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Linearized oscillations for neutral equations II: Even order

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ABSTRACT. We consider a nonlinear neutral delay differential equation of the form $\frac{d^n}{dt^n}[x(t) - p(t)g((t-\tau))] = q(t)h(x(t-\delta))$, $t \geq t_0$ with $p(t)$, $q(t)$ continuous, $\tau > 0$, $\delta \geq 0$ and n even. We obtain sufficient conditions for oscillation of all bounded solutions for the case when $p(t)$ takes values outside $(0,1)$, and thereby establish some criteria as proposed in an earlier open problem.

1. Introduction

In this paper we consider the nonlinear neutral delay differential equation

$$\frac{d^n}{dt^n}[x(t) - p(t)g(x(t-\tau))] = q(t)h(x(t-\delta)), \qquad t \geq t_o \tag{1.1}$$

where n is an even integer,

$$p, q \in C([t_0, \infty), \boldsymbol{R}), \qquad g, h \in C(\boldsymbol{R}, \boldsymbol{R}), \qquad \tau > 0 \qquad \text{and} \qquad \delta \geq 0. \tag{1.2}$$

Recently, the linearized oscillation theory for nonlinear neutral delay differential equations has been extensively developed, for example see [1–3, 5–10]; in particular, [3] deals with the case when n is odd. Roughly speaking, it has been proved that, under appropriate hypotheses, certain nonlinear neutral delay differential equations have the same oscillatory character as an associated linear equation. The following linearized oscillation result for the equation (1.1) was obtained in [7] (see also [5]):

THEOREM A. *Assume that* (1.2) *holds,*

$$\limsup_{t\to\infty} p(t) = P_0 \in (0,1), \qquad \liminf_{t\to\infty} p(t) = p_0 \in (0,1),$$

$$\lim_{t\to\infty} q(t) = q_0 \in (0,\infty), \tag{1.3}$$

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$$0 \le \frac{g(u)}{u} \le 1 \quad for\ u \ne 0, \qquad \lim_{u\to 0} \frac{g(u)}{u} = 1, \tag{1.4}$$

$$uh(u) > 0 \quad for\ u \ne 0 \qquad and \qquad \lim_{u\to 0} \frac{h(u)}{u} = 1. \tag{1.5}$$

Suppose that every bounded solution of the linearized equation

$$\frac{d^n}{dt^n}[y(t) - p_0 y(t-\tau)] = q_0 y(t-\delta) \tag{1.6}$$

oscillates. Then every bounded solution of (1.1) *also oscillates.*

The question naturally arises as to how one may establish the corresponding linearized oscillation results of (1.1) for the case when the coefficient $p(t)$ takes values outside the interval $(0, 1)$. Also see the open problem 10.10.4 in [5]. Our aim in this paper is to consider this problem.

Let $\rho = \max\{\tau, \delta\}$ and let $\phi \in C([t_0 - \rho, t_0], \boldsymbol{R})$. By a solution of (1.1) we mean a function $x \in C([t_1 - \rho, \infty), \boldsymbol{R})$ for some $t_1 \ge t_0$ such that $x(t) = \phi(t)$ on $[t_0 - \rho, t_0]$ and $x(t) - p(t)g(x(t-\tau))$ is n times continuously differentiable on $[t_1, \infty)$ and such that (1.1) is satisfied for $t \ge t_1$.

Throughout this paper, we set

$$\liminf_{t\to\infty} p(t) = p_0 \in \boldsymbol{R}, \qquad \lim_{t\to\infty} q(t) = q_0 \in (0, \infty)$$

$$E_1(t) = \{s \ge t : p(s) \le 0\}, \qquad E_2(t) = \{s \ge t : p(s) > 0\}.$$

2. The case $p_0 \le -1$

In this section we give a linearized oscillation result of (1.1) for the case $p_0 \le -1$. This result can be applied to the case when $p(t)$ itself is oscillatory.

