad \text{and} \quad u^{-1} + u\,.$$

Furthermore, Theorem B does not tell us if the positive solution of (II) with (C2) is unique. Inspired by the above-mentioned results, we will give a concise approach to establish some uniqueness theorems (in $C^1[\theta_1, \theta_2]$) which generalize Theorems A, C, D, the uniqueness results in [1–10, 12–13, 17–20] and confirm the uniqueness of Theorem B.

2. Main Results

Let $m \ge 0$ and $0 \le \theta_1 < \theta_2 < \infty$. In order to abbreviate our discussion, throughout this paper, we say $u(r)$ is a positive solution of (E) which means that $u \in C^1[\theta_1, \theta_2]$ satisfies (E) and is positive in (θ_1, θ_2), and suppose that the following assumptions hold:

(A_1) $f\colon (\theta_1, \theta_2) \times (0, \infty) \to (-\infty, \infty)$ is locally Lipschitz continuous in $u \in (0, \infty)$ for each fixed $r \in (\theta_1, \theta_2)$,

(A_2) $f(r, u)/u$ is strictly decreasing with respect to $u \in (0, \infty)$ for each fixed $r \in (\theta_1, \theta_2)$.

We now consider the uniqueness of positive solutions of

$$(E) \qquad u''(r) + \frac{m}{r}u'(r) + f(r, u(r)) = 0 \qquad \text{in } (\theta_1, \theta_2)$$

with boundary conditions

$$(BC_1) \qquad u(\theta_1) = \xi_1 \geq 0, \; u(\theta_2) = \xi_2 \geq 0$$

$$(BC_2) \qquad u'(\theta_1) = u(\theta_2) = 0$$

$$(BC_3) \qquad \begin{cases} \alpha_1 u(\theta_1) + \beta_1 u'(\theta_1) = 0, \\ \alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0, \end{cases}$$

where α_i, $\beta_i \in (-\infty, \infty)$ satisfy $\alpha_i^2 + \beta_i^2 \neq 0$, $i = 1, 2$.

In order to prove our main results, we introduce the following concepts: Suppose that u and v are two positive solutions of (E). Let

$$w(r) \equiv u(r)v'(r) - u'(r)v(r), \qquad 0 \leq \theta_1 \leq r \leq \theta_2 < \infty$$

be the wronskian of u and v. It is clear that $w(r)$ satisfies

$$(1) \qquad (r^m w(r))' = -r^m u(r)v(r)\left\{\frac{f(r, v(r))}{v(r)} - \frac{f(r, u(r))}{u(r)}\right\}$$

in (θ_1, θ_2). Assume that $u(r) > v(r)$ in $(a, b) \subseteq (\theta_1, \theta_2)$, it follows from (1) and (A_2) that $r^m w(r)$ is strictly decreasing in (a, b). Hence, we see that

$$(2) \quad u(b)v'(b) - u'(b)v(b) = w(b) < (a/b)^m w(a) = (a/b)^m (u(a)v'(a) - u'(a)v(a)).$$

Using the above-mentioned results, we establish the following uniqueness theorems which generalize the uniqueness results in [1–10, 12–13, 17–20].

REMARK 3.

(1*) If $m \geq 0$ is an even integer, then we can relax the restriction on θ_1, θ_2 to $-\infty \leq \theta_1 < \theta_2 \leq \infty$.

(2*) If $f(r, u)$ satisfies some stronger assumption than (A_2), for example,

"$f(r, u)$ is (strictly) decreasing with respect to $u \in (0, \infty)$",

then we can obtain the uniqueness results in $C^1(0, 1) \cap C[0, 1]$ similar to the following Theorems 1, 2 and 3, by using the Maximum principle with respect to the function $H(r) \equiv u(r) - v(r)$.

THEOREM 1. *(E) has at most one positive solution satisfying boundary condition* (BC_1).

PROOF. Assume, on the contrary, that u and v are two distinct positive solutions of (E) satisfying (BC_1). First, we claim that u and v intersect in (θ_1, θ_2). Suppose on the contrary that $u(r) > v(r)$ in (θ_1, θ_2). It follows from (2), $u'(\theta_1) \geq v'(\theta_1)$ and

$$u'(\theta_2) - v'(\theta_2) \leq 0$$

that

$$0 \leq \xi_2(v'(\theta_2) - u'(\theta_2)) < (\theta_1/\theta_2)^m \xi_1(v'(\theta_1) - u'(\theta_1)) \leq 0,$$

which gives a contradiction. Hence, there exists $\theta_3 \in (\theta_1, \theta_2)$ such that

