HIROSHIMA MATH. J.
26 (1996), 557–572

Linearized oscillations for neutral equations I: Odd order[1]

L. H. ERBE and J. S. YU

(Received January 31, 1994)
(Revised May 29, 1995)

ABSTRACT. In this paper we prove that, under appropriate hypotheses, the nonlinear neutral delay differential equation $\frac{d^n}{dt^n}[x(t) - p(t)g(x(t-\rho))] + q(t)h(x(t-\delta)) = 0$ has the same oscillatory character as its associate linear equation for the case when n is odd. Our result can also be applied to the case when $p(t)$ itself is allowed to change its sign.

1. Introduction

Neutral delay differential equations are differential equations in which the highest order derivative of the unknown function appears both with and without delays. The theory of neutral equations is of both theoretical and practical interest. Neutral delay differential equations appear in networks containing lossless transmission lines (as in high speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar and also as the Euler equations in some variational problems. See Driver [3], Hale [4] and the references cited therein.

Consider the n^{th}-order neutral delay differential equation

$$\frac{d^n}{dt^n}[x(t) - p(t)g(x(t-\tau))] + q(t)h(x(t-\delta)) = 0 \tag{1.1}$$

where n is an odd positive integer,

$$p, q \in C([t_0, \infty), \boldsymbol{R}), g, h \in C(\boldsymbol{R}, \boldsymbol{R}), \qquad \tau > 0 \quad \text{and} \quad \delta \geq 0. \tag{1.2}$$

The linearized oscillation theory for (1.1) was first studied in Ladas and Qian [8], see also Gyori and Ladas [5]. They proved

---

1991 *Mathematics Subject Classification.* Primary 34K40, Secondary 34C10.
Key words and phrases. Oscillation, neutral equation, linearized equation.
[1]Research partially supported by NSERC of Canada and NNSF of China. Research was done when the second author was visiting the University of Alberta.

THEOREM 1.1 [5, 8]. *Assume that* (1.2) *holds,*

$$\limsup_{t\to\infty} p(t) = P_0 \in (0,1), \qquad \liminf_{t\to\infty} p(t) = p_0 \in (0,1), \tag{1.3}$$

$$\lim_{t\to\infty} q(t) = q_0 \in (0,\infty)$$

$$0 \le \frac{g(u)}{u} \le 1 \quad \textit{for } u \ne 0, \qquad \lim_{u\to 0} \frac{g(u)}{u} = 1, \tag{1.4}$$

$$uh(u) > 0 \quad \textit{for } u \ne 0, \qquad |h(u)| \ge h_0 > 0 \quad \textit{for } |u| \textit{ sufficiently large} \tag{1.5}$$

and

$$\lim_{u\to 0} \frac{h(u)}{u} = 1. \tag{1.6}$$

Suppose that every solution of the linearized equation

$$\frac{d^n}{dt^n}[y(t) - p_0 y(t-\tau)] + q_0 y(t-\delta) = 0 \tag{1.7}$$

oscillates. Then every solution of (1.1) *also oscillates.*

The question naturally arises as to how one may establish the corresponding linearized oscillation results of (1.1) for the case when the coefficient $p(t)$ takes values outside the interval (0,1). Also see the open problem 10.10.4 in [5]. In the recent work [10], Yu, Shen and Qian investigated the case when $p(t) \ge 1$. But the case $p(t) \le 0$ has not yet been handled. This case appears to be more difficult. Our aim in this paper is to deal with the more general case when $p(t)$ may also be allowed to oscillate. In Section 2 we establish conditions for the oscillation of all solutions in (1.1) in terms of the oscillation of all solutions of its associated linearized equation. In Section 3 we show that, under appropriate hypotheses, the converse theorem is also true. To the best of our knowledge, this is the first paper dealing with the case when $p(t)$ itself is allowed to change its sign eventually.

For the special case when $n = 1$, the linearized oscillation problem of (1.1) has been studied in [1, 2, 5, 6, 9]. Similar results for even-order neutral delay differential equations have been obtained in [7]. For the general theory of neutral differential equations, we refer to [4].

Let $\rho = \max\{\tau, \delta\}$. By a solution of (1.1) we mean a function $x \in C([t_1 - \rho, \infty), \boldsymbol{R})$ for some $t_1 \ge t_0$ such that $x(t) - p(t)g(x(t-\tau))$ is n times continuously differentiable on $[t_1, \infty)$ and such that (1.1) is satisfied for $t \ge t_1$.

2. Main results

Throughout this section, we define for any $t \ge t_0$

$$E_1(t) = \{s \ge t : p(s) \le 0\} \qquad \text{and} \qquad E_2(t) = \{s \ge t : p(s) > 0\}.$$

Clearly, $E_1(t) \cup E_2(t) = [t, \infty)$. For any Lebesgue measurable subset E of the real number set $\boldsymbol{R}$, let $\mu(E)$ denote the Lebesgue measure of E.

