Boundary continuity of Dirichlet finite harmonic measures on compact bordered Riemannian manifolds

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ABSTRACT. Generalizing the notion of p-harmonic measures in the sense of Heins we consider $\mathscr{A}$-harmonic measures of exponent p on the interior M of a compact bordered Riemannian manifold $\overline{M} = M \cup \partial M$ with smooth border ∂M of class C^∞ for $1 < p < \infty$. It is shown that $\mathscr{A}$-harmonic measures of exponent p with finite p-Dirichlet integrals on M can always be extended to continuous functions on $\overline{M}$ which are constantly zero or one on each connected component of ∂M if and only if $2 \leq p < \infty$. In the appendix we consider an entirely arbitrary relatively compact subregion M of any Riemannian manifold of class C^∞ and it is shown that $\mathscr{A}$-harmonic measures of finite exponent $p > \dim M$ with finite p-Dirichlet integrals on M can always be extended to continuous functions on $\overline{M} = M \cup \partial M$ which are constantly zero or one on each connected component of the relative boundary ∂M of M.

0. Introduction

Take a compact bordered Riemannian manifold $\overline{M} = M \cup \partial M$ of dimension $d \geq 2$ of class C^∞ with smooth border ∂M of class C^∞ (cf. § 1.5 below) and fix a real number $1 < p < \infty$. Consider the quasilinear elliptic partial differential equation

$$(0.1) \qquad -\operatorname{div} \mathscr{A}_x(\nabla u) = 0$$

on the interior M of $\overline{M}$, where $\mathscr{A}_x(h) \cdot h \approx |h|^p$; the precise assumptions on $\mathscr{A}$ are listed in § 2.1 below. A typical example of the equation (0.1) is the so called p-Laplace equation

$$(0.2) \qquad -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

and thus of course the usual Laplace equation $-\Delta_2 u = -\Delta u = 0$ is included in our consideration.

A continuous weak solution of (0.1) on M is referred to as an $\mathscr{A}$-harmonic

function on M (cf. §2.1 below). An $\mathscr{A}$-harmonic measure w on M is by definition an $\mathscr{A}$-harmonic function w on M such that the greatest $\mathscr{A}$-harmonic minorant of w and $1 - w$ on M is zero (cf. §3.1 below). The function w is said to be p-Dirichlet finite if the so called p-Dirichlet integral

$$\int_M |\nabla w|^p dV$$

of w is finite, where dV is the Riemannian volume element on M. The purpose of this paper is to study the boundary behavior of p-Dirichlet finite $\mathscr{A}$-harmonic measures on M at the boundary ∂M of M.

Given an arbitrary function φ on the boundary ∂M of M such that φ is identically zero or one on each connected component of ∂M. It is easy to see that the Dirichlet solution w of (0.1) on M with boundary values φ on ∂M is a p-Dirichlet finite $\mathscr{A}$-harmonic measure on M. Our main concern is what happens to the converse of the above statement: can every p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M be obtained as a Dirichlet solution w of (0.1) on M with a certain boundary function φ on ∂M as described above? The main purpose of this paper is to show that the above question is settled in the affirmative for $2 \leqq p < \infty$ but in the negative for $1 < p < 2$. Namely, we will prove the following result.

THE MAIN THEOREM. *The following statement is true if and only if the exponent p lies in $2 \leqq p < \infty$: every p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M with exponent p is continuously extendable to $\overline{M} = M \cup \partial M$ and the extended function w takes the constant value zero or one on each connected component of ∂M.*

Thus, concerning Dirichlet finite harmonic measures, the situations differ drastically between the cases of $1 < p < 2$ and $2 \leqq p < \infty$. Suppose there are ℓ connected components $(\partial M)_j$ of ∂M $(j = 1, \ldots, \ell)$. When the exponent p lies in $2 \leqq p < \infty$, we will thus show that any p-Dirichlet finite $\mathscr{A}$-harmonic measure w with exponent p on M is extendable to a continuous function on $\overline{M}$, which we denote by the same notation w, and the restriction $w|(\partial M)_j$ of the extended w on $\overline{M}$ to $(\partial M)_j$ is either identically 0 or 1 $(j = 1, \ldots, \ell)$. Hence there are altogether 2^ℓ different p-Dirichlet finite $\mathscr{A}$-harmonic measures with exponent p on M. In particular, if ∂M is connected, i.e. $\ell = 1$, then there are only two different p-Dirichlet fintie $\mathscr{A}$-harmonic measures with exponent p on M which are the constant 0 and the constant 1 so that there exist no nonconstant p-Dirichlet finite $\mathscr{A}$-harmonic measures on M with exponent p. As an application we see that there are no nonconstant p-Dirichlet finite $\mathscr{A}$-harmonic measures with exponent p on any relatively compact subregion M of Euclidean space $\mathbf{R}^d$ with connected C^∞ relative boundary ∂M such as

ellipsoids or solid torii, and in particular, the open unit ball B^d (see [14] and also [4]). This result for B^d is a well known classical result in the theory of functions when $p = d = 2$ and $\mathscr{A}$ is the classical Laplace operator.

When the exponent p lies in $1 < p < 2$, we will first show, as is trivial in the case of $2 \leq p < \infty$ by the above assertion, that the essential range of the trace (a generalized "boundary values") on ∂M of any p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M with exponent p is contained in the two elements set $\{0, 1\}$. However, as the characteristic feature of the case of $1 < p < 2$, we will show the existence of a p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M with exponent p such that the actual boundary values of w are one on B and zero on $(\partial M) \backslash \bar{B}$ for any relatively compact smooth parametric ball B in the border manifold ∂M given in advance. Hence, if we take such a B in a component $(\partial M)_j$ of ∂M so that $(\partial M)_j \backslash \bar{B} \neq \varnothing$, then we see that w is discontinuous on $(\partial M)_j$ so that w cannot be continuously extendable to $\bar{M}$. Thus we also see that there are infinitely many different p-Dirichlet finite $\mathscr{A}$-harmonic measures on M with exponent p in $1 < p < 2$.

We assume in this paper that the border ∂M of M is smooth of class C^∞. Actually, we can easily show by giving some simple counterexample e.g. in the case $d > 2$ and $p = 2$ that the theorem is invalid unless a certain regularity condition is imposed upon ∂M. However, by examining the whole discussion in this paper it can be easily recognized that the regularity of the border ∂M of M may be weakened to being smooth of class C^2. Nevertheless, for the sake of simplicity, we still assume that ∂M is smooth of class C^∞ throughout this paper. On the other hand we must remark that the theorem reduces to a triviality without assuming any regularity condition on ∂M when the exponent p lies in $d = \dim M < p < \infty$. Namely we take an *arbitrary* relatively compact subregion M of any Riemannian manifold of class C^∞. Hence in particular the relative boundary ∂M of M may or may not consist of infinitely many connected components. Under this very general setting we will prove in Appendix at the end of the paper that $\mathscr{A}$-harmonic measures of finite exponent $p > \dim M$ with finite p-Dirichlet integrals on M can always be extended to continuous functions on $\bar{M} = M \cup \partial M$ which are constantly zero or one on each connected component of ∂M (Theorem A).

The paper consists of 7 sections and an appendix. Each sections are divided into several subsections. In §1 titled *Sobolev spaces on Riemannian manifolds* some properties of point norms of differential forms are considered and also traces for Sobolev functions are explained. The boundary behavior of p-Dirichlet finite $\mathscr{A}$-harmonic functions with given traces on the boundary are discussed in §2 with the title *Dirichlet problem for $\mathscr{A}$-harmonic functions*. A characterization of $\mathscr{A}$-harmonic measures is given in §3 under the title *$\mathscr{A}$-harmonic measures*. The title of §4 is *A property of 2-Dirichlet finite 2-*

harmonic measures. Here the orthogonality of differentials of any 2-Dirichlet finite 2-harmonic measures on M to the conjugate differentials of $(d-2)$-forms on neighborhoods of $\overline{M}$ in L_2 inner products is given. In §5 titled *A net of auxiliary forms* some technical lemma which is of fundamental importance in this paper is established. When $1 < p < 2$, we show in §6 with the title *A boundary characteristic function* that the characteristic function on ∂M of any smooth small parametric ball in ∂M is the trace of a Sobolev function on M. In the last §7 titled *Proof of the main theorem*, we divide the assertion of the main theorem into two parts: Theorem 7.1 is the assertion of the main theorem for $2 \leq p < \infty$ and Theorem 7.2 is that for $1 < p < 2$. These are proved separately in this last section. At the end of the paper there is Appendix titled *The case of $p > d$ on a general M* in which Theorem A mentioned above is proved.

1. Sobolev spaces on Riemannian manifolds

1.1. Throughout this paper we fix a Riemannian manifold N of class C^∞ of dimension $d \geq 2$, connected and orientable. For each $\xi \in N$ we denote by $B(\xi, R)$ $(0 < R < \infty)$ a relatively compact parametric ball at ξ with a local parameter $x = (x^1, \ldots, x^d)$ valid on a neighborhood of the closure $\overline{B}(\xi, R)$ of $B(\xi, R)$ such that $x(\xi) = 0$ and $B(\xi, R) = \{|x| < R\}$. Once $B(\xi, R)$ is fixed, we denote by $B(\xi, r)$ $(0 < r \leq R)$ the concentric ball $\{|x| < r\}$. We often use the same letter x to denote the generic point of N and also its local parameter.

Let (g_{ij}) be the metric tensor on N, $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. We denote by dV the volume element on N so that

$$dV(x) = \sqrt{g(x)}dx^1 \wedge \cdots \wedge dx^d$$

in terms of a local parameter $x = (x^1, \ldots, x^d)$. We also have $dV = *1$ where $*$ is the Hodge star operator. In each parametric ball $B(\xi, R)$ with a local parameter $x = (x^1, \ldots, x^d)$, the Riemannian measure $dV(x)$ and the Euclidean (Lebesgue) measure given by $dx = dx^1 \ldots dx^d$ are mutually absolutely continuous and the Radon-Nikodym densities $dV(x)/dx$ and $dx/dV(x)$ are essentially bounded. Hence a.e. dV and a.e. dx are identical and we can loosely use a.e. without referring to dV or dx.

For each fixed $x \in N$, the tangent space to N at x will be denoted by $T_x N$, and the tangent bundle, that is, the union of all tangent spaces to N, will be denoted by TN. We denote by $h \cdot k$ the inner product of two tangent vectors h and k in $T_x N$ and by $|h|$ the length of $h \in T_x N$ so that, if $(h_1, \ldots, h_d)$ and $(k_1, \ldots, k_d)$ are covariant components of h and k, then

$$h \cdot k = g^{ij}h_i k_j \qquad \text{and} \qquad |h| = \sqrt{h \cdot h} = (g^{ij}h_i h_j)^{1/2}.$$

Here and hereafter, we use the Einstein convention: whenever an index i appears both in the upper and lower positions, it is understood that summation for $i = 1, \ldots, d$ is carried out.

