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Radial symmetry of positive solutions for semilinear elliptic equations in a disc

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ABSTRACT. Symmetry and monotonicity properties of positive solutions of the problems $\Delta u + f(|x|, u) = 0$ in D and $u = 0$ on ∂D are considered, where D is the unit disc in $\boldsymbol{R}^2$. We give to D the Poincaré metric and then employ the moving plane method to obtain new theorems on symmetry. We also consider singular solutions.

1. Introduction

This paper is concerned with symmetry and monotonicity properties of positive solutions of the problems

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } D \setminus \{0\}, \\ u = 0 & \text{on } \partial D, \\ \lim\limits_{|x| \to 0} u(x) = \infty, \end{cases} \tag{1.2}$$

where $D = \{x \in \boldsymbol{R}^2 : |x| < 1\}$.

There is much current interest in the symmetry properties of solutions of the problems (1.1) and (1.2). Assume that $f(r, u)$ is decreasing in r. Then according to Gidas-Ni-Nirenberg's theorem [5], any nonnegative solution $u \in C^2(\bar{D})$ of (1.1) is rotationally symmetric. Their proof is based upon Alexandrov's moving plane method. Among other results, in [6], Lazer and McKenna have proved the following: assume that

$$\frac{\partial f}{\partial u}(r, u) < \lambda_2 \qquad \text{for } (r, u) \in ([0, 1] \times \boldsymbol{R}),$$

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where λ_2 denotes the second eigenvalue of the Laplacian with Dirichlet boundary conditions. Then all solutions $u \in C^2(D) \cap C^1(\bar{D})$ of (1.1) are radially symmetric. See, also [1, 4]. On the other hand, in [10], Veron considered the rotationally symmetric properties of singular solutions of the equation

$$\Delta u + \frac{Z}{|x|} u + h(u) = 0 \qquad \text{in } D \backslash \{0\},$$

where Z is a real number and h is a continuous function. For further studies of symmetric properties of solutions of (1.1) and (1.2), we refer to [2, 3, 8, 9].

In this paper we give to D the Poincaré metric and then employ the moving plane method to obtain new theorems on symmetry properties of positive solutions of (1.1) and (1.2). In Section 2 we state the main results. In Section 3 we present some preliminary lemmas and in Sections 4 and 5 we give the proofs of the theorems.

In a forthcomming paper, we shall study the higher dimensional version.

2. Statement of the results

2.1. First we consider the symmetric properties of positive solutions of the problem

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{2.1}$$

where $D = \{x \in \boldsymbol{R}^2 : |x| < 1\}$ and $f \in C^1([0,1] \times [0,\infty))$. We obtain the following theorem which extends the result of [5, Theorem 1′].

THEOREM 1. *Assume that* $(1 - r^2)^2 f(r, u)$ *is nonincreasing in* $r \in (0,1)$ *for each fixed* $u \in (0, \infty)$. *Let* $u \in C^2(D) \cap C(\bar{D})$ *be a positive solution of* (2.1). *Then* u *must be radially symmetric about the origin and* $u'(r) < 0$ *for* $0 < r = |x| < 1$.

As a corollary of Theorem 1, we have the following.

COROLLARY 1. *Let* $u \in C^2(D) \cap C(\bar{D})$ *be a positive solution of* (2.1), *and let* $u_\infty = \max\{u(x) : x \in D\}$. *Assume that* f *satisfies the following*:

(i) $(1 - r^2) f_u(r, u) < 8$ *for* $(r, u) \in ([0,1] \times [0, u_\infty])$;

(ii) *There exist constants* $r_0 \in (0,1]$ *and* $C_0 > 0$ *such that*

$$f_r(r, u) < C_0 r \qquad \textit{for } (r, u) \in ([0, r_0] \times [0, u_\infty]).$$

Then u *must be radially symmetric about the origin.*

PROOF. Let $v(x) = \frac{1}{4}(1 - |x|^2)$ and $w(x) = u(x) + Cv(x)$ for some constant $C > 0$. Then w is positive in D and satisfies where $D = \{x \in \boldsymbol{R}^2 : |x| < 1\}$ and $f \in C^1((0,1] \times [0,\infty))$. We obtain the following.

Theorem 2. *Assume that $(1-r^2)^2 f(r,u)$ is nonincreasing in $r \in (0,1)$ for each fixed $u \in (0,\infty)$. Let $u \in C^2(D\backslash\{0\}) \cap C(\bar{D}\backslash\{0\})$ be a positive solution of (2.2). Then u must be radially symmetric about the origin and $u'(r) < 0$ for $0 < r = |x| < 1$.*

This theorem is applicable to the problem investigated by [10]. We obtain the following.

Corollary 2. *Let $u \in C^2(D\backslash\{0\}) \cap C(\bar{D}\backslash\{0\})$ be a positive solution of*

$$\begin{cases} \Delta u + \dfrac{Z}{|x|}u + h(u) = 0 & \text{in } D\backslash\{0\}, \\ u = 0 & \text{on } \partial D, \\ \lim\limits_{|x| \to 0} u(x) = \infty, & \end{cases}$$

where Z is a nonnegative real number and h is a nonnegative continuous function. Then u must be radially symmetric about the origin.