The main result in this section is the following

THEOREM 2.1. *Assume that* (1.2) *holds and that there exists* $T > t_0$ *such that*

$$\{s \geq T : p(s+\tau) > 0\} \subseteq E_1(T). \tag{2.1}$$

Suppose further that

$$p(t) \text{ is bounded}, \quad \text{and} \quad \liminf_{t\to\infty} p(t) = -p_0 \in (-\infty, -1], \tag{2.2}$$

$$\lim_{t\to\infty} q(t) = q_0 \in (0, \infty), \tag{2.3}$$

$$ug(u) \geq 0 \quad \text{for } u \neq 0 \quad \text{and} \quad \lim_{u\to 0} \frac{g(u)}{u} = 1 \tag{2.4}$$

$$|h(u)| \geq h_0 > 0 \quad \text{for } |u| \text{ sufficiently large} \tag{2.5}$$

and

$$uh(u) > 0 \quad \text{for } u \neq 0 \quad \text{and} \quad \lim_{u\to 0} \frac{h(u)}{u} = 1 \tag{2.6}$$

and that every solution of the linearized equation

$$\frac{d^n}{dt^n}[y(t) + p_0 y(t-\tau)] + q_0 y(t-\delta) = 0 \tag{2.7}$$

oscillates. Then every solution of (1.1) *also oscillates.*

Since $p(t) \leq 0$ eventually implies that (2.1) holds, we immediately have the following corollary

COROLLARY 2.2. *Assume that* (1.2), (2.2)–(2.6) *hold. If*

$$p(t) \leq 0 \quad \text{for large } t, \tag{2.8}$$

then the oscillation of all solutions of (2.7) *implies the oscillation of all solutions of* (1.1).

PROOF OF THEOREM 2.1. Assume, for the sake of contradiction, that (1.1) has a nonoscillatory solution $x(t)$. We will assume that $x(t)$ is eventually positive. The case where $x(t)$ is eventually negative is similar and is omitted. Let $t_1 \geq t_0$ be such that $q(t) > 0$, $x(t-\rho) > 0$ for $t \geq t_1$, where $\rho = \max\{\tau, \delta\}$. Set

$$z(t) = x(t) - p(t)g(x(t-\tau)). \tag{2.9}$$

Then by (1.1),

$$z^{(n)}(t) = -q(t)h(x(t-\delta)) < 0 \qquad \text{for } t \geq t_1 \tag{2.10}$$

which means that the consecutive derivatives of $z(t)$ of order up to $n-1$ are strictly monotonic functions eventually. Since $z(t) > 0$ for $t \geq t_1$ with $t \in E_1(t_1)$, it follows that $z(t) > 0$ for all $t \geq t_1$. Furthermore by (2.10), $z^{(n-1)}(t) > 0$ for $t \geq t_1$. Set

$$\alpha = \lim_{t\to\infty} z^{(n-1)}(t).$$

Clearly, $0 \leq \alpha < \infty$. Thus, by integrating (2.10) from t_1 to ∞, we find

$$\int_{t_1}^{\infty} q(s)h(x(s-\delta))ds < \infty$$

which, together with (2.3), implies

$$\int_{t_1}^{\infty} h(x(s-\delta))ds < \infty. \tag{2.11}$$

We will prove that

$$\lim_{t\to\infty} x(t) = 0. \tag{2.12}$$

To this end, we first prove that $x(t)$ is bounded. Otherwise, $x(t)$ is unbounded. Set

$$\beta = \lim_{t\to\infty} z(t).$$

Then $0 \leq \beta \leq \infty$. We will show $\beta = \infty$. Suppose $\beta < \infty$. Then $z(t)$ must be decreasing on $[t_1, \infty)$. Choose an increasing real numbers sequence $\{s_m\}$ such that $s_m \geq T + \tau$, $s_m \to \infty$, $x(s_m) > z(T)$ and $x(s_m) \to \infty$ as $m \to \infty$. Hence

$$p(s_m)g(x(s_m - \tau)) = x(s_m) - z(s_m) > 0 \qquad \text{and goes to } \infty \text{ as } m \to \infty$$

which implies $s_m \in E_2(T)$ and $x(s_m - \tau) \to \infty$ as $m \to \infty$. By (2.1), $s_m - \tau \in E_1(T)$, i.e., $p(s_m - \tau) \leq 0$. Hence,

$$x(s_m - \tau) = z(s_m - \tau) + p(s_m - \tau)g(x(s_m - 2\tau)) < z(s_m - \tau) < z(T)$$

which yields a contradiction. And so $\beta = \infty$. From (2.5) and (2.6), let $N > 0$ be so big that $h(u) \geq h_0$ for $u \geq N$. Set

$$F_1 = \{t \geq t_1 : x(t) \geq N\} \qquad \text{and} \qquad F_2 = \{t \geq t_1 : x(t) \leq N\}.$$

