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Nonlinear perturbations of a class of integrated semigroups

Dedicated to Professor Hiroki Tanabe on the occasion of his sixtieth birthday

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ABSTRACT. Nonlinear perturbations of once integrated semigroups are treated in terms of nonlinear semigroup theory. Given an integrated semigroup $W(t)$ with generator A in a Banach space X, two general classes of nonlinear perturbations of the form $A + B$ are introduced. In order to define local quasidissipativity of $A + B$ and restrict the growth of solutions of the associated semilinear evolution equation (SE): $u'(t) = (A + B)u(t)$, a lower semicontinuous convex functional $\varphi: X \to [0, \infty]$ is employed. Necessary and sufficient conditions are given for a semilinear operator $A + B$ to generate a nonlinear semigroup $S(t)$ in X such that for $v \in D(B)$ the X-valued function $S(\cdot)v$ gives a unique weak solution. Application of the first main result to age-dependent population dynamics is discussed.

0. Introduction

The present paper is concerned with nonlinear semigroups which provide weak solutions to the semilinear problems of the form

$$\text{(SP)} \qquad \frac{d}{dt}u(t) = (A + B)u(t), \quad t > 0; \quad u(0) = v$$

in a real Banach space $(X, |\cdot|)$. Here A is assumed to be the generator of an integrated semigroup $\{W(t): t \geq 0\}$ in X and B is a nonlinear operator from a convex subset C of X into X.

The importance of semilinear problems of the type (SP) has constantly been recognized for many years in various branches of mathematical analysis. In this paper we introduce two general classes of nonlinear perturbations of linear integrated semigroups and discuss necessary and sufficient conditions on $A + B$ for the solutions of (SP) to exist in a global sense.

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The notion of integrated semigroup is a natural extension of the notion of semigroup of class (C_0) and the generation theory is applied to a variety of partial differential equations in an operator theoretic fashion. This new theory of operator semigroups developed in Arendt [1], Neubrander [21], Tanaka and Miyadera [26] is closely related to the theory of exponential distribution semigroups advanced in Lions [17] as well as that of semigroups of classes $(C_{(k)})$ introduced by Oharu [22], although it is an important feature that the generators of integrated semigroups need not be densely defined. In fact, Da Prato and Sinestrari [5] treated typical differential operators with non-dense domains which are eventually generators of integrated semigroups. Also, Kellermann and Hieber established in [15] a characterization theorem of a specific but natural class of integrated semigroups in terms of the corresponding generators in the sense of Arendt and showed that their results can be applied to various types of linear partial differential equations.

On the other hand, there has been a substantial development in the theory of nonlinear semigroups associated with semilinear problems (SP) in which A is assumed to be the infinitestimal generator of a (C_0)-semigroup in X. We here focus our attention to nonlinear perturbations of (C_0)-semigroups which are treated in Oharu and Takahashi [23]. In their treatise various types of characterizations of nonlinear semigroups providing weak solutions of semilinear problems of the form (SP) are obtained in terms of the corresponding semilinear infinitestimal generators. Their arguments contain three features: Firstly a lower semicontinuous functional $\varphi: X \to [0, \infty]$ is employed to define a local quasidissipativity of $A + B$ and the growth of solutions to (SP) is restricted in terms of the nonnegative function $\varphi(u(\cdot))$. In case of concrete partial differential equations the use of such functionals corresponds to a priori estimates or energy estimates which ensure the global existence of the solutions as well as their asymptotic properties. Secondly, the semilinear operator $A + B$ is assumed to be quasidissipative on φ-bounded sets. Thirdly, $A + B$ is assumed to satisfy the so-called implicit subtangential condition or explicit subtangential condition. Because of the simplicity and universality of these conditions it is possible to apply the perturbation theory to various evolution problems.

Therefore it is important from both theoretical and practical points of view to discuss semilinear problems (SP) in terms of nonlinear perturbations of integrated semigroups. Here we consider a class of integrated semigroups treated in [15] and investigate nonlinear perturbations of such integrated semigroups from the same point of view as in [23]. Locally Lipschitz perturbations of such integrated semigroups have been studied in [28] and the uniqueness of the associated weak solutions is discussed in [20]. Our results are affected by those works.

This paper is organized as follows: In Section 1 a class of integrated semigroups treated in [15] by Kellermann and Hieber is introduced and the basic results are outlined so that they may be directly applied to nonlinear perturbation problems. In Section 2, two generalized notions of solutions to (SP) and a notion of nonlinear semigroups associated with semilinear problems (SP) are discussed. Furthermore, a natural notion of semilinear infinitestimal generator is introduced for such nonlinear semigroups. Section 3 contains our first main result (Theorem 3.1) which gives a characterization of nonlinear semigroups providing weak solutions to (SP) under the implicit subtangential conditions. This result may be regarded as a nonlinear version of the characterization theorem for integrated semigroups due to Kellermann and Hieber. In Iwamiya [13], a time-dependent nonlinear perturbation theorem is given for (C_0)-semigroups. Section 4 discusses an extension of his result to the case of integrated semigroups. The extension is applied to obtain our second main result (Theorem 5.1) which gives a characterization of nonlinear semigroups providing weak solutions to (SP) under the explicit subtangential conditions. This result is given in Section 5 in connection with a perturbation result due to Thieme [28].

Section 6 is devoted to the application of the first main theorem to semilinear problems with semilinear constraints. The result may be regarded as an existence theorem for weak solutions to abstract initial-boundary problems for semilinear evolution equations. Finally, in Section 7, the result obtained in Section 6 is applied to a typical mathematical model arising in the study of population dynamics.

1. A class of integrated semigroups

In this section we introduce a class of integrated semigroups and discuss a characterization of such integrated semigroups as well as some basic facts on the corresponding generators.

In what follows, $(X^*, |\cdot|)$ denotes the dual space of X. For $x \in X$ and $f \in X^*$ the value of f at x is written as $\langle x, f \rangle$. The duality mapping of X is denoted by $\mathscr{F}$. For x, $y \in X$ the symbol $\langle x, y \rangle_i$ stands for the infimum of the set $\{\langle x, f \rangle : f \in \mathscr{F}(y)\}$. In (SP) the operator A is assumed to be a closed linear operator in X whose domain is not necessarily dense in X. We write A^* for the dual operator of A. If A is densely defined in X, then A^* is defined as a closed linear operator in X^*. If the domain $D(A)$ is not dense in X, then A^* is multi-valued and the identity $\langle Ax, f \rangle = \langle x, g \rangle$ holds for $x \in D(A)$, $f \in D(A^*)$ and $g \in A^*f$; hence the value $\langle x, g \rangle$ does not depend upon the choice of $g \in A^*f$. It follows immediately that $\langle x, g \rangle$ does not depend upon the choice of $g \in A^*f$ provided that $x \in Y$, where Y is the closure of $D(A)$:

$$Y \equiv \overline{D(A)}.$$

Hence for $y \in Y$ and $f \in D(A^*)$ it is justified and convenient to introduce the notation

$$\langle y, A^*f \rangle = \langle y, g \rangle, \quad g \in A^*f.$$

In concrete cases it may be difficult to characterize the set-valued operator A^*. In order to overcome this difficulty, we here introduce the notion of A-total set which is an essential part of $D(A^*)$ and makes it easier to formulate the notion of weak solution to the problem (SP).

A subset D of $D(A^*)$ is said to be A-total, if for λ sufficiently small the set

$$(I - \lambda A^*)D \equiv \{f - \lambda g \colon f \in D, g \in A^*f\}$$

separates points in Y in the sense that $x \in Y$ equals 0 whenever $\langle x, h \rangle = 0$ for $h \in (I - \lambda A^*)D$. In the case that a complex number λ is contained in the resolvent set $\rho(A)$, an A-total set can be easily constructed: Take any subset Z of X^* which separates points in Y. Then $D = [(I - \lambda A)^{-1}]^*(Z)$ is A-total. The point of this concept consists in the fact that one can equivalently replace $D(A^*)$ by an A-total subset of $D(A^*)$ in the notion of weak solution to (SP) which is discussed later. In many applications it will be possible to choose an A-total subset of $D(A^*)$ on which A^* has a nice representation while this is not the case on the whole of $D(A^*)$.

To describe the notion of once integrated semigroup in conjunction with linear evolution equations in X, we begin by considering the initial-value problem

$$(\mathrm{IVP})_\psi \qquad u'(t) = Au(t) + \psi(t), \quad t > 0; \quad u(0) = v \in X,$$

where A is a closed linear operator in X, $\psi(\cdot) \in \mathscr{C}((0, \infty); X) \cap L^1_{loc}(0, \infty; X)$ and v is an initial-value given in X. For the problem $(\mathrm{IVP})_\psi$ we may employ the following three kinds of notions of generalized solution. The first notion was first employed in Da Prato and Sinestrari [5].

DEFINITION 1.1. An X-valued function on $[0, \infty)$ is said to be an *integral solution of* $(\mathrm{IVP})_\psi$, if $u(\cdot) \in \mathscr{C}((0, \infty); X) \cap L^1_{loc}(0, \infty; X)$, $\int_0^t u(s)ds \in D(A)$ and

$$u(t) = v + A\int_0^t u(s)ds + \int_0^t \psi(s)ds \qquad \text{for } t \geq 0.$$

Moreover, a function $u(\cdot) \in \mathscr{C}([0, \infty); X)$ is called a $\mathscr{C}^1$-solution of $(\mathrm{IVP})_\psi$, if $u(\cdot) \in \mathscr{C}^1((0, \infty); X)$, $u'(t) = Au(t) + \psi(t)$ for $t > 0$, and $u(0) = v$.

The following notion of generalized solution to $(\mathrm{IVP})_\psi$ is due to J. Ball [2].

DEFINITION 1.2. A function $u(\cdot) \in \mathscr{C}([0, \infty); X)$ is said to be a *weak solution* of $(\mathrm{IVP})_\psi$, if $u(0) = v$ and for each $f \in D(A^*)$ the scalar-valued function $\langle u(\cdot), f\rangle$ is of class $\mathscr{C}^1(0, \infty)$ and satisfies

$$\frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A^*f\rangle + \langle \psi(t), f\rangle \qquad \text{for } t > 0.$$

The third notion is similar to that of weak solution but the last relation being restricted to f in an A-total subset of D of $D(A^*)$.

If $u(\cdot)$ is an integral solution of $(\mathrm{IVP})_\psi$ and is continuous at $t = 0$, then

$$u(t) = \lim_{h \downarrow 0} h^{-1} \int_t^{t+h} u(s)ds \in \overline{D(A)} \qquad \text{for } t \geq 0$$

and so $v = u(0) \in \overline{D(A)}$. Also, $u(\cdot)$ is an integral solution of $(\mathrm{IVP})_\psi$ if and only if the indefinite integral $w(t) = \int_0^t u(s)ds$ gives a $\mathscr{C}^1$-solution of the initial-value problem

$$\text{(1.1)} \qquad w'(t) = Aw(t) + v + \int_0^t \psi(s)ds, \quad t > 0; \quad w(0) = 0.$$

Let $\psi(t) \equiv 0$ in (1.1) and suppose that for each initial-value $v \in X$ there exists a unique integral solution $u(\cdot; v)$ of $(\mathrm{IVP})_\psi$ with $\psi(t) \equiv 0$. Now for each $t \geq 0$ we define an operator $W(t)$ on X by

$$W(t)v = \int_0^t u(r; v)dr \qquad \text{for } v \in X.$$

Since (1.1) is an autonomous linear equation, it follows that each $W(t)$ is linear. Therefore we get a one-parameter family $\mathscr{W} = \{W(t)\}$ of linear operators with the properties below:

$$\text{(w.1)} \qquad W(0)v = 0 \quad \text{and} \quad W(\cdot)v \in \mathscr{C}([0, \infty); X) \qquad \text{for } v \in X.$$

Since $w(\cdot) \equiv W(\cdot)v$ is the unique $\mathscr{C}^1$-solution of (1.1) with $\psi(t) \equiv 0$, we have the relation

$$\text{(1.2)} \qquad W(t)v = A \int_0^t W(r)vdr + tv \qquad \text{for } t \geq 0 \text{ and } v \in X.$$

Let $t \geq 0$ and $v \in X$. Then by assumptions on the existence and uniqueness of integral solutions of $(\mathrm{IVP})_\psi$ with $\psi(t) \equiv 0$ the function $W(\cdot)W(t)v$ is the unique solution of (1.1) with $\psi(t) \equiv 0$ and v replaced by $W(t)v$. We next consider the function

$$z(s) = \int_0^s [W(r+t)v - W(r)v]dr$$

defined on all of $[0, \infty)$. Then $z(0) = 0$ and (1.2) implies

$$\begin{aligned} z'(s) &= W(s+t)v - W(s)v \\ &= A\int_t^{t+s} W(r)vdr + A\int_0^t W(r)vdr + tv - A\int_0^s W(r)vdr \\ &= A\int_0^s [W(r+t)v - W(r)v]dr + W(t)v = Az(s) + W(t)v \end{aligned}$$

for $s > 0$. This shows that $z(\cdot)$ is also a $\mathscr{C}^1$-solution of (1.1) with $\psi(t) \equiv 0$ and v replaced by $W(t)v$. From this we see that $\mathscr{W}$ has the following property:

(*w*.2) $W(s)W(t)v = \int_0^s [W(r+t)v - W(r)v]dr$ for $s, t \geq 0$ and $v \in X$.

The above observation leads us to the following

DEFINITION 1.3. A one-parameter family $\mathscr{W} \equiv \{W(t): t \geq 0\}$ of bounded linear operators is said to be a *once integrated semigroup* on X if it has properties (*w*.1) and (*w*.2). We say that $\mathscr{W}$ is *non-degenerate* if $W(t)v \equiv 0$ for all $t > 0$ implies $v = 0$. If there exist constants $M > 1$ and $\omega \geq 0$ such that $|W(t)| \leq Me^{\omega t}$ for $t \geq 0$, the once integrated semigroup $\mathscr{W}$ is said to be *exponentially bounded.*

Throughout this paper any integrated semigroup is assumed to be non-degenerate, unless stated otherwise.

As shown in Thieme [27, Section 3] there exists a closed linear operator A in X such that $v \in D(A)$, and $w = Av$ are characterized by the property that the function $W(t)v$ is continuously differentiable and

$$\frac{d}{dt}W(t)v = v + W(t)w \qquad \text{for } t > 0.$$

The operator A is called the *generator* of $\mathscr{W}$ and an integrated semigroup is uniquely determined by its generator.