Remark. In [10, Theorem 2.1], solutions are not assumed to be positive, but additional hypotheses are needed to obtain the symmetric properties of solutions.

Furthermore, we have the following corollary by the same argument as in Subsection 2.1.

Corollary 3. *Let $u \in C^2(D\backslash\{0\}) \cap C(\bar{D}\backslash\{0\})$ be a positive solution of (2.2). Assume that f satisfies the following assumptions:*

(i) $(1-r^2) f_u(r,u) < 8$ *for* $(r,u) \in ((0,1] \times [0,\infty))$;

(ii) *There exist constants $r_0 \in (0,1]$ and $C_0 > 0$ such that*

$$f_r(r,u) < C_0 r \qquad \text{for } (r,u) \in ((0,r_0] \times [0,\infty)).$$

Then u must be radially symmetric about the origin.

3. Preliminaries

We give to D the Poincaré metric $ds_D^2 = (1-r^2)^{-2}|dx|^2$. Then, the space D is called *the Poincaré disc*. First we need to introduce a few nota-

$$\Delta w + g(|x|, w) = 0 \quad \text{in } D \qquad \text{and} \qquad w = 0 \quad \text{on } \partial D,$$

where

$$g(r, w) = f(r, w - Cv(r)) + C.$$

We see that

$$\frac{\partial}{\partial r}\{(1-r^2)^2 g(r, w)\} = (1-r^2)\left\{\frac{Cr}{2}\left((1-r^2)f_u - 8 - \frac{2f}{C}\right) + (1-r^2)f_r\right\}.$$

Then, for sufficiently large $C > 0$, we have $\frac{\partial}{\partial r}\{(1-r^2)^2 g(r, w)\} < 0$. Therefore, Theorem 1 can be applied to conclude that w is radially symmetric, which implies that u is radially symmetric. This completes the proof of Corollary 1. □

One might ask whether positive solutions u of the problem (2.1) are necessarily radially symmetric—even if f does not satisfy the conditions in Theorem 1 or Corollary 1. This is not the case in general. For example, we show the following. Let

$$w(r, \theta) = J_1(\lambda r)\cos\theta \qquad \text{for } (r, \theta) \in ([0, 1] \times [0, 2\pi]),$$

where J_1 is the Bessel function of first kind and λ is the first zero of $J_1(r)$ for $r > 0$. (We see that $\lambda = 3.83\ldots$.) Then we observe that

$$\Delta w + \lambda^2 w = 0 \quad \text{in } D \qquad \text{and} \qquad w = 0 \quad \text{on } \partial D.$$

For small $\varepsilon > 0$, the function $u(x) = 1 - |x|^2 + \varepsilon w(x)$ is positive in D and stisfies

$$\Delta u + \lambda^2 u + \lambda^2(r^2 - 1) + 4 = 0 \quad \text{in } D \qquad \text{and} \qquad u = 0 \quad \text{on } \partial D;$$

but u is not radially symmetric. Define $f(r, u) = \lambda^2 u + \lambda^2(r^2 - 1) + 4$. Then we see that

$$\frac{\partial}{\partial r}\{(1-r^2)^2 f(r, u) \;= r(1-r^2)\{-4(\lambda^2 u + 4) + 6\lambda^2(1-r^2)\} > 0$$

near $(r, u) = (0, 0)$, and that

$$(1-r^2)f_u(r, u) = (1-r^2)\lambda^2 > 8$$

near $r = 0$.

2.2. Next we investigate the symmetric properties of positive singular solutions of the problem

$$\text{(2.2)} \qquad \begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } D \, \{0\}, \\ u = 0 & \text{on } \partial D, \\ \lim\limits_{|x| \to 0} u(x) = \infty, & \end{cases}$$

tions. For each $\lambda \in (0,1)$, let T_λ be a geodesic which intersects x_1-axis orthogonally at $(\lambda, 0)$, i.e.,

$$T_\lambda = \left\{ (x_1, x_2) \in D : \left(x_1 - \frac{1+\lambda^2}{2\lambda} \right)^2 + x_2^2 = \left(\frac{1-\lambda^2}{2\lambda} \right)^2 \right\}.$$

Define $\Sigma_\lambda \subset D$ by

$$\Sigma_\lambda = \left\{ (x_1, x_2) \in D : \left(x_1 - \frac{1+\lambda^2}{2\lambda} \right)^2 + x_2^2 > \left(\frac{1-\lambda^2}{2\lambda} \right)^2 \right\}.$$

For $x \in \Sigma_\lambda$, let x^λ be the reflection of x with respect to T_λ, i.e.,

$$x^\lambda = e_\lambda + \left(\frac{1-\lambda^2}{2\lambda} \right)^2 J(x - e_\lambda),$$

where $e_\lambda = \left(\frac{1+\lambda^2}{2\lambda}, 0 \right)$ and $J(x) = x/|x|^2$. From Lemma A.1 in Appendix, we have $|x| < |x^\lambda|$ for $x \in \Sigma_\lambda$.

