HIROSHIMA MATH. J.
27 (1997), 159–175

L^q-mean limits for Taylor's expansion of Riesz potentials of functions in Orlicz classes

Tetsu SHIMOMURA

(Received January 17, 1996)

ABSTRACT. This paper deals with L^q-mean limits for Taylor's expansion of Riesz potentials $U_\alpha f$ of order α for functions f satisfying an Orlicz condition. We examine when

$$\lim_{r \to 0} \omega(r)\left(r^{-n}\int_{B(x_0,r)} |U_\alpha f(x) - P_{x_0}(x)|^q dx\right)^{1/q} = 0$$

holds for every $x_0 \in R^n$ possibly except that in a set of capacity zero, where ω is a weight function and P_{x_0} is a polynomial. If $\omega(r) = r^{-\ell}$, then this means that $U_\alpha f$ is L^q-differentiable of order ℓ at x_0.

1. Introduction

For $0 < \alpha < n$ and a nonnegative measurable function f on R^n, we define $U_\alpha f$ by

$$U_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy;$$

$U_\alpha f$ is called the Riesz potential of f of order α. Here it is natural to assume that $U_\alpha f \not\equiv \infty$, which is equivalent to

$$\int_{R^n} (1+|y|)^{\alpha-n} f(y)dy < \infty. \tag{1.1}$$

As in the previous papers [7], [8], we assume the condition

$$\int_{R^n} \Phi_p(f(y))dy < \infty, \tag{1.2}$$

where $\Phi_p(r) = r^p\varphi(r)$, $1 < p < \infty$, with a function φ on the interval $(0, \infty)$ having the following properties:

1991 *Mathematics Subject Classification.* 31B35, 46E35.

Key words and phrases. Mean limits, Taylor's expansion, Riesz potentials, Orlicz functions.

(φ1) φ is positive nondecreasing on $(0, \infty)$.
(φ2) φ is of logarithmic type, that is, there exists $A_1 > 0$ such that

$$A_1^{-1}\varphi(r) \leq \varphi(r^2) \leq A_1\varphi(r) \qquad \text{whenever} \quad r > 0.$$

In the previous paper [8], we discussed the existence of fine limits of the form

$$\lim_{x \to x_0, x \in R^n - E} \omega(|x - x_0|)[U_\alpha f(x) - P_{x_0}(x)] = 0 \tag{1.3}$$

for functions f satisfying (1.1) and (1.2), where E is an exceptional set, ω is a "weight function" and P_{x_0} is a polynomial.

In this paper, we prove that the L^q-mean satisfies

$$\lim_{r \to 0} \omega(r)\left(r^{-n}\int_{B(x_0,r)} |U_\alpha f(x) - P_{x_0}(x)|^q dx\right)^{1/q} = 0, \tag{1.4}$$

for $q > 0$ satisfying $1/q \geq 1/p - \alpha/n$, where $B(x_0, r)$ is the open ball centered at x_0 with radius r (see Theorem 3.1).

As in [8], $U_\alpha f(x) - P_{x_0}(x)$ is written as

$$U_{\alpha,\ell,x_0} f(x) = \int_{R^n} R_{\alpha,\ell,x_0}(x, y)f(y)dy$$

for some nonnegative integer ℓ, with the remainder term of Taylor's expansion of $R_\alpha(x - y) = |x - y|^{\alpha-n}$:

$$R_{\alpha,\ell,x_0}(x, y) = R_\alpha(x - y) - \sum_{|\mu| \leq \ell} \frac{(x - x_0)^\mu}{\mu!}[(D^\mu R_\alpha)(x_0 - y)],$$

provided

$$\int_{B(x_0,1)} |y - x_0|^{\alpha-n-\ell} f(y)dy < \infty. \tag{1.5}$$

If (φ1), (φ2) and

$$\int_0^1 [r^{n-\alpha p}\varphi(r^{-1})]^{-1/(p-1)} r^{-1} dr < \infty \tag{1.6}$$

hold, then $U_\alpha f$ is continuous everywhere on R^n (see [1, Theorem 5.4] and [6]). Furthermore we know (see [8]) that (1.3) holds for $E = \emptyset$ (the empty set) and hence (1.4) trivially holds (see also Theorem 3.2 below). Thus we are mainly concerned with the case where (1.6) does not necessarily hold.

In Section 4, we shall show that (1.4) holds as far as x_0 is not contained in a set of certain capacity zero (see Theorem 4.1 and Corollary 4.1 below).

In view of the behavior at the origin of Bessel kernels, our results can be considered as generalizations of the results by Meyers [3], [4] concerning Bessel potentials of functions in $L^p(R^n)$.

If (1.4) holds for $\omega(r) = r^{-\ell}$, then $U_\alpha f$ is said to be L^q-differentiable of order ℓ at x_0 (cf. Meyers [3], Stein [9] and Ziemer [10]), where ℓ is a positive integer such that $\ell \leq \alpha$. In the final section we discuss L^q-differentiability as a consequence of the proceeding results in case $\ell < \alpha$ (see Theorem 5.1 below). In case $\alpha = \ell$, we shall show that $U_\ell f$ is L^q-differentiable of order ℓ almost everywhere (see Theorem 5.2). Note that if (1.6) holds, then $U_\ell f$ is ℓ times differentiable almost everywhere (see [6, Theorem 2]).