Let G be an open subset of N. With a covariant tensor $a_{i_1 \ldots i_s}$ appearing in an s-form

$$\alpha = \sum_{i_1 < \cdots < i_s} a_{i_1 \ldots i_s} dx^{i_1} \wedge \cdots \wedge dx^{i_s}$$

on G we associate the contravariant tensor $a^{i_1 \cdots i_s}$ given by

$$a^{i_1 \cdots i_s} = g^{i_1 k_1} \ldots g^{i_s k_s} a_{k_1 \ldots k_s}$$

on G. We consider the inner product $\alpha \cdot \beta$ of two s-forms α and

$$\beta = \sum_{i_1 < \cdots < i_s} b_{i_1 \ldots i_s} dx^{i_1} \wedge \cdots \wedge dx^{i_s}$$

on G and the norm (point norm) $|\alpha|$ defined by

$$\alpha \cdot \beta = \sum_{i_1 < \cdots < i_s} a^{i_1 \cdots i_s} b_{i_1 \ldots i_s}$$

and

$$|\alpha| = \sqrt{\alpha \cdot \alpha} = \left(\sum_{i_1 < \cdots < i_s} a^{i_1 \cdots i_s} a_{i_1 \ldots i_s} \right)^{1/2}$$

so that $\alpha \cdot \beta$ and $|\alpha|$ are functions on G. Hence we have

$$\alpha \wedge *\beta = \alpha \cdot \beta dV = \alpha \cdot \beta * 1.$$

1.2. Let G be an open subset of N. In this paper we use the notation $L_p(G)$ ($1 \leqq p \leqq \infty$) in three ways. The first is the standard use: $L_p(G)$ is the Banach space of measurable functions f on G with finite norms $\|f; L_p(G)\| < \infty$ given by

$$\|f; L_p(G)\| = \left(\int_G |f|^p dV \right)^{1/p} \qquad (1 \leqq p < \infty)$$

and $\|f; L_\infty(G)\|$ is the essential supremum of $|f|$ on G. The second use: for a vector field X on G we write $X \in L_p(G)$ if $|X| \in L_p(G)$ in the first sense and we set

$$\|X; L_p(G)\| := \||X|; L_p(G)\|.$$

As the third use we write $\alpha \in L_p(G)$ for a differential form α on G if $|\alpha| \in L_p(G)$ in the first sense and we set

$$\|\alpha; L_p(G)\| := \||\alpha|; L_p(G)\|.$$

In any of these three senses the dual space

$$L_p(G)^* = L_q(G) \qquad (1/p + 1/q = 1, 1 \leqq p < \infty).$$

For example, if $L_p(G)$ is considered as the Banach space of pth integrable measurable differential s-forms on G, then, for any $\beta^* \in L_p(G)^*$, there is a unique s-form $\beta \in L_q(G)$ such that

$$\beta^*(\alpha) = \int_G \alpha \cdot \beta dV \qquad (\alpha \in L_p(G)).$$

The following elementary relation will be frequently made use of: let α be a d-form belonging to $L_1(G)$; then we have

(1.1)
$$\left| \int_G \alpha \right| \leqq \int_G |\alpha| dV.$$

In fact, let $\alpha = c_{1\ldots d} dx^1 \wedge \cdots \wedge dx^d$. Since

$$c^{1\cdots d} = g^{1i_1} \ldots g^{di_d} c_{i_1 \ldots i_d} = g^{1i_1} \ldots g^{di_d} \delta^{1\ldots d}_{1_1 \ldots i_d} c_{1\ldots d} = g^{-1} c_{1\ldots d},$$

where $\delta^{1\ldots d}_{i_1 \ldots i_d}$ is the generalized Kronecker delta, that is, it is 1 (-1, resp.) if $(i_1 \ldots i_d)$ is an even (odd, resp.) permutation of $(1\ldots d)$ and 0 if some two of $i_1, \ldots, i_d$ are identical, we see that $|\alpha| = (c^{1\cdots d} c_{1\ldots d})^{1/2} = |c_{1\ldots d}| g^{-1/2}$. Therefore we deduce

$$\left| \int_G \alpha \right| = \left| \int_G c_{1\ldots d} g^{-1/2} dV \right| \leqq \int_G |c_{1\ldots d}| g^{-1/2} dV = \int_G |\alpha| dV.$$

1.3. We are still assuming that G is an open subset of N. The following relation is useful in our later calculations.

LEMMA 1.1 (Wedge inequality). *If α is an s-form and β is a t-form on G, then the following inequality is valid on G:*

(1.2)
$$|\alpha \wedge \beta| \leqq \binom{d}{s+t}^{1/2} |\alpha||\beta|.$$

PROOF. If $s + t > d$, then $\alpha \wedge \beta = 0$ and $\binom{d}{s+t} = 0$ so that (1.2) is trivially valid. We can thus assume $s + t \leqq d$. If either $s = 0$ or $t = 0$, then $|\alpha \wedge \beta| = |\alpha\beta| = |\alpha||\beta|$ and $\binom{d}{s+t} \geqq 1$. Again (1.2) is trivially valid in this case. Hence we may assume that $s > 0$, $t > 0$ and $s + t \leqq d$ to prove (1.2). Take an arbitrary point $\xi \in G$. We only have to show that (1.2) is valid at ξ. Choose a local parameter $x = (x^1, \ldots, x^d)$ at ξ such that the corresponding components

of the metric tensor $g_{ij}(x)$ satisfies $g_{ij}(0) = \delta_{ij}$ $(x(\xi) = 0)$. Let

$$\alpha = \sum_{i_1 < \cdots < i_s} a_{i_1 \ldots i_s} dx^{i_1} \wedge \cdots \wedge dx^{i_s}$$

and

$$\beta = \sum_{j_1 < \cdots < j_t} b_{j_1 \ldots j_t} dx^{j_1} \wedge \cdots \wedge dx^{j_t}$$

in this coordinate. Then we have

$$|\alpha|^2 = \sum_{i_1 < \cdots < i_s} (a_{i_1 \ldots i_s})^2 \quad \text{and} \quad |\beta|^2 = \sum_{j_1 < \cdots < j_t} (b_{j_1 \ldots j_t})^2$$

at ξ. Observe that

$$\alpha \wedge \beta = \sum_{k_1 < \cdots < k_{s+t}} c_{k_1 \ldots k_{s+t}} dx^{k_1} \wedge \cdots \wedge dx^{k_{s+t}}$$

with $c_{k_1 \ldots k_{s+t}}$ being given by

$$c_{k_1 \ldots k_{s+t}} = \sum{}' \delta^{k_1 \ldots k_{s+t}}_{i_1 \ldots i_s j_1 \ldots j_t} a_{i_1 \ldots i_s} b_{j_1 \ldots j_t}$$

where the sum $\sum'$ is taken with respect to $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_t$ such that

$$\{i_1, \ldots, i_s\} \cup \{j_1, \ldots, j_t\} = \{k_1, \ldots, k_{s+t}\}$$

and δ^k_{ij} is the generalized Kronecker delta. By the Schwarz inequality

$$(c_{k_1 \ldots k_{s+t}})^2 \leq (\sum{}' |a_{i_1 \ldots i_s}||b_{j_1 \ldots j_t}|)^2 \leq (\sum{}' (a_{i_1 \ldots i_s})^2)(\sum{}' (b_{j_1 \ldots j_t})^2) \leq |\alpha|^2 |\beta|^2$$

at ξ and thus

$$|\alpha \wedge \beta|^2 = \sum_{k_1 < \cdots < k_{s+t}} (c_{k_1 \ldots k_{s+t}})^2 \leq \sum_{k_1 < \cdots < k_{s+t}} |\alpha|^2 |\beta|^2 = \binom{d}{s+t} |\alpha|^2 |\beta|^2$$

at ξ. Since $\xi \in G$ is arbitrary, we have obtained (1.2) on G. $\qquad \square$

1.4. Let G be an open subset of N. The *Sobolev space* $W_p^1(G)$ $(1 < p < \infty)$ is the family of functions $f \in L_p(G)$ whose distributional gradient $\nabla f \in L_p(G)$, where ∇f is determined by the relation

$$\int_G \nabla f \cdot \Psi dV = -\int_G f \operatorname{div} \Psi dV$$

for every C^∞ vector field Ψ on G with compact support in G. The Sobolev space $W_p^1(G)$ forms a Banach space equipped with the norm

$$\|f; W_p^1(G)\| = \|f; L_p(G)\| + \|\nabla f; L_p(G)\|.$$

The *Soblev null space* $W_{p,0}^1(G)$ is the closure of $C_0^\infty(G)$ in $W_p^1(G)$ with respect to the above norm. The spaces $W_p^1(G)$ and $W_{p,0}^1(G)$ are vector lattices with respect to the lattice operations $\cup$ and $\cap$ defined by

$$(f \cup h)(\xi) = \max(f(\xi), h(\xi)) \quad \text{and} \quad (f \cap h)(\xi) = \min(f(\xi), h(\xi)),$$

for every $\xi \in G$, for every pair of functions f and h on G.

1.5. A hypersurface S in N is said to be *smooth* if, for every $\xi \in S$, there exists a parametric ball $B(\xi, 1)$ with a local parameter x in N and a function $\Phi(x)$ of class C^1 of x such that $S \cap B(\xi, 1) = \{|x| < 1, \Phi(x) = 0\}$ and $\nabla \Phi \neq 0$ on $S \cap B(\xi, 1)$. We say that S is of class C^∞ if the above Φ is of class C^∞. In this case we can choose a parametric ball $B(\xi, 1)$ with the local parameter $x = (x^1, \ldots, x^d)$ such that

$$S \cap B(\xi, 1) = \{|x| < 1, x^d = 0\},$$

i.e. $x(S \cap B(\xi, 1))$ is a part of the hyperplane $x^d = 0$ in the unit ball of the Euclidean space $\mathbf{R}^d$ of dimension d.

Hereafter we fix a relatively compact subregion M of our fixed Riemannian manifold N such that $N \backslash M = \overline{N \backslash \overline{M}}$ and the relative boundary ∂M of M consists of a finite number of mutually disjoint smooth closed hypersurfaces of class C^∞. We may call $\overline{M} = M \cup \partial M$ to be a *compact bordered Riemannian manifold* of class C^∞ with C^∞ border ∂M. Actually any abstract compact bordered Riemannian manifold $\overline{M} = M \cup \partial M$ of class C^∞ with C^∞ border ∂M can always be represented as a relatively compact subregion of a certain Riemannian manifold N of class C^∞ such that $N \backslash M = \overline{N \backslash \overline{M}}$ and the relative boundary ∂M consists of a finite number of mutually disjoint smooth closed hypersurfaces of class C^∞.

The Riemannian metric on N induces a Riemannian surface element dS on ∂M. If we choose a parametric ball $B(\xi, 1)$ at any point ξ in ∂M with $(\partial M) \cap B(\xi, 1) = \{|x| < 1, x_d = 0\}$, then dS and $dx^1 \ldots dx^{d-1}$ are mutually absolutely continuous on $(\partial M) \cap B(\xi, 1)$. Unless otherwise is clearly stated, we understand that

$$L_p(\partial M) = L_p(\partial M, dS) \qquad (1 \leqq p \leqq \infty)$$

is the usual function space with respect to dS.