THEOREM 2.1. *Assume that* (1.2) *holds and that there exists* $T > t_0$ *such that*

$$\{s \ge T : p(s+\tau) > 0\} \subseteq E_1(T). \tag{2.1}$$

Suppose that

$$p(t)\ \textit{is bounded and}\ p_0 \le -1 \tag{2.2}$$

$$ug(u) \ge 0 \quad for\ u \ne 0 \qquad and \qquad \lim_{u\to 0} \frac{g(u)}{u} = 1 \tag{2.3}$$

$$uh(u) > 0 \quad for\ u \ne 0 \qquad and \qquad \lim_{u\to 0} \frac{h(u)}{u} = 1 \tag{2.4}$$

and that every bounded solution of the linearized equation

$$\frac{d^n}{dt^n}[y(t) - p_0 y(t-\tau)] = q_0 y(t-\delta) \tag{2.5}$$

oscillates. Then every bounded solution of (1.1) *also oscillates.*

Note the fact that if $p(t) \le 0$ eventually, then (2.1) holds automatically. We immediately have the following corollary.

COROLLARY 2.2. *Assume that* (1.2), (2.2), (2.3) *and* (2.4) *hold. If*

$$p(t) \le 0 \qquad \text{for large } t, \tag{2.6}$$

then the oscillation of all bounded solutions of (2.5) *implies the oscillation of all bounded solutions of* (1.1).

Before we prove Theorem 2.1, let us first examine the following two examples:

EXAMPLE 2.1. Consider the second order nonlinear neutral delay differential equation

$$\left[x(t) - \sin t \frac{x(t-\pi)}{1 + \frac{1}{2}\sin x(t-\pi)}\right]'' = \frac{x(t-2\pi)}{1 + x^2(t-2\pi)}, \qquad t \ge 0. \tag{2.7}$$

Let $T = \pi$. Then (2.1) is satisfied. The linearized equation of (2.7) takes the form

$$[y(t) + y(t-\pi)]'' = y(t-2\pi). \tag{2.8}$$

It is not difficult to prove that every bounded solution of (2.8) is oscillatory. Therefore, by Theorem 2.1 every bounded solution of (2.7) is also oscillatory.

EXAMPLE 2.2. For the delay equation

$$[x(t) - (-1 + \sin t)(x(t-\tau) + \sin x(t-\tau))]'' = 1 - e^{-x(t-\delta)} \tag{2.9}$$

set $p(t) = 2(-1 + \sin t)$, $q(t) = 1$, $g(x) = \frac{1}{2}(x + \sin x)$, $h(x) = 1 - e^{-x}$. Then the conditions (2.2), (2.3) and (2.4) are satisfied. The corresponding linearized equation is

$$[y(t) + 4y(t-\tau)]'' = y(t-\delta). \tag{2.10}$$

Thus by Corollary 2.2, if every bounded solution of (2.10) oscillates, then every bounded solution of (2.9) also oscillates.

PROOF OF THEOREM 2.1. Assume, for the sake of contradiction, that (1.1) has a bounded nonoscillatory solution $x(t)$. We assume that $x(t)$ is eventually positive. The case where $x(t)$ is eventually negative is similar and is omitted. Let $t_1 > T$ and $M > 0$ be such that

$$0 < x(t-\rho) \le M, \qquad p(t) \ge 2p_0 \qquad \text{and} \qquad q(t) > 0 \qquad \text{for } t \ge t_1 \tag{2.11}$$

where $\rho = \max\{\tau, \delta\}$. Set

$$z(t) = x(t) - p(t)g(x(t-\tau)). \tag{2.12}$$

Then $z(t)$ is bounded from above. By (1.1), we have

$$z^{(n)}(t) = q(t)h(x(t-\delta)) > 0 \qquad \text{for } t \ge t_1 \tag{2.13}$$

which means that the consecutive derivatives of $z(t)$ of order up to $n-1$ are strictly monotonic functions eventually. Since $z(t) > 0$ for $t \in E_1(t_1)$, it follows that $z(t)$ is eventually positive. Thus, we have eventually

$$z^{(n-1)}(t) < 0, \quad z^{(n-2)}(t) > 0, \dots, z''(t) > 0, \quad z'(t) < 0, \quad z(t) > 0. \tag{2.14}$$