2. The estimates of $U_{\alpha,\ell,x_0}f$

Throughout this paper, let M denote various constants independent of the variables in question.

First we collect properties which follow from conditions (φ1) and (φ2) (see [7] and [8, Section 2]).

(φ3) φ satisfies the doubling condition, that is, there exists $A > 1$ such that

$$(\varphi(r) \leq)\varphi(2r) \leq A\varphi(r) \qquad \text{whenever} \quad r > 0.$$

(φ4) For any $\gamma > 0$, there exists $A(\gamma) > 1$ such that

$$A(\gamma)^{-1}\varphi(r) \leq \varphi(r^\gamma) \leq A(\gamma)\varphi(r) \qquad \text{whenever} \quad r > 0.$$

(φ5) If $\gamma > 0$, then

$$s^\gamma \varphi(s^{-1}) \leq A t^\gamma \varphi(t^{-1}) \qquad \text{whenever} \quad 0 < s < t.$$

For an nonnegative integer ℓ, a point $x_0 \in R^n$ and a nonnegative measurable function f on R^n, we consider the potential

$$U_{\alpha,\ell,x_0}f(x) = \int_{R^n} R_{\alpha,\ell,x_0}(x, y)f(y)dy,$$

which is written as $U_{\alpha,\ell,x_0}f(x) = U_1(x) + U_2(x) + U_3(x)$ for $x \in R^n - \{x_0\}$, where

$$U_1(x) = \int_{R^n - B(x_0, 2|x-x_0|)} R_{\alpha,\ell,x_0}(x, y)f(y)dy,$$

$$U_2(x) = \int_{B(x_0, |x-x_0|/2)} R_{\alpha,\ell,x_0}(x, y)f(y)dy,$$

$$U_3(x) = \int_{B(x_0, 2|x-x_0|) - B(x_0, |x-x_0|/2)} R_{\alpha,\ell,x_0}(x, y)f(y)dy.$$

We know the following results (cf. [6] and [8, Section 3]).

LEMMA 2.1. *If* $y \in B(x_0, |x - x_0|/2)$, *then*

$$|R_{\alpha,\ell,x_0}(x, y)| \le M|x - x_0|^{\ell}|y - x_0|^{\alpha-n-\ell}.$$

LEMMA 2.2. *If* $y \in B(x_0, 2|x - x_0|) - B(x_0, |x - x_0|/2)$, *then*

$$|R_{\alpha,\ell,x_0}(x, y)| \le M|x - y|^{\alpha-n}.$$

LEMMA 2.3. *If* $y \in R^n - B(x_0, 2|x - x_0|)$, *then*

$$|R_{\alpha,\ell,x_0}(x, y)| \le M|x - x_0|^{\ell+1}|y - x_0|^{\alpha-n-\ell-1}.$$

Throughout this paper, let $\omega(r)$ be a positive nonincreasing function on $(0, \infty)$ satisfying the following doubling condition:
(ω1) There exists $A_1 > 0$ such that

$$\omega(r) \le A_1\omega(2r) \qquad \text{whenever} \quad r > 0.$$

LEMMA 2.4. *Suppose* ω *satisfies*
(ω2) $r^{\ell+1}\omega(r)$ *is nondecreasing on* $(0, \infty)$.
Let f *be a nonnegative measurable function on* R^n *satisfying*

$$\int_{R^n} |y - x_0|^{\alpha-n}\omega(|y - x_0|)f(y)dy < \infty. \tag{2.1}$$

Then

$$\omega(|x - x_0|)U_1(x) = O(1) \qquad \text{as} \quad x \to x_0.$$

If in addition, ω *satisfies*

$$\lim_{r\to 0} r^{\ell+1}\omega(r) = 0, \tag{ω3}$$

then

$$\omega(|x - x_0|)U_1(x) = o(1) \qquad \text{as} \quad x \to x_0.$$

PROOF. Let $\varepsilon > 0$. If $2|x - x_0| < \varepsilon$, then by Lemma 2.3 and condition (ω2) we have

$$\begin{aligned}|U_1(x)| &\le M|x - x_0|^{\ell+1}\int_{R^n - B(x_0, 2|x-x_0|)} |y - x_0|^{\alpha-n-\ell-1}f(y)dy \\ &\le M|x - x_0|^{\ell+1}[\varepsilon^{\ell+1}\omega(\varepsilon)]^{-1}\int_{R^n - B(x_0,\varepsilon)} |y - x_0|^{\alpha-n}\omega(|y - x_0|)f(y)dy \\ &\quad + M\omega(|x - x_0|)^{-1}\int_{B(x_0,\varepsilon) - B(x_0, 2|x-x_0|)} |y - x_0|^{\alpha-n}\omega(|y - x_0|)f(y)dy.\end{aligned}$$

Hence by (2.1) we obtain

$$|U_1(x)| \le \omega(|x-x_0|)^{-1}\left\{M_\varepsilon |x-x_0|^{\ell+1}\omega(|x-x_0|) + M\int_{B(x_0,\varepsilon)} |y-x_0|^{\alpha-n}\omega(|y-x_0|)f(y)dy\right\},$$

which implies that

$$U_1(x) = O(\omega(|x-x_0|)^{-1}) \qquad \text{as} \quad x \to x_0.$$

If in addition condition (ω3) holds, then

$$\limsup_{x\to x_0} \omega(|x-x_0|)|U_1(x)| \le M\int_{B(x_0,\varepsilon)} |y-x_0|^{\alpha-n}\omega(|y-x_0|)f(y)dy.$$

Since ε is arbitrary, we see that the left hand side is equal to zero.

