HIROSHIMA MATH. J.
27 (1997), 177–188

Higher Specht polynomials

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(Received September 14, 1995)
(Revised February 23, 1996)

ABSTRACT. A basis of the quotient ring S/J_+ is given, where S is the ring of polynomials and J_+ is the ideal generated by symmetric polynomials of positive degree. They are called higher Specht polynomials.

0. Introduction

The purpose of this paper is to give a detailed proof of the result announced in [4], and to give its generalization.

Let $S = \mathbf{C}[x_0, \dots, x_{n-1}]$ be the algebra of polynomials of n variables $x_0, \dots, x_{n-1}$ with complex coefficients, on which the symmetric group $\mathfrak{S}_n$ acts by the permutation of the variables:

$$(\sigma f)(x_0, \dots, x_{n-1}) = f(x_{\sigma(0)}, \dots, x_{\sigma(n-1)})(\sigma \in \mathfrak{S}_n)$$

Let $e_j(x_0, \dots, x_{n-1}) = \sum_{0 \le i_1 < \cdots < i_j \le n-1} x_{i_1} \dots x_{i_j}$ be the elementary symmetric polynomial of degree j and set $J_+ = (e_1, \dots, e_n)$, the ideal generated by $e_1, \dots, e_n$. The quotient ring $R = S/J_+$ has a structure of an $\mathfrak{S}_n$-module. Let $n_0, \dots, n_{r-1}$ be natural numbers such that $n = \sum_{i=0}^{r-1} n_i$. Then the product of symmetric groups $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}$ is naturally embedded in $\mathfrak{S}_n$. By restricting to this subgroup, R is an $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}$-module. We give a combinatorial procedure to obtain a basis of each irreducible component of R. In view of this construction, these polynomials such obtained might be called higher Specht polynomials. The case $n_0 = n$ is treated in [4]. When $n_0 = \cdots = n_{n-1} = 1$, this basis becomes the descent basis for R (see [3]).

As an application, we also give a similar basis for a complex reflection group $G_{r,n} = (\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$. Let S be the symmetric algebra of the natural $G_{r,n}$ representation over $\mathbf{C}$. The ring of invariants $S^{G_{r,n}}$ is known to be isomorphic to a polynomial ring $\mathbf{C}[e_1^{(r)}, \dots, e_n^{(r)}]$ generated by the elementary symmetric polynomials $e_1^{(r)}, \dots, e_n^{(r)}$ in x_i^r's. We put $R^{(r)} = S/J_+$, where $J_+ = (e_1^{(r)}, \dots, e_n^{(r)})$. As a $G_{r,n}$-module, it is equivalent to the regular representation. It is also known that the irreducible representations of $G_{r,n}$ are indexed

1991 *Mathematics Subject Classification.* 05E10, 20C30.
Key words and phrases. Reflection groups, Specht polynomials.

by r-tuples of Young diagrams $\Lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ with $\sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. We construct a basis for $R^{(r)}$ parametrized by the pairs of standard r-tuples of tableaux (S, T) of the same shape.

After completing this paper, we noticed that E. Allen published a similar construction of the basis for R ([1]). In the present paper, we give a different proof for the linear independence of the higher Specht polynomials.

The authors thank the referee for quite a few comments.

1. The index r-tableaux

A partition λ is a non-increasing finite sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_l$. We write $\lambda \vdash n$ when the sum $\sum_{i=1}^{l} \lambda_i$ equals n. Conversely, given a partition λ, $\sum_{i=1}^{l} \lambda_i$ is called the size of λ. As is usual, a partition is expressed by a Young diagram. Let r be a positive integer and $\Lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ be an r-tuple of Young diagrams. We call such a Λ an r-diagram. The sequence of integers $(n_0, \dots, n_{r-1}) = (|\lambda^{(0)}|, \dots, |\lambda^{(r-1)}|)$ is called the type of Λ and denoted by $type(\Lambda)$. The sum $n = \sum_{i=0}^{r-1} n_i$ is called the size of Λ. The irreducible representations of $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}$ are indexed by the set of r-diagrams of type $(n_0, \dots, n_{r-1})$. By filling each "box" with a non-negative integer, we obtain a tableau (resp. an r-tableau) from a diagram (resp. an r-diagram). The original r-diagram is called the shape of the r-tableau. An r-tableau $\mathbf{T} = (T^{(0)}, \dots, T^{(r-1)})$ is said to be standard if the written sequence on each column and each row of $T^{(i)}$ $(0 \leq i \leq r-1)$ is strictly increasing, and each number from 0 to $n-1$ appears exactly once. The set of all standard r-tableaux of shape Λ is denoted by $ST(\Lambda)$. The prime ($'$) denotes the transposition of a diagram or a tableau. For an r-diagram $\Lambda = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ and an r-tableau $\mathbf{T} = (T^{(0)}, \dots, T^{(r-1)})$, we define $\Lambda' = (\lambda^{(r-1)'}, \dots, \lambda^{(0)'})$ and $\mathbf{T}' = (T^{(r-1)'}, \dots, T^{(0)'})$, respectively.

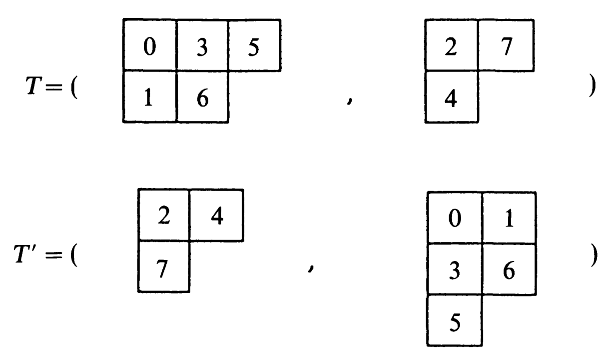

Figure 1

DEFINITION. A standard r-tableau is said to be natural if and only if the set of numbers written in $T^{(i)}$ is $\{n_0 + \cdots + n_{i-1}, \ldots, n_0 + \cdots + n_i - 1\}$. The set of natural standard r-tableaux of shape Λ is denoted by $NST(\Lambda)$.