The Laplace-Beltrami operator Δ_g of the Poincaré disc D is defined with $\Delta_g = (1-r^2)^2 \Delta$, where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$. Let u be a solution of the problem (2.1). Then u satisfies

$$\Delta_g u + (1 - |x|^2)^2 f(|x|, u) = 0 \qquad \text{in } D. \tag{3.1}$$

We prepare the following lemma.

Lemma 3.1. *Assume that $(1-r^2)^2 f(r,u)$ is nonincreasing in $r \in (0,1)$ for each fixed $u \in (0,\infty)$. Let $\lambda \in (0,1)$ and $v(x) = u(x) - u(x^\lambda)$ for $x \in \Sigma_\lambda$. Then, v satisfies*

$$\Delta v + c(x) v \le 0 \tag{3.2}$$

in Σ_λ, where

$$c(x) = \int_0^1 f_u\Big((|x|, u(x^\lambda) + t(u(x) - u(x^\lambda)) \Big) dt. \tag{3.3}$$

Proof. We observe that

$$\Delta_g u(x^\lambda) + (1 - |x^\lambda|^2)^2 f(|x^\lambda|, u(x^\lambda)) = 0 \qquad \text{for } x \in \Sigma_\lambda.$$

Then it follows from (3.1) that, for $x \in \Sigma_\lambda$,

$$\begin{aligned} 0 &= \Delta_g(u(x) - u(x^\lambda)) + (1 - |x|^2)^2 f(|x|, u(x)) - (1 - |x^\lambda|^2)^2 f(|x^\lambda|, u(x^\lambda)) \\ &\geq \Delta_g(u(x) - u(x^\lambda)) + (1 - |x|^2)^2 \Big(f(|x|, u(x)) - f(|x|, u(x^\lambda)) \Big) \\ &= \Delta_g v(x) + (1 - |x|^2)^2 c(x) v(x), \end{aligned}$$

where $c(x)$ is the function in (3.3). Therefore, v satisfies (3.2) in Σ_λ. $\square$

4. Proof of Theorem 1

We define

$$\Lambda = \left\{ \lambda \in (0,1) : u(x) - u(x^\lambda) > 0 \text{ in } \Sigma_\lambda \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } T_\lambda \right\},$$

where ν is the unit outer normal of $\partial \Sigma_\lambda$. We define u_∞ and f_∞ as follows:

$$\begin{aligned} u_\infty &= \max\{u(x) : x \in D\} \quad \text{and} \\ f_\infty &= \max\{f_u(r,u) : 0 \leq r \leq 1, 0 \leq u \leq u_\infty\}. \end{aligned} \tag{4.1}$$

We can choose $r_0 \in (0,1)$ so that there exists a function w_0 satisfying

$$\begin{aligned} w_0(x) > 0 \quad \text{on } r_0 \leq |x| \leq 1 \quad \text{and} \\ \Delta w_0 + f_\infty w_0 \leq 0 \quad \text{in } r_0 < |x| < 1. \end{aligned} \tag{4.2}$$

Since $u = 0$ on ∂D, there exists $r_1 \in (r_0, 1)$ such that

$$\max\{u(x) : r_1 \leq |x| \leq 1\} < \min\{u(x) : |x| \leq r_0\}. \tag{4.3}$$

LEMMA 4.1. *We have* $[r_1, 1) \subset \Lambda$.

PROOF. For each $\lambda \geq r_1$, let $v(x) = u(x) - u(x^\lambda)$. Define $B_0 = \{x \in D : |x| < r_0\}$. From (4.3), we have $v > 0$ in $\overline{B_0}$. By Lemma 3.1, we obtain

$$\begin{aligned} \Delta v + c(x) v \leq 0 \quad &\text{in } \Sigma_\lambda \backslash \overline{B_0}, \\ v > 0 \quad &\text{on } \partial B_0, \\ v = 0 \quad &\text{on } \partial \Sigma_\lambda. \end{aligned}$$

From (3.3) and (4.1), we find that $c(x) \leq f_\infty$ in Σ_λ. Then the positive function w_0 stated above satisfies

$$\Delta w_0 + c(x) w_0 \leq 0 \quad \text{in } \Sigma_\lambda \backslash \overline{B_0}.$$

Hence the maximal principle ([7, p. 73, Theorem 10]) implies that $v > 0$ in $\Sigma_\lambda \backslash \overline{B_0}$. Then we conclude that $v > 0$ in Σ_λ because $v > 0$ in $\overline{B_0}$.

Since v satisfies (3.2) with $v > 0$ in $\Sigma_\lambda \backslash \overline{B_0}$ and $v = 0$ on T_λ, the Hopf boundary lemma applies here and we have $\dfrac{\partial v}{\partial \nu} < 0$ on T_λ. We find that

$$\frac{\partial u}{\partial \nu} = \frac{1}{2} \frac{\partial v}{\partial \nu} < 0 \qquad \text{on } T_\lambda.$$

Therefore, we obtain $\lambda \in \Lambda$, which implies that $[r_1, 1) \subset \Lambda$. □

LEMMA 4.2. *Let* $\lambda_0 \in \Lambda$. *Then there exists* $\varepsilon > 0$ *such that* $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$.

