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Royden compactification of integers

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ABSTRACT. We present an explicit description of the Royden compactification of the discrete topological space $\boldsymbol{Z}$ of integers. It is defined to be the Gelfand space of the Royden algebra of bounded, Dirichlet finite functions. We identify it with a quotient space of the Čech-Stone compactification $\beta\boldsymbol{Z}$. The quotient map is expressed in terms of properties of subsets of $\boldsymbol{Z}$. Moreover, the quotient topology is also described in such a way.

1. Introduction

In the classification theory of Riemann surfaces, significant roles are played by different boundaries, invented by Wiener, Martin, Royden and others. The boundary defined by Royden (see [5]) is one of the most fruitful concepts in the theory. For a given Riemann surface R its Royden compactification R* is a space which satisfies the following conditions (see also [7], Chapter III.):

(R1) R* is a compact Hausdorff space.

(R2) R* contains R as an open dense subspace.

(R3) Every function from the Royden algebra BD(R) extends to a continuous function on R*.

(R4) The Royden algebra BD(R) separates points in R*.

It is known that the compactification R* of R exists and is unique up to a homeomorphism fixing R pointwise.

An analogous theory is developed in [8] and [9] for graphs instead of Riemann surfaces (or for more general structures called electrical networks), where the above is also true. The Gelfand theory of representations of commutative Banach algebras yields the existence of the Royden compactification for an infinite connected graph. It is identified with the Gelfand space of the Royden algebra BD. The general theory doesn't say, however, how the compactification looks. In general, there are no known examples of the Royden compactification for graphs.

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The purpose of this paper is to show a method of constructing the compactification for the discrete space $\boldsymbol{Z}$ of integers, which can be thought of as a graph consisting of one infinite line. The construction shows that the Royden compactification $\mathscr{R}\boldsymbol{Z}$ of $\boldsymbol{Z}$ is in a sense similar to the Čech-Stone compactification $\beta\boldsymbol{Z}$.

In section 2 we consider the Royden algebra BD of bounded, Dirichlet finite functions. We study there some basic examples of elements in it and some useful constructions.

Section 3 contains the construction of the Royden compactification $\mathscr{R}\boldsymbol{Z}$ as the Gelfand space of the commutative Banach algebra BD. It also shows some basic properties of its elements.

In section 4 we briefly recall basic facts about the Čech-Stone compactification $\beta\boldsymbol{Z}$.

In section 5 we define an equivalence relation on $\beta\boldsymbol{Z}$. As a main result we show in Theorem 5.3 how the quotient space may be identified with the Royden compactification $\mathscr{R}\boldsymbol{Z}$.

Section 6 contains an explicit description of the equivalence relation (Theorem 6.1) and of the quotient topology (Theorem 6.4).

2. Royden Algebra *BD*

We shall consider real-valued functions on the topological discrete space $\boldsymbol{Z}$ of integers (which are simply real doubly infinite sequences). For each such function f, we define the *Dirichlet sum* $D(f)$ *as:*

$$D(f) = \sum_{n \in \boldsymbol{Z}} |f(n+1) - f(n)|^2. \tag{1}$$

The space BD, introduced by H. L. Royden for open Riemann surfaces (see [4], Chapters III. IV), consists of bounded real-valued functions on $\boldsymbol{Z}$ which have finite Dirichlet sum. The operations of pointwise addition and multiplication of functions provide the structure of a real commutative unital algebra on BD, where the constant function $\mathit{1}$ is the identity element. We introduce the following norm on BD:

$$\|f\| = \|f\|_\infty + D(f)^{1/2}, \tag{2}$$

where $\|f\|_\infty$ is the usual sup norm.

Proposition 2.1. *With the* $\|\cdot\|$*-norm the space BD is a real commutative unital Banach algebra.*

The proof of the Proposition is a simple use of Minkowski's inequality and we omit it here (it can be found in [8], Theorem 6.2 or [9], Theorem 2.1).

Trivial examples of elements of BD are functions which are constant out of a finite subset of $\boldsymbol{Z}$. A more concrete example we get by defining the following function α. Let $J_0 = \{4^{|k|} : k \in \boldsymbol{Z}\}$, $J_1 = \{2 \cdot 4^{|k|} : k \in \boldsymbol{Z}\}$ and define

$$\alpha(n) = \begin{cases} 0 & \text{if } |n| \in J_0, \\ 1 & \text{if } |n| \in J_1, \\ \text{"linearly" between.} & \end{cases}$$

The graph of α is (discrete) piecewise linear between succesive values 0 and 1. One easily observes that the Dirichlet sum for α is a geometric progression, thus $\alpha \in BD$.