On the set $ST(\Lambda)$, we introduce the last letter order "$<$" as follows. For two r-tableaux $\mathbf{T}_1 = (T_1^{(0)}, \ldots, T_1^{(r-1)})$ and $\mathbf{T}_2 = (T_2^{(0)}, \ldots, T_2^{(r-1)})$ in $ST(\Lambda)$, we write $\mathbf{T}_1 < \mathbf{T}_2$ if and only if there exists m $(0 \leq m \leq n-1)$ such that if $m < p$, p is written in the same box and m is written either in

(1) $T_1^{(i)}$ and $T_2^{(j)}$ with $i < j$, or

(2) k-th row of $T_1^{(i)}$ and l-th row of $T_2^{(i)}$ with $k > l$.

REMARK. This definition of the last letter order is different from that in [2].

A sequence of non-negative integers $w = (w_0, \ldots, w_{n-1})$ is called a word. Set $|w| = \sum_{k=0}^{n-1} w_k$. For a word w, we associate a new word $\hat{w} = (\hat{w}_0, \ldots, \hat{w}_{n-1})$ arranging w into the non-decreasing order. A word is called a permutation if $\{w_0, \ldots, w_{n-1}\} = \{0, \ldots, n-1\}$. Let δ denote the permutation $(0, \ldots, n-1)$. We define the index $i(w)$ of a permutation w as follows.

(1) If $w_k = 0$, then $i_k = 0$.

(2) If $w_k = i$ and $w_l = i + 1$, then (a) $i_l = i_k$ if $k < l$, (b) $i_l = i_k + 1$ if $k > l$.

We put $w' = (w_{n-1}, \ldots, w_0)$ if $w = (w_0, \ldots, w_{n-1})$. The coindex $j(w)$ of w is defined by $i(w')'$. For a standard r-tableau $\mathbf{T}$, we associate a word $w(\mathbf{T})$ in the following way. First we read each column of the tableau $T^{(0)}$ from the bottom to the top starting from the left. We continue this procedure for the tableau $T^{(1)}$ and so on. Assigning the index $i(w)$ and the coindex $j(w)$ of $w(\mathbf{T})$ to the corresponding box, we get new r-tableaux $i(\mathbf{T})$ and $j(\mathbf{T})$ which are called the index r-tableau and the coindex r-tableau of $\mathbf{T}$, respectively.

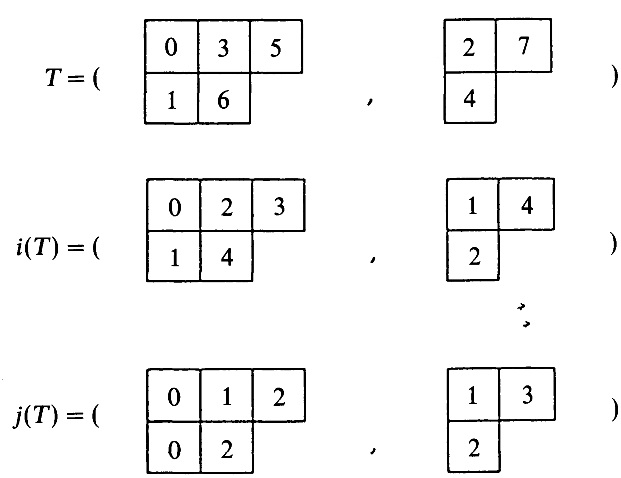

Figure 2

The following lemma is fundamental for the index and the coindex r-tableaux.

LEMMA 1. *Let* $\mathbf{T}$ *be a standard r-tableau of shape* Λ.

(1) *The index r-tableau* $i(\mathbf{T})$ *(resp. coindex r-tableau* $j(\mathbf{T})$*) is column strict (resp. row strict), i.e., if* $(p_1, \dots, p_l)$ *(resp.* $(q_1, \dots, q_m)$*) is a row (resp. column), then* $p_1 \le \dots \le p_l$ *(resp.* $q_1 \le \dots \le q_m$*) and if* $(q_1, \dots, q_m)$ *is (resp.* $(p_1, \dots, p_l)$*) a column (resp. row), then* $q_1 < \dots < q_m$ *(resp.* $p_1 < \dots < p_l$*).*

(2) $j(\mathbf{T}) = i(\mathbf{T}')'$.

(3) $i(\mathbf{T}) + j(\mathbf{T}) = \mathbf{T}$. *Here '+' denotes the elementwise summation.*

PROOF. (1) is obvious.
(2) It is obvious if the numbers i and $i+1$ appear in different components in $\mathbf{T}$. If they appear in the same component $T^{(j)}$, then $i+1$ is written in the box either right or lower to that filled with i. In the first case, $i+1$ is written in the upper row or the same. Therefore $i+1$ is read after i in $w(\mathbf{T})$ and before i in $w(\mathbf{T}')$. The latter case is similar. (3) If $w_k = i$ and $w_l = i+1$, then $i_l = i_k + 1$ and $j_l = j_k$ if $l < k$ and $j_l = j_k + 1$ if $k < \ell$. In any case, we have $i_l + j_l = i_k + j_k + 1$ and the statement.

2. Higher Specht polynomials and their independence

Let λ be a partition of n and T be a standard tableau of shape λ. We define the Young symmetrizer e_T of T by

$$e_T = \frac{f^\lambda}{n!} \sum_{\sigma \in C(T), \tau \in R(T)} sgn(\sigma)\sigma\tau \in \mathbf{C}[\mathfrak{S}_n],$$

where f^λ is the number of standard tableaux of shape λ and $C(T)$ (resp. $R(T)$) is the column (resp. row) stabilizer of T. It is an idempotent in $\mathbf{C}[\mathfrak{S}_n]$ ([2], p. 106, Theorem 3.10). For a subset I of $\{0, \dots, n-1\}$ of cardinality n_0 and a tableau T_0 of shape $\lambda_0 \vdash n_0$ filled with the numbers in the set I, denote the Young symmetrizer by $e_{T_0} \in \mathbf{C}[\mathfrak{S}(I)]$, where $\mathfrak{S}(I)$ is the symmetric group of the set I.