It is further shown in [27, Section 3] that (1.2) holds for integrated semigroups and their generators.

PROPOSITION 1.4. *Let $\mathscr{W}$ be an integrated semigroup and A the generator of $\mathscr{W}$. Then*

(a) $W(t)v \in D(A)$ *and* $AW(t)v = W(t)Av$ *for* $v \in D(A)$ *and* $t \geq 0$.

(b) $\int_0^t W(t)vdt \in D(A)$ *and* $W(t)v = A\int_0^t W(r)vdr + tv$ *for* $v \in X$ *and* $t \geq 0$.

As shown in [27, Section 3], it is seen that if $\mathscr{W}$ is exponentially bounded then the integral

$$R(\xi)v = \xi \int_0^\infty e^{-\xi t}W(t)vdt \tag{1.3}$$

exists for $\xi > \omega$ and $v \in X$, and that $\xi R(\xi) = (I - \lambda A)^{-1}$ for $\xi = 1/\lambda > \omega$. In what follows, we are mainly concerned with closed linear operators A in X satisfying the following condition:

(H1) There is a constant $\omega \in \boldsymbol{R}$ such that for $\lambda > 0$ with $\lambda\omega < 1$ the resolvent $(I - \lambda A)^{-1}$ exists and satisfies

$$|(I - \lambda A)^{-1}| \leq (1 - \lambda\omega)^{-1}.$$

The class of integrated semigroups treated in this paper is that of once integrated semigroups whose generators satisfy the above-mentioned conditions (H1). From condition (H1) we see that the part A_Y of A in the smaller Banach space $Y \equiv \overline{D(A)}$ has a dense domain in Y and generates a semigroup $\mathscr{T}_Y \equiv \{T_Y(t): t \geq 0\}$ of class (C_0) on Y by the Hille-Yosida theorem. Also, it has been shown in [11] that a closed linear operator A satisfying (H1) is the generator of an integrated semigroup $\mathscr{W} \equiv \{W(t): t \geq 0\}$ on X. The following result is a consequence of the results established in [1], [15], [26] and [28] and gives a characterization of the class of integrated semigroups under consideration.

THEOREM 1.5. *A closed linear operator A in X is the generator of a once integrated semigroup $\mathscr{W}$ on X such that*

$$|W(t+h) - W(t)| \leq \int_t^{t+h} e^{\omega s}ds \qquad \text{for } t, h \geq 0 \tag{1.4}$$

if and only if it satisfies condition (H1).

The above theorem follows from Theorem 1.6 below.

At the beginning of this section we observed that an integrated semigroup was derived from indefinite integrals of integral solutions of $(\mathrm{IVP})_\psi$ with $\psi(t) \equiv 0$. One obtains the following structure theorem for integrated semigroups which illustrates this observation.

THEOREM 1.6 (Structure Theorem). *Let A be a closed linear operator in X satisfying* (H1) *and $\mathscr{T}_Y$ the semigroup of class (C_0) on $Y \equiv \overline{D(A)}$ generated* *by the part A_Y of A in Y. Then the integrated semigroup $\mathscr{W}$ generated by A is represented as*

$$W(t)v = \lim_{\lambda\downarrow 0} \int_0^t T_Y(s)(I - \lambda A)^{-1} v ds \qquad \text{for } t \geq 0 \text{ and } v \in X. \tag{1.5}$$

PROOF. We give the proof of this theorem in such a way that the proof of Theorem 1.5 is also outlined. Let A be a closed linear operator in X. First assume that (H1) holds for A. Then, by [1, Theorem 4.1], A generates a once integrated semigroup $\mathscr{W}$ such that

$$\limsup_{h\downarrow 0} h^{-1} |W(t+h) - W(t)| \leq e^{\omega t} \qquad \text{for } t \geq 0.$$

Moreover, it is seen from [1, Proposition 3.3] that $\mathscr{W}$ satisfies (1.2). Let $\mathscr{T}_Y$ be the semigroup of class (C_0) on Y generated by A_Y. Taking any $\lambda > 0$ with $\lambda\omega < 1$ and applying the resolvent $(I - \lambda A)^{-1}$ to both sides of (1.2), we have

$$(I - \lambda A)^{-1} W(t) v = A_Y (I - \lambda A)^{-1} \int_0^t W(s) v ds + t(I - \lambda A)^{-1} v$$

for $t > 0$ and $v \in X$. Differentiating both sides with respect to t, we have

$$\frac{d}{dt}(I - \lambda A)^{-1} W(t) v = A_Y (I - \lambda A)^{-1} W(t) v + (I - \lambda A)^{-1} v.$$

Let $v \in X$. Since $W(0)v = 0$, this implies that the function $W(\cdot)v$ satisfies the variation of constants formula

$$(I - \lambda A)^{-1} W(t) v = \int_0^t T_Y(s)(I - \lambda A)^{-1} v ds \qquad \text{for } t \geq 0.$$

Since $W(t)v \in Y$ for $t \geq 0$ by Proposition 1.4(b), we obtain Formula (1.5) by letting $\lambda \downarrow 0$ in the above identity. Therefore the assertion of the theorem is obtained. Furthermore, $|T_Y(s)| \leq e^{\omega s}$ for $s \geq 0$ and (1.4) follows directly from (1.5). Next suppose that A is the generator of a once integrated semigroup $\mathscr{W}$ satisfying (1.4). Then we see from [1, Theorem 4.1] that (1.3) follows from (1.4) and A satisfies (H1). This means that the assertion of Theorem 1.5 is obtained. The proof is thereby complete. ■

The above structure theorem is contained in Thieme [28] as a characteristic property of locally Lipschitz once integrated semigroups, although the proof is different and contains another proof of Theorem 1.5. By means of this property, Theorem 1.5 may be regarded as a special case of [15, Theorem 2.4] and [26, Theorem 3.1], although to the best of the authors' knowledge there is no explicit mention of this fact. For other representations see Lumer [18]. In [19, Theorem 3.5], Lumer emphasizes the significant role of locally Lipschitz once integrated semigroups among (multiple) integrated semigroups. The following simple consequence suggests the terminology of *integrated* semigroup.

COROLLARY 1.7. *If $D(A)$ is dense in X, then $Y = X$ and $\mathscr{T}_X$ becomes a semigroup of class (C_0) on X and $\mathscr{W}$ is represented as*

$$W(t)v = \int_0^t T_X(r)v\,dr \qquad \textit{for } t \geq 0 \textit{ and } v \in X.$$

We now return to the inhomogeneous initial-value problem $(\mathrm{IVP})_\psi$ which is the starting point of our considerations. That $(\mathrm{IVP})_\psi$ has a unique integral solution was shown by Da Prato and Sinestrari [5] and Bénilan *et al.* [3]. We present several explicit formulas for the weak solutions and several alternative characterizations. At the same time, we give a new proof of the existence result. The cornerstone of the proof is the following observation.

PROPOSITION 1.8. *Let $\psi \in L^1(0, \tau; X)$, $\tau > 0$. Then the function*

$$w(t) = \int_0^t W(r)\psi(t-r)dr, \quad 0 \leq t < \tau$$

is continuously differentiable, takes its values in $D(A)$ and satisfies

$$\begin{aligned} w'(t) &= \int_0^t W(dr)\psi(t-r) \\ &= Aw(t) + \int_0^t \psi(r)dr \\ &= \lim_{\lambda \downarrow 0} \int_0^t T_Y(r)(I - \lambda A)^{-1}\psi(t-r)dr. \end{aligned} \tag{1.6}$$

where the Stieltjes integral makes sense as a uniform limit of Riemann-Stieltjes integrals. Moreover, we have the estimate

$$|w'(t)| \leq \int_0^t e^{\omega r}|\psi(t-r)|dr. \tag{1.7}$$

PROOF. We first assume that ψ is continuously differentiable on $[0, \tau)$. Then

$$w'(t) = \int_0^t W(r)\psi'(t-r)dr + W(t)\psi(0).$$

Integration by parts gives the first equality in (1.6), where the Stieltjes integral is understood to be the limit of Stieltjes sums in the usual way. The estimate (1.7) follows from (1.4). Moreover, we have

$$(1.8) \qquad (I - \lambda A)^{-1} w'(t) = \int_0^t T_Y(r)(I - \lambda A)^{-1} \psi(t - r) dr.$$

Finally, by Proposition 1.4, we find

$$w'(t) = \int_0^t \left(A \int_0^r W(s) ds + r \right) \psi'(t - r) dr.$$

The second equality is now obtained by integration by parts and the fact that A is closed. If $\psi \in L^1(0, \tau; X)$, we choose a sequence ψ_j of continuously differentiable functions on $[0, \tau]$ in such a way that

$$\int_0^\tau |\psi_j(s) - \psi(s)| ds \to 0, \quad j \to \infty.$$

Let w_j be given as in the statement of the proposition with ψ_j replacing ψ. Then

$$w_j'(t) = \int_0^t W(dr) \psi_j(t - r).$$

If follows from (1.7) that $(w_j'(t))_j$ is a Cauchy sequence uniformly in $t \in [0, \tau]$. Hence it has a continuous limit which we denote by

$$\int_0^t W(dr) \psi(t - r),$$

since the limit is independent of the choice of the sequence. Moreover the estimate (1.7) also holds in the limit. As $w_j(0) = 0 = w(0)$, we have uniform convergence of w_j to w and, at the same time, the continuous differentiability of w. Hence the first equality in (1.6) is obtained. The second equality in (1.6) now follows from

$$w_j'(t) = A w_j(t) + \int_0^t \psi_j(s) ds$$

and the fact that A is closed. Since (1.8) holds for w_j and $|T_Y(r)(I - \lambda A)^{-1}| \leq e^{\omega r}$, the relation (1.8) also holds for w. As $w'(t) \in Y$ we can take the limit as $\lambda \to 0$ and obtain the last equality in (1.6). This completes the proof. ■

The following results illustrate the main points of the above discussions.

THEOREM 1.9. *Let $\tau \in [0, \infty]$. Assume that ψ: $[0, \tau) \to X$ and u: $[0, \tau) \to Y$ be continuous. Let $v \in Y$. Then the following statements are equivalent*:

(i) *The function u is a weak solution of* $(\mathrm{IVP})_\psi$ *in the sense that* $u(0) = v$, $\langle u(t), f\rangle$ *is continuously differentiable on* $(0, \tau)$ *and*

$$\frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A^*f\rangle + \langle \psi(t), f\rangle$$

for all $0 < t < \tau$.
(ii) *The function u is a weak solution of* $(\mathrm{IVP})_\psi$ *in the same sense as in* (i), *but with f being restricted to elements in an A-total subset of* $D(A^*)$.
(iii) *The function u is an integral solution of* $(\mathrm{IVP})_\psi$ *in the sense that* $\int_0^t u(s)ds \in D(A)$ *and*

$$u(t) = v + A\int_0^t u(s)ds + \int_0^t \psi(s)ds, \quad 0 \le t < \tau.$$

(iv) $u(0) = v$ *and*

$$\lim_{h\downarrow 0} h^{-1}(u(t+h) - T_Y(t)u(t) - W(h)\psi(t)) = 0, \quad 0 < t < \tau.$$

(v)
$$u(t) = T_Y(t)v + \int_0^t W(dr)\psi(t-r), \quad 0 \le t < r.$$

(vi)
$$u(t) = T_Y(t)v + \lim_{\lambda\downarrow 0}\int_0^t T_Y(r)(I - \lambda A)^{-1}\psi(t-r)dr, \quad 0 \le t < \tau.$$

(vii)
$$u(t) = T_Y(t)v + \frac{d}{dt}\int_0^t W(r)\psi(t-r)dr, \quad 0 \le t < \tau.$$

(viii)
$$u(t) = T_Y(t)v + A\int_0^t W(r)\psi(t-r)dr + \int_0^t \psi(r)dr, \quad 0 \le t < \tau.$$

Moreover the expressions in formulas (v) *through* (viii) *are well-defined and, simultaneously, provide unique solutions as mentioned in* (i) *through* (iv).

Proof. It follows from Proposition 1.8 that the expressions in formulas (v) through (viii) are well-defined and equivalent. Proposition 1.8 also implies that, for w introduced there, $w'(t)$ is an integral solution of $(\mathrm{IVP})_\psi$ with $v = 0$. As $z(t) = T_Y(t)v$ is an integral solution of $(\mathrm{IVP})_\psi$ with $\psi = 0$, it follows that any of the equivalent formulas (v) through (viii) gives solutions to (iii). A solution in (iii) is also a solution in (i) and (ii). We now show that (v) implies (iv). By Theorem 1.6, we have

$$\frac{1}{h}(u(t+h) - T_Y(t)u(t) - W(h)\psi(t))$$
$$= \frac{1}{h}\lim_{\lambda\downarrow 0}\int_t^{t+h} T_Y(t+h-r)(I - \lambda A)^{-1}(\psi(r) - \psi(t))dr,$$

which converges to 0 as $h \downarrow 0$ because of (H1) and $|T_Y(s)| \le e^{\omega s}$. The proof of this theorem is complete if we show that any solution stated in (ii) or (iv) is given by one of the equivalent formulas (v) to (viii). Since we already know that (v) provides the desired solutions, we only need to show that the notions stated in (ii) and (iv) are sufficient to imply the uniqueness. Since the problems are linear, this follows from the following Lemmas. The proof is now complete. ■

LEMMA 1.10. *Let D be an A-total subset of $D(A^*)$ and $w: [0, \tau) \to Y$ a continuous function satisfying $w(0) = 0$ and $(d/dt)\langle w(t), f\rangle = \langle w(t), A^*f\rangle$ for $t \in (0, \tau)$ and $f \in D$. Then $w(t) \equiv 0$ on $[0, \tau)$.*

PROOF. Take any $\sigma \in (0, \tau)$ and define $u_\sigma: [0, \tau) \to Y$ by

$$u_\sigma(t) = w(t) \quad \text{for } t \in [0, \sigma] \quad \text{and} \quad u_\sigma(t) = T_Y(t - \sigma)w(\sigma) \quad \text{for } t \in [\sigma, \infty).$$

Then $u_\sigma(\cdot)$ is exponentially bounded and $(d/dt)\langle u_\sigma(t), f\rangle = \langle u_\sigma(t), A^*f\rangle$ for $t \ge 0$ and $f \in D$. Integrating by parts shows that $\langle \int_0^\infty \exp(-t/\lambda)u_\sigma(t)\,dt, f - \lambda g\rangle = 0$ for $\lambda > 0$ sufficiently small and $f \in D$. As D is A-total, we have $\int_0^\infty e^{-t/\lambda}u_\sigma(t)dt = 0$ for $\lambda > 0$ sufficiently small. This implies that $u_\sigma \equiv 0$ on $[0, \infty)$ for any $\sigma \in (0, \tau)$. The definition of u_σ now implies the assertion. ■

LEMMA 1.11. *Let $w: [0, \tau) \to Y$ be a continuous function satisfying $w(0) = 0$ and*

$$\lim_{h \downarrow 0} h^{-1}(w(t + h) - T_Y(t)w(t)) = 0.$$

Then $w(t) = 0$ for $t \in [0, \tau)$.