Without loss of generality, we may assume $\lambda_0 \leq r_1$. Let $B_1 = \{x \in D : |x| < r_1\}$. For convienience, we define

$$E(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \lambda < \lambda_2} T_\lambda \ \ (= \Sigma_{\lambda_2} \backslash \overline{\Sigma_{\lambda_1}}) \qquad \text{for } 0 < \lambda_1 < \lambda_2 < 1.$$

In order to prove Lemma 4.2, we prepare the following.

LEMMA 4.3. *Let* $\lambda_0 \in \Lambda$. *Then there exist* $\varepsilon_1 > 0$ *and* $\sigma > 0$ *such that, for each* $\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0)$,

$$u(x) - u(x^\lambda) > 0, \qquad x \in E(\lambda - \sigma, \lambda) \cap \overline{B_1}. \tag{4.4}$$

PROOF. Let $r_2 \in (r_1, 1)$ and $B_2 = \{x \in D : |x| < r_2\}$. Since $\dfrac{\partial u}{\partial \nu} < 0$ on T_{λ_0}, we can find $\delta_1 > 0$ such that

$$\frac{\partial u}{\partial \nu} < 0 \qquad \text{in } E(\lambda_0 - \delta_1, \lambda_0 + \delta_1) \cap \overline{B_2}. \tag{4.5}$$

There exists $\delta_2 \in (0, \delta_1)$ such that

$$x^{\lambda_0} \in E(\lambda_0, \lambda_0 + \delta_1) \cap \overline{B_2} \qquad \text{for } x \in E(\lambda_0 - \delta_2, \lambda_0) \cap \overline{B_1}. \tag{4.6}$$

Define ε_1 and σ as $\varepsilon_1 = \sigma = \frac{1}{2}\delta_2$. Let $\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0)$. We show that

$$x^\lambda \in E(\lambda, \lambda_0 + \delta_1) \cap \overline{B_2} \qquad \text{for } x \in E(\lambda - \sigma, \lambda) \cap \overline{B_1}. \tag{4.7}$$

Let $x \in E(\lambda - \sigma, \lambda) \cap \overline{B_1}$. Since $E(\lambda - \sigma, \lambda) \subset E(\lambda_0 - \delta_2, \lambda_0)$ and (4.6) holds, we have

$$x^{\lambda_0} \in E(\lambda_0, \lambda_0 + \delta_1) \cap \overline{B_2}.$$

We notice here that $x \in \Sigma_\lambda$ and $x, x^{\lambda_0} \in \overline{B_2}$. Then, by Lemma A.2, we have $x^\lambda \in \overline{B_2}$. Since $x^{\lambda_0} \in \Sigma_{\lambda_0 + \delta_1}$, by applying Lemma A.3 with $\mu = \lambda_0 + \delta_1$, we get $x^\lambda \in \Sigma_{\lambda_0 + \delta_1}$. Thus, we conculude that (4.7) holds.

Let $\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0)$ and $x \in E(\lambda - \sigma, \lambda) \cap \overline{B_1}$. From (4.7), we notice that $x, x^\lambda \in E(\lambda_0 - \delta_1, \lambda_0 + \delta_1) \cap \overline{B_2}$. By Lemma A.4, there exists an arc γ extending from x to x^λ such that γ is contained in $E(\lambda_0 - \delta_1, \lambda_0 + \delta_1) \cap \overline{B_2}$ and intersects T_μ orthogonally if $\gamma \cap T_\mu \neq \emptyset$. Since $\dfrac{\partial u}{\partial \nu} < 0$ on $\gamma \cap T_\mu$, we have $u(x) > u(x^\lambda)$ by employing the line integration. Therefore, we conclude that (4.4) holds. □

PROOF OF LEMMA 4.2. Define $F = \overline{\Sigma_{\lambda_0 - \sigma}}$, where σ is a constant appearing in Lemma 4.3. It follows from the assumption $\lambda_0 \in \Lambda$ that

$$u(x) - u(x^{\lambda_0}) > 0 \qquad \text{on } F \cap \overline{B_1}.$$

Since $F \cap \overline{B_1}$ is compact, we can find $\varepsilon_2 \in (0, \sigma)$ such that, if $|\lambda - \lambda_0| < \varepsilon_2$, then

$$u(x) - u(x^\lambda) > 0 \qquad \text{on } F \cap \overline{B_1}. \tag{4.8}$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$, where ε_1 is a constant appearing in Lemma 4.3. From (4.4) and (4.8), we have

$$u(x) - u(x^\lambda) > 0 \qquad \text{in } \Sigma_\lambda \cap \overline{B_1}. \tag{4.9}$$