Let $S = \mathbf{C}[x_0, \dots, x_{n-1}]$ be the polynomial ring in variables $x_0, \dots, x_{n-1}$ with complex coefficients, J_+ be the ideal generated by elementary symmetric functions $e_1(x_0, \dots, x_{n-1}), \dots, e_n(x_0, \dots, x_{n-1})$ and $R = S/J_+$. For words u and v, we define $x_v^u = x_{v_0}^{u_0} \dots x_{v_{n-1}}^{u_{n-1}}$. For standard r-tableaux $\mathbf{S}$, $\mathbf{T}$, we define $x_{\mathbf{T}}^{i(\mathbf{S})} = x_{w(\mathbf{T})}^{i(w(\mathbf{S}))}$ and $x_{\mathbf{T}}^{j(\mathbf{S})} = x_{w(\mathbf{T})}^{j(w(\mathbf{S}))}$.

DEFINITION. For a standard r-tableau $\mathbf{T} = (T^{(0)}, \dots, T^{(r-1)})$ of shape Λ, $e_{T^{(i)}}$ is defined in the same way as above, though each $T^{(i)}$ is not necessarily standard. (Note that $e_{T^{(i)}}$ is an element in the group ring of permutations of numbers which appear in $T^{(i)}$.) We set $e_{\mathbf{T}} = e_{T^{(0)}} \dots e_{T^{(r-1)}}$. For $\mathbf{T}, \mathbf{S} \in ST(\Lambda)$, we define the higher Specht polynomial for $(\mathbf{T}, \mathbf{S})$ by

$$F_{\mathbf{T}}^{\mathbf{S}} = F_{\mathbf{T}}^{\mathbf{S}}(x_0, \dots, x_{n-1}) = e_{\mathbf{T}}(x_{\mathbf{T}}^{i(\mathbf{S})}).$$

It is easy to see that $x_{\mathbf{T}'}^{i(\mathbf{S}')} = x_{\mathbf{T}}^{i(\mathbf{S})}$ by Lemma 1 (2). The first main result in this paper is as follows.

THEOREM 1. *Fix a sequence* $(n_0, \dots, n_{r-1})$ *such that* $\sum_{i=0}^{r-1} n_i = n$.

(1) *The collection*

$$\bigcup_{type(\Lambda)=(n_0,\dots,n_{r-1})} \{F_{\mathbf{T}}^{\mathbf{S}} | \mathbf{T} \in NST(\Lambda), \mathbf{S} \in ST(\Lambda)\}$$

forms a $\mathbf{C}$*-basis of* R.

(2) *For an r-diagram* Λ *of type* $(n_0, \dots, n_{r-1})$ *and* $\mathbf{S} \in ST(\Lambda)$, $\{F_{\mathbf{T}}^{\mathbf{S}} | \mathbf{T} \in NST(\Lambda)\}$ *forms a* $\mathbf{C}$*-basis of* $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}$*-submodule of* R *which affords the irreducible representation corresponding to* Λ.

(3) *If* $r = n$, $n_j = 1$ $(0 \leq j \leq n-1)$, *then* $\{x_\delta^{i(w)} | w$ *is a permutation*$\}$ *is a* $\mathbf{Z}$*-basis of* $\mathbf{Z}[x_0, \dots, x_{n-1}]/(e_1, \dots, e_n)$.

REMARK. Case $r = 1$ is treated in [4]. The basis given in (3) is called the descent basis (see [3]).

To prove (1) and (3), we introduce a pairing $\langle , \rangle$ on R and show that the matrix $(\langle F_{\mathbf{T}_1}^{\mathbf{S}_1}, F_{\mathbf{T}_2'}^{\mathbf{S}_2'} \rangle)_{(\mathbf{S}_1, \mathbf{T}_1), (\mathbf{S}_2, \mathbf{T}_2)}$ is non-singular. Here $\mathbf{T}_1, \mathbf{T}_2 \in NST(\Lambda)$ and $\mathbf{S}_1, \mathbf{S}_2 \in ST(\Lambda)$. For an element $f \in R$, we choose a lifting $\tilde{f} \in S$ of f. Define $\langle f, g \rangle$ by

$$\langle f, g \rangle = \left(\frac{1}{\Delta} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) \sigma(\tilde{f}\tilde{g}) \right) \Big|_{x_0 = \cdots = x_{n-1} = 0}.$$

Here Δ is the difference product $\prod_{j<i}(x_i - x_j)$. The right hand side is independent of the liftings $\tilde{f}$, $\tilde{g}$ since

$$\frac{1}{\Delta} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma)\sigma(e_i \tilde{f}) = e_i \frac{1}{\Delta} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma)\sigma(\tilde{f}), \qquad e_i|_{x_0 = \cdots = x_{n-1} = 0} = 0.$$

The following lemma is easy to see.

LEMMA 2.

(1) $\langle \sigma f, g \rangle = sgn(\sigma) \langle f, \sigma^{-1} g \rangle$ for $\sigma \in \mathfrak{S}_n$.

(2) $\langle e_{\mathbf{T}} f, g \rangle = \langle f, e_{\mathbf{T}'} g \rangle$ *for* $\mathbf{T} \in ST(\Lambda)$.

For two words $\alpha = (\alpha_0, \dots, \alpha_{n-1})$ and $\beta = (\beta_0, \dots, \beta_{n-1})$, we say that α is greater than β with respect to the lexicographic order, denoted by $\alpha > \beta$, if there exists an m $(0 \leq m \leq n-1)$ such that $\alpha_j = \beta_j$ for all $j = m+1, \dots, n-1$ and $\alpha_m > \beta_m$.