Now set

$$\alpha = \sup_{0<u\le M} \frac{g(u)}{u} \qquad \text{and} \qquad \beta = \inf_{0<u\le M} \frac{h(u)}{u}.$$

In light of (2.3) and (2.4), we see that

$$1 \le \alpha < \infty \qquad \text{and} \qquad 0 < \beta \le 1.$$

Thus, we obtain by (2.11)

$$g(x(t-\tau)) \le \alpha x(t-\tau) \qquad \text{and} \qquad h(x(t-\delta)) \ge \beta x(t-\delta), t \ge t_1. \tag{2.15}$$

Integrating (2.13) from t_1 to ∞, we find

$$\int_{t_1}^{\infty} q(t)h(x(t-\delta))dt < \infty$$

which, together with (2.15), implies

$$\int_{t_1}^{\infty} q(t)x(t-\delta)dt < \infty.$$

Noting $q(t) \to q_0 \in (0,\infty)$ as $t \to \infty$, we get

$$\int_{t_1}^{\infty} x(t-\delta)dt < \infty.$$

By (2.15), we have

$$\int_{t_1}^{\infty} g(x(t-\tau))dt < \infty.$$

This yields by (2.2) that

$$\int_{t_1}^{\infty} -p(t)g(x(t-\tau))dt < \infty$$

and hence

$$\int_{t_1}^{\infty} z(t)dt < \infty.$$

Since $z(t)$ is eventually positive and eventually decreasing, it follows that $\lim_{t\to\infty} z(t) = 0$. Next we will prove

$$\lim_{t\to\infty} x(t) = 0.$$

First since for $t \in E_1(t_1)$,

$$x(t) = z(t) + p(t)g(x(t-\tau)) \leq z(t)$$

we get

$$\lim_{\substack{t\in E_1(t_1)\\ t\to\infty}} x(t) = 0. \tag{2.16}$$

From (2.1), $t \in E_2(t_1+\tau)$ implies $t-\tau \in E_1(t_1)$. Thus, we have

$$\lim_{\substack{t\in E_2(t_1+\tau)\\ t\to\infty}} g(x(t-\tau)) = 0$$

and hence

$$\lim_{\substack{t\in E_2(t_1+\tau)\\ t\to\infty}} p(t)g(x(t-\tau)) = 0$$

which implies that

$$\lim_{\substack{t\in E_2(t_1+\tau)\\ t\to\infty}} x(t) = \lim_{\substack{t\in E_2(t_1+\tau)\\ t\to\infty}} [z(t) + p(t)g(x(t-\tau))] = 0. \tag{2.17}$$

Since $E_1(t_1) \cup E_2(t_1+\tau) \supseteq [t_1+\tau, \infty)$, it follows from (2.16) and (2.17) that

$$\lim_{t\to\infty} x(t) = 0. \tag{2.18}$$

Set

$$p^*(t) = -p(t)g(x(t-\tau))/x(t-\tau), \qquad q^*(t) = q(t)h(x(t-\delta))/x(t-\delta).$$

Then (1.1) becomes

$$\frac{d^n}{dt^n}[x(t) + p^*(t)x(t-\tau)] = q^*(t)x(t-\delta). \tag{2.19}$$

In view of (2.3), (2.4) and (2.18), we have

$$\limsup_{t\to\infty} p^*(t) = -p_0, \qquad \lim_{t\to\infty} q^*(t) = q_0. \tag{2.20}$$

By the definition of $z(t)$, (2.19) reduces to

$$z^{(n)}(t) + p^*(t-\delta)\frac{q^*(t)}{q^*(t-\tau)}z^{(n)}(t-\tau) = q^*(t)z(t-\delta). \tag{2.21}$$

For any $\varepsilon \in (0, q_0)$, let $\beta = \beta(\varepsilon) \in (0,1)$ be sufficiently near to 1 and such that $\beta q_0 > q_0 - \varepsilon$. Also choose $\alpha = \alpha(\varepsilon, \beta) > 1$ to be near to 1 such that $\beta q_0 > \alpha(q_0 - \varepsilon)$. By (2.20),

$$\limsup_{t\to\infty} p^*(t-\delta)\frac{q^*(t)}{q^*(t-\tau)} = -p_0 \geq 1.$$

Then there exists $t_2 > t_1 + \delta$ such that

$$p^*(t-\delta)\frac{q^*(t)}{q^*(t-\tau)} < -p_0 + \varepsilon, \qquad q^*(t) > \frac{q_0}{\alpha} \qquad \text{for } t \geq t_2.$$