Now for each $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$, let $v(x) = u(x) - u(x^\lambda)$. Then, v satisfies

$$\begin{aligned} \Delta v + c(x) v \le 0 \quad & \text{in } \Sigma_\lambda \backslash \overline{B_1}, \\ v > 0 \quad & \text{on } \Sigma_\lambda \cap \partial B_1, \\ v = 0 \quad & \text{on } \partial \Sigma_\lambda. \end{aligned}$$

Since $\overline{B_1} \supset \overline{B_0}$, the maximum principle ([7, p. 73 Theorem 10]) implies that $v > 0$ in $\Sigma_\lambda \backslash \overline{B_1}$, i.e., $u(x) - u(x^\lambda) > 0$ in $\Sigma_\lambda \backslash \overline{B_1}$. Then, by (4.9), we conclude that $u(x) - u(x^\lambda) > 0$ in Σ_λ. Since v satisfies (3.2) with $v > 0$ in Σ_λ and $v = 0$ on T_λ, we have $\dfrac{\partial u}{\partial \nu} < 0$ on T_λ by the same argument as in the proof of Lemma 4.1. Therefore, $\lambda \in \Lambda$. This implies that $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$. □

For $x = (x_1, x_2) \in D$, we define $x^0 = (-x_1, x_2)$. Let $D_+ = \{(x_1, x_2) \in D : x_1 > 0\}$.

LEMMA 4.4. *We have either*

$$\begin{aligned} & u(x) \equiv u(x^{\lambda_1}) \quad \textit{for some } \lambda_1 > 0 \qquad \textit{and} \\ & \frac{\partial u}{\partial \nu} < 0 \quad \textit{on } T_\lambda \textit{ for } \lambda \in (\lambda_1, 1) \end{aligned} \tag{4.10}$$

or

$$
(4.11) \qquad \begin{aligned} & u(x) \ge u(x^{\lambda_1}) \quad \text{for } x \in D_+ \qquad \text{and} \\ & \frac{\partial u}{\partial \nu} < 0 \quad \text{on } T_\lambda \text{ for } \lambda \in (0,1). \end{aligned}
$$

PROOF. Let

$$
(4.12) \qquad \lambda_1 = \inf\{\lambda > 0 : (\lambda, 1) \subset \Lambda\}.
$$

We distinguish the following two cases: (i) $\lambda_1 > 0$; (ii) $\lambda_1 = 0$.

(i) *The case where* $\lambda_1 > 0$. From the continuity of u, we have

$$
v(x) = u(x) - u(x^{\lambda_1}) \ge 0 \qquad \text{in } \Sigma_{\lambda_1}.
$$

It follows from Lemma 3.1 that

$$
\Delta v + c(x) v \le 0 \qquad \text{in } \Sigma_{\lambda_1},
$$

$$
v = 0 \quad \text{on } \partial\Sigma_{\lambda_1}, \qquad \text{and} \qquad v \ge 0 \quad \text{in } \Sigma_{\lambda_1}.
$$

Hence, we have that either

$$
(4.13) \qquad v \equiv 0 \quad \text{in } \Sigma_{\lambda_1}, \qquad \text{i.e.,} \quad u(x) \equiv u(x^{\lambda_1}) \quad \text{in } \Sigma_{\lambda_1},
$$

or

$$
(4.14) \qquad v > 0 \quad \text{in } \Sigma_{\lambda_1}, \qquad \text{i.e.,} \quad u(x) > u(x^{\lambda_1}) \quad \text{in } \Sigma_{\lambda_1}.
$$

If (4.13) occurs, we have $\dfrac{\partial u}{\partial \nu} < 0$ on T_λ since $\lambda \in \Lambda$ for $\lambda \in (\lambda_1, 1)$. Thus we obtain (4.10). On the other hand, if (4.14) occurs: then we have $\dfrac{\partial u}{\partial \nu} < 0$ on T_{λ_1} in a similar fashion as the proof of Lemma 4.1. Thus, $\lambda_1 \in \Lambda$. By Lemma 4.2, there exists $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1) \subset \Lambda$. This contradicts (4.12). Therefore, (4.14) cannot happen.

(ii) *The case where* $\lambda_1 = 0$. From the continuity of u, we have $u(x) \ge u(x^0)$ in D_+. Since $\lambda \in \Lambda$ for $\lambda \in (0,1)$, we have $\dfrac{\partial u}{\partial \nu} < 0$ on T_{λ_1}. Thus, (4.11) holds.

Therefore, we have either (4.10) or (4.11). □

PROOF OF THEOREM 1. If (4.11) occurs in Lemma 4.4, we can repeat the previous Lemmas 4.1–4.4 for the negative x_1-direction to conclude that either u is symmetric in the x_1 direction about some hyperplane in the Poincaré disc or

$$
(4.15) \qquad u(x) \le u(x^0) \qquad \text{for } x \in D_+.
$$

If (4.15) occurs, from (4.11) we have $u(x) \equiv u(x^0)$ for $x \in D_+$. Therefore, u must be symmetric in the x_1-direction about some hyperplane, and be strictly decreasing away from the hyperplane in the Pincaré disc. Since the equation in (2.1) is invariant under rotation, we may take any direction as the x_1-direction and conclude that u is radially symmetric about some point $x_0 \in D$ in the Poincaré disc. Since the equation is invariant under rotation, the point x_0 must be the origin. Therefore, u must be radially symmetric about the origin and $u_r < 0$ for $r = |x| > 0$. □