1.6. Take a compact bordered Riemannian manifold $\overline{M} = M \cup \partial M$ of class C^∞ with C^∞ border ∂M realized as a relatively compact subregion of N as described in 1.5. Consider the *trace* γ on ∂M (c.f. e.g. [9]). For each $1 < p < \infty$, the trace γ is the unique bounded linear operator from $W_p^1(M)$

to $L_p(\partial M)$ such that

(1.3) $$\gamma f = f|\partial M \qquad (f \in W_p^1(M) \cap C(\overline{M})).$$

If we denote by $\mathbf{n}_\xi$ the inner normal to ∂M at $\xi \in \partial M$, then we have

(1.4) $$(\gamma f)(\xi) = \lim_{x \in \mathbf{n}_\xi, x \to \xi} f(x)$$

for a.e. ξ in ∂M (cf. e.g. [10]). As a consequence of (1.4) we can conclude that

$$\gamma(f \cup h) = (\gamma f) \cup (\gamma h) \qquad \text{and} \qquad \gamma(f \cap h) = (\gamma f) \cap (\gamma h)$$

for all f and h in $W_p^1(M)$. In other words, γ preserves the lattice operations. Another consequence of (1.4) is that γ preserves the multiplication: if, f, h and fh belong to $W_p^1(M)$, then

$$\gamma(fh) = (\gamma f)(\gamma h).$$

Concerning the kernel Ker $\gamma = \gamma^{-1}(0)$ of γ and the image Im $\gamma = \gamma(W_p^1(M))$ of γ considered on $W_p^1(M)$ we have the following fundamental results. First, Ker γ characterizes the Sobolev null space (cf. (e.g. [10]):

(1.5) $$W_{p,0}^1(M) = \text{Ker } \gamma = \{f \in W_p^1(M): \gamma f = 0\}.$$

Second, we set

$$\Lambda_p(\partial M) = \text{Im } \gamma = \gamma(W_p^1(M)).$$

It is seen that the space $\Lambda_p(\partial M)$ forms a Banach space under the norm

$$\|\varphi; \Lambda_p(\partial M)\| = \|\varphi; L_p(\partial M)\| + \left(\iint_{(\partial M) \times (\partial M)} \frac{|\varphi(\xi) - \varphi(\eta)|^p}{(\text{dis}(\xi, \eta))^{p+d-2}} dS(\xi)dS(\eta) \right)^{1/p},$$

where $\text{dis}(\xi, \eta)$ is the Riemannian distance in N between two points ξ and η and dS is the Riemannian surface element on ∂M. The theorem of Gagliardo ([1]) assures the existence of a constant $C \geqq 1$ such that

(1.6) $$C^{-1} \|\varphi; \Lambda_p(\partial M)\| \leqq \inf_{\gamma f = \varphi} \|f; W_p^1(M)\| \leqq C \|\varphi; \Lambda_p(\partial M)\|$$

for every φ in $\Lambda_p(\partial M)$.

2. Dirichlet problems for $\mathscr{A}$-harmonic functions

2.1. We say that $\mathscr{A}$ is a strictly monotone elliptic operator M, a relatively compact subregion of N with smooth C^∞ boundary ∂M as described in 1.5, with exponent $1 < p < \infty$ if $\mathscr{A}$ is a mapping of the tangent bundle

TM to TM satisfying the following assumptions for some constants $0 < \alpha \leqq \beta < \infty$:

(2.1)   the mapping $\mathscr{A}_x = \mathscr{A}|T_xM: T_xM \to T_xM$ is continuous for almost every $x \in M$, and the mapping $x \mapsto \mathscr{A}_x(X)$ is measurable for all measurable vector field X on M;

for almost every $x \in M$ and for all $h \in T_xM$,

$$(2.2) \qquad \mathscr{A}_x(h) \cdot h \geqq \alpha |h|^p,$$

$$(2.3) \qquad |\mathscr{A}_x(h)| \leqq \beta |h|^{p-1},$$

$$(2.4) \qquad (\mathscr{A}_x(h_1) - \mathscr{A}_x(h_2)) \cdot (h_1 - h_2) > 0$$

whenever $h_1 \neq h_2$, and

$$(2.5) \qquad \mathscr{A}_x(\lambda h) = |\lambda|^{p-2} \lambda \mathscr{A}_x(h)$$

for all $\lambda \in \mathbf{R} \setminus \{0\}$, where $\mathbf{R}$ is the real number field.

   The class of all operators $\mathscr{A}$ on M satisfying (2.1)–(2.5) with the exponent $1 < p \leqq d$ will be denoted by $\mathscr{A}_p(M)$. Using an $\mathscr{A} \in \mathscr{A}_p(M)$ we consider a quasilinear elliptic partial differential equation

$$(2.6) \qquad -\operatorname{div} \mathscr{A}_x(\nabla u) = 0$$

on M. A function u on an open subset G of M is a weak solution of (2.6) if $u \in loc W_p^1(G)$ and

$$(2.7) \qquad \int_G \mathscr{A}_x(\nabla u) \cdot \nabla \varphi \, dV = 0$$

for every $\varphi \in C_0^\infty(G)$. If $u \in W_p^1(G)$, then it is easy to see by the Hölder inequality and $\mathscr{A}_x(\nabla u) \in L_q(G)$ $(1/p + 1/q = 1)$ as a consequence of (2.3) that u is a weak solution of (2.6) if and only if (2.7) is valid for every $\varphi \in W_{p,0}^1(G)$. As is well known, weak solutions of (2.6) (possibly modified on sets of zero measure dV) are actually continuous and in fact Hölder continuous (cf. e.g. [18], [19]).

   We say that a function u on an open subset G of M is $\mathscr{A}$-harmonic on G if $u \in C(G) \cap loc W_p^1(G)$ and u is a weak solution of (2.6) on G. We will denote by $H_{\mathscr{A}}(G)$ the class of all $\mathscr{A}$-harmonic functions on G. The simplest and the most typical operator $\mathscr{A}$ in $\mathscr{A}_p(M)$ is the p-Laplacian

$$\mathscr{A}_x(h) = |h|^{p-2} h$$

so that the corresponding elliptic partial differential equation is the p-Laplace equation

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

whose continuous weak solutions on G are in particular referred to as being *p-harmonic* on G. We denote by $H_p(G)$ the class of all *p*-harmonic functions on G. Observe that 2-harmonic functions are usual classical harmonic functions.

Fundamental properties of $\mathscr{A}$-harmonic functions on Riemannian manifolds are concisely compiled in e.g. [5] (cf. also [2]) and, in particular, the way how these properties originally obtained on open subsets in Euclidean space can be carried over Riemannian manifolds is explained. Among these properties we especially state the Harnack inequality: If K is a compact subset in a region D in M, then there is a constant $c = c(d, p, \alpha, \beta, D, K) \geqq 1$ such that

$$\sup_K u \leqq c \inf_K u$$

for every nonnegative $\mathscr{A}$-harmonic function u in D.

2.2. The existence of Sobolev Dirichlet solutions due to Maz'ya is of fundamental importance in our study. Let G be an open subset of M. Note that G is relatively compact in N and $\overline{G}$ and ∂G are considered in N. For any $f \in W_p^1(G)$ and any $\mathscr{A} \in \mathscr{A}_p(G)$ there exists a *unique* $u \in H_{\mathscr{A}}(G) \cap W_p^1(G)$ such that $u - f \in W_{p,0}^1(G)$ (cf. [6]). Since the function $u \in H_{\mathscr{A}}(G) \cap W_p^1(G)$ with $u - f \in W_{p,0}^1(G)$ is determined uniquely by $f \in W_p^1(G)$, we denote u by $\pi_{\mathscr{A}}^G f$, which will be referred to as the $\mathscr{A}$-*harmonic part* of $f \in W_p^1(G)$. We have thus obtained the direct sum decomposition (the Maz'ya decomposition) of $W_p^1(G)$:

$$(2.8) \qquad W_p^1(G) = (H_{\mathscr{A}}(G) \cap W_p^1(G)) \oplus W_{p,0}^1(G).$$

In the special case of the *p*-Laplace operator $\mathscr{A}_x(h) = |h|^{p-2}h$ we denote by $\pi_p^G f$ the *p*-harmonic part of $f \in W_p^1(G)$ in place of $\pi_{\mathscr{A}}^G f$.

A boundary point $\xi \in \partial G$ is said to be (Sobolev) $\mathscr{A}$-*regular* if

$$(2.9) \qquad \lim_{x \in G, x \to \xi} \pi_{\mathscr{A}}^G f(x) = f(\xi)$$

for every $f \in C(\overline{G}) \cap W_p^1(G)$. In this paper we only use the following sufficient condition: if there is a parametric ball $B(\xi, 1)$ about $\xi \in \partial G$ such that $B(\xi, 1) \cap \partial G$ is a smooth hypersurface of class C^∞, then ξ is $\mathscr{A}$-regular.

Let G be an open subset of M and $\{G_k\}_{k=1}^\infty$ be an exhaustion of G, that is, G_k is a finite union of mutually disjoint relatively compact subregions in G such that $M \backslash G_k = \overline{M \backslash \overline{G_k}}$ and ∂G_k consists of a finite number of mutually disjoint smooth closed hypersurfaces of class C^∞ $(k = 1, 2, \ldots)$ and $G = \bigcup_{k=1}^\infty G_k$. For any $f \in W_p^1(G)$, set

$$u = \pi_{\mathscr{A}}^G f$$

on G so that $u \in H_{\mathscr{A}}(G) \cap W_p^1(G)$ and also set

$$u_k = \begin{cases} \pi_{\mathscr{A}}^{G_k} f & \text{on } G_k, \\ f & \text{on } G \backslash G_k. \end{cases}$$

Since $\varphi \in W_{p,0}^1(G_k)$ belongs to $W_{p,0}^1(G)$ by putting $\varphi = 0$ on $G \backslash G_k$ (cf. e.g. [15]), we see that $f - u_k \in W_{p,0}^1(G)$ and $u_k \in W_p^1(G)$. We have the following consistency relations (cf. [6]): $\nabla u_k \to \nabla u$ ($k \to \infty$) weakly in $L_p(G)$; $\mathscr{A}_x(\nabla u_k) \to \mathscr{A}_x(\nabla u)$ ($k \to \infty$) weakly in $L_q(G)$ ($1/p + 1/q = 1$). We also have the following consistency relation (cf. [15]): $u_k \to u$ ($k \to \infty$) strongly in $L_p(G)$.

The most important property of the operator $\pi_{\mathscr{A}}^G$ of $W_p^1(G)$ to $H_{\mathscr{A}}(G) \cap W_p^1(G)$ is its *monotoneity* in two fashions (cf. [15]): if f_1 and f_2 belong to $W_p^1(G)$ and $f_1 \geqq f_2$ a.e. on G, then $\pi_{\mathscr{A}}^G f_1 \geqq \pi_{\mathscr{A}}^G f_2$ on G; if $f \in W_p^1(G)$ and $h \in H_{\mathscr{A}}(G)$ such that $f \leqq h$ ($f \geqq h$, resp.) a.e. on G, then $\pi_{\mathscr{A}}^G f \leqq h$ ($\pi_{\mathscr{A}}^G f \geqq h$, resp.) on G.