LEMMA 2.5. *Suppose ω satisfies*
(ω4) *$r^\ell\omega(r)$ is nonincreasing on* $(0, \infty)$.
If f is a nonnegative measurable function on R^n satisfying (2.1), *then*

$$\omega(|x-x_0|)U_2(x) = o(1) \qquad \text{as} \quad x \to x_0.$$

PROOF. By Lemma 2.1 and condition (ω4), we have

$$|U_2(x)| \le M|x-x_0|^\ell \int_{B(x_0,|x-x_0|/2)} |y-x_0|^{\alpha-n-\ell}f(y)dy$$
$$\le M\omega(|x-x_0|)^{-1}\int_{B(x_0,|x-x_0|/2)} |y-x_0|^{\alpha-n}\omega(|y-x_0|)f(y)dy,$$

which together with (2.1) implies the assertion of the lemma.

REMARK 2.1. If ω satisfies (ω4) and f satisfies (2.1), then (1.5) holds.

3. Mean limits

For $q > 0$, $x_0 \in R^n$ and $r > 0$, we define the L^q-mean of a measurable function u over $B(x_0, r)$ by

$$V_q(u, x_0, r) = \left(\frac{1}{\sigma_n r^n}\int_{B(x_0,r)} |u(x)|^q dx\right)^{1/q},$$

where σ_n denotes the volume of the unit ball $B(0, 1)$.

THEOREM 3.1. *Let $1 < p \le n/\alpha$ and $q > 0$ with $1/q \ge 1/p - \alpha/n$. Suppose ω satisfies* (ω2), (ω4) *and*

$$(\omega 5) \qquad \lim_{r\to 0} r^{\beta}\omega(r) = 0 \qquad \textit{for some} \quad \beta < \alpha.$$

If f is a nonnegative measurable function on R^n satisfying conditions (1.1), (2.1) *and*

$$(3.1) \qquad \lim_{r\to 0} [r^{n-\alpha p}\omega(r)^{-p}\varphi(r^{-1})]^{-1} \int_{B(x_0,r)} \Phi_p(f(y))dy = 0,$$

then

$$(3.2) \qquad \omega(r)V_q(U_{\alpha,\ell,x_0}f(x), x_0, r) = O(1) \qquad \text{as} \quad r\to 0.$$

If in addition condition (ω5) *holds for* $\beta \le \ell + 1$, *then*

$$(3.3) \qquad \omega(r)V_q(U_{\alpha,\ell,x_0}f(x), x_0, r) = o(1) \qquad \text{as} \quad r\to 0.$$

REMARK 3.1. By (ω4), $\beta > \ell$, and hence $\ell < \alpha$. If $\ell + 1 < \alpha$, then (ω2) implies (ω5) for β satisfying $\ell + 1 < \beta < \alpha$.

PROOF OF THEOREM 3.1. Note that if $\beta \le \ell + 1$, then (ω5) implies (ω3). Thus, in view of Lemmas 2.4 and 2.5, it suffices to treat only $U_3(x)$. For $\delta > 0$, we have by Lemma 2.2,

$$\begin{aligned}|U_3(x)| &\le M\int_{E(x)} |x-y|^{\alpha-n}f(y)dy \\ &= M\int_{\{y\in E(x): f(y) > |x-x_0|^{-\delta}\}} |x-y|^{\alpha-n}f(y)dy \\ &\quad + M\int_{\{y\in E(x): 0 < f(y) \le |x-x_0|^{-\delta}\}} |x-y|^{\alpha-n}f(y)dy \\ &= MU_{31}(x) + MU_{32}(x),\end{aligned}$$

where $E(x) = E(x; x_0) = B(x_0, 2|x-x_0|) - B(x_0, |x-x_0|/2)$. By condition ($\varphi$4), we see that if $f(y) > |x-x_0|^{-\delta}$, then

$$\varphi(f(y)) \ge \varphi(|x-x_0|^{-\delta}) \ge M\varphi(|x-x_0|^{-1}).$$

For q with $q > p$, let γ be a number such that $1/q = 1/p - \gamma/n$. Then $\alpha - \gamma = n(1/q - 1/p + \alpha/n) \ge 0$. If $|x-x_0| < r < 1$, then we have

$$\begin{aligned}U_{31}(x) &\le M[\varphi(|x-x_0|^{-1})]^{-1/p}\int_{E(x)} |x-y|^{\alpha-n}f(y)[\varphi(f(y))]^{1/p}dy \\ &\le M[\varphi(|x-x_0|^{-1})]^{-1/p}|x-x_0|^{\alpha-\gamma}\int_{E(x)} |x-y|^{\gamma-n}f(y)[\varphi(f(y))]^{1/p}dy \\ &\le M[\varphi(r^{-1})]^{-1/p}r^{\alpha-\gamma}\int_{B(x_0,2r)} |x-y|^{\gamma-n}f(y)[\varphi(f(y))]^{1/p}dy.\end{aligned}$$