The above example can be generalized in the following way. Let (a_n) be a bounded sequence of real numbers and $\mathbf{r} = (r_n)$ a strictly increasing sequence of integers satisfying the following "lacunary-like" condition:

$$S(\mathbf{r}) = \sum_{n \in \boldsymbol{Z}} \frac{1}{r_{n+1} - r_n} < +\infty. \tag{3}$$

Define

$$f(j) = \begin{cases} a_n & \text{if } j = r_n, \\ \text{"linearly" between } r_n\text{'s.} & \end{cases} \tag{4}$$

Then the sup norm of f is that of (a_n) and the Dirichlet sum of f is bounded by a constant multiple of $S(\mathbf{r})$. Hence we get an embedding of l^∞ into BD.

Another useful construction is given by the following LINEARIZATION process. Let f be a given BD-function and let $\mathbf{r}$ be a strictly increasing sequence of positive integers (which may, or may not, satisfy the above condition (3)). We define a new function $\hat{f}$ by assuming that it has the same values as f does at all integers of the form r_n for $n \in \boldsymbol{Z}$ and between these points we wish $\hat{f}$ to be linear. Then a simple use of the Schwarz inequality shows that $D(\hat{f}) \leq D(f)$. Therefore the mapping $f \mapsto \hat{f}$ is norm decreasing on BD.

3. Royden compactification of $\boldsymbol{Z}$

Let $\mathscr{R}\boldsymbol{Z} = \mathscr{R}$ be the space of all bounded complex-valued multiplicative linear functionals φ on BD normalized by $\varphi(\mathit{1}) = 1$. We call such a φ a *character*. We endow $\mathscr{R}$ with the weak*-topology, so the net $\{\varphi_i\}$ converges to φ if for every $f \in BD$ the net of numbers $\{\varphi_i(f)\}$ converges to $\varphi(f)$. This turns out to be the Gelfand topology and, by Tychonoff's Theorem, $\mathscr{R}$ becomes a compact Hausdorff space. Every integer $n \in \boldsymbol{Z}$ defines a character on BD by $\varphi_n(f) = f(n)$ and identifying n with φ_n we get $\boldsymbol{Z} \subset \mathscr{R}$ as an open dense subspace. Every $f \in BD$ may be considered as a continuous function defined on $\mathscr{R}$ by $f(\varphi) = \varphi(f)$. It follows easily that $\mathscr{R}$ is the Royden compactification of $\boldsymbol{Z}$.

REMARK 3.1. *The details of the proof that $\mathscr{R}$ is indeed the Royden compactification can be found in the monograph by Sario and Nakai* [7], *Chapter* III, §2B, (*for Riemann surfaces or manifolds*), *or in the book of Soardi* [8], *Theorem* 6.4. (*for graphs and electrical networks*).

We shall now study the properties of the characters on BD. First we observe that $\varphi(f)$ is always a real number.

PROPOSITION 3.2. *Let $\varphi \in \mathscr{R}$ and $f \in BD$, then $\varphi(f)$ is a real number.*

PROOF. To the contrary, assume that, for some $f \in BD$, we have $\varphi(f) = a + bi$ then for $F = \dfrac{f-a}{b}$ one gets $\varphi(F) = i$, hence $\varphi(F^2 + \boldsymbol{1}) = 0$. However, the element $F^2 + \boldsymbol{1}$ is invertible in BD because it is separated from 0 (is greater than or equal to 1). Therefore no $\varphi \in \mathscr{R}$ can vanish on it, hence a contradiction. □

It follows that for the point-mass functions δ_n it must be $\varphi(\delta_n) \in \{0, 1\}$. If $\varphi(\delta_n) = 1$, for some $n \in \boldsymbol{Z}$, then $\varphi(f) = \varphi(f)\varphi(\delta_n) = \varphi(f\delta_n) = \varphi(f(n)\delta_n) = f(n)$, for all $f \in BD$, and therefore $\varphi = \varphi_n$, which is the trivial case.

In what follows we shall assume that $\varphi(\delta_k) = 0$ for all $k \in \boldsymbol{Z}$. The set $\boldsymbol{Z}$ of integers splits into the positive part $\boldsymbol{N}$ and the negative part $\boldsymbol{Z} \backslash \boldsymbol{N}$ and the characteristic functions of these sets, say $\boldsymbol{1}_+$ and $\boldsymbol{1}_-$, are both BD-functions. By considering the value of φ on the sum and on the product of them, one easily gets that

$$\varphi(\boldsymbol{1}_+), \varphi(\boldsymbol{1}_-) \in \{0, 1\}$$

and

$$\varphi(\boldsymbol{1}_+) = 1 \quad \text{iff } \varphi(\boldsymbol{1}_-) = 0 \text{ (and vice versa).}$$

This implies that, for an arbitrary $f \in BD$, the value of $\varphi(f)$ is equal either to $\varphi(f_+)$ or to $\varphi(f_-)$, where we define $f_+ = f \cdot \boldsymbol{1}_+$ and $f_- = f \cdot \boldsymbol{1}_-$. Therefore we can restrict our considerations to $\boldsymbol{N}$ instead of $\boldsymbol{Z}$, and the Royden compactification of $\boldsymbol{Z}$ will be the sum of two homeomorphic copies of that of $\boldsymbol{N}$.