5. Proof of Theorem 2

Let

$$\Lambda = \left\{\lambda \in (0,1) : u(x) - u(x^\lambda) > 0 \text{ in } \Sigma_\lambda \backslash \{0\} \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } T_\lambda\right\}.$$

We define u_∞ and f_∞ as follows:

$$\begin{aligned} u_\infty &= \max\left\{u(x) : \frac{1}{2} \le |x| \le 1\right\} \qquad \text{and} \\ f_\infty &= \max\left\{f_u(r,u) : \frac{1}{2} \le r \le 1, 0 \le u \le u_\infty\right\}. \end{aligned} \tag{5.1}$$

We can choose $r_0 \in (\frac{1}{2}, 1)$ so that there exists a function w_0 satisfying (4.2). There exists $r_1 \in (r_0, 1)$ such that (4.3) holds.

By the same argument as in the proof of Lemma 4.1, we obtain the following.

LEMMA 5.1. *We have* $[r_1, 1) \subset \Lambda$.

LEMMA 5.2. *Let* $\lambda_0 \in \Lambda$. *Then there exists* $\varepsilon > 0$ *such that* $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$.

PROOF. Let $B_1 = \{x \in D : |x| < r_1\}$. By the same argument as in the proof of Lemma 4.3, we obtain the following: there exist $\varepsilon_1 > 0$ and $\sigma > 0$ such that (4.4) holds for each $\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0)$.

We can choose $r_3 \in (0, \lambda_0 - \sigma)$ so small that

$$\min\{u(x) : |x| \le r_3\} > \max\{u(x) : x \in D \backslash \Sigma_{\lambda_0 - \sigma}\}. \tag{5.2}$$

Let $B_3 = \{x \in D : |x| < r_3\}$. We define $F' = \overline{\Sigma_{\lambda_0-\sigma} \backslash B_3}$. It follows from the assumption $\lambda_0 \in \Lambda$ that

$$u(x) - u(x^{\lambda_0}) > 0 \qquad \text{on } F' \cap \overline{B_1}.$$

Since $F' \cap \overline{B_1}$ is compact, we can find $\varepsilon_2 \in (0, \sigma)$ such that, if $|\lambda - \lambda_0| < \varepsilon_2$, then

$$u(x) - u(x^\lambda) > 0 \qquad \text{on } F' \cap \overline{B_1}. \tag{5.3}$$

By virtue of (5.2), we find that $u(x) - u(x^\lambda) > 0$ in $\overline{B_3}\backslash\{0\}$. Therefore, we have, if $|\lambda - \lambda_0| < \varepsilon_2$,

$$u(x) - u(x^\lambda) > 0 \qquad \text{in } \overline{\Sigma_{\lambda_0-\sigma}}\backslash\{0\} \cap \overline{B_1}. \tag{5.4}$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$. From (4.4) and (5.4), we conclude that

$$u(x) - u(x^\lambda) > 0 \qquad \text{in } \Sigma_\lambda\backslash\{0\} \cap \overline{B_1}. \tag{5.5}$$

Then, by the same argument as in the proof of Lemma 4.2, for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$, we have $u(x) - u(x^\lambda) > 0$ in $\Sigma_\lambda\backslash\{0\}$ and $\dfrac{\partial u}{\partial \nu} < 0$ on T_λ. Therefore, $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$. $\square$

LEMMA 5.3. *We have*

$$u(x) \geq u(x^0) \quad \text{for } x \in D_+ \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} < 0 \quad \text{on } T_\lambda \text{ for } \lambda \in (0,1). \tag{5.6}$$

PROOF. Define λ_1 as (4.12). We show that $\lambda_1 = 0$. Assume to the contrary that $\lambda_1 > 0$. We observe that $v(x) = u(x) - u(x^{\lambda_1})$ satisfies

$$\Delta v + c(x)v \leq 0 \qquad \text{in } \Sigma_{\lambda_1}\backslash\{0\},$$

$$v = 0 \quad \text{on } \partial\Sigma_{\lambda_1}, \qquad v \geq 0 \quad \text{in } \Sigma_{\lambda_1}\backslash\{0\}, \qquad \text{and}$$

$$\lim_{|x|\to 0} v(x) = \infty.$$

Hence, the maximal principle implies that

$$v > 0 \quad \text{in } \Sigma_{\lambda_1}\backslash\{0\}, \qquad \text{i.e.,} \quad u(x) > u(x^{\lambda_1}) \quad \text{in } \Sigma_{\lambda_1}\backslash\{0\}. \tag{5.7}$$

Then we have $\dfrac{\partial u}{\partial \nu} < 0$ on T_{λ_1} by the same argument as in the proof of Lemma 4.1. Then, $\lambda_1 \in \Lambda$. By Lemma 5.2, there exists $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1) \subset \Lambda$. This contradicts the definition of λ_1. Thus, we conclude that $\lambda_1 = 0$.