On the other hand, we have

$$U_{32}(x) \le |x - x_0|^{-\delta} \int_{B(x_0, 2|x-x_0|)} |x - y|^{\alpha-n} dy \le M|x - x_0|^{\alpha-\delta} \le M r^{\alpha-\delta},$$

where $0 < \delta < \alpha$. We use Minkowski's inequality to obtain

$$V_q(U_3(x), x_0, r) \le M r^{\alpha-\delta} + M\left(\frac{1}{\sigma_n r^n}\right)^{1/q} [\varphi(r^{-1})]^{-1/p} r^{\alpha-\gamma}$$
$$\times \left\{\int_{B(x_0,r)} \left(\int_{B(x_0,2r)} |x - y|^{\gamma-n} f(y)[\varphi(f(y))]^{1/p} dy\right)^q dx\right\}^{1/q}.$$

Applying Sobolev's inequality to the last integral, we obtain

$$\omega(r) V_q(U_3(x), x_0, r) \le M[r^{n-\alpha p}\omega(r)^{-p}\varphi(r^{-1})]^{-1/p}\left(\int_{B(x_0,2r)} \Phi_p(f(y))dy\right)^{1/p}$$
$$+ M r^{\alpha-\delta}\omega(r).$$

Hence, by choosing $\delta > 0$ such that $\beta < \alpha - \delta$ it follows from (3.1) and (ω5)

$$\lim_{r\to 0} \omega(r) V_q(U_3(x), x_0, r) = 0.$$

Since $V_q(u, x_0, r)$ is nondecreasing with respect to q, Theorem 3.1 is obtained.

Set

$$\varphi^*(r) = \left(\int_0^r \varphi(t^{-1})^{-p'/p} t^{-1} dt\right)^{1/p'}.$$

In case $\alpha p = n$ and $q = \infty$, we shall establish the following result.

THEOREM 3.2. *Let $\alpha p = n$ and ω be as in Theorem 3.1. Let f be a nonnegative measurable function on R^n satisfying conditions* (1.1), (2.1) *and* (3.1). *If $\varphi^*(1) < \infty$, then*

(3.4) $$\sup_{x\in B(x_0,r)} |U_{\alpha,\ell,x_0} f(x)| = o(\omega(r)^{-1}\varphi(r^{-1})^{1/p}\varphi^*(r)) \quad \text{as} \quad r \to 0.$$

REMARK 3.2. Note that

$$\varphi^*(r) \ge \left(\int_{r^2}^r [\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'} \ge M[\varphi(r^{-1})]^{-1/p}[\log(1/r)]^{1/p'},$$

so that

$$\lim_{r\to 0} \varphi^*(r)[\varphi(r^{-1})]^{1/p} = \infty.$$

Hence (3.4) does not imply that

$$\omega(r) \sup_{x \in B(x_0, r)} |U_{\alpha, \ell, x_0} f(x)| = O(1) \qquad \text{as} \quad r \to 0.$$

PROOF OF THEOREM 3.2. In view of Lemmas 2.4 and 2.5, it suffices to treat only $U_3(x)$, as before. By [8, Lemma 4.1], we have

$$|U_3(x)| \leq M r^{\alpha-\delta} + M\varphi^*(r)\left(\int_{B(x_0, 2r)} \Phi_p(f(y))dy\right)^{1/p}$$

for $|x - x_0| < r$, where $0 < \delta < \alpha$. Consequently, it follows that

$$|U_3(x)| \leq M[\omega(r)^{-1}\varphi^*(r)\varphi(r^{-1})^{1/p}] \times \left\{r^{\alpha-\delta}\omega(r) + \left([\omega(r)^{-p}\varphi(r^{-1})]^{-1} \int_{B(x_0, 2r)} \Phi_p(f(y))dy\right)^{1/p}\right\}$$

for $|x - x_0| < r$. Hence we obtain by (3.1) and (ω5)

$$\lim_{r \to 0} [\omega(r)^{-1}\varphi^*(r)\varphi(r^{-1})^{1/p}]^{-1} \sup_{x \in B(x_0, r)} |U_3(x)| = 0.$$

This completes the proof of Theorem 3.2.

4. Quasi everywhere convergence of mean limits

Define

$$k(x) = |x|^{\alpha-n}\omega(|x|).$$

To evaluate the size of exceptional sets, for a set $E \subset R^n$ and an open set $G \subset R^n$, we consider

$$C_{k, \Phi_p}(E; G) = \inf_g \int_G \Phi_p(g(y))dy,$$

where the infimum is taken over all nonnegative measurable functions g on G such that $\int_{R^n} k(x - y)g(y)dy \geq 1$ for every $x \in E$ (cf. Meyers [2] and Mizuta [7]). For simplicity, we write $C_{k, \Phi_p}(E) = 0$ if

$$C_{k, \Phi_p}(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

In case $k(x) = |x|^{\beta-n}$, we write C_{β, Φ_p} for C_{k, Φ_p}. If a property holds except for a set E with $C_{k, \Phi_p}(E) = 0$, then we say that the property holds C_{k, Φ_p}-quasi everywhere.

LEMMA 4.1 (cf. [7, Lemma 7.1]). *If f is a nonnegative measurable function* *on* R^n *satisfying* (1.1) *and* (1.2), *then*

$$C_{k,\Phi_p}(E_f) = 0,$$

where

$$E_f = \left\{x: \int_{R^n} k(x-y)f(y)dy = \infty\right\}.$$

If h is a positive nondecreasing function on $(0, \infty)$ satisfying the doubling condition, then h is called a measure function. We denote by H_h the Hausdorff measure for the measure function h.