2.3. Besides the Dirichlet problem of Sobolev data, we consider the problem of finding an $\mathscr{A}$-harmonic function with a given trace. Given an arbitrary φ in $\Lambda_p(\partial M)$. Take any f in $W_p^1(M)$ such that $\gamma f = \varphi$. Let $u = \pi_{\mathscr{A}}^M f$. Since $u - f \in W_{p,0}^1(M)$, $\gamma(u - f) = 0$ by (1.5) and therefore $\gamma u = \gamma f = \varphi$. Thus we have found a $u \in H_{\mathscr{A}}(G) \cap W_p^1(G)$ with $\gamma u = \varphi$ on ∂M for any given $\varphi \in \Lambda_p(\partial M)$. Next we claim that such a u is unique. For the purpose take any $v \in H_{\mathscr{A}}(M) \cap W_p^1(M)$ such that $\gamma v = \varphi$. Observe that $w = u - v \in W_{p,0}^1(M)$ since $\gamma w = \gamma u - \gamma v = \varphi - \varphi = 0$ on ∂M (cf. (1.5)). Then we have the Maz'ya decomposition of u in two ways: $u = u + 0$ ($u \in H_{\mathscr{A}}(M) \cap W_p^1(M)$, $0 \in W_{p,0}^1(M)$) and $u = v + w$ ($v \in H_{\mathscr{A}}(M) \cap W_p^1(M)$, $w \in W_{p,0}^1(M)$). Since the decomposition is unique, we must have $u = v$ on M.

Since $u \in H_{\mathscr{A}}(M) \cap W_p^1(M)$ with $\gamma u = \varphi$ is uniquely determined by $\varphi \in \Lambda_p(\partial M)$, we denote u by $\tau_{\mathscr{A}}^M \varphi$. Then $\tau_{\mathscr{A}}^M$ gives rise to an operator

$$\tau_{\mathscr{A}}^M : \Lambda_p(\partial M) \to H_{\mathscr{A}}(M) \cap W_p^1(M),$$

which is clearly *bijective* and in fact $\tau_{\mathscr{A}}^M = (\gamma | H_{\mathscr{A}}(M) \cap W_p^1(M))^{-1}$. Moreover we have the following result (cf. [16]).

PROPOSITION 2.1. *The operator* $\tau_{\mathscr{A}}^M$ *is monotone, i.e. if* $\varphi_1 \geqq \varphi_2$ *a.e. on* ∂M *for any* φ_1 *and* φ_2 *in* $\Lambda_p(\partial M)$, *then* $\tau_{\mathscr{A}}^M \varphi_1 \geqq \tau_{\mathscr{A}}^M \varphi_2$ *everywhere on* M.

PROOF. Choose an arbitrary h_i in $W_p^1(M)$ with $\gamma h_i = \varphi_i$ ($i = 1, 2$). By the lattice property of $W_p^1(M)$, $(h_1 - h_2) \cup 0$ belongs to $W_p^1(M)$. Since γ preserves the lattice operations, we see that

$$\gamma((h_1 - h_2) \cup 0) = (\gamma(h_1 - h_2)) \cup 0 = (\varphi_1 - \varphi_2) \cup 0 = \varphi_1 - \varphi_2$$

on ∂M. If we set $f_2 = h_2$ and $f_1 = h_2 + (h_1 - h_2) \cup 0$, then $\gamma f_2 = \gamma h_2 = \varphi_2$ and

$$\gamma f_1 = \gamma h_2 + \gamma((h_1 - h_2) \cup 0) = \varphi_2 + (\varphi_1 - \varphi_2) = \varphi_1.$$

Then $\tau^M_{\mathscr{A}} \varphi_1 = \pi^M_{\mathscr{A}} f_1$, $\tau^M_{\mathscr{A}} \varphi_2 = \pi^M_{\mathscr{A}} f_2$ and $f_1 \geqq f_2$ on M imply that $\tau^M_{\mathscr{A}} \varphi_1 \geqq \tau^M_{\mathscr{A}} \varphi_2$ on M by the monotoneity of $\pi^M_{\mathscr{A}}$.

2.4. In addition to the defining boundary behavior $\gamma(\tau^M_{\mathscr{A}} \varphi) = \varphi$ of $\tau^M_{\mathscr{A}} \varphi$ we have the following more precise boundary behavior of $\tau^M_{\mathscr{A}} \varphi$ if an additional condition is imposed upon φ (cf. [16]). We say that a $\varphi \in L_\infty(\partial M)$ has an *essential limit* $\alpha \in \mathbf{R}$ at $\xi \in \partial M$ if

$$\lim_{r \downarrow 0} \| \varphi - \alpha; L_\infty(B(\xi, r) \cap \partial M) \| = 0,$$

where $B(\xi, R)$ is a parametric ball about $\xi \in \partial M$ in N.

PROPOSITION 2.2. *If* $\varphi \in L_\infty(\partial M) \cap \Lambda_p(\partial M)$ *has an essential limit* α *at* $\xi \in \partial M$, *then* $\tau^M_{\mathscr{A}} \varphi$ *has the boundary value* α *at* ξ, *i.e.*

(2.10)
$$\lim_{y \in M, y \to \xi} \tau^M_{\mathscr{A}} \varphi(y) = \alpha.$$

PROOF. Since $\tau^M_{\mathscr{A}}(\varphi - \alpha) = \tau^M_{\mathscr{A}} \varphi - \alpha$, we may suppose that φ has the essential limit 0 at $\xi \in \partial M$ and we only have to show (2.10) with α replaced by 0. Let $|\varphi| \leqq K$ a.e. on ∂M for a positive constant K. Fix a parametric ball $B(\xi, 1)$ in N at ξ with a local parameter x with $x(\xi) = 0$ and consider a function ρ on N defined by $\rho(x) = |x|$ in $B(\xi, 1)$ and $\rho = 1$ on $N \backslash B(\xi, 1)$. Clearly ρ belongs to the class $C(\overline{M}) \cap W^1_p(M)$ and $\tau^M_{\mathscr{A}}(\rho | \partial M) = \pi^M_{\mathscr{A}} \rho$, or more roughly, $\tau^M_{\mathscr{A}} \rho = \pi^M_{\mathscr{A}} \rho$. Since any point in ∂M is $\mathscr{A}$-regular,

$$\lim_{x \in M, x \to \xi} \tau^M_{\mathscr{A}} \rho(x) = \lim_{x \in M, x \to \xi} \pi^M_{\mathscr{A}} \rho(x) = \rho(\xi) = 0.$$

For any $\varepsilon > 0$ there is a $0 < \delta < 1$ such that $|\varphi(\eta)| < \varepsilon$ for a.e. η in $B(\xi, \delta) \cap \partial M$. Since $(K/\delta)\rho \geqq K$ in $(\partial M) \backslash B(\xi, \delta)$, we see that

$$-\frac{K}{\delta} \rho - \varepsilon \leqq \varphi \leqq \frac{K}{\delta} \rho + \varepsilon$$

a.e. on ∂M. By the monotoneity of $\tau^M_{\mathscr{A}}$, we have

$$-\frac{K}{\delta} \tau^M_{\mathscr{A}} \rho(y) - \varepsilon \leqq \tau^M_{\mathscr{A}} \varphi(y) \leqq \frac{K}{\delta} \tau^M_{\mathscr{A}} \rho(y) + \varepsilon \qquad (y \in M).$$

On letting y in M tend to ξ, we see by $\tau^M_{\mathscr{A}} \rho(y) \to 0$ that

$$-\varepsilon \leqq \liminf_{y \in M, y \to \xi} \tau^M_{\mathscr{A}} \varphi(y) \leqq \limsup_{y \in M, y \to \xi} \tau^M_{\mathscr{A}} \varphi(y) \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we finally conclude the required identity (2.10) with α replaced by 0.

3. 𝒜-harmonic measures

3.1. Take an $\mathscr{A} \in \mathscr{A}_p(M)$ $(1 < p < \infty)$, where M is as described in 1.5. We denote by

$$u \wedge v = u \wedge_{\mathscr{A}} v = u \wedge_{\mathscr{A}, M} v$$

the greatest $\mathscr{A}$-harmonic minorant of two $\mathscr{A}$-harmonic functions u and v on M. Thus $u \wedge v$ is characterized as the $\mathscr{A}$-harmonic function w on M with the following two properties. First, $w \leq u$ and $w \leq v$ on M. Second, if h is any $\mathscr{A}$-harmonic function on M such that $h \leq u$ and $h \leq v$ on M, then $h \leq w$. Needless to say, the greatest $\mathscr{A}$-harmonic minorant of u and v on M may or may not exist and once we use the notation $u \wedge v$, we understand that the existence of the greatest $\mathscr{A}$-harmonic minorant of u and v on M is assured. We also use the notation $u \vee v$ to indicate $-((-u) \wedge (-v))$, the least $\mathscr{A}$-harmonic majorant of u and v on M.

We say that w is an $\mathscr{A}$-*harmonic measure* on M if w is $\mathscr{A}$-harmonic on M and satisfies the condition

$$(3.1) \qquad\qquad w \wedge (1 - w) = 0$$

on M. Observe that $1 - w$ is an $\mathscr{A}$-harmonic measure on M along with w since we have $(1 - w) \wedge (1 - (1 - w)) = (1 - w) \wedge w = w \wedge (1 - w) = 0$ on M. The constant functions 0 and 1 are clearly $\mathscr{A}$-harmonic measures on M and actually these are only constant harmonic measures on M and any nonconstant $\mathscr{A}$-harmonic measure w on M satisfies $0 < w < 1$ on M. In fact, from (3.1) it follows that $0 \leq w \leq 1$ on M and hence both of w and $1 - w$ are nonnegative $\mathscr{A}$-harmonic functions on M. If $w \equiv c$, a constant, on M, then we see that

$$0 = w \wedge (1 - w) = c \cap (1 - c)$$

on M, which shows that $c = 0$ or $c = 1$. By the Harnack inequality we see that $w > 0$ and $1 - w > 0$ on M unless w is a constant on M.

The formulation (3.1) of harmonic measures was first introduced by Heins [3] for 2-harmonic functions on Riemann surfaces (cf. also [4], [11], [12], [13], [14], [15], [16], etc.).

3.2. Concerning the ranges $w(M) = \{w(x): x \in M\}$ of $\mathscr{A}$-harmonic measures w on M we have the following result. If w is a nonconstant $\mathscr{A}$-harmonic measure on M with $\mathscr{A} \in \mathscr{A}_p(M)$ $(1 < p < \infty)$, then the range $w(M)$ of w is the open interval $(0, 1)$:

$$w(M) = (0, 1) = \{\lambda \in \mathbf{R}: 0 < \lambda < 1\}.$$

In fact, since $w(M)$ is a connected subset of $(0, 1)$, we only have to show that $\inf_M w = 0$ and $\sup_M w = 1$. Set $a = \sup_M w$ so that $w \leq a$ on M and $0 < a \leq 1$. Clearly $1 - w \geq 1 - a$ on M. Since $0 \leq 1 - a < 1$ and $0 < w < 1$ on M, we have

$$w \geq (1 - a)w \qquad \text{and} \qquad 1 - a \geq (1 - a)w$$

on M. Hence we see that

$$0 = w \wedge (1 - w) \geq w \wedge (1 - a) \geq \{(1 - a)w\} \wedge \{(1 - a)w\} = (1 - a)w \geq 0$$

on M and a fortiori $(1 - a)w = 0$ on M, which implies that $a = 1$ so that $\sup_M w = 1$. Considering $1 - w$ instead of w in the above argument we see that $\sup_M(1 - w) = 1$ or $\inf_M w = 0$.