Thus by (2.5) and (2.11), we have

$$\mu(F_1) < \infty, \qquad \mu(F_2) = \infty.$$

For $t \in F_2$, we have

$$-p(t)g(x(t-\tau)) = z(t) - x(t) \geq z(t) - N \to \infty \qquad \text{as } t \to \infty$$

which implies that

$$\lim_{t \in F_2, t \to \infty} g(x(t-\tau)) = \infty$$

and hence

$$\lim_{t \in F_2, t \to \infty} x(t-\tau) = \infty.$$

Hence there exists $t_2 > t_1 + \tau$ such that

$$x(t-\tau) > N \qquad \text{for } t \geq t_2,\ t \in F_2$$

which means that $t \in [t_2, \infty) \cap F_2$ and so $t - \tau \in F_1$. Since $\mu([t_2, \infty) \cap F_2) = \infty$, it follows that $\mu(F_1) = \infty$, which is impossible and so $x(t)$ must be bounded. Let $M > 0$ be a positive constant such that

$$x(t-\rho) \leq M \qquad \text{for } t \geq t_1.$$

Now set

$$\beta_1 = \sup_{0<u\leq M} \frac{g(u)}{u} \qquad \text{and} \qquad \beta_2 = \inf_{0<u\leq M} \frac{h(u)}{u}.$$

Then by (2.4)–(2.6), $1 \leq \beta_1 < \infty$, $0 < \beta_2 \leq 1$. Thus, we get

$$g(x(t-\tau)) \leq \beta_1 x(t-\tau) \qquad \text{and} \qquad h(x(t-\delta)) \geq \beta_2 x(t-\delta), \quad t \geq t_1.$$

By (2.11), we have

$$\int_{t_1}^{\infty} x(s-\delta)ds < \infty$$

and hence

$$\int_{t_1}^{\infty} g(x(s-\tau))ds < \infty.$$

This yields by (2.2) that

$$\int_{t_1}^{\infty} -p(s)g(x(s-\tau))ds < \infty$$

and therefore

$$\int_{t_1}^{\infty} z(s)ds < \infty.$$

Since $z(t)$ is positive and eventually monotonic, it follows that $\lim_{t\to\infty} z(t) = 0$. Since for $t \in E_1(t_1)$,

$$x(t) = z(t) + p(t)g(x(t-\tau)) \leq z(t),$$

we get

$$\lim_{t \in E_1(t_1),\, t\to\infty} x(t) = 0. \tag{2.13}$$

Also by (2.1), we see that $t \in E_2(t_1+\tau)$ implies $t-\tau \in E_1(t_1)$. Thus, we have

$$\lim_{t \in E_2(t_1+\tau),\, t\to\infty} g(x(t-\tau)) = 0$$

and hence

$$\lim_{t \in E_2(t_1+\tau),\, t\to\infty} p(t)g(x(t-\tau)) = 0$$

which implies that

$$\lim_{t \in E_2(t_1+\tau),\, t\to\infty} x(t) = \lim_{t \in E_2(t_1+\tau),\, t\to\infty} [z(t) + p(t)g(x(t-\tau))] = 0. \tag{2.14}$$

Since $E_1(t_1) \cup E_2(t_1+\tau) \supseteq [t_1+\tau, \infty)$, it follows from (2.13) and (2.14) that

$$\lim_{t\to\infty} x(t) = 0$$

which completes the proof of (2.12). Set

$$p^*(t) = -p(t)g(x(t-\tau))/x(t-\tau), \qquad q^*(t) = q(t)h(x(t-\delta))/x(t-\delta).$$

Then (1.1) becomes

$$\frac{d^n}{dt^n}[x(t) + p^*(t)x(t-\tau)] + q^*(t)x(t-\delta) = 0. \tag{2.15}$$

In view of (2.2)–(2.4), (2.6) and (2.12) we obtain

$$\limsup_{t\to\infty} p^*(t) = p_0, \qquad \lim_{t\to\infty} q^*(t) = q_0. \tag{2.16}$$

By the definition of $z(t)$, (2.15) reduces to

$$z^{(n)}(t) + p^*(t-\delta)\frac{q^*(t)}{q^*(t-\tau)}z^{(n)}(t-\tau) + q^*(t)z(t-\delta) = 0. \tag{2.17}$$

For any $\varepsilon \in (0, q_0)$, let $\eta = \eta(\varepsilon) \in (0,1)$ be sufficiently near to 1 such that $\eta q_0 > q_0 - \varepsilon$. Also choose $\gamma = \gamma(\varepsilon, \eta) > 1$ to be near to 1 such that $\eta q_0 > \gamma(q_0 - \varepsilon)$. By (2.16),

$$\limsup_{t\to\infty} p^*(t-\delta)\frac{q^*(t)}{q^*(t-\tau)} = p_0 \geq 1.$$

Then there exists $t_2 > t_1 + \delta$ such that

$$p^*(t-\delta)\frac{q^*(t)}{q^*(t-\tau)} < p_0 + \varepsilon, \qquad q^*(t) > \frac{q_0}{\gamma} \quad \text{for } t \geq t_2.$$