From now on we shall consider only the subspace BD_+ of all BD-functions supported on $\boldsymbol{N}$. From the above assumption on φ it follows that the value $\varphi(f)$ does not depend on any finite number of terms of the sequence $\{f(n) : n \geq 0\}$. We are going to show that, as a matter of fact, it equals a limit of a subsequence $f(n_k)$. We start with the following

PROPOSITION 3.3. *If* $f \in BD_+$ *and* $\lim f = 0$, *then* $\varphi(f) = 0$.

PROOF. We begin the proof with a useful definition.

DEFINITION 3.4. *For an arbitrary* $n \in N$, *let* f_n *stand for the function equal to* f *on the interval* $[n, +\infty)$ *and zero elsewhere.*

Now take f and φ as in the hypothesis. Then $\varphi(f) = \varphi(f_n)$, independently of n. For every $\varepsilon > 0$, there exists an $n \in N$ such that both sup norm and the square root of the Dirichlet sum of f_n are less than $\frac{\varepsilon}{2}$. Hence also $\|f_n\| < \varepsilon$. This means that $\|f_n\| \to 0$ in BD and therefore, by continuity of φ, we get $\varphi(f) = \varphi(f_n) \to 0$. Hence $\varphi(f) = 0$. □

PROPOSITION 3.5. *If* $f \in BD_+$ *and* $f \geq 0$ *then* $\varphi(f) \geq 0$.

PROOF. For an $\varepsilon > 0$, define $g_\varepsilon = \sqrt{f + \varepsilon}$.

Then g_ε is a bounded function, which is bounded from below by $\sqrt{\varepsilon}$. Moreover,

$$D(g_\varepsilon) \leq \frac{1}{2 \cdot \sqrt{\varepsilon}} \cdot D(f)$$

implies that $g_\varepsilon \in BD_+$. Furthermore, $\varphi(g_\varepsilon)$ is a real number, so

$$0 \leq \varphi(g_\varepsilon)^2 = \varphi(g_\varepsilon^2) = \varphi(f + \varepsilon) = \varphi(f) + \varepsilon.$$

Hence $0 \leq \varphi(f) + \varepsilon$ for any positive ε, so finally we get $0 \leq \varphi(f)$. □

COROLLARY 3.6. 1. *If* $\limsup f = 0$ *then* $\varphi(f) \leq 0$.
2. *If* $\liminf f = 0$ *then* $\varphi(f) \geq 0$.

PROOF. Let us prove 1. For an arbitrary $\varepsilon > 0$ there exists $n \in N$ such that $f_n \leq \varepsilon$, whence $f_n - \varepsilon \leq 0$. Therefore, $\varphi(f) - \varepsilon = \varphi(f_n - \varepsilon) \leq 0$. This proves 1., and 2. is obtained by taking $-f$ instead of f. □

DEFINITION 3.7. *For a bounded function f, let* f^d *denote the set of all cluster points of the sequence* $\{f(n) : n \in N\}$. *In other words,* f^d *consists of all real numbers p which are limits of the form* $p = \lim_{k \to +\infty} f(n_k)$ *for some increasing sequence* (n_k) *of positive integers.*

Obviously, the set f^d is a closed subset of the real line and it turns out that, for a BD-function, it is a closed interval.

PROPOSITION 3.8. *For any* $f \in BD_+$ *we have* $f^d = [\liminf f, \limsup f]$.

PROOF. If there was a gap in f^d of length, say, $\varepsilon > 0$, then the Dirichlet sum for f would contain infinitely many terms greater than ε^2, because there would be infinitely many terms of the sequence $\{f(n)\}$ above as well as below the gap (at least those converging to $\liminf f$ and to $\limsup f$). Hence $D(f) = +\infty$, which would contradict the assumption $f \in BD$. □

To end this section we formulate a useful property of characters on BD_+, which is valid in the general setting of an alebra of bounded continuous real-valued functions on a completely regular space, which contains constant functions and separates points and closed sets. For this purpose we introduce a notation.

Definition 3.9. *Let $f \in BD$, then we define*

$$f^+(n) = \begin{cases} f(n) & \text{if } f(n) > 0, \\ 0 & \text{if } f(n) \leq 0. \end{cases}$$

and $f^- = f - f^+$.

Proposition 3.10. *If $\varphi(f) = 0$, then $\varphi(f^+) = 0$ and $\varphi(f^-) = 0$. In particular, $\varphi(|f|) = 0$.*

The proof follows the same arguments as used in the discussion of $\varphi(\mathbf{1}_+)$ and $\varphi(\mathbf{1}_-)$, so there is no need to repeat it here.