From the continuity of u, we have $u(x) \geq u(x^0)$ in D_+. Since $\lambda \in \Lambda$ for $\lambda \in (0,1)$, we have $\dfrac{\partial u}{\partial \nu} < 0$ on T_λ. $\square$

PROOF OF THEOREM 2. We can repeat the previous Lemmas 5.1–5.3 for the negative x_1-direction to conclude that

$$u(x) \leq u(x^0) \qquad \text{for } x \in D_+.$$

Hence, from (5.6), u must be symmetric in x_1-direction about the line $x_1 = 0$, and $\dfrac{\partial u}{\partial \nu} < 0$ on T_λ for $\lambda \in (0,1)$. We may take any direction as the x_1-direction and conclude that u is symmetric in every direction. Therefore, u must be radially symmetric about the origin and $u_r < 0$ for $r = |x| > 0$. $\square$

Appendix

Let $D = \{x \in \boldsymbol{R}^2 : |x| < 1\}$. We give to D the Poincaré metric $ds_D^2 = (1 - r^2)^{-2}|dx|^2$, where $r = |x|$. Then the space D is called *the Poincaré disc.*

For each $\lambda \in (0,1)$, let T_λ be a geodesic which intersects x_1-axis orthogonally at $(\lambda, 0)$, i.e.,

$$T_\lambda = \left\{(x_1, x_2) \in D : \left(x_1 - \frac{1+\lambda^2}{2\lambda}\right)^2 + x_2^2 = \left(\frac{1-\lambda^2}{2\lambda}\right)^2\right\}.$$

We define Σ_λ as

$$\Sigma_\lambda = \left\{(x_1, x_2) \in D : \left(x_1 - \frac{1+\lambda^2}{2\lambda}\right)^2 + x_2^2 > \left(\frac{1-\lambda^2}{2\lambda}\right)^2\right\}.$$

For $x \in \Sigma_\lambda$, let x^λ be a reflection of x with respect to T_λ, i.e.,

$$x^\lambda = e_\lambda + \left(\frac{1-\lambda^2}{2\lambda}\right)^2 J(x - e_\lambda),$$

where $e_\lambda = \left(\dfrac{1+\lambda^2}{2\lambda}, 0\right)$ and $J(x) = x/|x|^2$. Here we notice the following: we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$ in such a way that $(x_1, x_2) \in \boldsymbol{R}^2$ is $x_1 + ix_2 \in \boldsymbol{C}$. Then, for $z \in \Sigma_\lambda (z \in \boldsymbol{C})$, z^λ is represented as

$$z^\lambda = \frac{(1+\lambda^2)\bar{z} - 2\lambda}{2\lambda\bar{z} - (1+\lambda^2)}.$$

Let $H = \{z \in \boldsymbol{C} : \operatorname{Im} z > 0\}$. We give to H the Poincaré metric $ds_H^2 = |dx|^2/x_2^2$. Define $\Pi : D \to H$ as

$$\Pi(x) = i\,\frac{1 - x_1 - ix_2}{1 + x_1 + ix_2} \qquad \text{for } x = (x_1, x_2) \in D.$$

Then, Π is one-to-one and onto mapping.

For a subset E of D, we define $\Pi(E)$ as

$$\Pi(E) = \{z \in H : z = \Pi(x), x \in E\}.$$

Then, we observe that

$$\Pi(T_\lambda) = \left\{z \in H : |z| = \frac{1-\lambda}{1+\lambda}\right\} \qquad \text{and} \qquad \Pi(\Sigma_\lambda) = \left\{z \in H : |z| > \frac{1-\lambda}{1+\lambda}\right\}.$$

For $x \in \Sigma_\lambda$, let $z = \Pi(x)$ and $z^\lambda = \Pi(x^\lambda)$. Then,

$$z^\lambda = \left(\frac{1-\lambda}{1+\lambda}\right)^2 \frac{1}{\bar{z}}.$$

Hence, z and z^λ are on an Euclidean line containing the origin and satisfy

$$|z|\,|z^\lambda| = \left(\frac{1-\lambda}{1+\lambda}\right)^2.$$

LEMMA A.1. *For $x \in \Sigma_\lambda$, we have $|x| < |x^\lambda|$.*

PROOF. For $x = (x_1, x_2) \in \Sigma_\lambda$, let $r_0 = (x_1^2 + x_2^2)^{1/2}$ and $x^0 = (-x_1, x_2)$. Define z, z^λ, and z^0 as

$$z = \Pi(x), \qquad z^\lambda = \Pi(x^\lambda), \qquad \text{and} \qquad z^0 = \Pi(x^0),$$

respectively. Then, z, z^λ, and z^0 are on an Euclidean line containing the origin and satisfy

$$|z| > \frac{1-\lambda}{1+\lambda}, \qquad |z|\,|z^\lambda| = \left(\frac{1-\lambda}{1+\lambda}\right)^2, \qquad \text{and} \qquad |z|\,|z^0| = 1.$$