PROOF. Let $w_\lambda(t) = (I - \lambda A)^{-1}w(t)$. Then $w_\lambda'(t) = A_Y w_\lambda(t)$. Hence $w_\lambda(t) = T_Y(t)w_\lambda(0)$. As $w_\lambda(t) \to w(t)$ for $\lambda \downarrow 0$ and $w(0) = 0$, the assertion follows. ■

REMARK 1.12. The equivalence between (iii) and (v) through (viii) is verified in the case that $\psi \in L^1((0, \tau), Y)$, as seen from the proof of Proposition 1.8. The equivalence between (i), (ii) and (iv) remains true even if "continuously differentiable" is replaced by "absolutely continuous" and the relations are required to hold for almost all (instead of all) $t \in (0, \tau)$. Da Prato and Sinestrari [5] have also shown that $(\mathrm{IVP})_\psi$ can be solved in the sense of Friedrichs. This generalized notion of solution has the advantage that it generalizes to the situation where $A(t)$ depends on t. See Da Prato and Sinestrari [6].

2. Generalized solutions of (SP) and the associated semigroups

In this section we introduce a class of semilinear operators in X and formulate the associated semilinear problems of the form (SP). We then discuss generalized solutions to the semilinear problem (SP) and consider nonlinear semigroups providing such generalized solutions of (SP). Let A be a closed linear operator in X and write $Y = \overline{D(A)}$. Let B be a possibly nonlinear operator in X which is defined on a convex subset C of the closed linear subspace Y. If $D(A) \cap C \neq \phi$, then the sum $A + B$ defines an operator in X with domain $D(A + B) = D(A) \cap C$. Throughout this paper we call it a semilinear operator in X determined by A and B. The intersection $D(A) \cap C$ may be empty, but we use the symbol $A + B$ to represent the semilinear operator determined by a pair of operators A and B even though the domain may be empty. In order to impose a continuity condition and a localized quasi-dissipativity condition on B, we employ a lower semicontinuous convex functional $\varphi: X \to [0, \infty]$ such that $C \subset D(\varphi) \equiv \{v \in X : \varphi(v) < \infty\}$. Choosing a convex subset C of Y and a functional φ as mentioned above, we introduce a class of semilinear operators with which we are concerned in this paper. By a semilinear operator $A + B$ belonging to the class $\mathfrak{S}(C, \varphi)$ is meant the sum of a closed linear operator A and a possibly nonlinear operator B satisfying (H1) and the following conditions:

(H2) For each $\alpha > 0$, the level set $C_\alpha \equiv \{v \in C: \varphi(v) \leq \alpha\}$ is closed in X and the operator B is continuous on C_α.

(H3) For each $\alpha > 0$, the semilinear operator $A + B$ is locally quasi-dissipative in the sense that

$$\langle (A + B)v - (A + B)w, v - w \rangle_i \leq \omega_\alpha |v - w|^2$$

for v, $w \in D(A) \cap C_\alpha$ and some constant $\omega_\alpha \in \boldsymbol{R}$. The continuity condition (H2) on the level sets $\{C_\alpha: \alpha > 0\}$ is much weaker than the continuity on the whole domain C in general and considerably useful for the application to partial differential equations. In condition (H3) the intersections $D(A) \cap C_\alpha$ may be empty; condition (H3) states that the quasi-dissipativity of $A + B$ on C_α is assumed whenever $D(A) \cap C_\alpha \neq \phi$. Given a semilinear operator $A + B$ belonging to the class $\mathfrak{S}(C, \varphi)$ we consider the initial-value problem (SP) for the semilinear evolution equation in X

$$\text{(SE)} \qquad \frac{d}{dt} u(t) = (A + B)u(t), \quad t > 0.$$

DEFINITION 2.1. Let C be the convex set appearing in condition (H2) and $v \in C$. A strongly continuous function $u(\cdot): [0, \infty) \to X$ is said to be a *C-valued weak solution* of (SE) on $[0, \infty)$ with initial-value v, if $u(0) = v$, $u(t) \in C$ for $t > 0$, $Bu(\cdot) \in \mathscr{C}([0, \infty); X)$, and for each $f \in D(A^*)$ the scalar-valued function $\langle u(\cdot), f\rangle$ is continuously differentiable over $[0, \infty)$ and satisfies

$$\frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A^*f\rangle + \langle Bu(t), f\rangle \qquad \text{for } t \geq 0. \tag{2.1}$$

In concrete cases the notion of weak solution to (SE) as defined above may be rather cumbersome owing to the difficulty in characterizing the set-valued operator A^*. It follows from Theorem 1.9 that the set $D(A^*)$ can be replaced by A-total subsets.

Because of the localized quasi-dissipativity condition (H3), the semilinear problem (SP) may admit only local weak solutions. Hence it is necessary to restrict the growth of the weak solutions in order to discuss the weak solutions of (SP) on $[0, \infty)$. In this paper we employ a typical growth condition in terms of the real-valued function $\varphi(u(\cdot))$, namely,

$$\varphi(u(t)) \leq e^{at}(\varphi(v) + bt), \quad t \geq 0, \tag{EG}$$

where a and b are constants. This type of growth condition may be called the *exponential growth condition.* For weak solutions satisfying (EG) the following uniqueness theorem is valid (see Martin *et al.* [20]):

PROPOSITION 2.2. *Let $A + B$ be a semilinear operator of the class $\mathfrak{S}(C, \varphi)$. Let $v, w \in C_\alpha$ and let $u(\cdot; v)$, $u(\cdot; w)$ be the associated C-valued weak solutions satisfying the exponential growth condition. Then for each $\tau > 0$ and $\beta \geq e^{a\tau}(\alpha + b\tau)$ we have*

$$|u(\cdot; v) - u(\cdot; w)| \leq \exp(\omega_\beta t)|v - w| \qquad \text{for } t \in [0, \tau], \tag{2.2}$$

where ω_β is the constant provided for β by condition (H3).

A one-parameter family $\mathscr{S} \equiv \{S(t): t \geq 0\}$ of possibly nonlinear operators from C into itself is called a *semigroup* on C, if it has the two properties below:
(S1) $S(0)v = v$ and $S(t)S(s)v = S(t + s)v$ for $s, t \geq 0$ and $v \in C$.
(S2) For each $v \in C$, $S(\cdot)v \in \mathscr{C}([0, \infty]; X)$.
If in particular a semigroup $\mathscr{S}$ on C provides weak solutions of (SP) in the sense that for each $v \in C$ the function $u(\cdot; v)$ defined by

$$u(t; v) = S(t)v \qquad \text{for } t \geq 0, \tag{2.3}$$

is a C-valued weak solution of (SP) on $[0, \infty)$, then we say that $\mathscr{S}$ is *associated with* the semilinear evolution equation (SE). Under conditions (H1), (H2) and (H3) a family of solution operators in the sense of (2.3) gives rise to a semigroup $\mathscr{S}$ on C, as stated below.

PROPOSITION 2.3. *Let $A + B$ be a semilinear operator belonging to the class $\mathfrak{S}(C, \varphi)$. Suppose that for each $v \in C$ there is a C-valued weak solution*

$u(\cdot; v)$ of (SP) *on* $[0, \infty)$ *satisfying the exponential growth condition* (EG). *Then there is a semigroup* $\mathscr{S} \equiv \{S(t): t \geq 0\}$ *on* C *associated with* (SE) *such that for* $\alpha > 0$, $\tau > 0$ *and* $\beta \geq e^{a\tau}(\alpha + b\tau)$, $BS(\cdot)v \in C([0, \infty); X)$ *and*

$$|S(t)v - S(t)w| \leq \exp(\omega_\beta t)|v - w| \tag{2.4}$$

for $v, w \in C_\alpha$, $t \in [0, \tau]$ *and some constant* $\omega_\beta \in \boldsymbol{R}$.

PROOF. Proposition 2.2 states that $u(\cdot; v)$ is a unique C-valued weak solution of (SP) satisfying (EG). Also, $Bu(\cdot; v) \in \mathscr{C}([0, \infty); X)$ by (EG) and (H2). Therefore one can define for each $t \geq 0$ an operator $S(t)$ from C into itself by

$$S(t)v = u(t; v) \qquad \text{for } v \in C.$$

By the uniqueness of the C-valued weak solutions, the family $S \equiv \{S(t): t \geq 0\}$ of operators so defined have the properties (S1) and (S2). The local equi-Lipschitz continuity (2.4) of the semigroup $\mathscr{S}$ follows from (2.2). ■

In what follows, we say that a semigroup $\mathscr{S}$ on C is *locally equi-Lipschitz continuous* with respect to φ, if for each $\alpha > 0$ and $\tau > 0$ there is a number $\omega(\alpha, \tau)$ such that

$$|S(t)v - S(t)w| \leq e^{\omega(\alpha, \tau)t}|v - w| \qquad \text{for } t \in [0, \tau] \text{ and } v, w \in C_\alpha.$$

Proposition 2.3 states that if $A + B$ belongs to the class $\mathfrak{S}(C, \varphi)$ then the semigroup associated with (SE) is necessarily locally equi-Lipschitz continuous with respect to φ. In the remainder of this section we investigate the differentiability of the semigroup associated with (SE) and then show that such a semigroup can be characterized in several ways. In view of the characterization theorem below, we introduce a notion of semilinear infinitestimal generator.

THEOREM 2.4 (Differentiability Theorem). *Let A be a closed linear operator in X satisfying condition* (H1), $Y = \overline{D(A)}$, *and B a possibly nonlinear operator defined on a convex subset C of Y. Let $\mathscr{S} \equiv \{S(t): t \geq 0\}$ be a semigroup on C such that $BS(\cdot)v \in \mathscr{C}([0, \infty); X)$ for each $v \in C$. The following are equivalent*:

(a) *The semigroup $\mathscr{S}$ is associated with* (SE) *in the sense that the scalar-valued function $\langle S(\cdot)v, f\rangle$ is continuously differentiable over $[0, \infty)$ and*

$$\frac{d}{dt}\langle S(t)v, f\rangle = \langle S(t)v, A^*f\rangle + \langle BS(t)v, f\rangle$$

for $t \geq 0$, $v \in C$ and $f \in D(A^*)$.

(a′) *The semigroup $\mathscr{S}$ is associated with* (SE) *in the sense that for an arbitrarily given A-total subset D of $D(A^*)$ the scalar-valued function $\langle S(\cdot)v, f\rangle$*

is continuously differentiable over $[0, \infty)$ *and*

$$\frac{d}{dt}\langle S(t)v, f\rangle = \langle S(t)v, A^*f\rangle + \langle BS(t)v, f\rangle$$

for $t \geq 0$, $v \in C$, $f \in D$.

(b) *For* $v \in C$, $\int_0^t S(s)vds \in D(A)$ *and*

$$S(t)v = v + A\int_0^t S(s)vds + \int_0^t BS(s)vds \qquad \text{for } t \geq 0.$$

(c) *For* $v \in C$ *and* $t \geq 0$,

$$S(t)v = T_Y(t)v + \lim_{\lambda\downarrow 0}\int_0^t T_Y(t-s)(I-\lambda A)^{-1}BS(s)vds,$$

where the limit is taken with respect to the norm of X.

(c′) *For* $v \in C$ *and* $t \geq 0$,

$$S(t)v = T_Y(t)v + \int_0^t W(ds)BS(t-s)v$$

where the integral is taken in the sense of Stieltjes.

(d) *For* $v \in C$, $\lim_{h\downarrow 0} h^{-1}[(S(h)v - T_Y(h)v) - W(h)Bu] = 0$.

(e) *For* $v \in C$ *and* $f \in D(A^*)$

$$\lim_{h\downarrow 0}\langle h^{-1}(S(h)v - v), f\rangle = \langle v, A^*f\rangle + \langle Bv, f\rangle.$$

(e′) *If* D *is an arbitrarily given* A*-total subset of* $D(A^*)$ *and* $v \in C$, $f \in D$, *then*

$$\lim_{h\downarrow 0}\langle h^{-1}(S(h)v - v), f\rangle = \langle v, A^*f\rangle + \langle Bu, f\rangle.$$

PROOF. Set $u(t) = S(t)v$ and $\psi(t) = BS(t)v$. The equivalence of (a) to (d) then follows from Theorem 1.9. Clearly, (a) implies (e) and (e) implies (e′). It now remains to show that (a′) follows from (e′). Suppose that (e′) is valid. Let $v \in C$ and $f \in D$ and let D be an A-total subset of $D(A^*)$. Using the semigroup property of $\mathscr{S}$, we get

$$\frac{d^+}{dt}\langle S(t)v, f\rangle = \langle S(t)v, A^*f\rangle + \langle BS(t)v, f\rangle \qquad \text{for } t \geq 0,$$

where the left-hand side denotes the right-hand derivative of the function $\langle S(\cdot)v, f\rangle$. But the right-hand side of the above relation is continuous in $t \geq 0$, and so $\langle S(\cdot)v, f\rangle$ is of class $\mathscr{C}^1[0, \infty)$. This shows that (a′) holds. The proof of Theorem 2.1 is now complete. ■

The above theorem is a straightforward extension of [23, Theorem 3.1] to the case where A need not be densely defined. The equivalence of conditions (a), (b) and (e) is discussed in [3] in the context that A is a *multi-valued* linear operator. The formula given in condition (c) may be regarded as a straightforward extension of the so-called variation of constants formula for (SE). In fact, if A is densely defined, (c) is equivalent to the statement that for $v \in C$, $u(t) \equiv S(t)v$ becomes a usual mild solution.