LEMMA 3. *Let $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$, $\beta = (\beta_0, \ldots, \beta_{n-1})$ be words and w be a permutation such that $\langle x_w^\alpha, x_w^\beta \rangle \neq 0$. Then the following statements holds.*

(1) $|\alpha| + |\beta| = n(n-1)/2$, *and* $\{\alpha_0 + \beta_0, \ldots, \alpha_{n-1} + \beta_{n-1}\} = \{0, \ldots, n-1\}$.

(2) $\hat{\alpha} + \hat{\beta} \geq \delta$.

(3) *If* $\hat{\alpha} + \hat{\beta} = \delta$, *then for any* k $(0 \leq k \leq n-1)$, *there exists a unique* p *such that* $\alpha_p + \beta_p = k$ *and* $\alpha_p = \hat{\alpha}_k$, $\beta_p = \hat{\beta}_k$.

(4) *For a word* w, $\hat{i}(w) + \hat{j}(w) = \delta$.

PROOF. (1) If $|\alpha| + |\beta| < n(n-1)/2$, then $\sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) x_{\sigma(w)}^{\alpha+\beta}$ is an alternating polynomial of degree less than $n(n-1)/2$. It should be zero. If $|\alpha| + |\beta| > n(n-1)/2$, then $\frac{1}{\Delta} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) x_{\sigma(w)}^{\alpha+\beta}$ is a homogeneous polynomial of positive degree. Therefore it is zero if we put $x_0 = \cdots = x_{n-1} = 0$. Since $\alpha_i + \beta_i$ are distinct, we get the statement.

(2) Assume that there exists an m $(0 \leq m \leq n-1)$ such that $\hat{\alpha}_j + \hat{\beta}_j = j$ $(m+1 \leq j)$ and $\hat{\alpha}_m + \hat{\beta}_m < m$. If $\hat{\alpha}_{\sigma(j)} + \hat{\beta}_{\tau(j)} = j$ for $j = m+1, \ldots, n-1$, then $\hat{\alpha}_{\sigma(j)} = \hat{\alpha}_j$ and $\hat{\beta}_{\tau(j)} = \hat{\beta}_j$. Therefore there exist no k, $l = 0, \ldots, m$ such that $\hat{\alpha}_{\sigma(k)} + \hat{\beta}_{\tau(l)} = m$, which contradicts (1).

(3) Since $\{\alpha_0 + \beta_0, \ldots, \alpha_{n-1} + \beta_{n-1}\} = \{0, \ldots, n-1\}$, we find a unique $\sigma \in \mathfrak{S}_n$ such that $\alpha_{\sigma(i)} + \beta_{\sigma(i)} = i$ $(i = 0, \ldots, n-1)$. The inequality $\sigma\alpha \leq \hat{\alpha} = \delta - \hat{\beta} \leq \delta - \sigma\beta = \sigma\alpha$ implies $\sigma\alpha = \hat{\alpha}$, $\sigma\beta = \hat{\beta}$.

(4) If w, $i(w)$ and $j(w)$ are written as $w = (w_0, \ldots, w_{n-1})$, $i(w) = (i_0, \ldots, i_{n-1})$ and $j(w) = (j_0, \ldots, j_{n-1})$ respectively, then $w_k < w_l$ implies $i_k \leq i_l$ and $j_k \leq j_l$. This implies $\hat{i}(w) + \hat{j}(w) = \delta$.

Since the boxes in Λ are numbered by $\mathbf{T} \in NST(\Lambda)$, the symmetric group $\mathfrak{S}_n$ can be identified with the permutation group of boxes in diagram in Λ. For $\mathbf{S} \in ST(\Lambda)$, the group of permutations which stabilize $i(\mathbf{S})$ (resp. $j(\mathbf{S})$) can be identified with a subgroup $Stab_{\mathbf{T}}(i(\mathbf{S}))$ (resp. $Stab_{\mathbf{T}}(j(\mathbf{S}))$) of $\mathfrak{S}_n$ via the identification given above. Now we are ready to state the following properties for the pairing of higher Specht polynomials.

PROPOSITION 1.

(1) *Let* $\mathbf{S}_1$, $\mathbf{S}_2$ *be elements of* $ST(\Lambda)$ *such that* $\hat{i}(w(\mathbf{S}_1)) = \hat{i}(w(\mathbf{S}_2))$ *and* $\mathbf{S}_1 < \mathbf{S}_2$ *with respect to the last letter order. Then* $\langle F_{\mathbf{T}}^{\mathbf{S}_1}, F_{\mathbf{T}'}^{\mathbf{S}'_2} \rangle = 0$ *for* $\mathbf{T} \in NST(\Lambda)$.

(2) *Let* $h_c = \#(C(\mathbf{T}) \cap Stab_{\mathbf{T}}(j(\mathbf{S})))$ *and* $h_r = \#(R(\mathbf{T}) \cap Stab_{\mathbf{T}}(i(\mathbf{S})))$, *where* $C(\mathbf{T}) = C(T^{(0)}) \times \cdots \times C(T^{(r-1)})$, $R(\mathbf{T}) = R(T^{(0)}) \times \cdots \times R(T^{(r-1)})$. *Then we have*

$$\langle F_{\mathbf{T}}^{\mathbf{S}}, F_{\mathbf{T}'}^{\mathbf{S}'} \rangle = sgn(\mathbf{T}, \mathbf{S}) \frac{f^{\lambda^{(0)}} \cdots f^{\lambda^{(r-1)}}}{n_0! \ldots n_{r-1}!} h_r h_c$$

PROOF. For simplicity, $\hat{i}(w(\mathbf{S}))$ and $\hat{j}(w(\mathbf{S}))$ are denoted by $\hat{i}(\mathbf{S})$ and $\hat{j}(\mathbf{S})$ respectively. Since $x_{\mathbf{T}'}^{i(\mathbf{S}')} = x_{\mathbf{T}}^{i(\mathbf{S})}$, by the definition of higher Specht polynomials, we have

$$(2.2) \qquad \langle F_{\mathbf{T}}^{\mathbf{S}_1}, F_{\mathbf{T}'}^{\mathbf{S}'_2} \rangle = \frac{f^{\lambda(0)} \dots f^{\lambda(r-1)}}{n_0! \dots n_{r-1}!} \sum_{\sigma \in C(\mathbf{T}), \tau \in R(\mathbf{T})} \langle x_{\mathbf{T}}^{\tau^{-1} i(\mathbf{S}_1)}, x_{\mathbf{T}}^{\sigma^{-1} j(\mathbf{S}_2)} \rangle.$$