$$u(\theta_3) = v(\theta_3) > 0.$$

Next, we claim that u and v intersect in (θ_3, θ_2). Suppose to the contrary that $u(r) > v(r)$ in (θ_3, θ_2). It follows from $u'(\theta_3) \geq v'(\theta_3)$, $u'(\theta_2) - v'(\theta_2) \leq 0$ and (2) that

$$0 \leq \xi_2(v'(\theta_2) - u'(\theta_2)) < (\theta_3/\theta_2)^m u(\theta_3)(v'(\theta_3) - u'(\theta_3)) \leq 0,$$

which again gives a contradiction. Thus, there exists $\theta_4 \in (\theta_3, \theta_2)$ such that

$$u(\theta_4) = v(\theta_4) > 0.$$

Repeating the same argument, we obtain a strictly decreasing sequence

$$\{\theta_n\}_{n=5}^{\infty} \subseteq (\theta_3, \theta_4) \subseteq (\theta_1, \theta_2)$$

such that $\theta_n \in (\theta_3, \theta_{n-1})$ and $u(\theta_n) = v(\theta_n) > 0$ for all $n = 5, 6, \ldots$.

By the Bolazano-Weierstrass theorem, we see that $\{\theta_n\}_{n=5}^{\infty}$ has an accumulation point, say $\eta \in [\theta_3, \theta_4]$. It is clear that $u(\eta) = v(\eta) > 0$ and $u'(\eta) = v'(\eta)$, which combined with (A_1) implies $u(r) = v(r)$ in (θ_1, θ_2). (see, Hartman [9]). This gives a contradiction, thus the proof is complete.

THEOREM 2. (E) *has at most one positive solution satisfying boundary condition* (BC_2).

PROOF. Assume, on the contrary, that u and v are two distinct positive solutions of (E) satisfying (BC_2). Just as in the proof of Theorem 1, we claim that u and v intersect in (θ_1, θ_2). Suppose on the contrary that $u(r) > v(r)$ in (θ_1, θ_2). It follows from (BC_2) and (2) that

$$0 = u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) < (\theta_1/\theta_2)^m(u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1)) = 0,$$

which gives a contradiction. Hence, there exists $\theta_3 \in (\theta_1, \theta_2)$ such that

$$u(\theta_3) = v(\theta_3) > 0.$$

It follows from $u(\theta_3) = v(\theta_3) > 0$, $u(\theta_2) = v(\theta_2) = 0$ and Theorem 1 that $u(r) = v(r)$ in (θ_3, θ_2), which together with (A_1) implies $u(r) = v(r)$ in (θ_1, θ_2). This completes the proof.

REMARK 4.

(1*) If $\theta_1 = 0$ and $m > 0$, then Theorem 2 follows from Theorem D (Brezis and Oswald [3]).

(2*) Consider the equation (E) in case that $\theta_1 > 0$ *or* $m = 0$. It is clear that (E) can be reduced to an equation of the form

$$u''(r) + f(r, u(r)) = 0\,, \qquad \theta_1 < r < \theta_2\,.$$

For the case $m = 0$, this is obvious. For the case $\theta_1 > 0$, the change of variables

$$u(r) = v(s) \qquad \textit{and } r^{1-m} = s$$

reduces (E) to the equation

$$\frac{d^2 v}{ds^2} + \frac{1}{(1-m)^2} s^{2m/(1-m)} f(s^{1/(1-m)}, v) = 0\,, \qquad \theta_1^* < s < \theta_2^*\,,$$

where $\theta_1^* = \theta_2^{1-m}$ and $\theta_2^* = \theta_1^{1-m}$. Thus, if $\theta_1 > 0$ *or* $m = 0$, Theorem 2 follows from Theorem C (Kwong [10]).

THEOREM 3. (E) *has at most one positive solution satisfying boundary condition* (BC_3).

PROOF. Assume, on the contrary, that u and v are two distinct positive solutions of (E) satisfying (BC_3). We split the proof into the following cases:

Case (1). Suppose that $\alpha_1 = 0$. Thus, u and v satisfy $u'(\theta_1) = v'(\theta_1) = 0$. We claim that u and v intersect in (θ_1, θ_2). Suppose on the contrary that $u(r) > v(r)$ in (θ_1, θ_2). It follows from (2) that

(3) $$u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) < (\theta_1/\theta_2)^m (u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1)) = 0\,.$$

It is clear that $\alpha_2\beta_2 \neq 0$. In fact, if $\alpha_2 = 0$ *or* $\beta_2 = 0$, then u and v satisfy

$$u'(\theta_2) = v'(\theta_2) = 0 \qquad \textit{or} \qquad u(\theta_2) = v(\theta_2) = 0\ ,$$

respectively. Both of these equalities imply

$$u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) = 0\,,$$

which combined with (3) gives a contradition. Since $\alpha_2\beta_2 \neq 0$ and u, v satisfy (BC_3), $u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) = 0$. This and (3) gives a contradiction. Thus, there exists $\theta_3 \in (\theta_1, \theta_2)$ such that $u(\theta_3) = v(\theta_3) > 0$.