In the case that $D(A)$ is dense in X, Corollary 1.7 and assertion (d) together imply that for any $v \in C$ the formula

$$\lim_{h \downarrow 0} h^{-1}(S(h)v - T_X(h)v) = Bv \tag{2.5}$$

is valid, which means that $A + B$ is the full infinitestimal generator of $\mathscr{S}$ in the sense of [23]. However, the statement (d) does not seem to be appropriate for defining a semilinear operator of the semigroup $\mathscr{S}$. We here employ the statement (e) to introduce a notion of semilinear infinitesimal generator of $\mathscr{S}$.

Definition 2.5. Let $\mathscr{S} \equiv \{S(t): t \geq 0\}$ be a semigroup on C such that $BS(\cdot)v \in \mathscr{C}([0, \infty); X)$ for $v \in C$. Then $A + B$ is said to be the *full infinitesimal generator* of $\mathscr{S}$, if

$$\lim_{h \downarrow 0} \langle h^{-1}(S(h)v - v), f \rangle = \langle v, A^*f \rangle + \langle Bv, f \rangle \tag{2.6}$$

for $v \in C$ and $f \in D(A^*)$.

It should be noted that (2.6) holds on all of C and makes sense even if $D(A + B) = D(A) \cap C = \phi$. This fact motivates the terminology of *full* infinitesimal generator. Formula (2.6) may be interpreted as follows: The vector field generated by $A + B$ is tangential in a weak sense to the continuous curve $S(\cdot)v$ in X for any $v \in C$.

3. Characterization of nonlinearly perturbed integrated semigroups

In this section a necessary and sufficient condition is given for a semilinear operator $A + B$ in the class $\mathfrak{S}(C, \varphi)$ to generate a semigroup $\mathscr{S}$ on C associated with (SE) and satisfying the growth condition (EG).

The following theorem is our first main result;

Theorem 3.1. *Let $A + B$ be a semilinear operator belonging to the class $\mathfrak{S}(C, \varphi)$. Let a, $b \geq 0$. Then the following are equivalent*:

(I) *There exists a semigroup $\mathscr{S} \equiv \{S(t): t \geq 0\}$ on C such that for $v \in C$, $t \geq 0$ and $f \in D(A^*)$,* (I.a) *the* $\boldsymbol{R}$*-valued function* $\langle S(\cdot)v, f\rangle$ *is continuously differentiable over* $[0, \infty)$ *and*

$$\frac{d}{dt}\langle S(t)v, f\rangle = \langle S(t)v, A^*f\rangle + \langle BS(t)v, f\rangle,$$

(I.b) $$\varphi(S(t)v) \leq e^{at}(\varphi(v) + bt).$$

(II) $D(A)\cap C$ *is dense in* C; *for* $\alpha > 0$ *there exists* $\lambda_0 \equiv \lambda_0(\alpha) > 0$ *such that to each* $v \in C_\alpha$ *and each* $\lambda \in (0, \lambda_0)$ *there corresponds an element* $v_\lambda \in D(A)\cap C$ *satisfying*

(II.a) $$v_\lambda - \lambda(A + B)v_\lambda = v,$$

(II.b) $$\varphi(v_\lambda) \leq (1 - a\lambda)^{-1}(\varphi(v) + b\lambda).$$

(III) *For each* $v \in C$ *there exist a null sequence* (h_n) *of positive numbers and a sequence* (v_n) *in* $D(A)\cap C$ *such that*

(III.a) $$\lim_{n\to\infty} h_n^{-1}|v_n - h_n(A + B)v_n - v| = 0,$$

(III.b) $$\limsup_{n\to\infty} h_n^{-1}[\varphi(v_n) - \varphi(v)] \leq a\varphi(v) + b,$$

(III.c) $$\lim_{n\to\infty} |v_n - v| = 0.$$

The first statement (I) means the existence of a nonlinear semigroup $\mathscr{S}$ on C such that for $v \in C$ the X-valued function $u(\cdot) \equiv S(\cdot)v$ gives a C-valued weak solution of (SP) satisfying the exponential growth condition (EG). Condition (II) is usually called the range condition for the semilinear operator $A + B$ and guarantees the existence of the resolvents of $A + B$ in a local sense. The equivalence between (I) and (II) can be restated as a semilinear version of the Hille-Yosida Theorem. See Theorem 3.3 below. According to [23, Section 5], condition (III) is called an implicit subtangential condition. Although (III) is equivalent to the range condition (II), it is often easier to check (III) than (II). In this regard for instance we refer to a recent paper by Clément *et al.* [4].

PROOF. Applying the generation theorem for nonlinear evolution operators and the argument employed in the proof of the key lemma in [16, Lemma 3.3], one obtains the implication (III)$\Rightarrow$(I) in the same way as in [23, Section 3.3], one obtains the implication (III)$\Rightarrow$(I) in the same way as in [23, Section 5]. It is easy to check the implication (II)$\Rightarrow$(III). In fact, let $\alpha > 0$ and let λ_0 be a positive number determined for α by condition (II). Take any $v \in C_\alpha$ and any null sequence (h_n) in the interval $(0, \lambda_0)$. Then for each n there is $v_n \in D(A) \cap C$ such that $v_n - h_n(A + B)v_n = v$ and $\varphi(v_n) \leq (1 - ah_n)^{-1}(\varphi(v) + bh_n)$. Clearly, the sequence (v_n) satisfies (III.a) and (III.b). The third condition (III.c) is verified in the same way as in [23, Remark 5.1].

To show that (I) implies (II), we employ local Laplace transforms of the semigroup $\mathscr{S}$ defined as follows: For each $h > 0$ and each $\tau > 0$ we define an operator $J_{h,\tau}$ from C into X by

$$J_{h,\tau}v = (a_{h,\tau})^{-1} \int_0^\tau e^{-t/h} S(t)v\,dt \qquad \text{for } v \in C, \tag{3.1}$$

where

$$a_{h,\tau} = \int_0^\tau e^{-t/h}dt = h(1 - e^{-\tau/h}).$$

The operators $J_{h,\tau}$ have properties similar to those listed in [23, Proposition 6.1], as mentioned below:

(a) $J_{h,\tau}v \in D(A) \cap C$ and $(I - hA)J_{h,\tau}v$ can be written as

$$v + h(a_{h,\tau})^{-1} \int_0^\tau e^{-t/h} BS(t)v\,dt - he^{-\tau/h}(a_{h,\tau})^{-1}(S(\tau)v - v).$$

(b) $\lim_{h\downarrow 0} h^{-1}|(I - hA)J_{h,\tau}v - (v + hBv)| = 0$ and $\lim_{h\downarrow 0} |J_{h,\tau}v - v| = 0$.

(c) $\lim_{h\downarrow 0} h^{-1}[\varphi(J_{h,\tau}v) - \varphi(v)] \leq a\varphi(v) + b$ and $\lim_{h\downarrow 0} \varphi(J_{h,\tau}v) = \varphi(v)$.

Suppose that (I) holds. First the properties of $J_{h,\tau}$ stated above together imply

$$\bigcup_{\alpha > 0} \overline{D(A) \cap C_\alpha} = C,$$

and hence that $D(A) \cap C$ is dense in C. Next, let $\alpha > 0$ and define

$$\lambda_0 = \min\{(\omega_\alpha^*)^{-1}, (a(\alpha + 2) + b + 1)^{-1}\}, \tag{3.2}$$

where $\omega_\alpha^* = \max\{\omega_{\alpha+1}, 0\}$. Then for any $\lambda \in (0, \lambda_0)$ we have $\lambda\omega_{\alpha+1} < 1$ and $2\lambda a < 1$. Fix any $v \in C_\alpha$ and any $\lambda \in (0, \lambda_0)$. Let $\varepsilon \in (0, 1)$ and define a number $\beta \equiv \beta(\varepsilon)$ by

$$\beta = (1 - (1 + \varepsilon)a\lambda)^{-1}((1 - a\varepsilon\lambda)\varphi(v) + (b + \varepsilon)\lambda).$$

The numbers $\beta(\varepsilon)$ makes sense for $\varepsilon \in (0, 1]$ since $2a\lambda < 1$. Also, it is easily seen that $\beta(\varepsilon)$ is written as

$$\beta = \varphi(v) + (1 - a\varepsilon\lambda)^{-1}\lambda[a\beta + b + \varepsilon], \tag{3.3}$$

and that $\beta(\varepsilon)$ is monotone increasing and bounded by $\alpha + 1$ on $(0, 1]$ as a function of ε. Moreover, take $\tau > 0$, $w \in C_\beta$ and put

$$w_h = (1-h)J_{\lambda h,\tau}w + hJ_{\lambda h,\tau}v \qquad \text{for } h \in (0, 1].$$

Then the properties (a) through (c) of $J_{\lambda h,\tau}$ imply that

$$(3.4) \qquad w_h \in D(A) \cap C \quad \text{for } h \in (0, 1], \qquad \lim_{h \downarrow 0} |w_h - w| = 0$$

and that

$$\begin{aligned}(3.5) \qquad & |w_h - \lambda hAw_h - (w + \lambda hBw - hw + hv)| \\ & \le (1-h)|J_{\lambda h,\tau}w - \lambda hAJ_{\lambda h,\tau}w - (w + \lambda hBw)| \\ & \quad + h|J_{\lambda h,\tau}v - \lambda hAJ_{\lambda h,\tau}v - (v + \lambda hBv)| + \lambda h^2|Bv + Bw| \\ & = \lambda o(h) + \lambda h^2|Bv - Bw| \qquad \text{as } h \downarrow 0.\end{aligned}$$

We then demonstrate that

$$(3.6) \qquad w_h \in C_\beta \qquad \text{for sufficiently small } h \in (0, 1].$$

By the property (c) of $J_{\lambda h,\tau}$ and the convexity of φ we get

$$\begin{aligned}\varphi(w_h) & \le (1-h)\varphi(J_{\lambda h,\tau}w) + h\varphi(J_{\lambda h,\tau}v) \\ & \le (1-h)[\varphi(w) + (1 - a\varepsilon\lambda h)^{-1}\lambda h(a\varphi(w) + b + \varepsilon/2)] + h[\varphi(v) + \lambda\varepsilon/2] \\ & \le (1-h)\varphi(w) + h[\varphi(v) + (1 - a\varepsilon\lambda)^{-1}\lambda(a\varphi(w) + b + \varepsilon)] \\ & \le (1-h)\beta + h[\varphi(v) + (1 - a\varepsilon\lambda)^{-1}\lambda(a\beta + b + \varepsilon)] = \beta\end{aligned}$$

for $h \in (0, 1]$ sufficiently small. Hence it follows that (3.6) is valid.

We now take the restriction B_β of B to C_β and consider the semilinear operator $\lambda A + \lambda B_\beta - I + v$ from C_β into X, where $+v$ stands for the translation by v. Then one can show that the semilinear operator $\lambda A + \lambda B_\beta - I + v$ belongs to the class $\mathfrak{S}(C_\beta, 0)$,

$$\overline{D(\lambda A + \lambda B_\beta - I + v)} = C_\beta,$$

and that the subtangential condition

$$(3.7) \qquad \lim_{h \downarrow 0} h^{-1}d(w, R(I - h(\lambda A + \lambda B_\beta - I + v))) = 0$$

holds for $w \in C_\beta$. In fact, $\lambda B_\beta - I + v$ is continuous on its domain C_β and the semilinear operator $\lambda(A + B_\beta) - I + v - (\lambda\omega_\alpha^* - 1)I$ is dissipative on C_β by (3.2). This means that the semilinear operator satisfies conditions (H1) through (H3) with φ and ω_α replaced respectively by the trivial functional $\varphi \equiv 0$ and the negative number $\lambda\omega_\alpha^* - 1$. Therefore one can assert that the semilinear operator belongs to the class $\mathfrak{S}(C_\beta, 0)$. From (3.4) it follows that the domain $D(\lambda A + \lambda B_\beta - I + v) \equiv D(A) \cap C_\beta$ is dense in C_β. Furthermore, (3.5) implies

$$\lim_{h\downarrow 0} h^{-1} |w_h - h\lambda A w_h - (w + h\lambda B_\beta w - hw + hv)| = 0.$$

Since $w \in C_\beta \subset Y$, $(I - hA)^{-1} w \to w$ as $h \downarrow 0$. Hence $w_h \to w$ as $h \downarrow 0$, since $w_h - h\lambda A w_h - (w + h\lambda B_\beta w - hw + hv) \to 0$ or $w_h - h\lambda A w_h - w \to 0$ as $h \downarrow 0$. Thus, we obtain

$$\begin{aligned}
&\limsup_{h\downarrow 0} h^{-1} |w_h - h(\lambda A + \lambda B_\beta - I + v) w_h - w| \\
&\quad \le \lim_{h\downarrow 0} h^{-1} |w_h - h\lambda A w_h - (w + h\lambda B_\beta w - hw + hv)| \\
&\qquad + \lim_{h\downarrow 0} |\lambda B_\beta w_h - \lambda B_\beta w + w_h - w| = 0.
\end{aligned}$$

This shows that (3.7) is valid. Therefore the application of a generation theorem for nonlinear semigroup implies that $\lambda A + \lambda B_\beta - I + v$ generates a semigroup $\mathscr{S}_\lambda \equiv \{S_\lambda(t)\}$ on C_β such that

$$|S_\lambda(t)v - S_\lambda(t)w| \le \exp[(\lambda w_\alpha^* - 1)t] |v - w| \tag{3.8}$$

for v, $w \in C_\beta$ and $t \ge 0$. Because of (3.8) the semigroup $\mathscr{S}_\lambda$ has a common fixed point $v_{\lambda,\varepsilon} \in C_\beta$, namely, $S_\lambda(t) v_{\lambda,\varepsilon} = v_{\lambda,\varepsilon}$ for $t \ge 0$. Moreover it is seen in the same way as in the proof of the implication (II) $\Rightarrow$ (I) that $\mathscr{S}_\lambda$ is associated with the semilinear problem

$$(d/dt) u_\lambda(t) = (\lambda A + \lambda B_\beta - I) u_\lambda(t) + v, \quad t > 0; \quad u_\lambda(0) = w \in C_\beta.$$

This together with Theorem 2.4 (b) implies that

$$0 = \lim_{h\downarrow 0} h^{-1}(S_\lambda(h) v_{\lambda,\varepsilon} - v_{\lambda,\varepsilon}) = \lim_{h\downarrow 0} \lambda A h^{-1} \int_0^h S_\lambda(s) v_{\lambda,\varepsilon} ds + (\lambda B_\beta - I) v_{\lambda,\varepsilon} + v.$$

Since A is closed, we see that $v_{\lambda,\varepsilon}$ belongs to $D(A) \cap C_\beta$ and satisfies $\lambda(A + B) v_{\lambda,\varepsilon} - v_{\lambda,\varepsilon} + v = 0$. In view of (3.2), the relation $\lambda \omega_{\alpha+1} < 1$ and $\varphi(v_{\lambda,\varepsilon}) \le \beta \le \alpha + 1$, we see that $v_\lambda \equiv (I - \lambda(A + B_{\alpha+1}))^{-1} v$ exists and $v_{\lambda,\varepsilon} = v_\lambda$ for $\varepsilon \in (0, 1]$. Hence $v_\lambda - \lambda(A + B_{\alpha+1}) v_\lambda = v$ and $\varphi(v_\lambda) = \varphi(v_{\lambda,\varepsilon}) \le \beta(\varepsilon)$. Letting $\varepsilon \downarrow 0$, we obtain the estimate $\varphi(v_\lambda) \le (1 - a\lambda)^{-1} [\varphi(v) + b\lambda]$. Consequently, the element v_λ so obtained satisfies (II.a) and (II.b). The proof is now complete. ■

Combining Theorems 2.4 and 3.1, we obtain the following semilinear version of the Hille-Yosida theorem.