Substituting this into (2.21), we have

$$z^{(n)}(t) + (-p_0 + \varepsilon)z^{(n)}(t-\tau) > \frac{q_0}{\alpha}z(t-\delta), \qquad t \geq t_2. \tag{2.22}$$

Now set

$$Q(t) = (z^{(n)}(t) + (-p_0 + \varepsilon)z^{(n)}(t-\tau))/z(t-\delta). \tag{2.23}$$

Then by (2.22), we have

$$Q(t) > \frac{q_0}{\alpha} \qquad \text{for } t \geq t_2. \tag{2.24}$$

Rewrite (2.23) as

$$z^{(n)}(t) + (-p_0 + \varepsilon)z^{(n)}(t-\tau) = Q(t)z(t-\delta). \tag{2.25}$$

Clearly, $\lim_{t\to\infty} z^{(i)}(t) = 0$, for $i = 0, 1, \ldots, n-1$. Integrating (2.25) from t to infinity $n-1$ times and recalling that n is even, we get

$$z'(t) + (-p_0 + \varepsilon)z'(t-\tau) + \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}Q(s)z(s-\delta)ds = 0. \tag{2.26}$$

In what follows, for the sake of convenience, we set

$$p_1 = -p_0 + \varepsilon, \qquad Q^*(t) = \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}Q(s)z(s-\delta)ds. \tag{2.27}$$

Then (2.26) becomes

$$z'(t) + p_1 z'(t-\tau) + Q^*(t) = 0.$$

Integrating this from t to ∞ again, we have

$$z(t) + p_1 z(t-\tau) = \int_t^\infty Q^*(s)ds$$

or equivalently

$$z(t) = -\frac{1}{p_1} z(t+\tau) + \frac{1}{p_1}\int_{t+\tau}^\infty Q^*(s)ds. \tag{2.28}$$

By iteration, we have

$$z(t) = \sum_{i=1}^{m} (-1)^{i+1} p_1^{-i} \int_{t+i\tau}^\infty Q^*(s)ds + (-1)^m p_1^{-m} z(t+m\tau).$$

Since $p_1 > 1$ and $z(t) \to 0$ as $t \to \infty$, we let $m \to \infty$ to obtain

$$\begin{aligned}
z(t) &= \sum_{i=1}^{\infty} (-1)^{i+1} p_1^{-i} \int_{t+i\tau}^\infty Q^*(s)ds \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} (-1)^{j+1} p_1^{-j} \int_{t+i\tau}^{t+(i+1)\tau} Q^*(s)ds \\
&= \sum_{i=1}^{\infty} \int_{t+i\tau}^{t+(i+1)\tau} \frac{1-(-p_1)^{-i}}{1+p_1} Q^*(s)ds \\
&= \sum_{i=1}^{\infty} \int_{t+i\tau}^{t+(i+1)\tau} \frac{1}{1+p_1}(1-(-p_1)^{-[(s-t)/\tau]}) Q^*(s)ds \\
&= \frac{1}{1+p_1}\int_{t+\tau}^\infty (1-(-p_1)^{-[(s-t)/\tau]}) Q^*(s)ds
\end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. From (2.24),

$$Q^*(t) > \frac{q_0}{(n-2)!\alpha}\int_t^\infty (s-t)^{n-2} z(s-\delta)ds, \qquad t \ge t_2.$$

Thus, we obtain

$$z(t) > \frac{q_0}{(1+p_1)\alpha(n-2)!}\int_{t+\tau}^\infty (1-(-p_1)^{-[(s-t)/\tau]}) \int_s^\infty (u-s)^{n-2} z(u-\delta)du\, ds. \tag{2.29}$$

Since every bounded solution of (2.5) oscillates, it follows by [5] or [6] that the characteristic equation of (2.5)

$$f(\lambda) = -\lambda^n(1 - p_0 e^{-\lambda\tau}) + q_0 e^{-\lambda\delta} = 0$$

has no negative real roots. Consequently

$$\tau < \delta. \tag{2.30}$$

Next we will need the following claim, which can be proved by a slight modification in the proof of Lemma 5 of [7].