3.3. To describe the boundary behavior of general $\mathscr{A}$-harmonic measures on ∂M is a difficult problem but we can do it easily when w is moreover supposed to be of p-Dirichlet finite and the boundary behavior is considered in the sense of trace.

PROPOSITION 3.1. *A function $w \in H_{\mathscr{A}}(M) \cap W_p^1(M)$ is an $\mathscr{A}$-harmonic measure on M if and only if the essential range of γw is contained in $\{0, 1\}$.*

PROOF. Suppose first that $w \in H_{\mathscr{A}}(M) \cap W_p^1(M)$ is an $\mathscr{A}$-harmonic measure on M. Since $w(1 - w) \in W_p^1(M)$, we can consider $u = \pi_{\mathscr{A}}^M(w(1 - w))$ on M. By the monotoneity of $\pi_{\mathscr{A}}^M$, we see that $0 \leq u \leq w$ and $0 \leq u \leq 1 - w$ on M since $0 \leq w(1 - w) \leq w$ and $0 \leq w(1 - w) \leq 1 - w$ on M. Hence we deduce that

$$0 \leq u \leq w \wedge (1 - w) = 0$$

on M so that $\pi_{\mathscr{A}}^M(w(1 - w)) = 0$ on M. Since $w(1 - w) = w(1 - w) - \pi_{\mathscr{A}}^M(w(1 - w))$ belongs to $W_{p,0}^1(M) = \gamma^{-1}(0)$ (cf. (1.5)), we have

$$0 = \gamma(w(1 - w)) = (\gamma w)(\gamma(1 - w)) = (\gamma w)(1 - \gamma w)$$

dS-a.e., which proves that $\gamma w = 0$ or $1 - \gamma w = 0$ on ∂M dS-a.e. so that $\gamma w \in \{0, 1\}$ dS-a.e.

Conversely suppose that $w \in H_{\mathscr{A}}(M) \cap W_p^1(M)$ satisfies $\gamma w \in \{0, 1\}$ dS-a.e. Then $\gamma(w(1 - w)) = (\gamma w)(1 - \gamma w) = 0$ dS-a.e. on ∂M so that $w(1 - w) \in W_{p,0}^1(M)$ or $\pi_{\mathscr{A}}^M(w(1 - w)) = 0$. For any $h \in H_{\mathscr{A}}(M)$ with $h \leq w$ and $h \leq 1 - w$ on M, we deduce

$$h(1 - w) \leq w(1 - w) \qquad \text{and} \qquad wh \leq w(1 - w)$$

on M. Adding these two inequalities we obtain $h \leq 2w(1 - w)$ on M. Hence,

by the monotoneity of $\pi_{\mathscr{A}}^M$, we see that

$$h \leqq 2\pi_{\mathscr{A}}^M(w(1 - w)) = 0$$

on M. This proves that $w \wedge (1 - w) = 0$ on M and that w is an $\mathscr{A}$-harmonic measure on M.   □

COROLLARY 3.1. *The $\mathscr{A}$-harmonic part $\pi_{\mathscr{A}}^M f$ of any $f \in W_p^1(M)$ is an $\mathscr{A}$-harmonic measure on M if and only if $\gamma f \in \{0, 1\}$ on ∂M dS-a.e.*

PROOF. Since $\gamma f - \gamma(\pi_{\mathscr{A}}^M f) = \gamma(f - \pi_{\mathscr{A}}^M f) = 0$ on ∂M dS-a.e., $\gamma f \in \{0, 1\}$ dS-a.e. is equivalent to $\gamma(\pi_{\mathscr{A}}^M f) \in \{0, 1\}$ on ∂M dS-a.e., which means, by Proposition 3.1, that $\pi_{\mathscr{A}}^M f$ is an $\mathscr{A}$-harmonic measure.   □

4. A property of 2-Dirichlet finite 2-harmonic measures

4.1. In this section 4 we only consider 2-harmonic measures (i.e. classical harmonic measures) on a relatively compact subregion M of N with smooth relative boundary ∂M of class C^∞ as described in 1.5. We say that a differential form α on an open subset G of N is *smooth* if α is of class C^1 on G. The following will play a fundamental role in the proof of our main theorem in this paper.

LEMMA 4.1. *Any 2-Dirichlet finite 2-harmonic measure w on M has the following orthogonality relation:*

$$(4.1) \qquad\qquad \int_M dw \wedge d\alpha = 0$$

is valid for any smooth $(d - 2)$-form α defined on an open neighborhood U_α of $\overline{M}$ in N.

PROOF. We may suppose that w is not constant so that the range $w(M) = (0, 1)$, the open unit interval on the real line (cf. (3.2)). For two noncritical values λ and μ of w with $0 < \lambda < \mu < 1$ we set

$$W(\lambda, \mu) = \{\xi \in M : \lambda < w(\xi) < \mu\}.$$

By the Sard theorem (cf. e.g. [8]) that the set of critical values of w is of 1-dimensional Lebesgue measure zero, there exists a decreasing sequence $\{\lambda_k\}$ convergent to zero of noncritical values λ_k of w and an increasing sequence $\{\mu_k\}$ convergent to 1 of noncritical values μ_k of w with $\lambda_1 < \mu_1$. In view of

$$M = \bigcup_{k=1}^{\infty} W(\lambda_k, \mu_k),$$

we see that

$$\int_M dw \wedge d\alpha = \lim_{k\uparrow\infty} \int_{W(\lambda_k,\mu_k)} dw \wedge d\alpha = (-1)^{d-1} \lim_{k\uparrow\infty} \int_{W(\lambda_k,\mu_k)} d\alpha \wedge dw.$$

Therefore, in order to establish (4.1), we only have to show that

$$(4.2) \qquad\qquad \int_W d\alpha \wedge dw = 0 \qquad (W = W(\lambda,\mu))$$

for arbitrarily fixed noncritical values λ and μ of w with $0 < \lambda < \mu < 1$. Here W is an open subset of M with smooth relative boundary $\partial_M W$ relative to M each of whose components may or may not be compact in M.

We introduce a function $\varphi = ((w(1-w)) \cap c)/c$ on M where $c = \lambda(1-\mu)$. Since $w(1-w) > c$ on W, we see that $\varphi \equiv 1$ on $W \cup \partial_M W$. By Proposition 3.1, $\gamma w \in \{0,1\}$ dS-a.e. and hence $\gamma(w(1-w)) = 0$ or $w(1-w) \in W_{2,0}^1(M)$ by (1.5). Since $W_{2,0}^1(M)$ is a vector lattice, $\varphi \in W_{2,0}^1(M)$. Thus there exists a sequence $\{\varphi_k\}$ in $C_0^\infty(M)$ converging to φ in $W_2^1(M)$.

We also fix an exhaustion $\{M_k\}$ of M with supp $\varphi_k \subset M_k$, that is, M_k is a relatively compact subregion of M, $M\backslash M_k = \overline{M\backslash \overline{M}_k}$, ∂M_k is smooth, $\overline{M}_k \subset M_{k+1} \subset \overline{M}_{k+1} \subset M$ $(k = 1, 2, \ldots)$, and $M = \bigcup_{k=1}^\infty M_k$. For simplicity we set $\psi_k = \varphi - \varphi_k$ on M. Then

$$\|\psi_k; L_2(M)\| + \|d\psi_k; L_2(M)\| = \|\psi_k; L_2(M)\| + \|\nabla\psi_k; L_2(M)\| = \|\psi_k; W_2^1(M)\|$$

$$= \|\varphi - \varphi_k; W_2^1(M)\| \to 0 \qquad (k\uparrow\infty).$$

We first evaluate the integral

$$\int_W d(\psi_k\alpha) \wedge dw = \int_W d\psi_k \wedge \alpha \wedge dw + \int_W \psi_k d\alpha \wedge dw.$$

By the wedge inequality (1.2) we see that

$$|d\psi_k \wedge \alpha \wedge dw| = |d\psi_k \wedge (\alpha \wedge dw)| \leq |d\psi_k||d\alpha \wedge dw| \leq d|d\psi_k||\alpha||dw|,$$

where $d = \begin{pmatrix} d \\ d-1 \end{pmatrix}$. Hence by (1.1) and the Schwarz inequality we have

$$\left| \int_W d\psi_k \wedge \alpha \wedge dw \right|$$

$$\leq d \int_W |d\psi_k||\alpha||dw|\, dV \leq d\|\alpha; L_\infty(M)\| \int_W |d\psi_k||dw|\, dV$$

$$\leq d\|\alpha; L_\infty(M)\| \|d\psi_k; L_2(M)\| \|dw; L_2(M)\|$$

$$\leq d\|\alpha; L_\infty(M)\| \|\nabla w; L_2(M)\| \|\varphi - \varphi_k; W_2^1(M)\| \to 0 \qquad (k\uparrow\infty).$$

Here we have used the fact that $\|\alpha; L_\infty(M)\| < \infty$ which is a consequence of the smoothness of α and hence of the function $|\alpha|$ on $\overline{M}$. Similarly we have

$$\left| \int_W \psi_k d\alpha \wedge dw \right| \leq d \int_W |\psi_k| |d\alpha| |dw| dV \leq d \|d\alpha; L_\infty(M)\| \int_W |\psi_k| |dw| dV$$

$$\leq d \|d\alpha; L_\infty(M)\| \, \|\psi_k; L_2(M)\| \, \|dw; L_2(M)\|$$

$$\leq d \|d\alpha; L_\infty(M)\| \, \|\nabla w; L_2(M)\| \, \|\varphi - \varphi_k; W_2^1(M)\| \to 0 \qquad (k \uparrow \infty)$$

since $\|d\alpha; L_\infty(M)\| < \infty$ as a consequence of the smoothness of α on $\overline{M}$. Hence we can conclude that

$$(4.3) \qquad\qquad \lim_{k \uparrow \infty} \int_W d(\psi_k \alpha) \wedge dw = 0.$$

We next compute $\int_W d(\varphi_k \alpha) \wedge dw$ for each $k = 1, 2, \ldots$. Since $d((\varphi_k \alpha) \wedge dw) = d(\varphi_k \alpha) \wedge dw$, we have

$$\int_W d(\varphi_k \alpha) \wedge dw = \int_W d((\varphi_k \alpha) \wedge dw).$$

The following use of the Stokes formula is justified by the fact that the support of φ_k is contained in M_k:

$$\int_W d((\varphi_k \alpha) \wedge dw) = \int_{W \cap M_k} d((\varphi_k \alpha) \wedge dw) = \int_{\partial(W \cap M_k)} (\varphi_k \alpha) \wedge dw$$

$$= \int_{(\partial_M W) \cap M_k} (\varphi_k \alpha) \wedge dw + \int_{W \cap \partial M_k} (\varphi_k \alpha) \wedge dw.$$

Since w takes the constant value λ or μ on each component of $\partial_M W$, we see that $dw = 0$ along $\partial_M W$ and therefore the first term of the rightmost side of the above identity vanishes. The second term of the rightmost side of the above identity is clearly zero by the fact that $(\varphi_k \alpha) \wedge dw = 0$ on $W \cap \partial M_k$ since the support of φ_k is contained in M_k. Thus we have shown that

$$(4.4) \qquad\qquad \int_W d(\varphi_k \alpha) \wedge dw = 0 \qquad (k = 1, 2, \ldots).$$

Recall that $\varphi \equiv 1$ on W so that $\alpha = \varphi \alpha$ on W. Thus, by using (4.4) above, we proceed as follows:

$$\int_W d\alpha \wedge dw = \int_W d(\varphi \alpha) \wedge dw = \int_W d(\varphi \alpha) \wedge dw - \int_W d(\varphi_k \alpha) \wedge dw$$

$$= \int_W d((\varphi - \varphi_k)\alpha) \wedge dw = \int_W d(\psi_k \alpha) \wedge dw \to 0 \qquad (k \uparrow \infty).$$

Here the last relation follows from (4.3). Hence we have shown the validity of (4.2). □

4.2. The above lemma 4.1 can be used to describe the boundary behavior of w in terms of the trace γw of w on ∂M. Namely, we have the following result.