Suppose that $\mathbf{S}_1 < \mathbf{S}_2$ and $\langle x_{\mathbf{T}}^{\tau^{-1} i(\mathbf{S}_1)}, x_{\mathbf{T}}^{\sigma^{-1} j(\mathbf{S}_2)} \rangle \neq 0$ for $\sigma \in C(\mathbf{T})$, $\tau \in R(\mathbf{T})$. Assume that all the numbers from $m+1$ to $n-1$ are written in the same boxes of $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively, and the number m is written in the different places in $\mathbf{S}_1$ and $\mathbf{S}_2$. Let $b_{m+1}, \dots, b_{n-1}$ be the places where the numbers $m+1, \dots, n-1$ are written on $\mathbf{S}_1$ and $\mathbf{S}_2$. For $k \geq m$, let $i(\mathbf{S}_1^{(k)})$, $j(\mathbf{S}_2^{(k)})$ and $\mathbf{T}^{(k)}$ be the r-tableaux obtained by removing boxes $b_{k+1}, \dots, b_{n-1}$ from $i(\mathbf{S}_1)$, $j(\mathbf{S}_2)$ and $\mathbf{T}$, respectively. First we prove the following (A_k) for $m+1 \leq k \leq n-1$ by descending induction on k.

(A_k) the numbers written on b_k in r-tableaux $\tau^{-1}(i(\mathbf{S}_1))$ and $\sigma^{-1}(j(\mathbf{S}_2))$ equal the numbers $\hat{i}(\mathbf{S}_1)_k$ and $\hat{j}(\mathbf{S}_2)_k$, respectively. (Here $\sigma \in C(\mathbf{T})$ and $\tau \in R(\mathbf{T})$ act as permutations of boxes.)

For an r-tableau $\mathbf{S}$, $l \geq 0$, let $Supp(\mathbf{S}, l)$ be the boxes where l is written. Since

$$R(\mathbf{T})(Supp(i(\mathbf{S}_1), \hat{i}(\mathbf{S}_1)_{n-1})) \cap C(\mathbf{T})(Supp(j(\mathbf{S}_2), \hat{j}(\mathbf{S}_2)_{n-1})) = \{b_{n-1}\},$$

(A_{n-1}) holds by Lemma 3 (3). ($\hat{i}(\mathbf{S}_1) = \hat{i}(\mathbf{S}_2)$ implies $\hat{i}(\mathbf{S}_1) + \hat{j}(\mathbf{S}_2) = \delta$ by Lemma 1 (3) and Lemma 3 (4).) By the induction hypothesis, the numbers $\hat{i}(\mathbf{S}_1)_{k+1}, \dots, \hat{i}(\mathbf{S}_1)_{n-1}$ (resp. $\hat{j}(\mathbf{S}_2)_{k+1}, \dots, \hat{j}(\mathbf{S}_2)_{n-1}$) are already used to fill the places $b_{k+1}, \dots, b_{n-1}$ of $\tau^{-1}(i(\mathbf{S}_1))$ (resp. $\sigma^{-1}(j(\mathbf{S}_2))$). Therefore the r-tableaux $i(\mathbf{S}_1^{(k)})$ *and* $j(\mathbf{S}_2^{(k)})$ should be filled with the numbers $\hat{i}(\mathbf{S}_1)_1, \dots, \hat{i}(\mathbf{S}_1)_k$ and $\hat{j}(\mathbf{S}_2)_1, \dots, \hat{j}(\mathbf{S}_2)_k$, respectively. Since

$$R(\mathbf{T}^{(k)})(Supp(i(\mathbf{S}_1^{(k)}), \hat{i}(\mathbf{S}_1)_k)) \cap C(\mathbf{T}^{(k)})(Supp(j(\mathbf{S}_2^{(k)}), \hat{j}(\mathbf{S}_2)_k)) = \{b_k\},$$

(A_k) holds by Lemma 3 (3). This completes the proof of (A_k) for $m+1 \leq k \leq n-1$. By the inequality with respect to the last letter order, we have

$$R(\mathbf{T}^{(m)})(Supp(i(\mathbf{S}_1^{(m)}), \hat{i}(\mathbf{S}_1)_m)) \cap C(\mathbf{T}^{(m)})(Supp(j(\mathbf{S}_2^{(m)}), \hat{j}(\mathbf{S}_2)_m)) = \emptyset.$$

This contradicts the assumption $\langle x_{\mathbf{T}}^{\tau^{-1} i(\mathbf{S}_1)}, x_{\mathbf{T}\mathbf{T}}^{\sigma^{-1} j(\mathbf{S}_2)} \rangle \neq 0$ and completes the statement (1).

In the case $\mathbf{S}_1 = \mathbf{S}_2 = \mathbf{S}$, the summation (2.2) vanishes unless $\sigma \in C(\mathbf{T}) \cap Stab_{\mathbf{T}}(j(\mathbf{S}_1))$ and $\tau \in R(\mathbf{T}) \cap Stab_{\mathbf{T}}(i(\mathbf{S}_2))$. In this case, $\langle x_{\mathbf{T}}^{\tau^{-1} i(\mathbf{S})}, x_{\mathbf{T}}^{\sigma^{-1} j(\mathbf{S})} \rangle = sgn(\mathbf{S}, \mathbf{T})$. Thus we complete the proof the proposition.

The following two lemmas can be found in literature (e.g. [2]).