Hence, we have

$$\text{(A.1)} \qquad |z^\lambda| < |z^0| \qquad \text{and} \qquad |z^\lambda| < |z|.$$

Let $B_0 = \{y \in D : |y| < r_0\}$. Then, $x, x^0 \in \partial B_0$ and $z, z^0 \in \partial\Pi(B_0)$. Since $\Pi(\overline{B_0})$ is an Euclidean circular disc, $\Pi(\overline{B_0})$ includes a line segment zz^0. Then, from (A.1), $z^\lambda \notin \Pi(\overline{B_0})$. This implies that $x^\lambda \notin (\overline{B_0})$, i.e., $|x^\lambda| > r = |x|$. $\square$

Hereafter, we use the notation B as

$$B = \{y \in D : |y| < r\} \qquad \text{for some } r \in (0,1).$$

LEMMA A.2. *Let $0 < \lambda < \lambda_0 < 1$. Assume that $x \in \Sigma_\lambda (\subset \Sigma_{\lambda_0})$ and $x, x^{\lambda_0} \in \bar{B}$. Then $x^\lambda \in \bar{B}$.*

PROOF. Define z, z^λ, and z^{λ_0} as

$$\text{(A.2)} \qquad z = \Pi(x), \qquad z^\lambda = \Pi(x^\lambda), \qquad \text{and} \qquad z^{\lambda_0} = \Pi(x^{\lambda_0}),$$

respectively. Then, z, z^λ, and z^{λ_0} are on an Euclidean line containing the origin and satisfy $|z^\lambda| < |z|$ and

$$\text{(A.3)} \qquad |z|\,|z^\lambda| = \left(\frac{1-\lambda}{1+\lambda}\right)^2 > \left(\frac{1-\lambda_0}{1+\lambda_0}\right)^2 = |z|\,|z^{\lambda_0}|.$$

Hence we obtain $|z| > |z^\lambda| > |z^{\lambda_0}|$, which implies that z^λ is between z and z^{λ_0}. By the assumption, we find that $z, z^{\lambda_0} \in \Pi(\bar{B})$. Since $\Pi(\bar{B})$ is an Euclidean circular disc, we have $z^\lambda \in \Pi(\bar{B})$, i.e., $x^\lambda \in \bar{B}$. $\square$

LEMMA A.3. *Let $0 < \lambda < \lambda_0 < 1$, and let $x \in \Sigma_\lambda (\subset \Sigma_{\lambda_0})$. Assume that $x^{\lambda_0} \in \Sigma_\mu$ for some $\mu \in (\lambda_0, 1)$. Then $x^\lambda \in \Sigma_\mu$.*

Proof. Define z, z^{λ}, and z^{λ_0} as (A.2), respectively. Then, from (A.3), we have $|z^{\lambda}| > |z^{\lambda_0}|$. By virtue of $z^{\lambda_0} \in \Pi(\Sigma_\mu)$, we have $|z^{\lambda_0}| > \dfrac{1-\mu}{1+\mu}$. Then, $|z^{\lambda}| > \dfrac{1-\mu}{1+\mu}$. This implies that $z^{\lambda} \in \Pi(\Sigma_\mu)$, i.e., $x^{\lambda} \in \Sigma_\mu$. □

Lemma A.4. *Let $0 < \mu_1 < \lambda < \mu_2 < 1$. Define*

$$E = E(\mu_1, \mu_2) = \bigcup_{\mu_1 < \mu < \mu_2} T_\mu.$$

Let $x \in \Sigma_\lambda$. Assume that $x, x^{\lambda} \in E \cap \bar{B}$. Then, there exists an arc γ extending from x to x^{λ} such that γ is contained in $E \cap \bar{B}$ and intersects T_μ orthogonaly if $\gamma \cap T_\mu \neq \emptyset$.

Proof. Define z and z^{λ} as $z = \Pi(x)$ and $z^{\lambda} = \Pi(x^{\lambda})$, respectively. Then, z and z^{λ} are on an Euclidean line containing the origin. Let Γ be an Euclidean line segment zz^{λ} in H. We find that Γ and $\Pi(T_\mu)$ intersect orthogonally if $\Gamma \cap \Pi(T_\mu) \neq \emptyset$.

By virtue of $\Pi(E) = \left\{ w \in H : \dfrac{1-\mu_2}{1+\mu_2} < |w| < \dfrac{1-\mu_1}{1+\mu_1} \right\}$, we have $\Gamma \subset \Pi(E)$. Since $\Pi(\bar{B})$ is an Euclidean circular disc and $z, z^{\lambda} \in \Pi(\bar{B})$, we see that $\Gamma \subset \Pi(\bar{B})$. Therefore, we have $\Gamma \subset \Pi(E) \cap \Pi(\bar{B})$.

Let $\gamma = \{y \in D : w = \Pi(y), w \in \Gamma\}$. Then γ is an arc extending from x to x^{λ} such that $\gamma \subset E \cap \bar{B}$. Since Π is conformal, γ intersects T_μ orthogonally if $\gamma \cap T_\mu \neq \emptyset$. □

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