THEOREM 3.2. *Let a, $b \ge 0$. A semilinear operator $A + B$ in the class*

$\mathfrak{S}(C, \varphi)$ is the full infinitesimal generator of a semigroup $\mathscr{S}$ on C satisfying the growth condition (EG) *for a, $b \geq 0$ if and only if the domain $D(A) \cap C$ is dense in C and for each $\alpha > 0$ there exists a positive number $\lambda_0 \equiv \lambda_0(\alpha)$ such that for $\lambda \in (0, \lambda_0)$ and $\beta \geq (1 - a\lambda)^{-1}(\alpha + b\lambda)$ the resolvent $(I - \lambda(A + B_\beta))^{-1}$ exists, the range condition*

$$R(I - \lambda(A + B_\beta)) \supset C_\alpha \tag{3.9}$$

is satisfied and the growth condition

$$\varphi((I - \lambda(A + B_\beta))^{-1}v) \leq (1 - a\lambda)^{-1}(\varphi(v) + b\lambda) \tag{3.10}$$

holds for $v \in C_\alpha$, where B_β denotes the restriction of B to C_β.

This theorem is verified in a way parallel to the proof of [4, Theorem 6.1].

Moreover, the semigroup $\mathscr{S}$ is determined through the so-called exponential formula.

THEOREM 3.3 (Exponential Formula). *Let $A + B$ be a semilinear operator in the class $\mathfrak{S}(C, \varphi)$ and let $\mathscr{S}$ be a semigroup on C satisfying the growth condition* (EG) *for some a, $b \geq 0$. Suppose that $A + B$ is the full infinitesimal generator of $\mathscr{S}$. Then for $\alpha > 0$, $\tau > 0$ and $\beta > \exp(2a\tau)(\alpha + \beta\tau)$ the exponential formula*

$$S(t)v = \lim_{n \to \infty} \left(I - \frac{t}{n}(A + B_\beta)\right)^{-n} v, \tag{3.11}$$

holds for $t \in [0, \tau]$ and $v \in C_\alpha$, where B_β denotes the restriction of B to C_β.

The proof of Theorem 3.3 is obtained in the same way as in [23, Theorem 7.2].

4. Evolution equations under subtangential conditions

In this section we investigate variants of the implicit subtangential condition (III) stated in Theorem 3.1. We here treat a modification of the existence theory due to Iwamiya [13] from our point of view.

Let $0 \leq \sigma \leq \tau \leq +\infty$ and consider the nonautonomous semilinear problem

$$\frac{d}{dt}u(t) = Au(t) + B(t, u(t)), \quad \sigma < t < \tau; \quad u(\sigma) = v \tag{NSP}$$

in a Banach space $(X, |\cdot|)$. Here B is a nonlinear continuous operator from a subset C of $[\sigma, \tau] \times X$ into X. Assuming that A is the infinitesimal generator of a nonexpansive semigroup $\mathscr{T}_X$ of class (C_0) on X, Iwamiya advanced in [13] a general existence theory for the problem (NSP). We here outline a modification of his existence theorem in the case where A is the generator of a once integrated semigroup $\mathscr{W}$ such that

$$|W(t) - W(s)| \leq |t - s| \qquad \text{for } s, t \geq 0. \tag{4.1}$$

Condition (4.1) for the generator A is essential in our argument.. However, the semilinear operators $A + B(t, \cdot)$ are written as $(A - \omega) + (B(t, \cdot) + \omega)$ and the same results are obtained if there is $\omega \in \boldsymbol{R}$ such that $A - \omega$ is the generator of a once integrated semigroup satisfying (4.1) and such that the function $v \mapsto g(t, v) + wv$ satisfies condition (g.2).

As seen from Theorem 1.5, a closed linear operator A in X is the generator of a once integrated semigroup $\mathscr{W}$ satisfying (4.1) if and only if the condition below holds:

(H1)′ For $\lambda > 0$, $(I - \lambda A)^{-1}$ exists as a nonexpansive operator on X.

For the linear operator A in (NSP) we assume that the above condition (H1)′. As before, Y denotes the norm closure $\overline{D(A)}$ of the domain of A. We assume that $\boldsymbol{C} \subset [\sigma, \tau] \times Y$ and furthermore we impose the following conditions on the nonlinear operator B:

(C1) If $(t_n, v_n) \in \boldsymbol{C}$, $t_n \uparrow t$ in $[\sigma, \tau]$ and $v_n \to v$ in X, then $(t, v) \in \boldsymbol{C}$.

(C2) For each $t \in [\sigma, \tau]$, set $\boldsymbol{C}(t) = \{v \in X : (t, v) \in \boldsymbol{C}\}$. Then

$$\liminf_{h \downarrow 0} h^{-1} d(T_Y(h)v + W(h)B(t, v), \boldsymbol{C}(t + h)) = 0 \qquad \text{for } v \in \boldsymbol{C}(t).$$

(C3) For (t, v), $(t, w) \in \boldsymbol{C}$ and $h > 0$ we have

$$|v - w| \leq |(v - W(h)B(t, v)) - (w - W(h)B(t, w))| + hg(t, |v - w|),$$

where $g: [\sigma, \tau) \times \boldsymbol{R} \to \boldsymbol{R}$ is a given function such that

(g1) $g(t, w)$ satisfies the so-called Caratheodory's condition.

(g2) $g(t, 0) = 0$ and the function $w(t) \equiv 0$ is a maximal solution to the initial-value problem

$$w'(t) = g(t, w(t)), \quad \sigma < t < \tau; \quad w(\sigma) = 0. \tag{4.2}$$

Condition (C2) is nothing but the subtangential condition (ST) introduced by Thieme [28]. If in particular A is densely defined in X, then it is the infinitesimal generator of a nonexpansive semigroup $\mathscr{T}_X$ of class (C_0) on X and, by Corollary 1.7, condition (C2) is equivalent to the condition

(C2)′ $$\liminf_{h \downarrow 0} h^{-1} d(T_X(h)v + hB(t, v), \boldsymbol{C}(t + h)) = 0 \qquad \text{for } (t, v) \in \boldsymbol{C}.$$

Moreover the denseness of $D(A)$ and (C3) together imply that

(C3)′ $$|v - w| \leq |(v - hB(t, v)) - (w - hB(t, w))| + hg(t, |v - w|)$$

for (t, v), $(t, w) \in \boldsymbol{C}$. Namely, for each $t \in [\sigma, \tau)$, the operator $B(t, \cdot)$: $\boldsymbol{C}(t) \to X$ is quasidissipative. The converse is not always true, even though A is densely defined. If in particular the operator $B: \boldsymbol{C} \to X$ is Lipschitz continuous in the sense that there is a constant $\omega \geq 0$ and

$$|B(t, v) - B(t, w)| \leq \omega |v - w| \qquad \text{for } (t, v),\ (t, w) \in \boldsymbol{C},$$

then condition (C3) holds for the function $g(t, |v - w|) = 3\omega |v - w|$.

In view of Theorem 2.4 (c), we introduce a notion of mild solution (NSP).

DEFINITION 4.1. Let $s \in [\sigma, \tau)$ and $v \in \boldsymbol{C}(s)$. A continuous function $u(\cdot)$: $[s, \tau) \to X$ is said to be a *mild solution* of (NSP) on the interval $[s, \tau)$, if $(t, u(t)) \in \boldsymbol{C}$ for $t \in [s, \tau)$, the function $t \mapsto B(t, u(t))$ is continuous from $[s, \tau)$ into X, and $u(\cdot)$ satisfies the equation

$$u(t) = T_Y(t - s)v + \lim_{\lambda \downarrow 0} \int_s^t T_Y(t - \xi)(I - \lambda A)^{-1} B(\xi, u(\xi)) d\xi \tag{4.3}$$

for $t \in [s, \tau)$, where the limit is taken in X and in the sense of the norm topology.

Under the conditions as mentioned above, the following modified version of Iwamiya's existence theorem is obtained.

THEOREM 4.2. *Suppose that conditions* (H1)′, (C1), (C2) *and* (C3) *are fulfilled. If $\boldsymbol{C}$ is a connected subset of $[\sigma, \tau) \times X$ such that $\boldsymbol{C}(t) \neq \phi$ for all $t \in [\sigma, \tau)$, then for each initial-value $(s, v) \in \boldsymbol{C}$, the problem* (NSP) *has a unique mild solution $u(\cdot)$ on $[s, \tau)$ satisfying $u(s) = v$.*

In Iwamiya's argument most of the estimates are very precise and the denseness of $D(A)$ is used in many parts of his proof, since he uses the property that $\lim_{t \downarrow 0} T(t)v = v$ for all $v \in X$. Therefore it is necessary to change every part of the proof where this property is employed. Although we need much more delicate estimates, it is possible to overcome this difficulty with the aid of Theorem 1.6.

From Theorem 4.2 we obtain a generation theorem for nonlinear evolution operators associated with nonautonomous semilinear problem of the form (NSP), under the explicit subtangential condition (C2).

5. Explicit subtangential conditions

In this section we establish a characterization of nonlinear semigroups providing the mild solutions of (SP) through a variant of the explicit subtangential condition (C2) for $A + B$. In what follows, let A be a closed linear operator satisfying (H1)′ and B a nonlinear operator in X defined on a convex subset C satisfying (H2).

In addition to (H1)′ and (H2) we assume the following condition which is a stronger form than (C3):

(H3)′ For each $\alpha > 0$ there exists $\omega_\alpha \in \boldsymbol{R}$ such that

$$(1 - h\omega_\alpha)|v - w| \le |(v - W(h)Bv) - (w - W(h)Bw)| \qquad \text{for } h > 0 \text{ and } v,\ w \in C_\alpha.$$

Condition (H1)′ and (H3)′ together imply (H3). Under the above conditions we obtain the characterization theorem below:

THEOREM 5.1. *Let a, $b \ge 0$. Assume that A and B satisfy conditions* (H1)′, (H2) *and* (H3)′. *Then the following are equivalent*:

(I)′ *There exists a nonlinear semigroup $\mathscr{S} \equiv \{S(t): t \ge 0\}$ on C such that*

(I.a)′ $$S(t)v = T_Y(t)v + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I - \lambda A)^{-1} BS(s)v\,ds,$$

(I.b)′ $$\varphi(S(t)v) \le e^{at}[\varphi(v) + bt], \qquad \text{for each } v \in C \text{ and } t \ge 0.$$

(IV) *For each $v \in C$ there exist a null sequence $\{h_n\}$ of positive numbers and a sequence $\{v_n\}$ in C such that*

(IV.a) $$\lim_{n \to \infty} h_n^{-1} |T_Y(h_n)v - W(h_n)Bv - v_n| = 0,$$

(IV.b) $$\limsup_{n \to \infty} h_n^{-1} [\varphi(v_n) - \varphi(v)] \le a\varphi(v) + b.$$

Prior to giving the proof of the theorem we make a remark on the relation between the theorem and Theorem 3.1 and prepare a uniqueness theorem for mild solutions of (SP) satisfying (EG). It would be interesting to compare the uniqueness theorems: Proposition 2.2 and Proposition 5.3 below.

REMARK 5.2. If in particular B is assumed to be locally Lipschitz continuous in the sense that

(H3)″ for each $\alpha > 0$ there exists $\omega_\alpha > 0$ such that

$$|Bv - Bw| \le \omega_\alpha |v - w| \qquad \text{for } v,\ w \in C_\alpha,$$

then it follows from Theorem 2.4 and Theorem 3.1 that the statements (I)–(III) in Theorem 3.1 are equivalent to (IV).

Since $|W(h)| \le h$ for $h > 0$ by (4.1), it is readily seen that (H3)″ implies (H3)′. Also, (H3)″ implies (H3) under condition (H1)′. Hence the application of Theorem 2.3 implies that condition (I) in Theorem 3.1 is equivalent to condition (I)′ in Theorem 5.1. This shows that conditions (I) through (IV) are all equivalent.

PROPOSITION 5.3. *Assume that A and B satisfy conditions* (H1)′, (H2) *and* (H3)′. *Then given* $v \in C$ *there exists at most one mild solution* $u(\cdot)$ *of the problem* (SP) *satisfying* (EG).