LEMMA 4.2 (cf. [7, Lemma 7.2]). *Let* h *be a measure function on* $[0, \infty)$ *for which*

$$\lim_{r \to 0} r^{-n}h(r) = \infty.$$

For a locally integrable function g *on* R^n, *set*

$$E_{g,h} = \left\{x: \limsup_{r \to 0} [h(r)]^{-1} \int_{B(x,r)} |g(y)|dy > 0\right\}.$$

Then $H_h(E_{g,h}) = 0$.

LEMMA 4.3 (cf. [7, Corollary 7.2]). *If* G *and* G' *are bounded open sets in* R^n *such that* $\overline{G'} \subset G$, *then there exists* $M > 0$, *depending on the distance between* $\partial G'$ *and* ∂G, *such that*

$$C_{k,\Phi_p}(E; G) \le MH_h(E)$$

for any set $E \subset G'$, *where*

$$h(r) = \left(\int_r^1 [t^{n-\alpha p}\omega(t)^{-p}\varphi(t^{-1})]^{-p'/p}t^{-1}dt\right)^{-p/p'}, \qquad 0 < r \le 2^{-1},$$

and $h(r) = h(2^{-1})$ *for* $r > 2^{-1}$.

PROOF. First we show that for any $a > 0$, there exists $M > 1$ such that

$$C_{k,\Phi_p}(B(0, r); B(0, a)) \le M[\kappa_a(r)]^{-p}$$

whenever $0 < r < a/2$, where

$$\kappa_a(r) = \left(\int_r^a [t^{n-\alpha p}\omega(t)^{-p}\varphi(t^{-1})]^{-p'/p}t^{-1}dt\right)^{1/p'}.$$

Let $0 < r < a/2$ and consider the function

$$f_r(y) = \begin{cases} |y|^{-\alpha}\omega(|y|)^{-1}[|y|^{n-\alpha p}\omega(|y|)^{-p}\varphi(|y|^{-1})]^{-p'/p}, & \text{if } y \in B(0, a) - B(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

If $x \in B(0, r)$, then $|x - y| \le 2|y|$ for $y \in B(0, a) - B(0, r)$, so that

$$\begin{aligned} &\int |x - y|^{\alpha - n}\omega(|x - y|)f_r(y)dy \\ &\ge M \int_{B(0,a)-B(0,r)} |y|^{-n}[|y|^{n-\alpha p}\omega(|y|)^{-p}\varphi(|y|^{-1})]^{-p'/p}dy \\ &= M[\kappa_a(r)]^{p'}. \end{aligned}$$

Hence it follows that

$$C_{k,\Phi_p}(B(0, r); B(0, a)) \le \int \Phi_p\left(\frac{f_r(y)}{M[\kappa_a(r)]^{p'}}\right)dy.$$

For $\beta = \alpha + np'/p - \alpha p'$, we see that

$$\frac{f_r(y)}{[\kappa_a(r)]^{p'}} \le M \frac{|y|^{-\beta}\omega(|y|)^{p'-1}}{[\kappa_a(a/2)]^{p'}}$$

whenever $y \in B(0, a)$. Here note by the doubling condition on ω that

$$\omega(r) \le Mr^{-\delta}, \qquad 0 < r < 1,$$

for some $\delta > 0$. Thus $f_r(y)[\kappa_a(r)]^{-p'} \le M|y|^{-\gamma}$ for $\gamma = \beta + \delta(p' - 1) > 0$. Hence, we find by conditions (φ3) and (φ4)

$$\begin{aligned} &\Phi_p\left(\frac{f_r(y)}{M[\kappa_a(r)]^{p'}}\right) \\ &\le M\left(\frac{f_r(y)}{M[\kappa_a(r)]^{p'}}\right)^p \varphi(|y|^{-1}) \\ &\le M[\kappa_a(r)]^{-pp'}|y|^{-\alpha p}\omega(|y|)^{-p}[|y|^{n-\alpha p}\omega(|y|)^{-p}\varphi(|y|^{-1})]^{-p'}\varphi(|y|^{-1}) \\ &= M[\kappa_a(r)]^{-pp'}[|y|^{n-\alpha p}\omega(|y|)^{-p}\varphi(|y|^{-1})]^{-p'+1}|y|^{-n}. \end{aligned}$$

Consequently we establish

$$\begin{aligned} &C_{k,\Phi_p}(B(0, r); B(0, a)) \\ &\le M[\kappa_a(r)]^{-pp'} \int_{B(0,a)-B(0,r)} [|y|^{n-\alpha p}\omega(|y|)^{-p}\varphi(|y|^{-1})]^{-p'/p}|y|^{-n}dy \\ &= M[\kappa_a(r)]^{-p}. \end{aligned}$$

Let $a' = \mathrm{dist}(\partial G', \partial G)$. For any $x \in G'$,

$$C_{k,\Phi_p}(B(x,r); B(x,a')) \le M[\kappa_{a'}(r)]^{-p} \le Mh(r)$$

whenever $0 < r < a'/2$. Hence, we have

$$C_{k,\Phi_p}(B(x,r); G) \le Mh(r).$$

If $E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j)$, $r_j < a'/2$, then we obtain

$$\begin{aligned} C_{k,\Phi_p}(E; G) &\le C_{k,\Phi_p}\left(\bigcup_{j=1}^{\infty} B(x_j, r_j); G\right) \\ &\le \sum_{j=1}^{\infty} C_{k,\Phi_p}(B(x_j, r_j); G) \\ &\le M \sum_{j=1}^{\infty} h(r_j), \end{aligned}$$

which proves

$$C_{k,\Phi_p}(E; G) \le MH_h(E).$$

LEMMA 4.4. *For a nonnegative measurable function f on R^n satisfying* (1.2), *set*

$$F = \left\{x_0 \colon \limsup_{r \to 0}\, [r^{n-\alpha p}\omega(r)^{-p}\varphi(r^{-1})]^{-1} \int_{B(x_0,r)} \Phi_p(f(y))dy > 0\right\}.$$

If (ω5) *holds, then* $C_{k,\Phi_p}(F) = 0$.