Next, we claim that u and v intersect in (θ_1, θ_3). Assume, on the contrary, that $u(r) > v(r)$ in (θ_1, θ_3). If follows from (2) and $u'(\theta_3) \le v'(\theta_3)$ that

$$0 \le u(\theta_3)(v'(\theta_3) - u'(\theta_3)) < (\theta_1/\theta_3)^m(u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1)) = 0\,,$$

which gives a contradiction. Hence, there exists $\theta_4 \in (\theta_1, \theta_3)$ such that $u(\theta_4) = v(\theta_4) > 0$. By Theorem 1, we see that $u(r) = v(r)$ in (θ_4, θ_3), this and (A_1) imply $u(r) = v(r)$ in (θ_1, θ_2).

Case (2). Suppose that $\beta_1 = 0$. Hence, u and v satisfy $u(\theta_1) = v(\theta_1) = 0$, and so $u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1) = 0$. The rest of proof is the same as in Case (1), so we omit the details.

Case (3). Suppose that $\alpha_2 = 0$. Thus, u and v satisfy $u'(\theta_2) = v'(\theta_2) = 0$. Just as in the proof of Case (1), we claim that u and v intersect in (θ_1, θ_2). Assume, on the contrary, that $u(r) > v(r)$ in (θ_1, θ_2). It follows from (2) that

$$(4) \quad 0 = u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) < (\theta_1/\theta_2)^m(u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1)) = 0\,.$$

It is clear that $\alpha_1\beta_1 \neq 0$. In fact, if $\alpha_1 = 0$ *or* $\beta_1 = 0$, then u and v satisfy $u'(\theta_1) = v'(\theta_1) = 0$ *or* $u(\theta_1) = v(\theta_1) = 0$, respectively. Both of these equalities imply $u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1) = 0$, which together with (4) gives a contradiction. Since $\alpha_1\beta_1 \neq 0$ and u, v satisfy (BC_3), we see that $u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1) = 0$. This contradicts (4). Hence, there exists $\theta_3 \in (\theta_1, \theta_2)$ such that $u(\theta_3) = v(\theta_3) > 0$.

Next, we claim that u and v intersect in (θ_3, θ_2). Assume, on the contrary, that $u(r) > v(r)$ in (θ_3, θ_2). It follows from (2) and $u'(\theta_3) \ge v'(\theta_3)$ that

$$0 = u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) < (\theta_3/\theta_2)^m u(\theta_3)(v'(\theta_3) - u'(\theta_3)) \le 0\,,$$

which gives a contradiction. Thus, there exists $\theta_4 \in (\theta_3, \theta_2)$ such that $u(\theta_4) = v(\theta_4) > 0$. Using Theorem 1, we see that $u(r) = v(r)$ in (θ_3, θ_4), which combined with (A_1) implies $u(r) = v(r)$ in (θ_1, θ_2).

Case (4). Suppose that $\beta_2 = 0$. Hence, u and v satisfy $u(\theta_2) = v(\theta_2) = 0$, and so $u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) = 0$. The rest of the proof is the same as in Case (3), so we omit the details.

Case (5). Suppose that $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$. Thus, u and v satisfy

$$u(\theta_2)v'(\theta_2) - u'(\theta_2)v(\theta_2) = u(\theta_1)v'(\theta_1) - u'(\theta_1)v(\theta_1) = 0\,.$$

The rest of the proof is quite similar to Case (1) or Case (3), so we omit the details.

By Cases (1)–(5), the proof is complete.