PROOF. Let $\alpha > 0$, $\tau > 0$, $\beta = e^{a\tau}[\alpha + b\tau]$ and let ω_β a constant provided for the β by condition (H3)′. Let $w, z \in C_\alpha$ and let $u(\cdot)$, $v(\cdot)$ be the corresponding global mild solutions of (SP) satisfying (EG). Then $u(t), v(t) \in C_\beta \subset Y$ for $t \in [0, \tau]$. By (H3)′ we have

$$(1 - h\omega_\beta)|u(t+h) - v(t+h)| \le |(I - W(h)B)u(t+h) - (I - W(h)B)v(t+h)|$$

for $t \in [0, \tau)$ and $h > 0$ with $t + h \le \tau$. But $(I - W(h)B)u(t+h)$ is written as

$$T_Y(h)u(t) + \lim_{\lambda \downarrow 0} \int_t^{t+h} T_Y(t+h-s)(I - \lambda A)^{-1}[Bu(s) - Bu(t+h)]ds$$

and $(I - W(h)B)v(t+h)$ is also written in the same form. Hence we have

$$(1 - h\omega_\beta)|u(t+h) - v(t+h)| \le |u(t) - v(t)| + \int_t^{t+h} |Bu(s) - Bu(t+h)|ds$$
$$+ \int_t^{t+h} |Bu(s) - Bu(t+h)|ds$$

for $t \in [0, \tau)$ and $h \in (0, \tau - t]$. From this it follows that $D^+|u(t) - v(t)| \le \omega_\beta |u(t) - v(t)|$ for $t \in [0, \tau)$, where $D^+\phi(t)$ stands for the Dini upper right derivative of an $\boldsymbol{R}$-valued function ϕ on $[0, \tau)$ at t. Solving this differential inequality, one obtains

$$|u(t) - v(t)| \le e^{\omega_\beta t}|w - z| \qquad \text{for } t \in [0, \tau) \text{ and } w, z \in C_\alpha.$$

This implies the desired assertion. ■

PROOF OF THEOREM 5.1. In view of Proposition 5.3 it suffices to show that for any $\tau > 0$ and any $z \in C$ there exists an X-valued continuous function $u(\cdot)$ on $[0, \tau]$ such that for each $t \in [0, \tau]$, $u(t) \in C$,

$$u(t) = T_Y(t)z + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I - \lambda A)^{-1}Bu(s)ds$$

and

$$\varphi(u(t)) \le e^{at}[\varphi(z) + bt].$$

Let $\tau > 0$, $\varepsilon \in (0, 1]$ and $z \in C$. Set $\alpha = e^{a\tau}[\varphi(z) + (b + \varepsilon)\tau]$ and let ω_α denote the constant given by (H3)′. Also, for each $t \in [0, \tau]$, we write $D(t)$ for the set $\{v \in C: \varphi(v) \le e^{at}[\varphi(z) + (b + \varepsilon)t]\}$ and define an operator $B(t)$ from $D(t)$

into X by $B(t)v = Bv$ for $v \in D(t)$. Then the following are valid for the operators $B(t)$, $t \in [0, \tau]$:

(i) Each of $D(t)$ is closed and $D(s) \subset D(t)$ for $0 \le s \le t \le \tau$.

(ii) The mapping $(t, v) \to B(t)v$ is continuous from $\mathscr{D} \equiv \bigcup_{s \in [0,\tau]}(\{s\} \times D(s))$ into X.

(iii) For each $t \in [0, \tau]$, $h > 0$ and v, $w \in D(t)$, we have

$$(1 - h\omega_\alpha)|v - w| \le |(I - W(h)B(t))v - (I - W(h)B(t))w|.$$

Let $t \in [0, \tau)$ and $v \in D(t)$. Then by (IV) there exist sequences $\{h_n\}$ and $\{v_n\}$ satisfying (IV.a) and (IV.b). Hence, by (IV.b), we have

$$\varphi(v_n) \le \varphi(v) + h_n[a\varphi(v) + b + \varepsilon/2] \le e^{ah_n}[\varphi(v) + (b + \varepsilon)h_n],$$

for n sufficiently large. Since $\varphi(v) \le e^{at}[\varphi(z) + (b + \varepsilon)t]$, it follows that

$$\begin{aligned}\varphi(v_n) &\le e^{a(t+h_n)}[\varphi(z) + (b + \varepsilon)t] + e^{ah_n}(b + \varepsilon)h_n \\ &\le e^{a(t+h_n)}[\varphi(z) + (b + \varepsilon)(t + h_n)]\end{aligned}$$

for n sufficiently large. From this and (IV.a) we infer that for $t \in [0, \tau)$ and $v \in D(t)$

(iv) $$\liminf_{h\downarrow 0} h^{-1}d(T_Y(h)v + W(h)B(t)v, D(t + h)) = 0.$$

By virtue of the facts (i)–(iv) mentioned above, we apply Theorem 4.2 to conclude that there exists a function $u(\cdot) \in \mathscr{C}([0, \tau]; X)$ such that $u(t) \in D(t)$ and (I.a)′ holds for $t \in [0, \tau]$. Since $u(t) \in C_\alpha$ for $t \in [0, \tau]$, it follows from the uniqueness of the mild solutions that $u(\cdot)$ is independent of ε. The fact that $u(t) \in D(t)$ for $t \in [0, \tau]$ means that

$$\varphi(u(t)) \le e^{at}[\varphi(z) + (b + \varepsilon)t] \qquad \text{for } t \in [0, \tau] \text{ and } \varepsilon \in (0, 1].$$

Taking the limit as $\varepsilon \downarrow 0$, we have $\varphi(u(t)) \le e^{at}[\varphi(z) + bt]$ for $t \in [0, \tau]$. This completes the proof. ■

REMARK 5.4. Let C be a fixed closed convex subset of Y. In Thieme [28] it is assumed that $\boldsymbol{C} = [\sigma, \tau] \times C$, and that the nonlinear operator $B: \boldsymbol{C} \to X$ in (NSP) is locally Lipschitz continuous and of linear growth in the following sense:

(LL) For any $t \ge 0$ and $v \in C$ there exist positive numbers $\delta > 0$, $\Gamma > 0$ such that

$$|B(s, w) - B(s, z)| \le \Gamma |w - z|$$

for $t \le s \le t + \delta$, w, $z \in C$ with $|w - v| \le \delta$ and $|z - v| \le \delta$.

(LG) For any $\tau > 0$ there exists a positive number $\kappa \equiv \kappa(\tau) > 0$ such that

$$|B(t, v)| \leq \kappa(1 + |v|) \qquad \text{for } 0 \leq t \leq \tau \text{ and } v \in C_0.$$

We infer from (4.3) that under the linear growth condition (LG) the following *a priori* estimates are obtained for mild solutions $u(\cdot)$ of (NSP):

$$|u(t)| \leq e^{\kappa(t-\sigma)}[|v| + \kappa(t - \sigma)] \qquad \text{for } t \in [\sigma, \tau],$$

where $(\sigma, v) \in C$ stands for the initial-value given in (NSP).

6. Evolution equations with semilinear constraints

In our main results, Theorems 3.1 and 5.1, the linear operator A is assumed to be non-densely defined in X. This enables us to treat semilinear evolution equations with semilinear boundary conditions in the framework of our theory. We here refer to the work done by Greiner [8, 9] and Thieme [28] and discuss the construction of semigroup solutions of semilinear problems subject to semilinear constraints. In the next section, we apply the result to a mathematical model which describes a specific type of nonlinear age-dependent population dynamics.

In this section we consider a pair of Banach spaces $(X, |\cdot|_X)$ and $(Z, |\cdot|_Z)$, a convex subset C of X and a lower semicontinuous convex functional $\varphi(\cdot): X \to [0, \infty]$ with $C \subset D(\varphi) = \{x \in X: \varphi(x) < \infty\}$. Let A be a linear operator in X, L a linear operator from $D(A)$ onto Z, F a nonlinear operator from C into X and let B be a nonlinear operator from C into Z. Then one can formulate the initial-value problem for the semilinear equation

$$\text{(SP)} \qquad u'(t) = Au(t) + Fu(t), \quad t > 0; \quad u(0) = v,$$

subject to the semilinear constraint

$$\text{(SB)} \qquad Lu(t) = Bu(t) \qquad \text{for } t > 0.$$

On the operators A, L, F and B we put the following conditions:

(s.1) The linear operator A is closed and densely defined in X and the restriction A_K of A to Ker L is also densely defined in X. Furthermore, there exists $\omega_K \in \boldsymbol{R}$ such that

$$|(I - \lambda A_K)^{-1} v|_X \leq (1 - \lambda\omega_K)^{-1} |v|_X \qquad \text{for } \lambda \in (0, 1/\omega_K) \text{ and } v \in X.$$

(s.2) The operator $L: D(A) \to Z$ is surjective and there exists a constant $\omega_L \in \boldsymbol{R}$ such that

$$\lambda |Lz|_Z \geq (1 - \lambda\omega_L)|z|_X \qquad \text{for } z \in \text{Ker}(I - \lambda A) \text{ and } \lambda \in (0, 1/\omega_L).$$

(s.3) For each $\alpha > 0$, the level set $C_\alpha \equiv \{v \in C: \varphi(v) \le \alpha\}$ is closed in X, the nonlinear operators F and B are continuous on C_α, and there is $\omega_\alpha \in \boldsymbol{R}$ such that

$$\langle Bv - Bw, f\rangle + \langle Fv - Fw, g\rangle \le \omega_\alpha |v - w|_X^2$$

for v, $w \in C_\alpha$, $f \in Z^*$ and $g \in \mathscr{F}(v - w)$ satisfying $|f|_{Z^*} \le |v - w|_X$, where $\mathscr{F}$ is the duality mapping of X.

(s.4) There exist nonnegative constants a and b such that for each $v \in C$ there exists a null sequence (h_n) in $(0, \infty)$ and a sequence (v_n) in $D(A) \cap C$ satisfying

$$\lim_{n\to\infty} h_n^{-1} |v_n - h_n(A + F)v_n - v|_X = 0,$$

$$\lim_{n\to\infty} |Lv_n - Bv_n|_Z = 0, \quad \lim_{n\to\infty} |v_n - v|_X = 0,$$

$$\limsup_{n\to\infty} h_n^{-1}(\varphi(v_n) - \varphi(v)) \le a\varphi(v) + b.$$

Under the above conditions we introduce a product Banach space $\mathscr{X} \equiv Z \times X$ equipped with the norm $\|(\cdot, \cdot)\|$ defined by $\|(z, v)\| = |z|_Z + |v|_X$ for $(z, v) \in \mathscr{X}$ and convert the semilinear problem (SP) subject to (SB) to the semilinear problem in $\mathscr{X}$. To this end, we define a linear operator $\mathscr{A}$ in $\mathscr{X}$ by

$$D(\mathscr{A}) = \{0\} \times D(A) \quad \text{and} \quad \mathscr{A}(0, v) = (-Lv, Av) \qquad \text{for } v \in D(A)$$

and a nonlinear operator $\mathscr{B}$ in $\mathscr{X}$ by

$$D(\mathscr{B}) = \{0\} \times C \quad \text{and} \quad \mathscr{B}(0, v) = (Bu, Fu) \qquad \text{for } v \in C.$$

By means of the operators $\mathscr{A}$ and $\mathscr{B}$ the semilinear problem (SP) with semilinear constraint (SB) can be rewritten as the semilinear problem in $\mathscr{X}$:

$$\textbf{(SP)} \qquad \mathbf{u}'(t) = \mathscr{A}\mathbf{u}(t) + \mathscr{B}\mathbf{u}(t), \quad t > 0; \quad \mathbf{u}(0) = \mathbf{v} = (0, v).$$

Define a lower semicontinuous functional $\boldsymbol{\varphi}: Z \times X \to [0, \infty]$ by $\boldsymbol{\varphi}(\mathbf{v}) = \varphi(v)$ for $\mathbf{v} = (z, v)$ and set $\mathscr{C}_\alpha = \{\mathbf{v} \in \mathscr{C}: \boldsymbol{\varphi}(\mathbf{v}) \le \alpha\}$.

We shall check that the conditions (H1), (H2) and (H3) stated in Sections 1 and 2 hold for the operator $\mathscr{A}$ and $\mathscr{B}$. To this end, we need the following lemma:

LEMMA 6.1 ([8], [28]). *Under* (s.1) *and* (s.2) *we have the following.*

(a) $D(A) = \operatorname{Ker} L \oplus \operatorname{Ker}(I - \lambda A)$ *for* $\lambda^{-1} \in \rho(A_K)$.

(b) *Let* $\omega \equiv \max\{\omega_1, \omega_2\}$. *Then we have*

$$\textbf{(H1)} \qquad \|(I - \lambda\mathscr{A})^{-1}\| \le (1 - \lambda\omega)^{-1} \qquad \text{for } \lambda > 0 \text{ with } \lambda\omega < 1.$$

PROOF. (a): Let $v \in D(A)$ and $\lambda^{-1} \in \rho(A_K)$. By the definition, $(I - \lambda A_K)^{-1}(I - \lambda A)v \in \operatorname{Ker} L$ and $v - (I - \lambda A_K)^{-1}(I - \lambda A)v \in \operatorname{Ker}(I - \lambda A)$. Moreover, condition (s.1) implies that $\operatorname{Ker} L \cap \operatorname{Ker}(I - \lambda A) = \{0\}$. Hence (a) is obtained.

(b): Let $(f, g) \in Z \times X$, $\lambda > 0$, and let $\lambda\omega < 1$. Put $v = (I - \lambda A_K)^{-1}g$, $w = (L_{|\operatorname{Ker}(I-\lambda A)})^{-1}(\lambda^{-1}f)$ and $u = v + w$. Then $v \in \operatorname{Ker} L$ and $w \in \operatorname{Ker}(I - \lambda A)$. We now apply (a) to see that $u \in D(A)$ and

$$(I - \lambda\mathscr{A})(0, u) = (f, g).$$

By the definition of the norm of $Z \times X$, we have

$$\begin{aligned}\|(0, u)\| &= |u|_X \le |v|_X + |w|_X \\ &\le (1 - \lambda\omega_K)^{-1}|g|_X + (1 - \lambda\omega_L)^{-1}|f|_Z \\ &\le (1 - \lambda\omega)^{-1}\|(f, g)\|,\end{aligned}$$

which implies (b). The proof is now complete. ■

Rewriting (s.3) and (s.4) we have the following:

(H2) For each $\alpha > 0$ there exists an $\omega_\alpha \in \boldsymbol{R}$ such that

$$\langle \mathscr{B}\mathbf{v} - \mathscr{B}\mathbf{w}, \mathbf{f} \rangle \le \omega_\alpha \|\mathbf{v} - \mathbf{w}\|^2$$

for $\mathbf{v}, \mathbf{w} \in \mathscr{C}_\alpha$ and $\mathbf{f} \in \mathscr{F}(\mathbf{v} - \mathbf{w})$, where $\mathscr{F}$ denotes the duality mapping of $\mathscr{X}$.