PROOF. Letting $\rho(x)$ denote the distance of x from the boundary ∂G, we define $G_j = \{x \in G \colon \rho(x) > j^{-1}\}$ for each positive integer j. Since $F \cap G = \bigcup_{j=1}^{\infty} (F \cap G_j)$, we have

$$C_{k,\Phi_p}(F \cap G; G) \le \sum_{j=1}^{\infty} C_{k,\Phi_p}(F \cap G_j; G).$$

Let h be defined as in Lemma 4.3. By the doubling conditions on ω and φ we see that

$$h(r) \le M[r^{n-\alpha p}\omega(r)^{-p}\varphi(r^{-1})].$$

Since (ω5) implies

$$\lim_{r \to 0} r^{-n}[r^{n-\alpha p}\omega(r)^{-p}\varphi(r^{-1})] = \lim_{r \to 0} r^{(\beta-\alpha)p}[r^{\beta}\omega(r)]^{-p}\varphi(r^{-1}) = \infty,$$

we have $H_h(F) = 0$ by Lemma 4.2. Hence it follows from Lemma 4.3 that $C_{k,\Phi_p}(F) = 0$.

Now, with the aid of Lemmas 4.1 and 4.4, we obtain the following result from Theorem 3.1.

THEOREM 4.1. *Let* $1 < p \le n/\alpha$, $q > 0$ *with* $1/q \ge 1/p - \alpha/n$ *and* ω *be as in Theorem* 3.1. *If* f *is a nonnegative measurable function on* R^n *satisfying conditions* (1.1) *and* (1.2), *then* (3.2) *holds for* C_{k,Φ_p}*-quasi every* x_0. *If in addition* (ω5) *holds for* $\beta \le \ell + 1$, *then* (3.3) *holds for* C_{k,Φ_p}*-quasi every* x_0.

REMARK 4.1. Let $\alpha p \le n$, $0 \le a < 1$ and $\alpha - \ell - a > 0$. If $\omega(r) = r^{-(\ell+a)}$, then conditions ($\omega$1) $\sim$ (ω5) are all satisfied.

COROLLARY 4.1. *Let* $\ell + a < \alpha \le n/p$ *and* $0 \le a < 1$. *If* f *is a nonnegative measurable function on* R^n *satisfying conditions* (1.1) *and* (1.2), *then*

$$\lim_{r \to 0} r^{-\ell-a} V_q(U_{\alpha,\ell,x_0} f(x), x_0, r) = 0 \tag{4.1}$$

holds for $C_{\alpha-\ell-a,\Phi_p}$*-quasi every* x_0 *and* $q > 0$ *with* $1/q \ge 1/p - \alpha/n$.

REMARK 4.2. Meyers [3] obtained a result similar to Corollary 4.1 for Taylor's expansion of Bessel potentials of L^p-functions.

5. L^q-differentiability

We say that u is L^q-differentiable of order ℓ at x_0 if

$$\lim_{r \to 0} r^{-\ell} V_q(u(x) - P(x), x_0, r) = 0$$

for some polynomial P (see Meyers [3], Stein [9] and Ziemer [10]).

In view of Corollary 4.1, we have the following result.

THEOREM 5.1. *Let* $\alpha p \le n$. *Let* f *be a nonnegative measurable function on* R^n *satisfying conditions* (1.1) *and* (1.2). *If* ℓ *is a nonnegative integer smaller than* α, *then* $U_\alpha f$ *is* L^q*-differentiable of order* ℓ $C_{\alpha-\ell,\Phi_p}$*-quasi everywhere for* $q > 0$ *with* $1/q \ge 1/p - \alpha/n$.

For similar results for Bessel potentials of L^p-functions, see Meyers [3].

In case $\ell = \alpha$, we show the following result.

THEOREM 5.2. *Let* ℓ *be a positive integer with* $\ell p \le n$. *Let* f *be a nonnegative function in* $L^p_{loc}(R^n)$ *satisfying condition* (1.1) *with* $\alpha = \ell$. *Then* $U_\ell f$ *is* L^q*-differentiable of order* ℓ *almost everywhere for* $q > 0$ *with* $1/q \ge 1/p - \ell/n$.

REMARK 5.1. For L^p-differentiability of Bessel potentials, we refer the reader to Ziemer [10, Theorem 3.4.2]. In case $\ell = \alpha = 1$ and $p < n$, Theorem 5.2 implies the result by Stein [9, Theorem 1, Chapter 8].

For the reader's convenience, we give a proof of Theorem 5.2. First we recall the following result from the singular integral theory (see Stein [9]; Theorem 4 in Chapter 2).