LEMMA 4.2. *Let w be any 2-Dirichlet finite 2-harmonic measure on M. Then the surface integral*

$$(4.5) \qquad \int_{\partial M} (\gamma w) d\alpha = 0$$

for any smooth $(d-2)$-form α defined on an open neighborhood U_α of $\overline{M}$ in N.

PROOF. Since $\overline{M}$ is compact in N with $\overline{N \setminus M} = \overline{N \setminus \overline{M}}$ and ∂M is smooth of class C^∞, we can show that $C^\infty(\overline{M}) = \{f | M : f \in C^\infty(N)\}$ is dense in $W_2^1(M)$ by exactly the same fashion as in the case of $N = \mathbf{R}^d$ (cf. e.g. [7], [10], etc.). Hence we can find a sequence $\{w_k\}$ in $C^\infty(\overline{M})$ such that

$$\lim_{k \to \infty} \|w - w_k; W_2^1(M)\| = 0.$$

Recall that the function space $L_2(\partial M)$ is considered with respect to the area element dS on ∂M induced by the Riemannian metric on N. Since $\gamma: W_2^1(M) \to L_2(\partial M)$ is continuous, there exists a finite constant $C > 0$ such that

$$\|\gamma(w - w_k); L_2(\partial M)\| \leq C \|w - w_k; W_2^1(M)\|.$$

In view of $w_k \in C^\infty(\overline{M}) \subset C(\overline{M})$ we have $\gamma w_k = w_k$ on ∂M and therefore we deduce

$$(4.6) \qquad \lim_{k \uparrow \infty} \|\gamma w - w_k; L_2(\partial M)\| = 0.$$

Considering $d\alpha$ as a $(d-1)$-form on ∂M we can write $d\alpha = a \, dS$ on M with a smooth function a defined on ∂M. By the Schwarz inequality and (4.6) we see that

$$\left| \int_{\partial M} (\gamma w - w_k) d\alpha \right| = \left| \int_{\partial M} (\gamma w - w_k) a \, dS \right| \leq \int_{\partial M} |\gamma w - w_k| |a| \, dS$$

$$\leq \|a; L_2(\partial M)\| \, \|\gamma w - w_k; L_2(\partial M)\| \to 0 \qquad (k \uparrow \infty),$$

i.e. we have shown that

$$(4.7) \qquad \lim_{k \uparrow \infty} \int_{\partial M} (\gamma w - w_k) d\alpha = 0.$$

Next, using the Stokes formula and the property (4.1): $\int_M dw \wedge d\alpha = 0$, we proceed as follows:

$$\int_{\partial M} w_k d\alpha = \int_M d(w_k d\alpha) = \int_M dw_k \wedge d\alpha$$

$$= \int_M dw_k \wedge d\alpha - \int_M dw \wedge d\alpha = \int_M (dw_k - dw) \wedge d\alpha.$$

Hence by the wedge inequality (Lemma 1.1) and the Schwarz inequality we see that

$$\left| \int_{\partial M} w_k d\alpha \right| = \left| \int_M (dw_k - dw) \wedge d\alpha \right| \leqq \int_M |dw_k - dw| |d\alpha| dV$$

$$\leqq \|d\alpha; L_2(M)\| \|dw_k - dw; L_2(M)\|.$$

Since $\|dw_k - dw; L_2(M)\| = \|V(w_k - w); L_2(M)\| \leqq \|w_k - w; W_2^1(M)\|$, we deduce

$$\left| \int_{\partial M} w_k d\alpha \right| \leqq \|d\alpha; L_2(M)\| \|w_k - w; W_2^1(M)\| \to 0 \qquad (k \uparrow \infty),$$

where we have used $\|d\alpha; L_2(M)\| < \infty$ as a consequence of the smoothness of α in a neighborhood of $\overline{M}$ in N. We have thus shown that

$$(4.8) \qquad \qquad \lim_{k \uparrow \infty} \int_{\partial M} w_k d\alpha = 0.$$

Finally, by using (4.8) and (4.7) in this order, we deduce (4.5) as follows:

$$\int_{\partial M} (\gamma w) d\alpha = \int_{\partial M} (\gamma w) d\alpha - \lim_{k \uparrow \infty} \int_{\partial M} w_k d\alpha = \lim_{k \uparrow \infty} \int_{\partial M} (\gamma w - w_k) d\alpha = 0. \qquad \square$$

5. A net of auxiliary forms

5.1. Fix an arbitrary point $\xi \in \partial M$. Choose a parametric ball $B_\xi = B = B(\xi, 1)$ about ξ with a local parameter $x = (x^1, \ldots, x^d)$ such that $x(\xi) = 0$, $B = B(\xi, 1) = \{|x| < 1\}$,

$$B \cap M = \{|x| < 1, x^d > 0\} \qquad \text{and} \qquad B \cap \partial M = \{|x| < 1, x^d = 0\}.$$

We may identify the parametric ball B with the unit ball $B^d = \{x \in \mathbf{R}^d : |x| < 1\}$ in the Euclidean space $\mathbf{R}^d$ of dimension d and the boundary fragment

$$P_\xi = P = B \cap \partial M$$

with the unit ball $\{|x| < 1, x^d = 0)$ in the hypersurface $\mathbf{R}^{d-1} = \{x \in \mathbf{R}^d : x^d = 0\}$

in $\mathbf{R}^d$. Hence we can consider the Euclidean area element ($(d-1)$-dimensional Lebesgue measure)

$$d\sigma(x) = dx^1 dx^2 \ldots dx^{d-1}$$

on P in addition to the proper Riemannian surface element dS on ∂M considered on P. These two measures on P are absolutely continuous to each other and Radon-Nikodym densities $dS/d\sigma$ and $d\sigma/dS$ are locally essentially bounded on P.

We take a $(d-1)$-dimensional open interval $Q_\xi = Q$ about ξ considered in P given by

$$Q = \{|x^i| < 1/4\sqrt{d} \quad (i = 1, \ldots, d-1), x^d = 0\} \subset P.$$

For each point $a = (a^1, \ldots, a^{d-1}, 0) \in Q$ and each real number $\delta \in (0, 1/40\sqrt{d})$ we set

$$S(a, \delta) = \{|x^i - a^i| < \delta \quad (i = 1, \ldots, d-1), x^d = 0\}$$

which is an open interval contained in P. The following highly technical lemma will play a crucial role in the proof of our main theorem.

LEMMA 5.1. *For any pair of points a and b in Q and any δ in $(0, 1/40\sqrt{d})$ there exists a net $\{\alpha_\varepsilon\}_{\varepsilon \downarrow 0}$ of smooth $(d-2)$-forms α_ε in N with compact supports contained in a fixed compact subset in B such that*

$$(5.1) \qquad \int_{S(b,\delta)} f d\sigma - \int_{S(a,\delta)} f d\sigma = \lim_{\varepsilon \downarrow 0} \int_{\partial M} f d\alpha_\varepsilon$$

for any locally dS-integrable function f defined on ∂M.

Since the proof is long, it will be given in 5.2–5.7 divided into 6 steps.

5.2. Let $b - a = \eta = (\eta^1, \ldots, \eta^{d-1}, 0)$ and set

$$\eta_i = (\eta^1 \delta_{1i}, \ldots, \eta^{d-1} \delta_{d-1,i}, 0) \qquad (i = 1, \ldots, d-1),$$

where δ_{ki} ($k = 1, \ldots, d-1$) is the Kronecker delta. We take d points $a_i \in Q$ ($i = 0, 1, \ldots, d-1$) determined by

$$a_0 = a \qquad \text{and} \qquad a_i = a_{i-1} + \eta_i \qquad (i = 1, \ldots, d-1).$$

If we can show the existence of nets $\{\alpha_{i\varepsilon}\}_{\varepsilon \downarrow 0}$ of smooth $(d-2)$-forms $\alpha_{i\varepsilon}$ on N with compact supports contained in a fixed compact subset in B such that

$$(5.2) \qquad \int_{S(a_{i-1},\delta)+\eta_i} f d\sigma - \int_{S(a_{i-1},\delta)} f d\sigma = \lim_{\varepsilon \downarrow 0} \int_{\partial M} f d\alpha_{i\varepsilon}$$

$(i = 1, \ldots, d - 1)$, then $\alpha_\varepsilon = \sum_{i=1}^{d-1} \alpha_{i\varepsilon}$ will satisfy (5.1). The proof of the existence of $\{\alpha_{i\varepsilon}\}$ satisfying (5.2) for an arbitrary i $(1 \leq i < d)$ is reduced to that for $i = 1$ by interchanging the order of the components of the coordinate. Thus we only have to prove (5.2) for $i = 1$, i.e. we only have to prove the existence of a net $\{\alpha_\varepsilon\}_{\varepsilon \downarrow 0}$ of $(d - 2)$-forms α_ε on N with compact supports contained in a fixed compact subset in B such that

$$(5.3) \qquad \int_{S(a,\delta)+\eta_1} f d\sigma - \int_{S(a,\delta)} f d\sigma = \lim_{\varepsilon \downarrow 0} \int_{\partial M} f d\alpha_\varepsilon$$

for a and $a + \eta_1$ belonging to Q, where $\eta_1 = (\eta^1, 0, \ldots, 0) \in \mathbf{R}^d$.