4. Čech-Stone compactification βN

In this section, we briefly recall basic facts about the Čech-Stone compactification of the discrete topological space N of positive integers. For the details we encourage the reader to see the nice paper by W. Rudin ([6]). As a general source book for the theory of ultrafilters we recommend the monograph ([1]).

Recall first that the Čech-Stone compactification βN of N can be identified with the space of all ultrafilters on N with the topology given by the following basis of open sets:

$$V(E) = \{\Omega \in \beta N : E \in \Omega\} \tag{5}$$

where $E \subset N$ is arbitrary. An ultrafilter on N is a family Ω of nonempty subsets of N with the following properties:

1. If $A, B \in \Omega$, then $A \cap B \in \Omega$.
2. If $A \in \Omega$ and $B \supset A$, then $B \in \Omega$.
3. For every $A \subset N$, we have $A \in \Omega \Longleftrightarrow N \setminus A \notin \Omega$.

A non-empty family satisfying only 1. and 2. is called a *filter* on N and the condition 3. is equivalent to maximality of Ω with respect to the ordering given by set inclusion. An ultrafilter is called *free* if it does not contain any finite set. On the other hand, if an ultrafilter Ω contains a finite subset, then it must contain also exactly one single point set $\{n\}$ for some $n \in N$; in this case it consists of all subsets of N that contain $\{n\}$ and is denoted by Ω_n. By identifying every $n \in N$ with Ω_n one gets an inclusion $N \subset \beta N$ as a dense discrete open subspace.

With the above topology the Čech-Stone compactification βN can be also identified with the Gelfand space of the commutative Banach algebra l^∞ of all bounded sequences with the sup norm ([3] Theorem 3.2.11. and example A.2.1). The action of an ultrafilter Ω on a bounded sequence f is the generalized limit defined in the following way.

DEFINITION 4.1. *$\Omega(f) = p$ if, for every positive ε, all the sets $\{n : |f(n) - p| < \varepsilon\}$ belong to Ω.*

REMARK 4.2. *It is known that the generalized limit $\Omega(f)$ exists for every $f \in l^\infty$ and $\Omega \in \beta N$.*

REMARK 4.3. *A family $\mathscr{P}$ of subsets of N can be extended to an ultrafilter if and only if it has the* finite intersection property, *that is every finite collection of elements of $\mathscr{P}$ has non-empty intersection* (*see* [1], *Theorem* 2.2).

5. Royden compactification $\mathscr{R}_+$ as a quotient space of the Čech-Stone compactification βN

Now we are going to define an equivalence relation on βN which will be essential in the description of the Royden compactification.

DEFINITION 5.1. *For $\Omega, \Omega' \in \beta N$ we write $\Omega \sim \Omega'$ if $\Omega(f) = \Omega'(f)$ for all $f \in BD$.*

Observe that $\Omega(f)$ is well defined because the function f is bounded. The following Proposition is considered as a standard topological fact and we give it without proof.

PROPOSITION 5.2. *The relation "$\sim$" is an equivalence on βN, so the quotient $S = \beta N/_\sim$ is a compact Hausdorff space including N* as an open dense subset. *The topology on S is given by the basis*:

$$W(E) = \{[\Omega]_\sim : E \in \Omega\}, \qquad E \subset N. \tag{6}$$

The main result of this section is the following.

THEOREM 5.3. *The Royden compactification $\mathscr{R}_+$ is homeomorphic to the quotient space S.*

Before starting the proof we would like to justify this approach. It is important, although quite easy, to see that the restriction of $\Omega \in \beta N$ to BD_+ is a character and that two equivalent ultrafilters have the same restrictions to BD_+. Therefore, an equivalence class $[\Omega]_\sim \in S$ defines a unique element $\varphi_\Omega \in \mathscr{R}_+$ by

$$\varphi_\Omega(f) = \Omega(f) \qquad \text{for } f \in BD, \tag{7}$$

which does not depend on the choice of a representative from $[\Omega]_\sim$. What we need to prove Theorem 5.3 is to show that the mapping

$$S \ni [\Omega]_\sim \mapsto \varphi_\Omega \in \mathscr{R}_+ \tag{8}$$

is $1-1$, onto and continuous, because both spaces are compact! Observe also that $[\Omega_n]_\sim = \{\Omega_n\}$ can be identified with φ_n for every $n \in N$, so we may restrict our attention to free ultrafilters only.

PROOF. (i) We start with a simple proof of injectivity:

If $\Omega \not\sim \Omega'$, then for some $f \in BD$ we have $\Omega(f) \neq \Omega'(f) \Longleftrightarrow \varphi_\Omega(f) \neq \varphi_{\Omega'}(f)$ which implies that $\varphi_\Omega \neq \varphi_{\Omega'}$.