LEMMA 5.1. *Let f be a locally integrable function on R^n satisfying condition* (1.1). *Then there exists a set E_1 with n-dimensional measure zero such that*

$$A_\nu(x_0) = A_{\nu,\ell}(x_0) = \lim_{r\to 0} \int_{R^n - B(x_0,r)} D^\nu R_\ell(x_0 - y) f(y)dy$$

exists and is finite for every $x_0 \in R^n - E_1$ and every multi-index ν with $|\nu| \le \ell$.

Set

$$U(x) = \int_{B(x_0,1)} R_\ell(x - y)dy.$$

Then U is infinitely differentiable on $B(x_0, 1)$ (see e.g., [5, Lemma 4]). Define

$$B_\nu = D^\nu U(x_0)$$

for any multi-index ν with $|\nu| \le \ell$. Note here that B_ν does not depend on x_0, that is,

$$B_\nu = D^\nu \int_{B(0,1)} R_\ell(x - y)dy \bigg|_{x=0}.$$

The following lemma is elementary (cf. [5, Lemma 1]).

LEMMA 5.2. *For a nonnegative function $g \in L^1_{loc}(R^n)$, set*

$$\varepsilon(r) = \sup_{0<t<r} t^{-n} \int_{B(0,t)} g(y)dy.$$

If $\gamma > 0$, then

$$\int_{B(0,r)} |y|^{\gamma-n} g(y)dy \le M r^\gamma \varepsilon(r) \tag{5.1}$$

and

$$r^\gamma \int_{B(0,s)-B(0,r)} |y|^{-\gamma-n} g(y)dy \le M\varepsilon(s) \tag{5.2}$$

whenever $0 < r < s$.

PROOF OF THEOREM 5.2. By Lemma 5.1, we can find a set E_1 with n-dimensional measure zero such that $A_{\nu,\ell}(x_0)$ exists and is finite for every $x_0 \in R^n - E_1$ and every multi-index ν with $|\nu| \le \ell$. Consider the set

$$E_2 = \left\{x_0 \colon \limsup_{r \to 0} r^{-n} \int_{B(x_0, r)} |f(y) - f(x_0)|^p dy > 0\right\};$$

note that E_2 has n-dimensional measure zero since $f \in L^p_{loc}(R^n)$. We show that $U_\ell f$ is L^q-differentiable of order ℓ at $x_0 \in R^n - (E_1 \cup E_2)$. For simplicity, we assume that $x_0 = 0$. For $|\nu| \le \ell$, set

$$C_\nu = \begin{cases} A_{\nu,\ell}(0) & \text{if } |\nu| < \ell, \\ A_{\nu,\ell}(0) + f(0) B_\nu & \text{if } |\nu| = \ell. \end{cases}$$

For $x \in B(0, 1/2) - \{0\}$, we write $K_\ell(x, y) = R_{\ell,\ell,0}(x, y)$ and

$$\begin{aligned} |x|^{-\ell} & \left\{U_\ell f(x) - \sum_{|\nu| \le \ell} \frac{C_\nu}{\nu!} x^\nu\right\} \\ &= |x|^{-\ell} \int_{R^n - B(0,1)} K_\ell(x, y) f(y) dy \\ &+ |x|^{-\ell} \int_{B(0,1) - B(0, 2|x|)} K_\ell(x, y)\{f(y) - f(0)\} dy \\ &- |x|^{-\ell} \sum_{|\nu| \le \ell} \frac{x^\nu}{\nu!} \lim_{r \to 0} \int_{B(0, 2|x|) - B(0, r)} D^\nu R_\ell(-y)\{f(y) - f(0)\} dy \\ &+ f(0)|x|^{-\ell} \left(\lim_{r \to 0} \int_{B(0,1) - B(0,r)} K_\ell(x, y) dy - \sum_{|\nu| = \ell} \frac{B_\nu}{\nu!} x^\nu\right) \\ &+ |x|^{-\ell} \int_{\{y \in B(0, 2|x|); |x - y| \ge |x|/2\}} R_\ell(x - y)\{f(y) - f(0)\} dy \\ &+ |x|^{-\ell} \int_{\{y \in B(0, 2|x|); |x - y| < |x|/2\}} R_\ell(x - y)\{f(y) - f(0)\} dy \\ &= u_1(x) + u_2(x) - u_3(x) + f(0) u_4(x) + u_5(x) + u_6(x), \end{aligned}$$

if the limits exist.

With the aid of Lemma 2.3, it is easy to see that

$$\lim_{x \to 0} u_1(x) = 0.$$

For $a > 0$, set

$$\varepsilon_a(r) = \sup_{0 < t < r} \left(t^{-n} \int_{B(0,t)} |f(y) - f(0)|^a dy\right)^{1/a}.$$

Then note that $\lim_{r \to 0} \varepsilon_p(r) = 0$, since we assumed that $0 \notin E_2$. Hölder's inequality gives