Substituting this into (2.17), we have

$$z^{(n)}(t) + (p_0+\varepsilon)z^{(n)}(t-\tau) + \frac{q_0}{\gamma}z(t-\delta) < 0 \qquad t \geq t_2. \tag{2.18}$$

Now set

$$Q(t) = -[z^{(n)}(t) + (p_0+\varepsilon)z^{(n)}(t-\tau)]/z(t-\delta). \tag{2.19}$$

Then by (2.18), we have

$$Q(t) > \frac{q_0}{\gamma} \qquad \text{for } t \geq t_2. \tag{2.20}$$

Rewrite (2.19) as

$$z^{(n)}(t) + (p_0+\varepsilon)z^{(n)}(t-\tau) + Q(t)z(t-\delta) = 0. \tag{2.21}$$

It is easy to see that $\lim_{t\to\infty} z^{(i)}(t) = 0$, for $i = 0, 1, \ldots, n-1$. In what follows, for the sake of convenience, we define

$$Q^*(t) = \begin{cases} \dfrac{1}{(n-2)!}\displaystyle\int_t^\infty (s-t)^{n-2}Q(s)z(s-\delta)ds, & \text{when } n > 1 \\ Q(t)z(t-\delta), & \text{when } n = 1. \end{cases}$$

Integrating (2.21) from t to infinity $n-1$ times and recalling that n is odd, we get

$$z'(t) + (p_0+\varepsilon)z'(t-\tau) + Q^*(t) = 0$$

which yields

$$z(t) + (p_0+\varepsilon)z(t-\tau) = \int_t^\infty Q^*(s)ds$$

or equivalently

$$z(t) = -\frac{1}{p_0+\varepsilon}z(t+\tau) + \frac{1}{p_0+\varepsilon}\int_{t+\tau}^\infty Q^*(s)ds. \tag{2.22}$$

By iteration, we have

$$z(t) = \sum_{i=1}^{m}(-1)^{i+1}(p_0+\varepsilon)^{-i}\int_{t+i\tau}^{\infty} Q^*(s)ds + (-1)^m(p_0+\varepsilon)^{-m}z(t+m\tau).$$

Since $p_0+\varepsilon > 1$ and $z(t) \to 0$ as $t \to \infty$, we let $m \to \infty$ to have

$$\begin{aligned}
z(t) &= \sum_{i=1}^{\infty}(-1)^{i+1}(p_0+\varepsilon)^{-i}\int_{t+i\tau}^{\infty} Q^*(s)ds \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}(-1)^{j+1}(p_0+\varepsilon)^{-j}\int_{t+i\tau}^{t+(i+1)\tau} Q^*(s)ds \\
&= \sum_{i=1}^{\infty}\int_{t+i\tau}^{t+(i+1)\tau}\frac{1-(-p_0-\varepsilon)^{-i}}{1+p_0+\varepsilon}Q^*(s)ds \\
&= \sum_{i=1}^{\infty}\int_{t+i\tau}^{t+(i+1)\tau}\frac{1}{1+p_0+\varepsilon}(1-(-p_0-\varepsilon)^{-[(s-t)/\tau]})Q^*(s)ds \\
&= \frac{1}{1+p_0+\varepsilon}\int_{t+\tau}^{\infty}(1-(-p_0-\varepsilon)^{-[(s-t)/\tau]})Q^*(s)ds
\end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. From (2.20), we have for $t \geq t_2$,

$$Q^*(t) > \begin{cases} \dfrac{q_0}{(n-2)!\gamma}\displaystyle\int_t^{\infty}(s-t)^{n-2}z(s-\delta)ds, & \text{when } n > 1 \\ \dfrac{q_0}{\gamma}z(t-\delta), & \text{when } n = 1. \end{cases}$$

Thus, we obtain

$$z(t) > \begin{cases} \dfrac{q_0}{(1+p_0+\varepsilon)\gamma(n-2)!}\displaystyle\int_{t+\tau}^{\infty}(1-(-p_0-\varepsilon)^{-[(s-t)/\tau]}) \\ \qquad \times \displaystyle\int_s^{\infty}(u-s)^{n-2}z(u-\delta)du\,ds, & \text{if } n > 1 \\ \dfrac{q_0}{(1+p_0+\varepsilon)\gamma}\displaystyle\int_{t+\tau}^{\infty}(1-(-p_0-\varepsilon)^{-[(s-t)/\tau]})z(s-\delta)ds, & \text{if } n = 1. \end{cases} \tag{2.23}$$

Since every solution of (2.7) oscillates, it follows by [5] that the characteristic equation of (2.7)

$$f(\lambda) = \lambda^n(1+p_0e^{-\lambda\tau}) + q_0e^{-\lambda\delta} = 0$$

has no real roots. Consequently

$$\tau < \delta. \tag{2.24}$$

Next we will need the following claim, which can be proved by a slight modification in the proof of Lemma 2 of [6].