REMARK 5. If $f(r, u) \equiv f(u)$ satisfies a stronger assumption than (A_1), for example, assume that (A_1^*) f: $[0, \infty) \to (-\infty, \infty)$ is locally Lipschitz continuous in $u \in$ "$[0, \infty)$" (that is, f is well-defined at the point $u = 0$) and consider the equation (E) for the case $\theta_1 = 0$ *and* $m > 0$. Then, we can show that the positive solution u of (E) satisfies $u'(0) = 0$. Since $u \in C^1[0, \theta_2]$, $\lim_{r \to 0} r^m u'(r) = 0$. Integrating the above equation on $[0, r]$, we obtain

$$r^m u'(r) + \int_0^r s^m f(u(s))ds = 0$$

and therefore

$$u'(r) = -r^{-m} \int_0^r s^m f(u(s))ds .$$

It follows that

$$|u'(r)| \leq Mr^{-m} \int_0^r s^m ds = \frac{Mr}{m+1}, \qquad 0 < r < \theta_2 ,$$

where $M \equiv \{\max |f(u(s))|: 0 \leq s \leq \theta_2\}$ (Since f is continuous at $u = 0$, M exists). Thus, $u'(0) = 0$. Therefore, the boundary condition $\alpha_1 u(0) + \beta_1 u'(0) = 0$ is equivalent to $u(0) = 0$ *if* $\alpha_1 \neq 0$. For the case $f(0) = 0$, the boundary value problem (E)-(BC_3) with $\alpha_1 \neq 0$ has no positive solutions.

EXAMPLES:

(1) It follows from Theorem 1 that $u(r) \equiv r(1 - r)$ is the unique positive solution (in $C^1[0, 1]$) of

$$(A) \qquad \begin{cases} u''(r) + 2[r(1-r)]^p u^{-p} = 0 & \textit{in } (0, 1) \\ u(0) = u(1) = 0 , \end{cases}$$

in case $p \in (0, \infty)$.

(2) It follows from Theorem 2 that $u(r) \equiv 1 - r^2$ is the unique positive solution (in $C^1[0, 1]$) of

$$(B) \qquad \begin{cases} u''(r) + (m/r)u'(r) + [2(1+m)/(1-r^2)^p]u^p = 0 & \textit{in } (0, 1) \\ u'(0) = u(1) = 0 , \end{cases}$$

in case $p \in (-\infty, 1)$ *and* $m \geq 0$.

(3) It follows from Theorem 3 that $u(r) \equiv r$ is the unique positive solution (in $C^1[0, 1]$) of

$$(C) \qquad \begin{cases} u''(r) + (m/r)u'(r) - [m/r^{p+1}]u^p = 0 & \textit{in } (0, 1) \\ u(0) = u(1) - u'(1) = 0 , \end{cases}$$

in case $m \geq 0$ *and* $p \in (1, \infty)$.

REMARK 6. We apply the results of this paper and the other known results concerning the uniqueness of positive solutions of Emden-Fowler type equations

$$(V) \qquad u''(r) + \frac{m}{r}u'(r) + f(u(r)) = 0$$

with various boundary conditions:

	Relations of $f(u)$ and m	Boundary conditions	Results
(a)	$\begin{cases} f(u) = u^p, \\ 1 < p < (m+3/m-1), \\ m > 1 \end{cases}$	$\begin{cases} u'(0) = 0, \\ u(\theta_2) = 0, \\ \theta_2 \in (0, \infty) \end{cases}$	*Uniqueness* *Ni* [14, 15]
(b)	$\begin{cases} f(u) = u^p, \\ 1 \le p \le (m+3/m-1), \\ m > 1 \end{cases}$	$\begin{cases} u(\theta_1) = 0, \\ u(\theta_2) = 0, \\ \theta_1, \theta_2 \in (0, \infty) \end{cases}$	*Uniqueness* *Ni* [14, 15]
(c)	$\begin{cases} f(u) = u^p, \\ 1 \le p < \infty \\ m = 1 \end{cases}$	$\begin{cases} u(\theta_1) = 0, \\ u(\theta_2) = 0, \\ \theta_1, \theta_2 \in (0, \infty) \end{cases}$	*Uniqueness* *Ni* [14, 15]
(d)	$\begin{cases} f(u) = u^p + u^{(p+1)/2}, \\ (m+3/m-1) < p < (m+7/m-1), \\ m > 1 \end{cases}$	$\begin{cases} u'(0) = 0, \\ u(\theta_2) = 0, \\ \theta_2 > 0 \text{ is large} \end{cases}$	*Non-uniqueness* *Lin and Ni* [11]
(e)	$\begin{cases} f(u) = u^p, \\ -\infty < p < 1, \\ m \ge 0 \end{cases}$	$\begin{cases} (BC_1), & or \\ (BC_2), & or \\ (BC_3) \end{cases}$	*Uniqueness* *Our main results* (*cf.* [3, 10])
(f)	$\begin{cases} f(u) = u - u^p, \\ p > 1, \\ m \ge 0 \end{cases}$	$\begin{cases} (BC_1), & or \\ (BC_2), & or \\ (BC_3) \end{cases}$	*Uniqueness* *Our main results* (*cf.* [10])

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