Claim: There exists $\varepsilon_0 > 0$ such that for every $\eta \in [0, \varepsilon_0]$ every bounded solution of the equation

$$\frac{d^n}{dt^n}[y(t) + (-p_0 + \eta)y(t-\tau)] = (q_0 - \eta)y(t-\delta) \tag{2.31}$$

also oscillates.

We choose the above $\varepsilon > 0$ to be in $(0, \varepsilon_0]$. Our aim is to prove that the equation

$$\frac{d^n}{dt^n}[y(t) + (-p_0 + \varepsilon)y(t-\tau)] = (q_0 - \varepsilon)y(t-\delta)$$

has a bounded positive solution. To this end, we consider the Banach space X of all bounded and continuous functions defined on $[t_2 + \tau - \delta, \infty)$ with the sup-norm. Let

$$A = \{w \in X : 0 \le w(t) \le 1 \qquad \text{for } t \ge t_2 + \tau - \delta\}.$$

Then A is a bounded, closed and convex subset of X. Define a mapping $S\colon A \to X$ as follows:

$$(Sw)(t) = \begin{cases} \dfrac{q_0 - \varepsilon}{(1+p_1)(n-2)!z(t)} \displaystyle\int_{t+\tau}^{\infty} \left(1 - (-p_1)^{-[(s-t)/\tau]}\right) & \\ \quad \times \displaystyle\int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\,du\,ds, & t \ge t_2 \\ (Sw)(t_2) + e^{h(t_2 - t)} - 1, & t_2 + \tau - \delta \le t \le t_2 \end{cases}$$

where $h = \ln(2 - \beta)/(\delta - \tau) > 0$.

We first show that S maps A into itself. It is easy to see by (2.29) that for any $w \in A$

$$0 \le (Sw)(t) \le 1 \qquad \text{for } t \ge t_2 + \tau - \delta$$

and $(Sw)(t)$ is continuous on $[t_2 + \tau - \delta, t_2]$. We will prove that $(Sw)(t)$ is continuous on $[t_2, \infty)$. In fact, for any $s_1, s_2 \in [t_2, \infty)$,

$$
\begin{aligned}
&|z(s_1)(Sw)(s_1) - z(s_2)(Sw)(s_2)| \\
&\quad = \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!} \left| \int_{s_1+\tau}^{\infty} (1 - (-p_1)^{-[(s-s_1)/\tau]}) \right. \\
&\qquad \times \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \\
&\qquad \left. - \int_{s_2+\tau}^{\infty} (1 - (-p_1)^{-[(s-s_2)/\tau]}) \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right| \\
&\quad = \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!} \left| \int_{s_1+\tau}^{s_2+\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right. \\
&\qquad - \sum_{i=1}^{\infty} (-p_1)^{-i} \int_{s_1+i\tau}^{s_1+(i+1)\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \\
&\qquad \left. + \sum_{i=1}^{\infty} (-p_1)^{-i} \int_{s_2+i\tau}^{s_2+(i+1)\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right| \\
&\quad \le \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!} \left\{ \sum_{i=1}^{\infty} p_1^{-i} \left| \int_{s_1+i\tau}^{s_1+(i+1)\tau} \right. \right. \\
&\qquad \times \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \\
&\qquad \left. - \int_{s_2+i\tau}^{s_2+(i+1)\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right| \\
&\qquad \left. + \left| \int_{s_1+\tau}^{s_2+\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right| \right\} \\
&\quad = \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!} \left\{ \sum_{i=1}^{\infty} p_1^{-i} \left| \int_{s_1+i\tau}^{s_2+i\tau} \right. \right. \\
&\qquad \times \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \\
&\qquad \left. - \int_{s_1+(i+1)\tau}^{s_2+(i+1)\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right| \\
&\qquad \left. + \left| \int_{s_1+\tau}^{s_2+\tau} \int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds \right| \right\}
\end{aligned}
$$