Claim: There is an $\varepsilon_0 \in (0, q_0)$ such that for every $\bar{\eta} \in [0, \varepsilon_0]$ every solution of the equation

$$\frac{d^n}{dt^n}[y(t) + (p_0 + \bar{\eta})y(t-\tau)] + (q_0 - \bar{\eta})y(t-\delta) = 0 \tag{2.25}$$

also oscillates.

We choose the above $\varepsilon > 0$ to be in $(0, \varepsilon_0]$. Our aim is to prove the equation

$$\frac{d^n}{dt^n}[y(t) + (p_0 + \varepsilon)y(t-\tau)] + (q_0 - \varepsilon)y(t-\delta) = 0 \tag{2.26}$$

has a positive solution. The method used here is the Banach contraction principle. We consider the Banach space X of all bounded functions defined on $[t_2 + \tau - \delta, \infty)$ with the sup-norm. Let

$$A = \{\omega \in X : 0 \le \omega(t) \le 1 \text{ for } t \ge t_2 + \tau - \delta\}.$$

Then A is a bounded, closed and convex subset of X. Define a mapping $S\colon A \to X$ as follows for $t \ge t_2$

$$(S\omega)(t) = \begin{cases} \dfrac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)(n-2)!z(t)} \displaystyle\int_{t+\tau}^{\infty} \left(1 - (p_0 - \varepsilon)^{-[(s-t)/\tau]}\right) \\ \quad \times \displaystyle\int_s^{\infty} (u-s)^{n-2} z(u-\delta)\omega(u-\delta)\,du\,ds, & \text{if } n > 1 \\ \dfrac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)z(t)} \displaystyle\int_{t+\tau}^{\infty} \left(1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}\right) z(s-\delta)\omega(s-\delta)\,ds, & \text{if } n = 1 \end{cases} \tag{2.27}$$

and for $t_2 + \tau - \delta \le t \le t_2$

$$(S\omega)(t) = (S\omega)(t_2) + e^{h(t_2 - t)} - 1 \tag{2.28}$$

where $h = \ln(2 - \eta)/(\delta - \tau) > 0$. It is very easy to see by (2.22) that S maps A into itself. Next we prove that S is a contraction on A. In fact, for any $\omega_1, \omega_2 \in A$ and $t \ge t_2$, we have if $n > 1$

$$\begin{aligned} &|(S\omega_1)(t) - (S\omega_2)(t)| \\ &\quad \le \frac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)(n-2)!z(t)} \int_{t+\tau}^{\infty} \left(1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}\right) \\ &\quad\quad \times \int_s^{\infty} (u-s)^{n-2} z(u-\delta)|\omega_1(u-\delta) - \omega_2(u-\delta)|\,du\,ds \end{aligned}$$

$$\leq \frac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)(n-2)!z(t)} \|\omega_1 - \omega_2\| \int_{t+\tau}^{\infty} (1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]})$$

$$\times \int_s^{\infty} (u-s)^{n-2} z(u - \delta) du\, ds$$

$$\leq \|\omega_1 - \omega_2\| \frac{\gamma(q_0 - \varepsilon)}{q_0} \qquad \text{(by (2.23))}$$

$$\leq \eta \|\omega_1 - \omega_2\|$$

and if $n = 1$

$$|(S\omega_1(t) - (S\omega_2)(t)|$$

$$\leq \frac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)z(t)} \int_{t+\tau}^{\infty} (1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]})$$

$$\times z(s - \delta)|\omega_1(s - \delta) - \omega_2(s - \delta)| ds$$

$$\leq \frac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)z(t)} \|\omega_1 - \omega_2\| \int_{t+\tau}^{\infty} (1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}) z(s - \delta) ds$$

$$\leq \|\omega_1 - \omega_2\| \frac{\gamma(q_0 - \varepsilon)}{q_0} \leq \eta \|\omega_1 - \omega_2\|.$$

And for $t_2 + \tau - \delta \leq t \leq t_2$,

$$|(S\omega_1)(t) - (S\omega_2)(t)| = |(S\omega_1)(t_2) - (S\omega_2)(t_2)| \leq \eta \|\omega_1 - \omega_2\|.$$

Hence

$$\|S\omega_1 - S\omega_2\| = \sup_{t \geq t_2 + \tau - \delta} |(S\omega_1)(t) - (S\omega_2)(t)| \leq \eta \|\omega_1 - \omega_2\|.$$

Since $0 < \eta < 1$, this shows that S is a contraction on A. By the Banach Contraction Principle, S has a fixed point $\omega \in A$. That is for $t \geq t_2$

$$\omega(t) = \begin{cases} \dfrac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)(n-2)!z(t)} \displaystyle\int_{t+\tau}^{\infty} (1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}) & \\ \qquad \times \displaystyle\int_s^{\infty} (u-s)^{n-2} z(u - \delta)\omega(u - \delta) du\, ds, & \text{if } n > 1 \\ \dfrac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)z(t)} \displaystyle\int_{t+\tau}^{\infty} (1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}) z(s - \delta)\omega(s - \delta) ds, & \text{if } n = 1 \end{cases} \tag{2.29}$$