To prove (5.3) we may assume that

$$(\overline{S(a,\delta)} + \eta_1) \cap \overline{S(a,\delta)} = \varnothing.$$

If not, we can find an $a + \bar{\eta}_1$ $(\bar{\eta}_1 = (\bar{\eta}^1, 0, \ldots, 0) \in \mathbf{R}^d)$ in Q such that $\overline{S(a,\delta)} + \bar{\eta}_1$ is disjoint from $\overline{S(a,\delta)} + \eta_1$ and $\overline{S(a,\delta)}$ in view of the choice of δ. If we can show the existence of $\{\alpha'_\varepsilon\}_{\varepsilon \downarrow 0}$ and $\{\alpha''_\varepsilon\}_{\varepsilon \downarrow 0}$ of smooth $(d - 2)$-forms α'_ε and α''_ε on N with compact supports contained in a fixed compact subset in B such that

$$\int_{S(a,\delta)+\bar{\eta}_1+(\eta_1-\bar{\eta}_1)} f d\sigma - \int_{S(a,\delta)+\bar{\eta}_1} f d\sigma = \lim_{\varepsilon \downarrow 0} \int_{\partial M} f d\alpha'_\varepsilon$$

and similarly

$$\int_{S(a,\delta)+\bar{\eta}_1+(-\bar{\eta}_1)} f d\sigma - \int_{S(a,\delta)+\bar{\eta}_1} f d\sigma = \lim_{\varepsilon \downarrow 0} \int_{\partial M} f da_\varepsilon A,$$

then $\alpha_\varepsilon = \alpha'_\varepsilon - \alpha''_\varepsilon$ satisfies (5.3). Moreover we can assume that $\eta^1 > 0$. Otherwise we only have to replace $S(a,\delta) + \eta_1$ by $S(a,\delta)$, i.e. $S(a,\delta) = (S(a,\delta) + \eta_1) + (-\eta_1)$ and $-\eta^1 > 0$. Hence we may assume that $a^1 + \delta < a^1 - \delta + \eta^1$ so that $\eta^1 > 2\delta$.

5.3. We will prove (5.3) under the assumption that $\eta^1 > 2\delta$. For simplicity we denote by c the midpoint of the interval $[a^1 + \delta, a^1 - \delta + \eta^1]$ and also the interval $[a^1 - \delta, a^1 + \delta + \eta^1]$ so that $c = a^1 + \eta^1/2$, and by ρ_1 (ρ_2, resp.) the half of the length of the interval $[a^1 + \delta, a^1 - \delta + \eta^1]$ ($[a^1 - \delta, a^1 + \delta + \eta^1]$, resp.) so that $\rho_1 = \eta^1/2 - \delta$ and $\rho_2 = \eta^1/2 + \delta$.

We now define $d - 1$ functions $\varphi_1, \ldots, \varphi_{d-1}$ on $\mathbf{R}^d$ with their supports in B^d as follows. Considering their restrictions on B^d we may also view them as being defined on B. The first φ_1 of these $d - 1$ functions $\varphi_1, \ldots, \varphi_{d-1}$ is defined simply by

$$\varphi_1(x) + \rho_2 = \begin{cases} \rho_1 & ((x^1 - c)^2 + (x^d)^2 \leqq \rho_1^2), \\ ((x^1 - c)^2 + (x^d)^2)^{1/2} & (\rho_1^2 \leqq (x^1 - c)^2 + (x^d)^2 \leqq \rho_2^2), \\ \rho_2 & (\rho_2^2 \leqq (x^1 - c)^2 + (x^d)^2). \end{cases}$$

It is a piecewise smooth continuous function on $\mathbf{R}^d$ with $|\nabla\varphi_1| \leqq 1$ on $\mathbf{R}^d$, where ∇ is the Euclidean gradient so that $\nabla\varphi_1 = (\partial\varphi_1/\partial x^1, \ldots, \partial\varphi_1/\partial x^d)$. More concretely we have

$$(5.4) \quad d\varphi_1(x) = \begin{cases} \dfrac{\partial\varphi_1}{\partial x^1}(x)dx^1 + \dfrac{\partial\varphi_1}{\partial x^d}(x)dx^d & (\rho_1^2 < (x^1 - c)^2 + (x^d)^2 < \rho_2^2), \\[2mm] 0 & ((x^1 - c)^2 + (x^d)^2 < \rho_1^2 \quad \text{or} \quad > \rho_2^2), \end{cases}$$

where the coefficients of dx^1 and dx^d are given by

$$\frac{\partial\varphi_1}{\partial x^1}(x) = \frac{x^1 - c}{((x^1 - c)^2 + (x^d)^2)^{1/2}} \quad \text{and} \quad \frac{\partial\varphi_1}{\partial x^d}(x) = \frac{x^d}{((x^1 - c)^2 + (x^d)^2)^{1/2}}.$$

The rest $\varphi_2, \ldots, \varphi_{d-1}$ of φ_1 are simpler:

$$\varphi_i(x) = \begin{cases} a^i - \delta & (x^i \leqq a^i - \delta), \\ x^i & (a^i - \delta \leqq x^i \leqq a^i + \delta), \\ a^i + \delta & (a^i + \delta \leqq x^i) \end{cases}$$

for $1 < i < d$. Clearly these are also piecewise smooth continuous functions on $\mathbf{R}^d$ with $|\nabla\varphi_i| \leqq 1$ on $\mathbf{R}^d$. More precisely we see that

$$(5.5) \qquad dr_i(x) = \frac{\partial\varphi_i}{\partial x^i}(x)dx^i = \begin{cases} dx^i & (|x^i - a^i| < \delta), \\ 0 & (|x^i - a^i| > \delta) \end{cases}$$

for $1 < i < d$.

5.4. Using $d - 1$ functions φ_i $(1 \leqq i < d)$ defined in 5.3 we define a $(d - 2)$-form α on the unit ball B^d, which is also viewed as being defined on the parametric ball $B_\varsigma = B$ in the following fashion:

$$\alpha = \varphi_1 d(\varphi_2 d(\varphi_3 d(\cdots d(\varphi_{d-2}d\varphi_{d-1})\cdots))) = \varphi_1 d\varphi_2 \wedge d\varphi_3 \wedge \cdots \wedge d\varphi_{d-1}$$

$$= \left(\varphi_1 \prod_{i=2}^{d-1} \frac{\partial\varphi_i}{\partial x^i}\right) dx^2 \wedge dx^3 \wedge \cdots \wedge dx^{d-1}.$$

Thus α is a measurable form on B^d or on B. Since $\varphi_1 = 0$ on $(x^1 - c)^2 + (x^d)^2 \geqq \rho_2^2$ and $d\varphi_i = (\partial\varphi_i/\partial x^i)dx^i = 0$ on $|x^i| > \delta$ $(1 < i < d)$, we see that α has a compact support in B^d or in B, and therefore, by setting $\alpha = 0$ on $N\backslash B$, α may be viewed as a $(d - 2)$-form on N. By computing $d\alpha$ using (5.4) and (5.5) we have

$$d\alpha = d(\varphi_1 d(\varphi_2 d(\varphi_3 d(\cdots d(\varphi_{d-2}d\varphi_{d-1})\cdots)))) = d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge \cdots \wedge d\varphi_{d-1}$$

$$= \left(\prod_{i=1}^{d-1} \frac{\partial\varphi_i}{\partial x^i}\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{d-1}$$

$$+ (-1)^{d-2}\left(\prod_{i=2}^{d} \frac{\partial\varphi_i}{\partial x^i}\right) dx^2 \wedge dx^3 \wedge \cdots \wedge dx^d,$$

where we have set $\varphi_d = \varphi_1$.

We will consider $d\alpha$ along ∂M and in reality only along $P_\xi = P$ because $d\alpha = 0$ on $(\partial M)\backslash P$. Along P we have $x^d = 0$ and also $dx^d = 0$. Hence

$$(5.6) \qquad d\alpha = \left(\prod_{i=1}^{d-1} \frac{\partial\varphi_i}{\partial x^i}\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{d-1}$$

along P. By using (5.4) and (5.5) we see that, along ∂M,

$$d\alpha = \begin{cases} \dfrac{x^1 - c}{|x^1 - c|} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{d-1} & \text{on } S(a, \delta)\cup(S(a, \delta) + \eta_1), \\[2mm] 0 & \text{elsewhere on } \partial M. \end{cases}$$

Since $|x^1 - c| = -(x^1 - c)$ on $S(a, \delta)$ and $|x^1 - c| = x^1 - c$ on $S(a, \delta) + \eta_1$, we finally conclude that

$$d\alpha = \begin{cases} -dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{d-1} & \text{on } S(a, \delta), \\ dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{d-1} & \text{on } S(a, \delta) + \eta_1, \\ 0 & \text{elsewhere on } \partial M \end{cases}$$

along ∂M. In terms of the measure $d\sigma$, $d\alpha$ along P has an expression as a signed measure on P as follows:

$$(5.7) \qquad d\alpha(x) = (\chi_{S(a,\delta)+\eta_1}(x) - \chi_{S(a,\delta)}(x))d\sigma(x)$$

along P, where χ_E is the characteristic function of a subset E of P considered on P.

5.5. As usual we take $m_\varepsilon(x) = \varepsilon^{-d}m(\varepsilon^{-1}x)$ $(\varepsilon > 0)$ on $\mathbf{R}^d$, where

$$m(x) = \left(\int_{B^d} e^{-1/(1-|x|^2)}dx\right)^{-1} e^{-1/(1-|x|^2)}$$

for $|x| < 1$ and $m(x) = 0$ for $|x| \geqq 1$, and we form the regularization (mollifier)

$$\varphi_{i\varepsilon}(x) = (\varphi_i * m_\varepsilon)(x) = \int_{\mathbf{R}^d} \varphi_i(x - y)m_\varepsilon(y)dy$$

for every ε in $0 < \varepsilon < \rho_1 \cap (1/40\sqrt{d})$ $(1 \leqq i < d)$. Clearly $\varphi_{1\varepsilon}(x) =$

$\varphi_{1\varepsilon}(x^1, \ldots, x^d)$ is a function of only x^1 and x^d so that

(5.8)
$$d\varphi_{1\varepsilon} = \frac{\partial \varphi_{1\varepsilon}}{\partial x^1} dx^1 + \frac{\partial \varphi_{1\varepsilon}}{\partial x^d} dx^d$$

and $\varphi_{i\varepsilon}(x) = \varphi_{i\varepsilon}(x^1, \ldots, x^d)$ is the function of only x^i for each $i = 2, \ldots, d-1$ so that

(5.9)
$$d\varphi_{i\varepsilon} = \frac{\partial \varphi_{i\varepsilon}}{\partial x^i} dx^i \qquad (1 < i < d).$$

Since φ_i is piecewise smooth, we have

$$|\nabla \varphi_{i\varepsilon}(x)|^2 = \left| \int_{\mathbf{R}^d} \nabla \varphi_i(x - y) m_\varepsilon(y) dy \right|^2 \leqq \int_{\mathbf{R}^d} |\nabla \varphi_i(x - y)|^2 m_\varepsilon(y) \, dy$$

and hence we see that $|\nabla \varphi_{i\varepsilon}| \leqq 1$ along with $|\nabla \varphi_i| \leqq 1$ $(1 \leqq i < d)$. Observe that $\nabla \varphi_{i\varepsilon}$ converges to $\nabla \varphi_i$ as $\varepsilon \downarrow 0$ a.e. on $B_\xi = B$ and even on $P_\xi = P$ except for a set of $d\sigma$-measure zero $(1 \leqq i < d)$.