(ii) To prove the surjectivity, we will show that every $\varphi \in \mathscr{R}_+$ can be extended to an ultrafilter $\Omega \in \beta N$ with the restriction to BD_+ equal to φ again. For this purpose we show that every φ uniquely determines a family $\mathscr{P}_\phi$ of subsets of N which has the finite intersection property.

DEFINITION 5.4. *For $\varphi \in \mathscr{R}_+$ we define*

$$\mathscr{P}_\phi = \{\{n : |f(n)| < \varepsilon\} : f \in BD,\ \varepsilon > 0,\ \varphi(f) = 0\}. \tag{9}$$

The family $\mathscr{P}_\phi$ is non-empty:

(1) if $f \to 0$, then $\varphi(f) = 0$, hence $[n, +\infty) \in \mathscr{P}_\phi$ for $n \in N$;

(2) if $\varphi(f) = c$ then $\varphi(f - c) = 0$ and all the sets $\{n : |f(n) - c| < \varepsilon\}$ belong to $\mathscr{P}_\phi$.

The next Lemma is crucial in the proof of the surjectivity.

LEMMA 5.5. *Let $\varphi \in \mathscr{R}_+$. Then for any $A, B \in \mathscr{P}_\phi$ there exists $C \in \mathscr{P}_\phi$ such that $C \subset A \cap B$.*

PROOF. Let $A = \{n : |f(n)| < \varepsilon\}$ and $B = \{n : |g(n)| < \varepsilon'\}$ for some positive ε and ε' and $f, g \in BD_+$ with $\varphi(f) = \varphi(g) = 0$. Then also $\varphi(|f|) = \varphi(|g|) = 0$; in particular, for $h = |f| + |g|$ we have $\varphi(h) = 0$, $h \in BD_+$ and $h \geq 0$. Let $\varepsilon_0 \leq \min\{\varepsilon, \varepsilon'\}$, then for any $n \in N$

$$h(n) < \varepsilon_0 \Longleftrightarrow |f(n)| + |g(n)| < \varepsilon_0 \Rightarrow |f(n)| < \varepsilon \wedge |g(n)| < \varepsilon' \Rightarrow n \in A \cap B.$$

Hence $C = \{n : h(n) < \varepsilon_0\} \subset A \cap B$, and obviously $C \in \mathscr{P}_\phi$. $\square$

COROLLARY 5.6. *For every $\varphi \in \mathscr{R}_+$, the family $\mathscr{P}_\phi$ has the finite intersection property.*

Actually, if we define

$$\mathscr{F}_\phi = \{D \subset N : \exists A \in \mathscr{P}_\phi \quad A \subset D\}, \tag{10}$$

then $\mathscr{F}_\phi$ is a filter on N. Any of its maximal extensions can serve as an ultrafilter containing $\mathscr{P}_\phi$. This proves the surjectivity.

Let us also observe that, for two distinct φ_0 and φ_1, the families $\mathscr{P}_{\varphi_0}$ and $\mathscr{P}_{\varphi_1}$ are different. To see this assume that there exists $f \in BD_+$ such that $p_0 = \varphi_0(f) \neq \varphi_1(f) = p_1$. By taking $f' = \dfrac{f - p_0}{p_1 - p_0}$ we may assume that $p_i = i$ for $i = 0, 1$. Then, for sufficiently small ε, the set $\{n : |f(n)| < \varepsilon\} \cap \{n : |f(n) - 1| < \varepsilon\}$ is finite (possibly empty). However, $\{n : |f(n)| < \varepsilon\} \in \mathscr{P}_{\varphi_0}$ and for $g = f - 1$ also $\{n : |g(n)| < \varepsilon\} \in \mathscr{P}_{\varphi_1}$. In view of the above Lemma the families $\mathscr{P}_{\varphi_0}$ and $\mathscr{P}_{\varphi_1}$ differ.

(iii) To prove the continuity, let us take an open neighbourhood $\mathscr{U}$ of some $\psi \in \mathscr{R}_+$, say:

$$\mathscr{U} = \{\varphi \in \mathscr{R}_+ : |\varphi(f_1) - \psi(f_1)| < \varepsilon_1, \ldots, |\varphi(f_k) - \psi(f_k)| < \varepsilon_k\}, \tag{11}$$

where $f_1, \ldots, f_k \in BD$ are given and $\varepsilon_1, \ldots, \varepsilon_k > 0$ are fixed. Let $g_i = f_i - \psi(f_i)$ for $i = 1, \ldots, k$, then also $\psi(g_i) = 0$.