and for $t_2 + \tau - \delta \leq t \leq t_2$

$$\omega(t) = \omega(t_2) + e^{h(t_2 - t)} - 1. \tag{2.30}$$

Set

$$y(t) = \omega(t)z(t).$$

Then $y(t)$ satisfies for $t \geq t_2$

$$y(t) = \begin{cases} \dfrac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)(n-2)!} \displaystyle\int_{t+\tau}^{\infty} \left(1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}\right) \\ \qquad \times \displaystyle\int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds, & \text{if } n > 1 \\ \dfrac{q_0 - \varepsilon}{(1 + p_0 + \varepsilon)} \displaystyle\int_{t+\tau}^{\infty} (1 - (-p_0 - \varepsilon)^{-[(s-t)/\tau]}) y(s-\delta)\,ds, & \text{if } n = 1. \end{cases} \tag{2.31}$$

We are now going to prove that $y(t)$ is continuous on $[t_2 + \tau - \delta, \infty)$. Clearly, $y(t)$ is continuous on $[t_2 + \tau - \delta, t_2]$. Next we show that $y(t)$ is continuous on $[t_2, \infty)$. We suppose $n > 1$. The case $n = 1$ is exactly similar and the proof is omitted. For any $s_1, s_2 \in [t_2, \infty)$,

$$\begin{aligned}
&\frac{(1 + p_0 + \varepsilon)(n-2)!}{q_0 - \varepsilon} |y(s_1) - y(s_2)| \\
&= \left| \int_{s_1+\tau}^{\infty} \left(1 - (-p_0 - \varepsilon)^{-[(s-s_1)/\tau]}\right) \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right. \\
&\quad \left. - \int_{s_2+\tau}^{\infty} \left(1 - (-p_0 - \varepsilon)^{-[(s-s_2)/\tau]}\right) \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right| \\
&= \left| \int_{s_1+\tau}^{s_2+\tau} \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds - \sum_{i=1}^{\infty} (-p_0 - \varepsilon)^{-i} \int_{s_1+i\tau}^{s_1+(i+1)\tau} \int_{s}^{\infty} (u-s)^{n-2} \right. \\
&\quad \left. \times y(u-\delta)\,du\,ds + \sum_{i=1}^{\infty} (-p_0 - \varepsilon)^{-i} \int_{s_2+i\tau}^{s_2+(i+1)\tau} \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right| \\
&\leq \left| \int_{s_1+\tau}^{s_2+\tau} \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right| + \sum_{i=1}^{\infty} (p_0 + \varepsilon)^{-i} \left| \int_{s_1+i\tau}^{s_1+(i+1)\tau} \int_{s}^{\infty} (u-s)^{n-2} \right. \\
&\quad \left. \times y(u-\delta)\,du\,ds - \int_{s_2+i\tau}^{s_2+(i+1)\tau} \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right| \\
&= \left| \int_{s_1+\tau}^{s_2+\tau} \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right| + \sum_{i=1}^{\infty} (p_0 + \varepsilon)^{-i} \left| \int_{s_1+i\tau}^{s_2+i\tau} \int_{s}^{\infty} (u-s)^{n-2} \right. \\
&\quad \left. \times y(u-\delta)\,du\,ds - \int_{s_1+(i+1)\tau}^{s_2+(i+1)\tau} \int_{s}^{\infty} (u-s)^{n-2} y(u-\delta)\,du\,ds \right|
\end{aligned}$$

$$\le |s_2 - s_1| \int_{t_2}^{\infty} (u - t_2)^{n-2} z(u - \delta) du \left(1 + 2 \sum_{i=1}^{\infty} (p_0 + \varepsilon)^{-i} \right)$$
$$= \left(1 + \frac{2}{p_0 + \varepsilon - 1} \right) |s_1 - s_2| \int_{t_2}^{\infty} (u - t_2)^{n-2} z(u - \delta) du$$

which shows that $y(t)$ is continuous on $[t_2, \infty)$. Finally, we show that $y(t) > 0$ for $t \ge t_2 + \tau - \delta$. Obviously, $y(t) > 0$ for $t_2 + \tau - \delta \le t < t_2$. It suffices to prove that $y(t) > 0$ for all $t \ge t_2$. In fact, if there is a $t^* \in [t_2, t_2 - \tau + \delta)$ such that $y(t^*) = 0$, then by (2.31),

$$\int_{t^*+\tau}^{\infty} \left(1 - (-p_0 - \varepsilon)^{-[(s-t^*)/\tau]}\right) \int_{s}^{\infty} (u - s)^{n-2} y(u - \delta) du\, ds = 0$$