$$\leq \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!}\left\{\sum_{i=1}^{\infty} p_1^{-i} 2|s_1 - s_2| + |s_1 - s_2|\right\}$$

$$\times \int_{t_2}^{\infty} (u-t_2)^{n-2} z(u-\delta) w(u-\delta) du$$

$$\leq \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!}\left(1 + \frac{2}{p_1 - 1}\right)|s_1 - s_2| \int_{t_2}^{\infty} (u-t_2)^{n-2} z(u-\delta) du$$

which shows that $z(t)(Sw)(t)$ is continuous on $[t_2, \infty)$ and so $(Sw)(t)$ is continuous on $[t_2, \infty)$. Hence S maps A into A.

Next we prove that S is a contraction on A. In fact for any $w_1, w_2 \in A$ and $t \geq t_2$, we have

$$|(Sw_1)(t) - (Sw_2)(t)|$$

$$\leq \frac{q_0 - \varepsilon}{(1+p_1)(n-2)! z(t)} \int_{t+\tau}^{\infty} (1 - (-p_1)^{-[(s-t)/\tau]})$$

$$\times \int_{s}^{\infty} (u-s)^{n-2} z(u-\delta) |w_1(u-\delta) - w_2(u-\delta)| du\, ds$$

$$\leq \frac{q_0 - \varepsilon}{(1+p_1)(n-2)! z(t)} \|w_1 - w_2\| \int_{t+\tau}^{\infty} (1 - (-p_1)^{-[(s-t)/\tau]})$$

$$\times \int_{s}^{\infty} (u-s)^{n-2} z(u-\delta) du\, ds$$

$$\leq \|w_1 - w_2\| \frac{\alpha(q_0 - \varepsilon)}{q_0} \qquad \text{(by (2.29))}$$

$$\leq \beta \|w_1 - w_2\|$$

and for $t_2 + \tau - \delta \leq t \leq t_2$,

$$|(Sw_1)(t) - (Sw_2)(t)| = |(Sw_1)(t_2) - (Sw_2)(t_2)| \leq \beta \|w_1 - w_2\|.$$

Hence

$$\|Sw_1 - Sw_2\| = \sup_{t \geq t_2 + \tau - \delta} |(Sw_1)(t) - (Sw - 2)(t)|$$

$$\leq \beta \|w_1 - w_2\|.$$

Since $0 < \beta < 1$, this shows that S is a contraction on A. By the Banach contraction principle, S has a fixed point $w \in A$. That is

$$w(t) = \begin{cases} \dfrac{q_0 - \varepsilon}{(1+p_1)(n-2)!z(t)} \displaystyle\int_{t+\tau}^{\infty} \left(1 - (-p_1)^{-[(s-t)/\tau]}\right) & \\ \qquad \times \displaystyle\int_s^{\infty} (u-s)^{n-2} z(u-\delta) w(u-\delta)\, du\, ds, & t \geq t_2 \\ w(t_2) + e^{h(t_2 - t)} - 1, & t_2 + \tau - \delta \leq t \leq t_2 . \end{cases} \tag{2.32}$$

Clearly, $w(t) > 0$ for $t_2 + \tau - \delta \leq t < t_2$. We now are going to prove that $w(t) > 0$ for all $t \geq t_2$. In fact, if there exists $t^* \in [t_2, t_2 - \tau + \delta)$ such that $w(t^*) = 0$, then by (2.32), $w(t) \equiv 0$ for $t \geq t^* + \tau - \delta$. Since $t^* + \tau - \delta \in [t_2 + \tau - \delta, t_2)$, it follows that $w(t^* + \tau - \delta) > 0$, which is a contradiction. Therefore, $w(t) > 0$ for $t_2 \leq t < t_2 - \tau + \delta$. In general, by induction we have $w(t) > 0$ for $t \in [t_2 - i(\tau - \delta), t_2 - (i+1)(\tau - \delta))$, $i = 0, 1, 2, \ldots$ which shows that $w(t) > 0$ for all $t \geq t_2 + \tau - \delta$. Set

$$y(t) = w(t) z(t).$$

Then $y(t)$ is a positive and continuous function on $[t_2 + \tau - \delta, \infty)$ and satisfies

$$y(t) = \frac{q_0 - \varepsilon}{(1+p_1)(n-2)!} \int_{t+\tau}^{\infty} \left(1 - (-p_1)^{-[(s-t)/\tau]}\right) \int_s^{\infty} (u-s)^{n-2} y(u-\delta)\, du\, ds, \quad t \geq t_2.$$