(5.3) $$\varepsilon_1(r) \le M\varepsilon_p(r) \qquad \text{for} \quad r > 0.$$

Hence we have by Lemma 2.3 and (5.2),

$$\begin{aligned}\limsup_{x\to 0} |u_2(x)| &\le M \limsup_{x\to 0} |x| \int_{B(0,1)-B(0,2|x|)} |y|^{-n-1}|f(y)-f(0)|dy \\ &= M \limsup_{x\to 0} |x| \int_{B(0,\delta)-B(0,2|x|)} |y|^{-n-1}|f(y)-f(0)|dy \\ &\le M\varepsilon_1(\delta)\end{aligned}$$

for any $\delta > 0$, which proves

$$\lim_{x\to 0} u_2(x) = 0.$$

Similarly, if $|\nu| < \ell$, then (5.1) and (5.3) give

$$\begin{aligned}&\limsup_{x\to 0} |x|^{|\nu|-\ell}\left|\int_{B(0,2|x|)} D^\nu R_\ell(-y)\{f(y)-f(0)\}dy\right| \\ &\le M \limsup_{x\to 0} |x|^{|\nu|-\ell} \int_{B(0,2|x|)} |y|^{\ell-n-|\nu|}|f(y)-f(0)|dy \\ &\le M \limsup_{x\to 0} \varepsilon_1(2|x|) = 0.\end{aligned}$$

If $|\nu| = \ell$, then, since

(5.4) $$\int_{B(0,s)-B(0,r)} D^\nu R_\ell(-y)dy = 0, \qquad 0 < r < s$$

(see [5, Proof of Theorem 3]), we see that by the assumption that $0 \notin E_1$ and (5.4)

$$\begin{aligned}&\lim_{r\to 0} \int_{B(0,2|x|)-B(0,r)} D^\nu R_\ell(-y)\{f(y)-f(0)\}dy \\ &= \lim_{r\to 0}\left\{\int_{B(0,2|x|)-B(0,r)} D^\nu R_\ell(-y)f(y)dy - f(0)\int_{B(0,2|x|)-B(0,r)} D^\nu R_\ell(-y)dy\right\} \\ &= \lim_{r\to 0}\left\{\int_{R^n-B(0,r)} D^\nu R_\ell(-y)f(y)dy - \int_{R^n-B(0,2|x|)} D^\nu R_\ell(-y)f(y)dy\right\}\end{aligned}$$

tends to zero as $x \to 0$, so that $u_3(x)$ is well-defined and

$$\lim_{x\to 0} u_3(x) = 0.$$

Noting that

$$\int_{B(0,1)} D^{\nu}R_{\ell}(-y)dy = D^{\nu}U(0)$$

for $|\nu| < \ell$, we see by (5.4) that $u_4(x)$ is well-defined and

$$u_4(x) = |x|^{-\ell}\left\{U(x) - \sum_{|\nu|<\ell} \frac{x^{\nu}}{\nu!}\int_{B(0,1)} D^{\nu}R_{\ell}(-y)dy - \sum_{|\nu|=\ell} \frac{B_{\nu}}{\nu!}x^{\nu}\right\}$$

$$= |x|^{-\ell}\left\{U(x) - \sum_{|\nu|\le\ell} \frac{x^{\nu}}{\nu!}(D^{\nu}U)(0)\right\}.$$

Since U is infinitely differentiable at 0,

$$\lim_{x\to 0} u_4(x) = 0.$$

As to u_5, we see by (5.1) that

$$|u_5(x)| \le M|x|^{-n}\int_{B(0,2|x|)} |f(y) - f(0)|dy \le M\varepsilon_1(2|x|),$$

which tends to zero as $x \to 0$ in view of (5.3).

In case $\ell p < n$, note that

$$|u_6(x)| \le |x|^{-\ell}\int_{B(x,|x|/2)} |x-y|^{\ell-n}|f(y) - f(0)|dy$$

$$\le M\int_{B(x,|x|/2)} |x-y|^{\ell-n}|y|^{-\ell}|f(y) - f(0)|dy.$$

Hence, letting $1/q = 1/p - \ell/n$, Sobolev's inequality yields

$$V_q(u_6, 0, r) \le Mr^{-n/q}\left(\int_{B(0,2r)} [|y|^{-\ell}|f(y) - f(0)|]^p dy\right)^{1/p}.$$

Consequently, (5.1) gives

$$V_q(u_6, 0, r) \le M\varepsilon_p(2r),$$

which shows that

(5.5) $$\lim_{r\to 0} V_q(u_6, 0, r) = 0.$$

In case $\ell p = n$, for $q > p$, take γ such that $1/q = 1/p - \gamma/n$. Then $0 < \gamma < \ell$ and

$$|u_6(x)| \le M\int_{B(x,|x|/2)} |x-y|^{\gamma-n}|y|^{-\gamma}|f(y) - f(0)|dy,$$

so that

$$V_q(u_6, 0, r) \le Mr^{-n/q}\left(\int_{B(0,2r)} [|y|^{-\gamma}|f(y) - f(0)|]^p dy\right)^{1/p} \le M\varepsilon_p(2r).$$

Therefore, (5.5) also follows. Hence we have established that

$$\lim_{r \to 0} r^{-\ell} V_q(U_\ell f(x) - P(x), 0, r) = 0$$

holds for $q > 0$ with $1/q \ge 1/p - \ell/n$, where

$$P(x) = \sum_{|\nu| \le \ell} [C_\nu/\nu!] x^\nu.$$

Acknowledgement

The author would like to express his deep gratitude to Prof. Yoshihiro Mizuta for his valuable advices and encouragement. The author also thanks Prof. Fumi-Yuki Maeda for his kind comments and suggestions.

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The Division of Environmental and Material Sciences
Graduate School of Biosphere Sciences
Hiroshima University
Higashi-Hiroshima 739, *Japan*