which implies

$$\int_{t^*+\tau}^{\infty} (u - t^* - \tau)^{n-2} y(u - \delta) du = 0$$

and hence $y(u) = 0$ for all $u \ge t^* + \tau - \delta$. But since $t^* + \tau - \delta \in [t_2 + \tau - \delta, t_2)$, it follows that $y(t^* + \tau - \delta) > 0$. This is a contradiction and so $y(t)$ is positive on $[t_2, t_2 - \tau + \delta)$. In general, by induction we have $y(t) > 0$ for $t \in [t_2 + i(-\tau + \delta),\ t_2 + (i + 1)(-\tau + \delta))$, $i = 0, 1, 2, \ldots$, which shows that $y(t) > 0$ for all $t \ge t_2$. Thus, we have proved that $y(t)$ is positive and continuous on $[t_2 + \tau - \delta, \infty)$. From (2.31), we have for $t \ge t_2 + \tau$

$$y(t) + (p_0 + \varepsilon) y(t - \tau) = \begin{cases} \dfrac{q_0 - \varepsilon}{(n-2)!} \displaystyle\int_{t}^{\infty} \int_{s}^{\infty} (u - s)^{n-2} y(u - \delta) du\, ds, & \text{if } n > 1 \\ (q_0 - \varepsilon) \displaystyle\int_{t}^{\infty} y(u - \delta) du, & \text{if } n = 1. \end{cases}$$

Differentiating both sides n times, we have

$$\frac{d^n}{dt^n} [y(t) + (p_0 + \varepsilon) y(t - \tau)] + (q_0 - \varepsilon) y(t - \delta) = 0, \qquad t \ge t_2 + \tau$$

which contradicts the Claim and hence the proof is complete.

Since (2.8) together with $n = 1$ implies that every nonoscillatory solution of (1.1) is bounded, it follows that (2.5) can be removed in this case. Thus, we immediately have the following Corollary, which is an essential improvement of Theorem 1 in [1].

COROLLARY 2.3. *Assume that* (1.2), (2.2)–(2.4), (2.6) *and* (2.8) *hold. If* *every solution of the linearized equation*

$$\frac{d}{dt}[y(t) + p_0 y(t-\tau)] + q_0 y(t-\delta) = 0 \tag{2.32}$$

oscillates, then every solution of (1.1) *with* $n = 1$ *also oscillates.*

3. The existence of positive solutions

Consider the n^{th} order neutral delay differential equation

$$\frac{d^n}{dt^n}[x(t) + px(t-\tau)] + q(t)h(x(t-\delta)) = 0 \tag{3.1}$$

where n is an odd integer

$$p > 1, \quad \tau > 0, \quad \delta \geq 0, \quad q \in C([t_0, \infty), \boldsymbol{R}^+) \quad \text{and} \quad h \in C(\boldsymbol{R}, \boldsymbol{R}). \tag{3.2}$$

The next theorem is a partial converse of Theorem 2.1 and shows that, under appropriate hypotheses, (3.1) has a positive solution provided that an associated linear equation has a positive solution.

THEOREM 3.1. *Assume that* (2.24) *and* (3.2) *hold and that there exist positive constants* q_0 *and* M *such that*

$$0 \leq q(t) \leq q_0, \tag{3.3}$$

and either

$$0 \leq h(u) \leq u \qquad \text{for } 0 \leq u \leq M \tag{3.4}$$

or

$$0 \geq h(u) \geq 0 \qquad \text{for } -M \leq u \leq 0. \tag{3.5}$$

Suppose also that $h(u)$ *is nondecreasing in* u *and that the linear equation*

$$\frac{d^n}{dt^n}[y(t) + py(t-\tau)] + q_0 y(t-\delta) = 0 \tag{3.6}$$

has a nonoscillatory solution. Then (3.1) *has also a nonoscillatory solution.*

PROOF. Suppose that (3.4) holds. The case when (3.5) holds is similar and will be omitted. Let $y(t)$ be a positive solution of (3.6). It is easy to see that $\lim_{t\to\infty} y(t) = 0$. Thus, there exists a $T \geq t_0$ such that $0 < y(t) \leq M$ for $t \geq T - \delta$. Now integrate (3.6) from $t \geq T$ to infinity n times, we have

$$y(t) + py(t-\tau) = \frac{q_0}{(n-1)!}\int_t^\infty (s-t)^{n-1} y(s-\delta)ds$$

which yields for $t \geq T - \tau$

$$y(t) = \begin{cases} \dfrac{q_0}{(1+p)(n-2)!}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]}) \\ \qquad \times \displaystyle\int_s^{\infty}(u-s)^{n-2}y(u-\delta)\,du\,ds, & \text{if } n > 1 \\ \dfrac{q_0}{1+p}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]})y(s-\delta)\,ds, & \text{if } n = 1 \end{cases} \tag{3.7}$$

where $[\cdot]$ is the greatest integer function. In light of (3.3) and (3.4), (3.7) yields for $t \geq T - \tau$

$$y(t) \geq \begin{cases} \dfrac{1}{(1+p)(n-2)!}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]}) \\ \qquad \times \displaystyle\int_s^{\infty}(u-s)^{n-2}q(u)h(y(u-\delta))\,du\,ds, & \text{if } n > 1 \\ \dfrac{1}{1+p}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]})q(s)h(y(s-\delta))\,ds, & \text{if } n = 1. \end{cases} \tag{3.8}$$