LEMMA 4. *For tableaux T_1, T_2, we define the last letter order in the same way. Let T_1, T_2 be standard tableaux of the same shape λ of size n. If $T_1 < T_2$ with respect to the last letter order, then $e_{T_1}e_{T_2} = 0$.*

PROOF. For a standard tableau T, set $H_T = \sum_{\sigma \in R(T)} \sigma$ and $V_T = \sum_{\sigma \in C(T)} sgn(\sigma)\sigma$. We prove $H_{T_1}V_{T_2} = 0$ by induction on the size n. For $n = 1$, it is obvious since there is only one tableau. We assume the case where the size is $n - 1$. By taking off the box filled with the number n from T_1 and T_2, we get tableaux T_1^* and T_2^*. If the shape of T_1^* and T_2^* are the same, then, by the induction hypothesis, we have $H_{T_1^*}V_{T_2^*} = 0$. Note that

$$H_{T_1} = (1 + (p_1, n) + \cdots + (p_t, n))H_{T_1^*},$$
$$V_{T_2} = V_{T_2^*}(1 - (q_1, n) - \cdots - (q_s, n)),$$

where $p_1, \ldots, p_t$ (resp. $q_1, \ldots, q_s$) are all the numbers which appear in the same row (resp. column) as n in T_1 (resp. T_2). If the shapes of T_1^* and T_2^* are different, by the definition of the last letter order, $T_1^* > T_2^*$ with respect to the lexicographic order. Therefore there exists (p, q) which belongs to the same row in T_1^* and the same column in T_2^* ([5] p. 94, combinatorial lemma). Hence, we have

$$H_{T_1^*}V_{T_2^*} = H_{T_1^*}(p, q)V_{T_2^*} = -H_{T_1^*}V_{T_2^*}$$

As a consequence, we have

$$H_{T_1^*}V_{T_2^*} = 0.$$

LEMMA 5. *Let $\{T_i\}_{1 \le i \le f^\lambda}$ be the set of standard tableaux such that $e_{T_i}e_{T_j} = 0$ if $i < j$. We write $T_i = \sigma_i T_1$ ($\sigma_i \in \mathfrak{S}_n$). Then $\{\sigma_i e_{T_1}\}$ is a basis of $\mathbf{C}[\mathfrak{S}_n]e_{T_1}$.*

PROOF. Since the dimension of $\mathbf{C}[\mathfrak{S}_n]e_{T_1}$ and the number of standard tableaux of shape λ are both f^λ ([2]), it is sufficient to prove the independence. Suppose $\sum_{i=1}^{f^\lambda} c_i\sigma_i e_{T_1} = 0$. We prove that $c_1 = \cdots = c_k = 0$ by induction on k. Under the induction hypothesis, we have the equation $0 = e_{T_{k+1}}(\sum c_i\sigma_i e_{T_1}) = \sum c_i e_{T_{k+1}} e_{T_i}\sigma_i = c_{k+1} e_{T_{k+1}}\sigma_{k+1}$.

Now we return to the properties of higher Specht polynomials.

PROPOSITION 2. *Let $\mathbf{T}_1$, $\mathbf{T}_2$ be elements in $NST(\Lambda)$. If $\mathbf{T}_1 > \mathbf{T}_2$ with respect to the last letter order, then*

$$\langle F_{\mathbf{T}_1}^{\mathbf{S}_1}, F_{\mathbf{T}_2'}^{\mathbf{S}_2'} \rangle = 0.$$

PROOF. By the definition of natural standard tableaux and the last letter order, there exists a number m such that $T_1^{(m)} > T_2^{(m)}$ with respect to the last letter order. Note that $e_{\mathbf{T}_2}e_{\mathbf{T}_1} = e_{T_2^{(m)}}e_{T_1^{(m)}}\prod_{j \ne m} e_{T_2^{(j)}}e_{T_1^{(j)}} = 0$.

PROOF OF THEOREM 1. (1) To compute the "Gramian" of the pairing $\langle , \rangle$ with respect to $\{F_{\mathbf{T}}^{\mathbf{S}}\}$ and $\{F_{\mathbf{T}'}^{\mathbf{S}'}\}$, we introduce a total order "<" on the set $NST(\Lambda) \times ST(\Lambda)$. For two elements $(\mathbf{T}_1, \mathbf{S}_1)$ and $(\mathbf{T}_2, \mathbf{S}_2)$ of $NST(\Lambda) \times ST(\Lambda)$, $(\mathbf{T}_1, \mathbf{S}_1) < (\mathbf{T}_2, \mathbf{S}_2)$ if and only if

(1) $\mathbf{T}_1 > \mathbf{T}_2$ with respect to the last letter order, or

(2) $\mathbf{T}_2 = \mathbf{T}_2$ and $\hat{i}(\mathbf{S}_1) < \hat{i}(\mathbf{S}_2)$ with respect to the lexicographic order, or

(3) $\mathbf{T}_1 = \mathbf{T}_2$, $\hat{i}(\mathbf{S}_1) = \hat{i}(\mathbf{S}_2)$ and $\mathbf{S}_1 < \mathbf{S}_2$ with respect to the last letter order.

Then by Proposition 1 and 2, we have $\langle F_{\mathbf{T}_1}^{\mathbf{S}_1}, F_{\mathbf{T}'_2}^{\mathbf{S}'_2} \rangle = 0$ if $(\mathbf{T}_1, \mathbf{S}_1) < (\mathbf{T}_2, \mathbf{S}_2)$ and $\langle F_{\mathbf{T}}^{\mathbf{S}}, F_{\mathbf{T}'}^{\mathbf{S}'} \rangle$ is a non-zero rational number. Thus the Gramian with respect to $\{F_{\mathbf{T}}^{\mathbf{S}}\}$ and $\{F_{\mathbf{T}'}^{\mathbf{S}'}\}$ is a non-zero rational number.