(H3) For each $\mathbf{v} \in \mathscr{C}$ there exists a null sequence $\{h_n\}$ of positive numbers and a sequence $\{\mathbf{v}_n\} \in D(\mathscr{A}) \cap \mathscr{C}$ satisfying

$$\lim_{n\to\infty} h_n^{-1}\|\mathbf{v}_n - h_n(\mathscr{A} + \mathscr{B})\mathbf{v}_n - \mathbf{v}\| = 0, \quad \lim_{n\to\infty} \|\mathbf{v}_n - \mathbf{v}\| = 0,$$

$$\limsup_{n\to\infty} h_n^{-1}[\varphi(\mathbf{v}_n) - \varphi(\mathbf{v})] \le a\varphi(\mathbf{v}) + b.$$

Applying Theorem 2.4 and Theorem 3.1 to the semilinear problem **(SP)**, we obtain the theorem below.

THEOREM 6.2. *Suppose* (s.1)–(s.4). *Then there exists a nonlinear semigroup* $\mathscr{S} \equiv \{S(t): t \ge 0\}$ *on* C *such that for each* $v \in C$ *and each* $t \ge 0$, $\int_0^t S(s)xds \in D(A)$ *and*

$$S(t)v = v + A\int_0^t S(s)vds + \int_0^t FS(s)vds,$$

$$L\int_0^t S(s)vds = \int_0^t BS(s)vds,$$

$$\varphi(S(t)v) \le e^{at}[\varphi(v) + bt].$$

REMARK 6.3. It is seen in the same way as in [4] that if F and B are continuously Fréchet differentiable on each C_α, then $S(\cdot)v$ gives a strong solution to the problem (SP)–(SB) provided that $v \in D(A) \cap C$ and $Lv = Bv$.

7. An application to population dynamics with pair formation

In this section we make an attempt to apply the result obtained in Section 6 to a semilinear system modeling nonlinear age-dependent population dynamics.

Let $u(t, a)$ represent the single male individuals with age a at time t, $v(t, b)$ the single female individuals with age b at time t, and $p(t, a, b, c)$ the pairs of age c at time t consisting of a male with age a and a female of age b. Obviously the age of a pair (i.e., the time that has passed since the pair was formed) is less than both the age of the male and the age of the female individual constituting the pair.

The differential equations take the form

$$\text{(DE)}\quad \begin{aligned} &u_t(t, a) + u_a(t, a) + d_1(t, a)u(t, a) + \int_0^\infty p(t, a, b, 0)db \\ &\quad = \int_0^\infty \int_0^\infty \sigma(t, a, b, c)p(t, a, b, c)\, dbdc, \\ &v_t(t, b) + v_b(t, b) + d_2(t, b)v(t, b) + \int_0^\infty p(t, a, b, 0)da \\ &\quad = \int_0^\infty \int_0^\infty \sigma(t, a, b, c)p(t, a, b, c)dadc, \\ &p_t(t, a, b, c) + p_a(t, a, b, c) + p_b(t, a, b, c) + p_c(t, a, b, c) \\ &\quad = -(d_1(t, a) + d_2(t, b) + \sigma(t, a, b, c))p(t, a, b, c). \end{aligned}$$

Here $d_1(t, a)$ and $d_2(t, b)$ denote the respective per capita mortality rates of males of age a and females of age b at time t, $\sigma(t, a, b, c)$ denotes the separation rate of a pair of age c at time t where the male has age a and the female has age b, and $\int_0^\infty p(t, a, b, 0)db$, e.g., is identical to the rate at which single males of age a are lost at time t due to the formation of new pairs. Actually we want to handle an autonomous problem and the time dependence in d_1, d_2, σ enters by assuming that these rates depend on the age densities $u(t, \cdot)$, $v(t, \cdot)$, $p(t, \cdot)$,

$$d_j(t, a) = \mu_j(a, u(t, \cdot), v(t, \cdot), p(t, \cdot)),\ j = 1,\ 2,$$
$$\sigma(t, a, b, c) = \rho(a, b, c, u(t, \cdot), v(t, \cdot), p(t, \cdot)).$$

The differential equations are supplemented by nonlinear boundary conditions

$$
\text{(BC)}\qquad
\begin{aligned}
u(t, 0) &= \int_{(0,\infty)^3} \beta_1(t, a, b, c)p(t, a, b, c)dadbdc,\\
v(t, 0) &= \int_{(0,\infty)^3} \beta_2(t, a, b, c)p(t, a, b, c)dadbdc,\\
&p(t, 0, b, c) = p(t, a, 0, c) = 0.\\
&p(t, a, b, 0) = \psi(t, a, b),
\end{aligned}
$$

and initial conditions

$$
\text{(IC)}\qquad u(0, a) = u_0(a), \quad v(0, b) = v_0(b), \quad p(0, a, b, c) = p_0(a, b, c).
$$

The function $\beta_1(t, a, b, c)$ denotes the rate at which male offspring is born at time t from a pair of age c where the male partner has age a and the female partner has age b, the function $\beta_2(t, a, b, c)$ denotes the analogous rate of female offspring; $\psi(t, a, b)$ gives the rate at which pairs are formed at time t where the male partner has age a and the female partner has age b. Again, the time dependence of β_1, β_2 and ψ originates from dependence on the various age densities

$$
\beta_j(t, a, b, c) = \alpha_j(a, b, c, u(t, \cdot), v(t, \cdot), p(t, \cdot)),
$$
$$
\psi(t, a, b) = \Psi(u(t, \cdot), v(t, \cdot), p(t, \cdot))(a, b).
$$

This model, without the pair age c, was apparently introduced by Hoppensteadt [12] while Staroverov [25] observed that one could also keep track of pair age. Interest has been rekindled in this model as a prerequisite for studying the spread of sexually transmitted diseases by Hadeler [10, 11]. A result concerning the global existence and uniqueness of solutions (similar to ours, but with the rates β_j, σ, d_j being independent of the solution) has been announced by Prüss and Schappacher [24]. For applications to age-structured population models involving additional structures we refer to Thieme [29].

We treat the semilinear differential equation (DE) with the semilinear boundary condition (BC) in the Banach space

$$
X = L^1(0, \infty) \times L^1(0, \infty) \times L^1((0, \infty)^3),
$$

more precisely, in the non-negative convex cone

$$
C = L^1_+(0, \infty) \times L^1_+(0, \infty) \times L^1_+((0, \infty)^3).
$$

We make the following assumptions:

(A1) There exists a monotone increasing function $\phi: [0, \infty) \to [0, \infty)$ such that for all a, b, $c \geq 0$ and all $\boldsymbol{w} \in C$,

$$0 \leq \mu_j(a, \boldsymbol{w}) \leq \phi(|\boldsymbol{w}|_X),$$

where, for $\boldsymbol{w} = (u, v, p) \in X$, the norm of $\boldsymbol{w}$ is defined by

$$|\boldsymbol{w}|_X = \max\left\{\int_0^\infty |u(a)|\, da, \int_0^\infty |v(b)|\, db, \int_{(0,\infty)^3} |p(a, b, c)|\, dadbdc\right\}.$$

There exists a constant $\nu > 0$ such that, for a, b, $c \geq 0$ and $\boldsymbol{w} \in C$,

$$0 \leq \rho(a, b, c, \boldsymbol{w}) \leq \nu, \quad 0 \leq \alpha_j(a, b, c, \boldsymbol{w}) \leq \nu.$$

Furthermore, μ_j, ρ and α_j are uniformly locally Lipschitz in the following sense: For any $\gamma > 0$ there exists Λ_γ such that, for $\boldsymbol{v}$, $\boldsymbol{w} \in C$ with $|\boldsymbol{v}|_X$, $|\boldsymbol{w}|_X \leq \gamma$,

$$|\mu_j(a, \boldsymbol{v}) - \mu_j(a, \boldsymbol{w})| \leq \Lambda_\gamma |\boldsymbol{v} - \boldsymbol{w}|_X,$$

$$|\rho(a, b, c, \boldsymbol{v}) - \rho(a, b, c, \boldsymbol{w})| \leq \Lambda_\gamma |\boldsymbol{v} - \boldsymbol{w}|_X,$$

$$|\alpha_j(a, b, c, \boldsymbol{v}) - \alpha_j(a, b, c, \boldsymbol{w})| \leq \Lambda_\gamma |\boldsymbol{v} - \boldsymbol{w}|_X.$$

(A2) The operator $\Psi: C \to L^1_+((0, \infty)^2)$ is locally Lipschitz continuous in the sense that

$$\int_0^\infty \int_0^\infty |\Psi(\boldsymbol{v})(a, b) - \Psi(\boldsymbol{w})(a, b)|\, dadb \leq \Lambda_\gamma |\boldsymbol{v} - \boldsymbol{w}|_X$$

for $\boldsymbol{v}$, $\boldsymbol{w} \in C$ with $|\boldsymbol{v}|_X$, $|\boldsymbol{w}|_X \leq \gamma$, and there exists a constant $M > 0$ such that

$$\int_0^\infty \int_0^\infty |\Psi(\boldsymbol{w})(a, b)|\, dadb \leq M|\boldsymbol{w}|_X, \quad \boldsymbol{w} \in C.$$

Let

$$Z \equiv \boldsymbol{R}^1 \times \boldsymbol{R}^1 \times L^1((0, \infty)^2), \ |z|_Z = |x| + |y| + \int_0^\infty \int_0^\infty |f(a, b)|\, dadb,$$

for $z = (x, y, f) \in Z$. We denote by Γp the trace of $p \in W^{1,1}((0, \infty)^3)$ and denote the restriction of Γp to $\{0\} \times (0, \infty) \times (0, \infty)$, $(0, \infty) \times \{0\} \times (0, \infty)$ and $(0, \infty) \times (0, \infty) \times \{0\}$ by $\Gamma_1 p$, $\Gamma_2 p$ and $\Gamma_3 p$, respectively.

PROPOSITION 7.1. *For $\phi \in L^1((0, \infty)^2)$ there exists $p \in W^{1,1}((0, \infty)^3)$ such that $\Gamma_1 p = \Gamma_2 p = 0$ and $\Gamma_3 p = \phi$.*

The proof is obtained in a way similar to Giusti [7, Proposition 2.15]. We then define a linear operator T in $L^1((0, \infty)^3)$ by

$$Tp = -p_a - p_b - p_c, \quad D(T) = \{p \in W^{1,1}((0, \infty)^3): \Gamma_1 p = \Gamma_2 p = 0\}.$$

Then the operator T is closable. We denote the closure of T by $\overline{T}$. In view of the definition of T, we have

$$\int_{(0,\infty)^3} (\operatorname{sgn} p(a, b, c)) Tp(a, b, c) dadbdc$$
$$= -\int_{(0,\infty)^3} (\partial_a + \partial_b + \partial_c)|p(a, b, c)| dadbdc = |\Gamma_3 p|_1.$$

This shows that Γ_3 is continuous with respect to the graph norm of T, and that it can be extended to $D(\overline{T})$. We denote the extension of Γ_3 to $D(\overline{T})$ by $\overline{\Gamma}_3$ and define a linear operator by

$$\overline{T}_0 p = \overline{T} p, \quad D(\overline{T}_0) = \{p \in D(\overline{T}) : \overline{\Gamma}_3 p = 0\}.$$

LEMMA 7.2. *The operator $\overline{T}_0$ is the infinitesimal generator of a strongly continuous semigroup on $L^1((0, \infty)^3)$.*

PROOF. Let $p \in D(\overline{T}_0)$, $\lambda > 0$, and take a sequence $\{p_n\} \in D(T)$ such that $p_n \to p$, $Tp_n \to \overline{T}_0 p$ and $\Gamma_3 p_n \to 0$ as $n \to \infty$. Let $f_n = (\operatorname{sgn} p_n)|p_n|_1$. Then we have

$$|(I - \lambda T)p_n|_1 |p_n|_1 \geq |\langle (I - \lambda T)p_n, f_n \rangle|$$
$$= \left| \int_{(0,\infty)^3} |p_n|_1 \{1 + \lambda(\partial_a + \partial_b + \partial_c)\} |p_n(a, b, c)| dadbdc \right|$$
$$= \| p_n|_1 - \lambda |\Gamma_3 p_n|_1 \| p_n|_1.$$

Passing to the limit as $n \to \infty$, we obtain

$$|(I - \lambda \overline{T}_0)p|_1 |p|_1 \geq |p|_1^2.$$

Hence, for each $\lambda > 0$, we have

$$|(I - \lambda \overline{T}_0)p|_1 \geq |p|_1 \qquad \text{for } p \in D(\overline{T}_0). \tag{7.1}$$

This means that the range of $(I - \lambda \overline{T}_0)$ is closed. Let $g \in C_0^\infty(\boldsymbol{R}^3)$, $\operatorname{supp} g \subset (0, \infty)^3$, and set

$$p(a, b, c) = \lambda^{-1} \int_0^c e^{-\lambda^{-1}(c-\xi)} g(a - c + \xi, b - c + \xi, \xi) d\xi \qquad \text{for } \lambda > 0. \tag{7.2}$$

Then we infer that $p \in W^{1,1}((0, \infty)^3)$, $\Gamma_1 p = \Gamma_2 p = \Gamma_3 p = 0$, and that $p - \lambda Tp = g$. This means that the range of $(I - \lambda \overline{T}_0)$ is dense. This together with (7.1) implies that $\overline{T}_0$ is the infinitesimal generator of a strongly continuous contraction semigroup. ∎

We next define a closed linear operator A in X by

$$Aw = \left[-u_a - \int_0^\infty \overline{\Gamma}_3 p(a, b)db,\ -v_b - \int_0^\infty \overline{\Gamma}_3 p(a, b)da,\ \overline{T}p\right]$$

$$D(A) = \{w = (u, v, p) \colon u, v \in W^{1,1}(0, \infty), p \in D(\overline{T})\},$$

and define a linear operator L from $D(A)$ into Z by

$$Lw = [u(0), v(0), \overline{\Gamma}_3 p].$$

Let A_K be the restriction of A to $\operatorname{Ker} L$. Then

$$A_K w = [-u_a, -v_b, \overline{T}_0 p], \quad D(A_K) = W_0^{1,1}(0, \infty) \times W_0^{1,1}(0, \infty) \times D(\overline{T}_0).$$

The next proposition follows immediately from Lemma 7.2 and shows that the linear operator A satisfies condition (s.1) in the previous section.