LEMMA 5.7. *Let $g \in BD_+$, $\Omega \in \beta N$, $\varphi \in \mathscr{R}_+$, $\varepsilon > 0$ be arbitrary. Then:*

1. *If $\{n : |g(n)| < \varepsilon\} \in \Omega$, then $|\Omega(g)| \leq \varepsilon$. In particular, if $\{n : |g(n)| < \varepsilon\} \in \mathscr{P}_\phi$, then $|\varphi(g)| \leq \varepsilon$.*
2. *If $|\Omega(g)| < \varepsilon$, then $\{n : |g(n)| < \varepsilon\} \in \Omega$. In particular, if $|\varphi(g)| \leq \varepsilon$, then $\{n : |g(n)| < \varepsilon\} \in \mathscr{P}_\phi$.*

PROOF. To prove 1. assume, to the contrary, that $|\Omega(g)| = p > \varepsilon$. There exists $\delta > 0$ such that the intervals $(p - \delta, p + \delta)$ and $(-\varepsilon, \varepsilon)$ are disjoint. Then also the sets $\{n : |g(n) - p| < \delta\}$ and $\{n : |g(n)| < \varepsilon\}$ are disjoint. However, both of them should be in Ω, by the assumptions. This is a contradiction, which proves the first part of 1. The second part follows by taking any extension Ω of φ.

To prove 2. observe that, by definition, if $|\Omega(g)| = p < \varepsilon$ then for sufficiently small δ we have the inclusion $\{n : |g(n) - p| < \delta\} \subset \{n : |g(n)| < \varepsilon\}$. The smaller set is in Ω so the bigger one is there as well. This proves the first part of 2. and the second part follows by the same argument as in 1. Thus the lemma is proved. $\square$

Now, all the sets $\left\{n : |g_i(n)| < \frac{\varepsilon_i}{2}\right\}$ are in $\mathscr{P}_\psi$, so, by Lemma 5.5, there exists an element E of $\mathscr{P}_\psi$ included in the intersection of the sets $\left\{n : |g_i(n)| < \frac{\varepsilon_i}{2}\right\}$. We shall show that the open subset $W(E)$ of S is mapped into $\mathscr{U}$. For this purpose take $\Omega \in \beta N$ such that $E \in \Omega$, (i.e. $[\Omega]_\sim \in W(E)$). Then all sets containing E, in particular $\left\{n : |g_i(n)| < \frac{\varepsilon_i}{2}\right\}$, are in Ω. By the above Lemma, $\Omega(g_i) \leq \frac{\varepsilon_i}{2} < \varepsilon_i$. Hence, $\varphi(g_i) < \varepsilon_i$ for every $i = 1, \dots, k$. This proves the continuity and finishes the proof of the theorem. $\square$

6. An explicit description of $\mathscr{R}_+$

In this final section we describe more explicitly the Royden compactification $\mathscr{R}_+$ of N and its relation with the Čech-Stone compactification. We show how the equivalence relation "$\sim$" on βN can be expressed in terms of properties of subsets of N. This will help us to show how to construct ultrafilters equivalent to a given one. We also describe more explicitly the topology on $\mathscr{R}_+$.

Let us start with some simple remarks on subsets of N. If $A, B \subset N$ are infinite and disjoint, $A \cap B = \varnothing$, then we can uniquely represent A and B as disjoint unions

$$A = \bigcup_{j=1}^{\infty} A_j, \qquad B = \bigcup_{j=1}^{\infty} B_j, \tag{12}$$

where

$$\dots < \min A_j \leq \max A_j < \min B_j \leq \max B_j < \min A_{j+1} \leq \dots \tag{13}$$

Moreover, the natural numbers

$$k_j = \min B_j - \max A_j, \qquad k_j' = \min A_{j+1} - \max B_j \tag{14}$$

are then also uniquely determined.

It follows that the sets A_j, B_j are finite and ordered in the above sense. We will call this representation the *disjoint representation* of the pair (A, B).

The next theorem, the main one in this section, describes which ultrafilters *are not* equivalent, which is more natural to formulate than when they *are* equivalent.

THEOREM 6.1. *Let $\Omega, \Omega' \in \beta N$ be two arbitrary ultrafilters. Then the following two conditions are equivalent*:

1. $\Omega \not\sim \Omega'$;
2. *There exist two disjoint subsets $A \in \Omega$ and $B \in \Omega'$ such that for the*

disjoint representation of the pair (A, B)

$$\sum_{j=1}^{\infty} \frac{1}{k_j} < +\infty, \qquad \textit{and} \qquad \sum_{j=1}^{\infty} \frac{1}{k'_j} < +\infty. \tag{15}$$

PROOF. 2. $\Rightarrow$ 1. Assume that A, B are given as in 2. and define a function f on N by

$$f(n) = \begin{cases} 0 & \text{if } n \in A, \\ 1 & \text{if } n \in B, \\ \text{“linear between”}. \end{cases}$$

Then f is bounded, non-negative, $f \equiv 0$ on $[\min A_j, \max A_j]$, $f \equiv 1$ on $[\min B_j, \max B_j]$ and f is linear between A_j's and B_j's. One easily checks that the Dirichlet sum for f is

$$D(f) = \sum_{j=1}^{\infty} \left(\frac{1}{k_j} + \frac{1}{k'_j} \right)$$

so $f \in BD_+$. Now, $A \subset \{n : f(n) < \varepsilon\}$ implies that $\{n : f(n) < \varepsilon\} \in \Omega$ and $B \subset \{n : |f(n) - 1| < \varepsilon\}$ implies that $\{n : |f(n) - 1| < \varepsilon\} \in \Omega'$ for every $\varepsilon > 0$, so that $\Omega(f) = 0$ and $\Omega'(f) = 1$. Hence (1) follows.