Since if the shapes of $\mathbf{T}_1$ and $\mathbf{T}_2$ are different, $\langle F_{\mathbf{T}_1}^{\mathbf{S}_1}, F_{\mathbf{T}'_2}^{\mathbf{S}'_2} \rangle = 0$ and the cardinality of $\coprod_{\Lambda} NST(\Lambda) \times ST(\Lambda)$ equals $n!$, the collection

$$\cup_{type(\Lambda)=(n_0,\dots,n_{r-1})}\{F_{\mathbf{T}}^{\mathbf{S}} | \mathbf{T} \in NST(\Lambda), \mathbf{S} \in ST(\Lambda)\}$$

forms a basis for R.

(2) We use Lemma 5 and

$$\begin{aligned}\sigma F_{\mathbf{T}}^{\mathbf{S}} &= \sigma e_{\mathbf{T}} x_{\mathbf{T}}^{i(\mathbf{S})} \\ &= \sigma e_{\mathbf{T}} \sigma^{-1} x_{\sigma\mathbf{T}}^{i(\mathbf{S})} \\ &= e_{\sigma\mathbf{T}} x_{\sigma\mathbf{T}}^{i(\mathbf{S})} \\ &= F_{\sigma\mathbf{T}}^{\mathbf{S}}\end{aligned}$$

to conclude that $\sum_{\mathbf{T} \in NST(\Lambda)} \mathbf{C} F_{\mathbf{T}}^{\mathbf{S}} = \mathbf{C}[\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}] F_{\mathbf{T}_1}^{\mathbf{S}}$, where $\mathbf{T}_1$ is the minimum element in $NST(\Lambda)$ with respect to the last letter order.

(3) In this case, the values $\langle F_{\mathbf{T}}^{\mathbf{S}}, F_{\mathbf{T}'}^{\mathbf{S}'} \rangle$ are ± 1 by Proposition 1 (2). Hence we can see that $\{F_{\mathbf{T}_1}^{\mathbf{S}}\}$ forms a $\mathbf{Z}$-basis of $\mathbf{Z}[x_0, \dots, x_{n-1}]/(e_1, \dots, e_n)$.

3. An application to wreath products

Let $T = (\mathbf{Z}/r\mathbf{Z})^n$ and $\varphi_a \in Hom(\mathbf{Z}/r\mathbf{Z}, \mathbf{C}^\times)$ be a character defined by $\varphi_a(x(\mathrm{mod}\ r)) = exp(2\pi i x a/r)$. Then an element $\varphi \in \hat{T} = Hom(T, \mathbf{C}^\times)$ can be written as

$$\varphi = \varphi_{a_0 \dots a_{n-1}} = \varphi_{a_0} \boxtimes \cdots \boxtimes \varphi_{a_{n-1}}.$$

Let n_j be the cardinality of $\{p | a_p = j\}$. We call the sequence $(n_0, \dots, n_{r-1})$ the type of the character $\varphi_{a_0 \dots a_{n-1}} \in \hat{T}$. Conversely, for a given sequence $(n_0, \dots, n_{r-1})$ such that $\sum_{i=0}^{r-1} n_i = n$, the character $\varphi^{(n_0, \dots, n_{r-1})}$ is defined as $\varphi_{a_0 \dots a_{n-1}}$, where $a_i = j$ if $\sum_{p=0}^{j-1} n_p \le i \le \sum_{p=0}^{j} n_p - 1$. The wreath product $G_{r,n} = (\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ is defined as the semi-direct product $\mathfrak{S}_n \ltimes T$. The group $(\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}) \ltimes T$ is regarded as a subgroup of $G_{r,n}$ by identifying the group $\mathfrak{S}_{n_j}$ with the permutation group for the set of numbers $\{i|\sum_{p=0}^{j-1} n_p \leq i \leq \sum_{p=0}^{j} n_p - 1\}$. Let $\lambda^{(0)}, \ldots, \lambda^{(r-1)}$ be Young diagrams of size $n_0, \ldots, n_{r-1}$, respectively. For representations $V^{\lambda^{(0)}}, \ldots, V^{\lambda^{(r-1)}}$ of $\mathfrak{S}_{n_0}, \ldots, \mathfrak{S}_{n_{r-1}}$, respectively and a character $\varphi^{(n_0,\ldots,n_{r-1})}$, set

$$V_\Lambda = Ind_{\mathfrak{S}_{n_0}\times\cdots\times\mathfrak{S}_{n_{r-1}}\ltimes T}^{G_{r,n}}(V^{\lambda^{(0)}} \boxtimes \cdots \boxtimes V^{\lambda^{(r-1)}} \boxtimes \varphi^{(n_0,\ldots,n_{r-1})}),$$

where $\Lambda = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$. It is known that all the irreducible representations of $G_{r,n}$ are obtained in this way, and that two representations V_{Λ_1} and V_{Λ_2} are isomorphic if and only if $\Lambda_1 = \Lambda_2$. A representation space W of $G_{r,n}$ is decomposed as $W = \oplus_{\varphi\in\hat{T}} W_\varphi$, where $W_\varphi = \{v \in W | tv = \varphi(t)v \text{ for all } t \in T\}$. The symmetric group $\mathfrak{S}_n$ acts on the character group $\hat{T}$. It is easy to see that V_Λ is decomposed into

$$V_\Lambda = \oplus_{g\in\mathfrak{S}_n/\mathfrak{S}_{n_0}\times\cdots\times\mathfrak{S}_{n_{r-1}}} g(V_{\Lambda,\varphi}),$$

with $g(V_{\Lambda,\varphi}) = V_{\Lambda,g\varphi}$. By the definition of the induced module, for an element $g \in \mathfrak{S}_n$, $V_{\Lambda,g\varphi}$ becomes a $g(\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}})g^{-1}$-module and the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}} & \longrightarrow & g(\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}})g^{-1} \\
\downarrow & & \downarrow \\
Aut(V_{\Lambda,\varphi}) & \longrightarrow & Aut(V_{\Lambda,g\varphi}).
\end{array}$$

DEFINITION. Let $\mathbf{T}$, $\mathbf{S}$ be elements in $ST(\Lambda)$. We define the higher Sprecht polynomial $\hat{F}_{\mathbf{T}}^{\mathbf{S}}$ for $G_{r,n}$ by

$$\hat{F}_{\mathbf{T}}^{\mathbf{S}}(x_0, \ldots, x_{n-1}) = F_{\mathbf{T}}^{\mathbf{S}}(x_0^r, \ldots, x_{n-1}^r) \cdot \prod_{j=0}^{r-1} \left(\prod_{m\in T^{(j)}} x_m \right)^j.$$

Here $F_{\mathbf{T}}^{\mathbf{S}}$ is the higher Specht polynomial defined in §2.