LEMMA 7.3. *The restriction A_K of A to* Ker L *is the infinitesimal generator of a strongly continuous contraction semigroup on X.*

We next show that L satisfies (s.2). The surjectivity of L follows from Proposition 7.1. Let $\lambda > 0$ and let $w = (u, v, p) \in \operatorname{Ker}(I - \lambda A)$. Then direct computations yield

$$(7.3) \qquad \begin{aligned} u(a) &= e^{-\lambda^{-1}a} u(0) - \lambda^{-1} \int_0^a \int_0^\infty e^{-\lambda^{-1}(a-\xi)} \overline{\Gamma}_3 p(\xi, b) db d\xi, \\ v(b) &= e^{-\lambda^{-1}b} v(0) - \lambda^{-1} \int_0^b \int_0^\infty e^{-\lambda^{-1}(b-\xi)} \overline{\Gamma}_3 p(a, \xi) da d\xi. \end{aligned}$$

By the definition of $\overline{T}$ there exists a sequence $p_n \in D(T)$ such that $p_n \to p$ and $Tp_n \to \overline{T}p$ as $n \to \infty$. Let $p_n - Tp_n = f_n$ for $n = 1, 2, \ldots$. Multiplying both sides of these equations by $\operatorname{sgn} p_n$ and integrating the resultant identities over $(0, \infty)^3$, we have

$$|p_n|_1 + \lambda \int_{(0,\infty)^3} (\partial_a |p_n| + \partial_b |p_n| + \partial_c |p_n|) da db dc = \int_{(0,\infty)^3} \operatorname{sgn} p_n f_n da db dc.$$

Applying the boundary conditions $\Gamma_1 p_n = \Gamma_2 p_n = 0$ and passing to the limit as $n \to \infty$, we have

$$(7.4) \qquad |p|_1 - \lambda |\overline{\Gamma}_3 p|_1 = 0.$$

It follows from (7.3) and (7.4) that

$$|u|_1 \le \lambda(|u(0)| + |\overline{\Gamma}_3 p|_1), \quad |v|_1 \le \lambda(|v(0)| + \overline{\Gamma}_3 p|_1), \quad |p|_1 = \lambda |\overline{\Gamma}_3 p|_1.$$

This means that

$$\lambda |Lw|_Z \geq |w|_X \qquad \text{for } w = [u, v, p] \in \mathrm{Ker}(I - \lambda A).$$

Hence L satisfies (s.2).

We next define nonlinear operators F from C into X and B from C into Z by

$$Fw = [F_1 w, F_2 w, F_3 w], \qquad Bw = [B_1 w, B_2 w, B_3 w],$$

$$[F_1 w](a) = -\mu_1(a, w)u(a) + \int_0^\infty \int_0^\infty \rho(a, b, c, w)p(a, b, c)dbdc,$$

$$[F_2 w](b) = -\mu_2(b, w)v(b) + \int_0^\infty \int_0^\infty \rho(a, b, c, w)p(a, b, c)dadc,$$

$$[F_3 w](a, b, c) = -(\mu_1(a, w) + \mu_2(b, w) + \rho(a, b, c, w))p(a, b, c),$$

$$B_1 w = \int_{(0,\infty)^3} \alpha_1(a, b, c, w)p(a, b, c)dadbdc,$$

$$B_2 w = \int_{(0,\infty)^3} \alpha_2(a, b, c, w)p(a, b, c)dadbdc,$$

$$[B_3 w](a, b) = \Psi(w)(a, b),$$

respectively. For each $\gamma > 0$ we set $C_\gamma = \{w \in C: |w|_X \leq \gamma\}$.

LEMMA 7.4. *Under conditions* (**A.1**) *and* (**A.2**) *we have the following*:

(a) *For* $\gamma > 0$, *B and F are Lipschitz continuous on the level sets* C_γ.

(b) *For* $\lambda > 0$ *with* $\lambda(2\phi(\gamma) + \nu) < 1$ *and* $w \in C_\gamma$ *we have*

$$|(I + \lambda F)w|_X \leq (1 + \lambda \nu)|w|_X.$$

(c) *Let* $\kappa = \max\{\nu, M\}$. *Then* $\max\{|B_1 w|, |B_2 w|, |B_3 w|_1\} \leq \kappa |w|_X$ *for* $w \in C$.

PROOF. Let w and $\hat{w} \in C_\gamma$. Then we have

$$\begin{aligned}
|F_1 w - F_1 \hat{w}|_1 \leq& \int_0^\infty |\mu_1(a, w)\{u(a) - \hat{u}(a)\}| da \\
&+ \int_0^\infty |\{\mu_1(a, w) - \mu_1(a, \hat{w})\}\hat{u}(a)| da \\
&+ \int_{(0,\infty)^3} |\rho(a, b, c, w)\{p(a, b, c) - \hat{p}(a, b, c)\}| dadbdc \\
&+ \int_{(0,\infty)^3} |\{\rho(a, b, c, w) - \rho(a, b, c, \hat{w})\}\hat{p}(a, b, c)| dadbdc \\
&\leq \phi(\gamma)|u - \hat{u}|_1 + \Lambda_\gamma |\hat{u}|_1 |w - \hat{w}|_X + \nu |p - \hat{p}|_1 + \Lambda_\gamma |\hat{p}|_1 |w - \hat{w}|_X \\
&\leq (\phi(\gamma) + \nu + 2\gamma \Lambda_\gamma)|w - \hat{w}|_X.
\end{aligned}$$

Similarly, we have

$$|F_2 w - F_2 \hat{w}|_1 \le (\phi(\gamma) + \nu + 2\gamma \Lambda(\gamma))|w - \hat{w}|_X,$$

$$|F_3 w - F_3 \hat{w}|_1 \le (2\phi(\gamma) + \nu + 3\gamma \Lambda(\gamma))|w - \hat{w}|_X.$$

Hence we obtain

$$|Fw - F\hat{w}|_X \le (2\phi(\gamma) + \nu + 3\gamma \Lambda(\gamma))|w - \hat{w}|_X.$$

In a similar way, one can show that

$$|Bw - B\hat{w}|_Z \le (\nu + (1 + \gamma)\Lambda(\gamma))|w - \hat{w}|_X.$$

We next show that (b) holds. Let $\lambda > 0$ with $\lambda(2\phi(\gamma) + \nu) < 1$. Then we have

$$\begin{aligned} |u + \lambda F_1 w|_1 &\le \int_0^\infty |(1 - \lambda \mu_1(a, w))u(a)| da \\ &\quad + \lambda \int_{(0,\infty)^3} |\rho(a, b, c, w) p(a, b, c)| dadbdc \\ &\le |u|_1 + \lambda \nu |p|_1 \\ &\le (1 + \lambda \nu)|w|_X. \end{aligned}$$

Similarly, we have

$$|v + \lambda F_2 w|_1 \le (1 + \lambda \nu)|w|_X,$$

$$|p + \lambda F_3 w|_1 \le |w|_X.$$

Therefore we obtain (b). Finally, we show that (c) holds. Let $w \in C$. Then we have

$$|B_j w| \le \int_{(0,\infty)^3} |\alpha_j(a, b, c, w) p(a, b, c)| dadbdc \le \nu |w|_X$$

for $j = 1, 2$, and

$$|B_3 w|_1 \le \int_0^\infty \int_0^\infty |\Psi(w)(a, b)| dadb \le M |w|_X.$$

Thus (c) is obtained and the proof is thereby complete. ∎

LEMMA 7.5. *Under the assumption* **(A.1)** *and* **(A.2)**, *the following hold*:
(i) $(I - \lambda A_K)^{-1} w \in C$ *for* $\lambda > 0$ *and* $w \in C$.
(ii) $w + \lambda F w \in C$ *for* $w \in C_\gamma$ *and* $\lambda > 0$ *with* $\lambda(2\phi(\gamma) + \nu) < 1$.

PROOF. Let $(f, g, h) \in C$ and $(u, v, p) \in D(A_K)$. Suppose that

$$u + \lambda u_a = f, \quad v + \lambda v_b = g \quad \text{and} \quad p + \lambda \overline{T}_0 p = h.$$

Then we have

$$u(a) = \lambda^{-1} \int_0^a e^{-\lambda^{-1}(a-\xi)} f(\xi) d\xi \quad \text{and} \quad v(b) = \lambda^{-1} \int_0^b e^{-\lambda^{-1}(b-\xi)} g(\xi) d\xi.$$

This implies that u, $v \geq 0$ a.e. Choose $h_n \in C_0^\infty(\boldsymbol{R}^3)$ so that $h_n \geq 0$, supp $h_n \subset (0, \infty)^3$ and $h_n \to h$ in $L^1((0, \infty)^3)$. Using (7.2) with p and g replaced, respectively, by p_n and h_n, we have $p_n \in D(\overline{T}_0)$, $p_n - \lambda \overline{T}_0 p_n = h_n$ and $p_n \geq 0$. Letting $n \to \infty$, we obtain $p(a, b, c) \geq 0$ a.e. We next show that (ii) holds. Let $w = (u, v, p) \in C$. Let λ be a positive number such that $\lambda(2\phi(\gamma) + \nu) < 1$. Then we have

$$\begin{aligned}
&u(a) - \lambda \mu_1(a, w) u(a) + \lambda \int_0^\infty \int_0^\infty \rho(a, b, c, w) p(a, b, c) db dc \\
&\quad \geq (1 - \lambda \phi(\gamma)) u(a) \geq 0, \\
&v(b) - \lambda \mu_2(b, w) v(b) + \lambda \int_0^\infty \int_0^\infty \rho(a, b, c, w) p(a, b, c) da dc \\
&\quad \geq (1 - \lambda \phi(\gamma)) v(b) \geq 0, \\
&(1 - \lambda(\mu_1(a, w) + \mu_2(b, w) + \rho(a, b, c, w))) p(a, b, c) \\
&\quad \geq (1 - \lambda(2\phi(\gamma) + \nu) p(a, b, c) \geq 0.
\end{aligned}$$

These inequalities together imply the desired result. ■

For $w \in C$ and $n \geq 1$, we set

$$(7.5) \qquad \begin{aligned}
w_n &= (I - n^{-1} A_K)^{-1} (w + n^{-1} F w) + \Phi_n w, \\
\Phi_n w &= [e^{-na} B_1 w, e^{-nb} B_2 w, e^{-nc} \phi_n(a - c, b - c)],
\end{aligned}$$

where

$$\phi_n \geq 0, \ \phi_n \in \mathscr{C}_0^\infty(\boldsymbol{R}^2), \ \text{supp}\, \phi_n \subset (0, \infty)^2, \ \text{and} \ \phi_n \to B_3 w \ \text{in} \ L^1((0, \infty)^2).$$

By Lemma 7.5, we have $w_n \in D(A) \cap C$ for n sufficiently large. It follows from (7.5) that

$$w_n - w = (I - n^{-1} A_K)^{-1} w - w + n^{-1} (I - n^{-1} A_K)^{-1} F w + \Phi_n w.$$

This together with Lemma 7.4 yields

$$\begin{aligned}
|w_n - w|_X &\leq |(I - n^{-1} A_K)^{-1} w - w|_X + n^{-1} |F w|_X + |\Phi_n w|_X \\
&\leq |(I - n^{-1} A_K)^{-1} w - w|_X + n^{-1} |F w|_X \\
&\quad + n^{-1} (\kappa |w|_X + |\phi_n - B_3 w|_1).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} |w_n - w|_X = 0. \tag{7.6}$$

Noting that $(I - n^{-1}A)\Phi_n w = 0$, we infer from (7.5) that

$$w_n - n^{-1}(A + F)w_n - w = n^{-1}(Fw - Fw_n).$$

In view of (7.6) and the continuity of F, we have

$$\lim_{n\to\infty} n|w_n - n^{-1}(A + F)w_n - w|_X = 0. \tag{7.7}$$

Since $\operatorname{Ker} L = D(A_K)$, it follows from (7.5) that

$$Lw_n = [B_1 w, B_2 w, \phi_n(a, b)].$$

Hence we see from (7.6) that

$$\lim_{n\to\infty} |Lw_n - Bw_n|_Z = 0. \tag{7.8}$$

Also, (7.5) and Lemma 7.4 together imply

$$\begin{aligned}|w_n|_X &\le |w + n^{-1}Fw|_X + |\Phi_n w|_X \\ &\le (1 + n^{-1}\nu)|w|_X + n^{-1}(\kappa|w|_X + |\phi_n - B_3 w|_1).\end{aligned}$$

This shows that

$$\limsup_{n\to\infty} n(|w_n|_X - |w|_X) \le (\kappa + \nu)|w|_X. \tag{7.9}$$

Thus, in view of Lemma 7.3, Lemma 7.4, (7.4) and (7.6) through (7.9), we have shown that conditions (s.1) through (s.4) stated in Section 6 are all satisfied. Theorem 6.2 and Remark 6.3 can now be applied to obtain the following theorem:

THEOREM 7.6. *Let* $\omega = \kappa + \nu$. *Suppose that* (A1) *and* (A2) *hold. Then there exists a nonlinear semigroup* $\mathcal{S} \equiv \{S(t): t \ge 0\}$ *on* C *such that*

$$S(t)w = w + A\int_0^t S(s)w ds + \int_0^t FS(s)w ds,$$

$$L\int_0^t S(s)w ds = \int_0^t BS(s) ds,$$

$$|S(t)w|_X \le e^{\omega t}|w|_X, \qquad \textit{for each } w \in C \textit{ and } t \ge 0.$$

If in particular, B *and* F *are continuously Fréchet differentiable, then* $S(t)w$ *gives a strong solution provided that* $w \in D(A) \cap C$ *and* $Lw = Bw$.

REMARK 7.7. The previous proof shows that the results in Thieme [28] provide a local semiflow which is global in case that the rates μ_j is a bounded function of $w \in C$. The approach presented in this paper allows to remove the boundedness of this rate which is an unnatural restriction as one should like to have a generalization of logistic growth in unstructured population models. Alternatively one can combine the local existence results with a priori estimates like in Theorem 7.6 to obtain global existence of the semiflow.

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