1. $\Rightarrow$ 2. Let $\Omega \not\to \Omega'$. Then there exists $f \in BD_+$ for which $p = \Omega(f) \neq \Omega'(f) = p'$. By taking $f' = \dfrac{f - p}{p' - p}$ instead of f we may assume that $p = 0$ and $p' = 1$. For an $\varepsilon < \frac{1}{3}$ define $A = \{n : |f(n)| < \varepsilon\}$, $B = \{n : |f(n) - 1| < \varepsilon\}$. Then $A \cap B = \varnothing$ and both are infinite (recall that Ω, Ω' are free ultrafilters!). Take the disjoint decomposition of the pair (A, B) and consider the strictly increasing sequence $\mathbf{r}$ of positive integers naturally constructed from the following sequence:

$$\ldots \min A_j \leq \max A_j < \min B_j \leq \max B_j < \min A_{j+1} \leq \ldots. \tag{16}$$

Let $\hat{f}$ be defined by the LINEARIZATION process of f for $\mathbf{r}$ (described at the end of section 2). Then $\hat{f} \in BD_+$ and $D(\hat{f}) \leq D(f) < +\infty$. However, the Dirichlet sum for $\hat{f}$ gives the following estimation:

$$\begin{aligned} D(\hat{f}) &\geq \sum_{j=1}^{\infty} \left\{ \sum_{n=\max A_j}^{\min B_j} |\hat{f}(n+1) - \hat{f}(n)|^2 + \sum_{n=\max B_j}^{\min A_{j+1}} |\hat{f}(n+1) - \hat{f}(n)|^2 \right\} \\ &\geq (1 - 2\varepsilon)^2 \sum_{j=1}^{\infty} \left(\frac{1}{k_j} + \frac{1}{k'_j} \right); \end{aligned}$$

hence 2. follows. □

It is natural ask the following question: For a given ultrafilter Ω, how does one find other elements of its equivalence class $[\Omega]_\sim$? As we have seen, the question is trivially answered in the case of non-free ultrafilters Ω_n. For free ultrafilters, however, we can only give here a method of constructing a set of the power of the continuum of ultrafilters equivalent to a given one. This method is not sufficient to describe *all* such ultrafilters, so the problem of a complete description remains open.

Our construction is the following. For an arbitrary bijection $\phi : \boldsymbol{N} \to \boldsymbol{N}$ and an ultrafilter $\Omega \in \beta\boldsymbol{N}$ we define

$$\phi(\Omega) = \{\phi(A) : A \in \Omega\}, \tag{17}$$

where $\phi(A) = \{\phi(n) : n \in A\}$. It is seen at once that $\phi(\Omega)$ is an ultrafilter on $\boldsymbol{N}$. For special ϕ's it turns out to be equivalent to Ω.

PROPOSITION 6.2. *Let $\Omega \in \beta\boldsymbol{N}$ be a free ultrafilter. If*

$$\sup_{n \in \boldsymbol{N}} |\varphi(n) - n| < +\infty. \tag{18}$$

then $\phi(\Omega) \sim \Omega$.

PROOF. Assume, to the contrary, that $\phi(\Omega) \not\sim \Omega$. Then we can find $A \in \Omega$ and $B \in \Omega' = \phi(\Omega)$ such that the disjoint representation of the pair (A, B) satisfies condition 2. of Theorem 6.1. In particular,

$$k_j = \min B_j - \max A_j \to +\infty.$$

We may assume that $B = \phi(D)$ for some $D \in \Omega$ such that $D \subset A$, because a subset of B also satisfies the same conditions when paired with A. Let $\min B_j = \phi(d_j)$. Then the sequence $|d_j - \varphi(d_j)|$ is bounded, by the property of ϕ. On the other hand, it tends to $+\infty$ because $|d_j - \min B_j| \geq |\max A_j - \min B_j| = k_j$. This is impossible, hence, by contradiction, the proposition follows. □