Let STP be the union $\cup_\Lambda ST(\Lambda) \times ST(\Lambda)$.

THEOREM 2.

(1) *The ring of invariants* $\mathbf{C}[x_0, \ldots, x_{n-1}]^{G_{r,n}}$ *of* $S = \mathbf{C}[x_0, \ldots, x_{n-1}]$ *under the natural action of* $G_{r,n}$ *is the polynomial ring of* $e_1^{(r)}, \ldots, e_n^{(r)}$, *where* $e_j^{(r)}$ *is the j-th elementary symmetric function of* $x_0^r, \ldots, x_{n-1}^r$.

(2) *The set* $\{\hat{F}_{\mathbf{T}}^{\mathbf{S}} | (\mathbf{T}, \mathbf{S}) \in STP\}$ *is a basis for* $R^{(r)} = S/(e_1^{(r)}, \ldots, e_n^{(r)})$, *and for a fixed* $\mathbf{S} \in ST(\Lambda)$, *the set* $\{\hat{F}_{\mathbf{T}}^{\mathbf{S}} | \mathbf{S} \in ST(\Lambda)\}$ *spans an irreducible representation of* $G_{r,n}$ *over* $\mathbf{C}$.

(3) *The set* $\{\hat{F}_{\mathbf{T}}^{\mathbf{S}} | (\mathbf{T}, \mathbf{S}) \in STP\}$ *forms a free basis of* S *over* $S^{G_{r,n}}$, *and for a fixed* $\mathbf{S} \in ST(\Lambda)$, *the set* $\{\hat{F}_{\mathbf{T}}^{\mathbf{S}} | \mathbf{S} \in ST(\Lambda)\}$ *spans an irreducible representation of* $G_{r,n}$ *over* $S^{G_{r,n}}$.

PROOF. (1) Since $\mathbf{C}[x_0,\ldots,x_{n-1}]^T=\mathbf{C}[x_0^r,\ldots,x_{n-1}^r]$, it reduces to the fundamental theorem of symmetric functions. The statement (3) is a direct consequence of (2). Therefore we prove (2).

The space $R^{(r)}=S/(e_1^{(r)},\ldots,e_n^{(r)})$ is known to be isomorphic to the regular representation, and for a character $\varphi_{a_0\ldots a_{n-1}}$ of T of type $(n_0,\ldots,n_{r-1})$, $R^{(r)}_{\varphi_{a_0\cdots a_{n-1}}}$ is the subspace $\mathbf{C}[x_0^r,\ldots,x_{n-1}^r]/(e_1^{(r)},\ldots,e_n^{(r)})\cdot\prod_{i=0}^{n-1}x_i^{a_i}$ of $R^{(r)}$. Let $g\in\mathfrak{S}_n$ be given by (a) $a_{g(i)}=j$ for $\sum_{p=0}^{j-1}n_p\le i\le\sum_{p=0}^{j}n_p-1$ and (b) $g(i)<g(k)$ if $\sum_{p=0}^{j-1}n_p\le i<k\le\sum_{p=0}^{j}n_p-1$. Since $g\varphi^{(n_0,\ldots,n_{r-1})}=\varphi_{a_0\ldots a_{n-1}}$, $R^{(r)}_{\varphi_{a_0\cdots a_{n-1}}}$ becomes a $g(\mathfrak{S}_{n_0}\times\cdots\times\mathfrak{S}_{n_{r-1}})g^{-1}$-module. In Theorem 1, we considered the action of $\mathfrak{S}_{n_0}\times\cdots\times\mathfrak{S}_{n_{r-1}}$ on $\mathbf{C}[x_0^r,\ldots,x_{n-1}^r]/(e_1^{(r)},\ldots,e_n^{(r)})$. To apply this theorem, we should consider the action of $g(\mathfrak{S}_{n_0}\times\cdots\times\mathfrak{S}_{n_{r-1}})g^{-1}$. Let $NST(g,\Lambda)$ be the set of r-tableaux $\mathbf{T}=(T^{(0)},\ldots,T^{(r-1)})$ such that the number j is filled in the tableau $T^{(i)}$ if $j=g(k)$ with $\sum_{p=0}^{i-1}n_p\le k\le\sum_{p=0}^{i}n_p-1$. Then by Theorem 1, we have the following.

(1) The collection

$$\bigcup_{type(\Lambda)=(n_0,\ldots,n_{r-1})}\{\hat{F}_{\mathbf{T}}^{\mathbf{S}}\mid\mathbf{T}\in NST(g,\Lambda),\ \mathbf{S}\in ST(\Lambda)\}$$

forms a basis for $R^{(r)}_{\varphi_{a_0\cdots a_{n-1}}}$.

(2) For a fixed $\mathbf{S}\in ST(\Lambda)$, $\{\hat{F}_{\mathbf{T}}^{\mathbf{S}}\mid\mathbf{T}\in NST(g,\Lambda)\}$ spans the irreducible representation $V^{\lambda^{(0)}}\boxtimes\cdots\boxtimes V^{\lambda^{(r-1)}}$ of $g(\mathfrak{S}_{n_0}\times\cdots\times\mathfrak{S}_{n_{r-1}})g^{-1}$. Therefore the collection

$$\bigcup_{type(\Lambda)=(n_0,\ldots,n_{r-1})}\{\hat{F}_{\mathbf{T}}^{\mathbf{S}}\mid\mathbf{T}\in ST(\Lambda),\ \mathbf{S}\in ST(\Lambda)\}$$

spans the irreducible representation V_Λ.

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