Define the set of functions

$$W = \{\omega \in C([T-\delta, \infty), \boldsymbol{R}^+) : 0 \leq \omega(t) \leq y(t) \qquad \text{for } t \geq T - \delta\}$$

and define the mapping S on W as follows for $t \geq T$

$$(S\omega)(t) = \begin{cases} \dfrac{1}{(1+p)(n-2)!}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]}) \\ \qquad \times \displaystyle\int_s^{\infty}(u-s)^{n-2}q(u)h(\omega(u-\delta))\,du\,ds, & \text{if } n > 1 \\ \dfrac{1}{1+p}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]})q(s)h(\omega(s-\delta))\,ds, & \text{if } n = 1 \end{cases}$$

and for $T - \delta \leq t \leq T$

$$(S\omega)(t) = \frac{t+h}{T+h}(S\omega)(T)\frac{y(t)}{y(T)} + \left(1 - \frac{t+h}{T+h}\right)y(t)$$

where $h > 0$ is a constant with $T + h > \delta$.

It is easy to prove that S maps W into itself. Now define the following sequence of functions on W:

$$z_0(t) = y(t) \qquad \text{and} \qquad z_m(t) = (Sz_{m-1})(t) \qquad \text{for } t \geq T - \delta,\ m = 1, 2, \ldots.$$

By induction, we have

$$0 \le z_m(t) \le z_{m-1}(t) \le y(t) \qquad \text{for } t \ge T - \delta.$$

Set $x(t) = \lim_{m\to\infty} z_m(t)$ for $t \ge T - \delta$. It follows by Lebesgue's dominated convergence theorem that $x(t)$ satisfies for $t \ge T$

$$x(t) = \begin{cases} \dfrac{1}{(1+p)(n-2)!}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]}) \\ \qquad \times \displaystyle\int_{s}^{\infty}(u-s)^{n-2}q(u)h(x(u-\delta))du\,ds, & \text{if } n > 1 \\ \dfrac{1}{1+p}\displaystyle\int_{t+\tau}^{\infty}(1-(-p)^{-[(s-t)/\tau]})q(s)h(x(s-\delta))ds, & \text{if } n = 1 \end{cases} \tag{3.9}$$

and for $T - \delta \le t \le T$,

$$x(t) = \frac{t+h}{T+h}x(T)\frac{y(t)}{y(T)} + \left(1 - \frac{t+h}{T+h}\right)y(t) > 0.$$

Then by a similar method to the proof of Theorem 2.1, we can easily prove that $x(t)$ is continuous and positive. By (3.9), we have

$$x(t) + px(t-\tau) = \frac{1}{(n-1)!}\int_{t}^{\infty}(s-t)^{n-1}q(s)h(x(s-\delta))ds, \qquad t \ge T + \tau.$$

Clearly, $x(t)$ is a positive solution of (2.1) on $[T+\tau, \infty)$. The proof is complete.

By combining Theorems 2.1 and 3.1, we obtain the following necessary and sufficient condition for the oscillation of all solutions of (3.1).

Corollary 3.2. *Assume that* (2.6), (2.24) *and* (3.2) *hold and that $h(u)$ is nondecreasing in u. In addition, assume that there exist positive constants q_0 and M such that either* (3.4) *or* (3.5) *are satisfied and*

$$0 \le q(t) \le q_0 = \lim_{t\to\infty} q(t).$$

Then all solutions of (3.1) *oscillate if and only if all solutions of* (3.6) *oscillate.*

References

[1] Ming-Po Chen and J. S. Yu, Linearized oscillations for first order neutral delay differential equations, Panamerican Math. J. **3(3)** (1993), 33–45.

[2] Q. Chuanxi and G. Ladas, Linearized oscillations for equations with positive and negative coefficients, Hiroshima Math. J. **20** (1990), 331–340.

[3] R. D. Driver, Existence and continuous dependence of solutions of a neutral functional differential equation, Arch. Ration. Mech. Anal. **19** (1965), 149–166.

[4] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[5] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[6] G. Ladas, Linearized oscillations for neutral equations, Proceedings of Equadiff 87, Vol. **118**, pp. 379–387, Lecture Notes in Pure and Applied Mathematics, Differential Equations, Marcel Dekker, Inc., 1989.

[7] G. Ladas and C. Qian, Linearized oscillations for even order neutral differential equations, J. Math. Anal. Appl. **159** (1991), 237–250.

[8] G. Ladas and C. Qian, Linearized oscillations for odd order neutral delay differential equations, J. Differential Equations **88** (1990), 238–247.

[9] J. S. Yu and Z. Wang, A linearized oscillation result for neutral delay differential equations, Math. Nachr. **163** (1993), 101–107.

[10] J. S. Yu, J. H. Shen and X. Qian, A linearized oscillation theorem for odd order neutral delay differential equations, J. Math. Anal. Appl., to appear.

Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1
and
Department of Applied Mathematics
Hunan University
Changsha, Hunan 410082
China