From this, we get

$$y(t) + p_1 y(t-\tau) = \frac{q_0 - \varepsilon}{(n-2)!} \int_t^{\infty} \int_s^{\infty} (u-s)^{n-2} y(u-\delta)\, du\, ds, \qquad t \geq t_2 + \tau.$$

Differentiating both sides n times, we have

$$\frac{d^n}{dt^n}[y(t) + p_1 y(t-\tau)] = (q_o - \varepsilon) y(t-\delta)$$

i.e.,

$$\frac{d^n}{dt^n}[y(t) + (-p_0 + \varepsilon) y(t-\tau)] = (q_o - \varepsilon) y(t-\delta), \qquad t \geq t_2 + \tau$$

which contradicts the claim and hence the proof of Theorem 2.1 is complete.

3. The case $p_0 > 1$

In this section we discuss the case $p_0 > 1$. The main result is

THEOREM 3.1. *Assume that* (1.2) *holds and*

$$p_0 \geq 1 \quad \textit{and} \quad p(t) \textit{ is bounded.} \tag{3.1}$$

$$\frac{g(u)}{u} \geq 1 \qquad \textit{for } u \neq 0 \tag{3.2}$$

$$uh(u) > 0 \qquad \textit{for } u \neq 0. \tag{3.3}$$

Then every bounded solution of (1.1) *oscillates.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1.1). We assume that $x(t)$ is eventually positive. Choose $t_1 > t_0$ and $M > 0$ to be such that

$$0 < x(t-\rho) \leq M, \qquad q(t) > 0 \qquad \text{for } t \geq t_1. \tag{3.4}$$

Set

$$y(t) = x(t) - p(t)g(x(t-\tau)). \tag{3.5}$$

Then $y(t)$ is bounded from above. Since

$$y^{(n)}(t) = q(t)h(x(t-\delta)) > 0, \qquad t \geq t_1$$

the consecutive derivatives of $y(t)$ of order up to $n-1$ are strictly monotonic.

There are two cases to consider:

Case 1: $y(t) > 0$ eventually, i.e.,

$$x(t) > p(t)g(x(t-\tau))$$

which, together with (3.1) and (3.2), implies that there exist $t_2 > t_1$ and $N > 0$ such that $x(t) \geq N$ for $t \geq t_2$. Set

$$\alpha = \inf_{N \leq u \leq M} h(u).$$

Clearly, $\alpha > 0$. Thus, we get

$$h(x(t-\delta)) \geq \alpha \qquad \text{for } t \geq t_2 + \delta$$

and so

$$y^{(n)}(t) \geq \alpha q(t) \qquad \text{for } t \geq t_2 + \delta$$

which implies

$$y(t) \to \infty \qquad \text{as } t \to \infty.$$

This is a contradiction.

Case 2: $y(t) < 0$ eventually.

For this case, $y(t)$ must be eventually decreasing. Therefore, there exists

$t_3 > t_1$ such that

$$y(t) \leq y(t_3) < 0 \qquad \text{for } t \geq t_3.$$

That is

$$x(t) - p(t)g(x(t-\tau)) \leq y(t_3) \qquad \text{for } t \geq t_3.$$

Since $p(t)$ is bounded, it follows that there exists $m > 0$ such that

$$g(x(t-\tau)) \geq m, \qquad t \geq t_3$$

and so there exists $m_0 > 0$ such that $x(t-\tau) \geq m_0$, for $t \geq t_3$. By using a similar argument as in Case 1, we can easily derive a new contradiction. The proof is complete.

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