Another natural and important question is how big is the space $\mathscr{R}_+$. It will be shown that, in the set-theoretical sense, it has the same power as the Čech-Stone compactification. For this purpose observe, that, if $A \in \Omega$ then $\Omega_A = \{B \cap A : B \in \Omega\}$ is an ultrafilter on A and then $\Omega = \{D \subset \boldsymbol{N} : \exists B \in \Omega_A \ B \subset D\}$. In general, any free ultrafilter on a given infinite subset A of $\boldsymbol{N}$ can be uniquely extended to an ultrafilter on $\boldsymbol{N}$, in the same way as above. On the other hand, if $A = \{r_j\}$ is a sequence which satisfies the condition $S(\mathbf{r}) < +\infty$ (see (3) in section 2), then two different ultrafilters on A are not equivalent, by Theorem 6.1. However, there are as many free ultrafilters on an infinite (countable) set as on $\boldsymbol{N}$. Hence we are led to

PROPOSITION 6.3. *There are* 2^{c} *elements in* $\mathscr{R}_{+}$, *where* c = continuum *is the power of the set of real numbers.*

Now we are going to describe the topology on $\mathscr{R}_{+}$. Recall that its basis is given by the family of sets

$$W(E) = \{\varphi = [\Omega]_{\sim} : E \in \Omega\},$$

where $E \subset N$. These sets are described in a more explicit way in the following.

THEOREM 6.4. *Let* $E \subset N$ *be arbitrary. Then*

$$W(E) = \left\{\varphi \in \mathscr{R}_{+} : \forall f \in BD_{+}\left(\lim_{n \in E} f(n) = 0 \Rightarrow \varphi(f) = 0\right)\right\}. \tag{19}$$

The theorem is trivial for finite E, so we only need to prove it for infinite sets.

PROOF.

(a) "$\subseteq$" Let $\varphi \in W(E)$ and $f \in BD_{+}$ satisfy $\lim_{n \in E} f(n) = 0$. Then, by the definition of $W(E)$, there exsists $\Omega \in \varphi$ such that $E \in \Omega$. For every $\varepsilon > 0$ the set $\{n : |f(n)| < \varepsilon\}$ is in Ω because it contains E (up to, possibly, a finite number of elements). Hence $\Omega(f) = 0$ and therefore $\varphi(f) = \Omega(f) = 0$.

(b) "$\supseteq$" Let $E \in N$ and assume that a $\varphi \in \mathscr{R}_{+}$ satisfies

$$\lim_{n \in E} f(n) = 0 \Rightarrow \varphi(f) = 0. \tag{20}$$

Our goal is to find $\Omega \in \varphi$ such that $E \in \Omega$. This will be done if we show that the family $\mathscr{P}_{\varphi} \cup \{E\}$ has the finite intersection property. This, in turn, is equivalent to showing that $E \cap A \neq \varnothing$ for all $A \in \mathscr{P}_{\varphi}$, by Lemma 5.5.

Assume, to the contrary, that this is not true, so that there exists $g \in BD_{+}$ such that

1. $\lim_{n \in E} g(n) \neq 0$,
2. $\varphi(g) = 0$,
3. $E \cap \{n : |g(n)| < \varepsilon\} = \varnothing$, for some $\varepsilon > 0$.

We may assume that $g \geq 0$ (by taking $|g|$, if not), and that $g^{d} = [0, 1]$. Then for $A_{\varepsilon} = \{n : |g(n)| < \varepsilon\}$ the pair (A_{ε}, E) has the disjoint representation of the form

$$E = \bigcup_{j} E_{j}, \qquad A_{\varepsilon} = \bigcup_{j} A_{\varepsilon,j}.$$

Let $A_{\varepsilon/2} = \left\{n : |g(n)| < \frac{\varepsilon}{2}\right\}$ and let $A_{\varepsilon/2,j} = A_{\varepsilon/2} \cap A_{\varepsilon,j}$. For the sequences $k_j = |\min A_{\varepsilon/2,j} - \min A_{\varepsilon,j}|$, $k_j' = |\max A_{\varepsilon/2,j} - \max A_{\varepsilon,j}|$, we have

$$\sum_{j=1}^{\infty} \frac{1}{k_j} < +\infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} \frac{1}{k_j'} < +\infty.$$

However, $r_j = |\max E_j - \min A_{\varepsilon/2,j+1}| \geq k_j$ and $r_j' = |\min E_j - \max A_{\varepsilon/2,j}| \geq k_j'$ so the series of inverses of the numbers r_j and r_j' converge. Therefore, if we define

$$f(n) = \begin{cases} 1 & \text{if } n \in A_{\varepsilon/2}, \\ 0 & \text{if } n \in E, \\ \text{“linear between”}, & \end{cases}$$

then $f \in BD_+$ and $\lim_{n \in E} f(n) = 0$. So $\varphi(f) = 0$ and hence $\varphi(f+g) = 0$. On the other hand, $f + g \geq \frac{\varepsilon}{2}$ implies that $\varphi(f+g) \geq \frac{\varepsilon}{2}$, by Proposition 3.5. This gives a contradiction, so that (b) follows. $\square$

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