5.6. Using $\varphi_{i\varepsilon} = \varphi_i * m_\varepsilon$ $(1 \leqq i < d)$ we define a $(d-2)$-form α_ε on B^d or on $B_\xi = B$ by

$$\alpha_\varepsilon = \varphi_{1\varepsilon} d(\varphi_{2\varepsilon} d(\varphi_{3\varepsilon} d(\cdots d(\varphi_{d-2,\varepsilon} d\varphi_{d-1,\varepsilon}) \cdots))) = \varphi_{1\varepsilon} d\varphi_{2\varepsilon} \wedge d\varphi_{3\varepsilon} \wedge \cdots \wedge d\varphi_{d-1,\varepsilon}$$

$$= \left(\varphi_{1\varepsilon} \prod_{i=2}^{d-1} \frac{\partial \varphi_{i\varepsilon}}{\partial x^i} \right) dx^2 \wedge dx^3 \wedge \cdots dx^{d-1}.$$

Since $\varphi_{1\varepsilon} = 0$ on $(x^1 - c)^2 + (x^d)^2 \geqq (\rho_2 + \varepsilon)^2$ and $d\varphi_{i\varepsilon} = (\partial \varphi_{i\varepsilon}/\partial x^i) dx^i = 0$ on $|x^i - c| > \delta + \varepsilon$ $(1 < i < d)$ by the similar properties of φ_j $(1 \leqq j < d)$ and the definitions of $\varphi_{j\varepsilon}$ $(1 \leqq j < d)$, we see that α_ε has a compact support in B^d or in $B_\xi = B$ and therefore, by setting $\alpha_\varepsilon = 0$ and $N \backslash B$, α_ε may be viewed as a *smooth* $(d-2)$-form on N. By computing $d\alpha_\varepsilon$ by using (5.8) and (5.9) we have

$$d\alpha_\varepsilon = d(\varphi_{1\varepsilon} d(\varphi_{2\varepsilon} d(\varphi_{3\varepsilon} d(\cdots d(\varphi_{d-2,\varepsilon} d\varphi_{d-1,\varepsilon}) \cdots)))) = d\varphi_{1\varepsilon} \wedge d\varphi_{2\varepsilon} \wedge \cdots \wedge d\varphi_{d-1,\varepsilon}$$

$$= \left(\prod_{i=1}^{d-1} \frac{\partial \varphi_{i\varepsilon}}{\partial x^i} \right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{d-1}$$

$$+ (-1)^{d-2} \left(\prod_{i=2}^{d} \frac{\partial \varphi_{i\varepsilon}}{\partial x^i} \right) dx^2 \wedge dx^3 \wedge \cdots \wedge dx^d,$$

where we have set $\varphi_{d\varepsilon} = \varphi_{1\varepsilon}$. Hence $d\alpha_\varepsilon$ along $P_\xi = P$ has a compact support in P and an expression as a signed measure on P as follows (cf. (5.6)):

(5.10)
$$d\alpha_\varepsilon(x) = \left(\prod_{i=1}^{d-1} \frac{\partial \varphi_{i\varepsilon}}{\partial x^i}(x) \right) d\sigma(x) \quad \text{along } P.$$

5.7. As we have seen in 5.6, the function $\prod_{i=1}^{d-1}(\partial\varphi_{i\varepsilon}/\partial x^i)$ $(0 < \varepsilon < \rho_1 \cap (1/40\sqrt{d}))$ considered on $P_\xi = P$ has a compact support in a fixed compact subset of P, is bounded by 1 in its absolute values, and converges to the function $\prod_{i=1}^{d-1}(\partial\varphi_i/\partial x^i)$ as $\varepsilon \downarrow 0$ $d\sigma$-almost everywhere on P. Hence by the Lebesgue dominated convergence theorem we conclude that

$$\int_P f(x)\left(\prod_{i=1}^{d-1}\frac{\partial\varphi_i}{\partial x^i}(x)\right)d\sigma(x) = \lim_{\varepsilon\downarrow 0}\int_P f(x)\left(\prod_{i=1}^{d-1}\frac{\partial\varphi_{i\varepsilon}}{\partial x^i}(x)\right)d\sigma(x)$$

for every f in $locL_1(\partial M)$. Hence, by (5.6), $d\alpha$ along ∂M is expressed as a signed measure on P as follows:

$$d\alpha(x) = \left(\prod_{i=1}^{d-1}\frac{\partial\varphi_i}{\partial x^i}(x)\right)d\sigma(x) \quad \text{along } P$$

and therefore

$$\int_{\partial M} fd\alpha = \int_P f(x)\left(\prod_{i=1}^{d-1}\frac{\partial\varphi_i}{\partial x^i}(x)\right)d\sigma(x),$$

and similarly, (5.10) implies that

$$\int_{\partial M} fd\alpha_\varepsilon = \int_P f(x)\left(\prod_{i=1}^{d-1}\frac{\partial\varphi_{i\varepsilon}}{\partial x^i}(x)\right)d\sigma(x).$$

Hence we have shown that

$$\int_{\partial M} fd\alpha = \lim_{\varepsilon\downarrow 0}\int_{\partial M} fd\alpha_\varepsilon.$$

On the other hand, (5.7) implies that

$$\int_{\partial M} fd\alpha = \int_{S(a,\delta)+\eta_1} fd\sigma - \int_{S(a,\delta)} fd\sigma.$$

We can thus conclude the validity of (5.3), and the proof of Lemma 5.1 started from 5.2 is herewith complete. □

6. A boundary characteristic function

6.1. As before we denote by $B(\xi, 2)$ a relatively compact parametric ball in N at $\xi \in N$ with a local parameter $x = (x^1, \ldots, x^d)$ such that $x(\xi) = 0$ and $B(\xi, 2) = \{|x| < 2\}$. We also denote by $B(\xi, r)$ the concentric parametric ball $\{|x| < r\}$ $(0 < r \leq 2)$. The closed parametric ball $\overline{B(\xi, r)}$ is simply denoted by $\bar{B}(\xi, r)$. Since ∂M consists of a finite number of mutually disjoint smooth closed hypersurfaces of class C^∞ with $N\backslash M = N\backslash\overline{M}$, we can find a $B(\xi, 2)$

for any $\xi \in \partial M$ such that

$$B(\xi, 2) \cap M = \{|x| < 2, x^d > 0\} \quad \text{and} \quad B(\xi, 2) \cap \partial M = \{|x| < 2, x^d = 0\}.$$

We will show that the characteristic function on ∂M of $B(\xi, 1) \cap \partial M$ for each $\xi \in \partial M$ is a Sobolev boundary values on ∂M of a function in $W_p^1(M)$ when $1 < p < 2$.

LEMMA 6.1. *There exists a bounded continuous function f in $W_p^1(M)$ $(1 < p < 2)$ for any ξ in ∂M such that f has the boundary values 1 on $B(\xi, 1) \cap \partial M$ and 0 on $(\partial M) \backslash \bar{B}(\xi, 1)$ so that the trace γf on ∂M of f takes either 0 or 1 a.e. on ∂M.*

The proof of this lemma (cf. e.g. [13], [14], [4]) will be given in 6.2 and 6.3 below.

6.2. We denote by γ_1 the spherical surface fragment $B(\xi, 1) \cap \partial M$ and by γ_0 the spherical cone $\{x^d = 1 - |x'| (|x'| < 1)\}$, where $x = (x^1, \ldots, x^{d-1}, x^d) = (x', x^d)$ so that $x' = (x^1, \ldots, x^{d-1})$. Consider the region V bounded by γ_0 and γ_1: $\partial V = \bar{\gamma}_0 \cup \bar{\gamma}_1$. Define a function f on M by

$$f(\zeta) = \begin{cases} 1 - \dfrac{x^d(\zeta)}{1 - |x'(\zeta)|} & (\zeta \in V), \\ 0 & (\zeta \in M \backslash V). \end{cases}$$

We see that $f \in C(M)$, $0 \leq f < 1$ on M and f has the boundary values 1 on γ_1 and 0 on $(\partial M) \backslash \gamma_1$. Since $\bar{\gamma}_1 \backslash \gamma_1$ is of surface measure zero considered on ∂M, the trace γf of f on ∂M is either 1 or 0 a.e. on ∂M if we know that $f \in W_p^1(M)$.

The proof is over if we can show that $f \in W_p^1(M)$. Considering $X = B(\xi, 2) \cap M$ as a Riemannian submanifold of M, we see that f vanishes in a neighborhood of the relative boundary $\partial_M X$ of X relative to M and on $M \backslash X$. Hence it suffices to show that $f \in W_p^1(X)$ in order to maintain $f \in W_p^1(M)$. We view $Y = \{x \in \mathbf{R}^d : |x| < 2, x^d > 0\}$ as an Euclidean subspace of $\mathbf{R}^d$. Let $(g_{ij}(x))$ be the components of the metric tensor in $B(\xi, 2)$ in the Riemannian manifold N with respect to the local parameter $x = (x^1, \ldots, x^d)$ in $B(\xi, 2)$. Since $B(\xi, 2)$ is relatively compact in N, there exists a finite constant $C \geq 1$ such that

$$C^{-1}(\delta_{ij}) \leq (g_{ij}(x)) \leq C(\delta_{ij})$$

on $B(\xi, 2)$ so that

$$C^{-1}(\delta^{ij}) \leq (g^{ij}(x)) \leq C(\delta^{ij}) \quad \text{and} \quad C^{-d/2} dx \leq dV \leq C^{d/2} dx$$

on $B(\xi, 2)$, where $(g^{ij}(x)) = (g_{ij}(x))^{-1}$, $g(x) = \det(g_{ij}(x))$ and $dV(x) = \sqrt{g(x)} dx$.

We can identify X with $(Y, g_{ij}(x))$. Thus the above inequalities imply that a function φ on X (and hence on Y) belongs to $W_p^1(X)$ if and only if $\varphi \in W_p^1(Y)$, and

$$C^{-(1/2+d/2p)}\|\varphi; W_p^1(Y)\| \leq \|\varphi; W_p^1(X)\| \leq C^{1/2+d/2p}\|\varphi; W_p^1(Y)\|.$$

In view of this we only have to prove that $f \in W_p^1(U)$ viewing f as the function on $U = (\mathbf{R}^d)^+ = \{x \in \mathbf{R}^d : x^d > 0\}$ defined by

$$f(x) = \begin{cases} 1 - \dfrac{x^d}{1 - |x'|} & (x \in Z), \\ 0 & (x \in U \backslash Z), \end{cases}$$

where $Z = \{x \in \mathbf{R}^d : 0 < x^d < 1 - |x'|(|x'| < 1)\}$. Clearly $f \in C(U) \cap L_\infty(U)$ and f is absolutely continuous on all lines in U parallel to coordinate axes. We will see below in 6.3 that the ordinary gradient ∇f of f is p^{th} integrable on U. Hence by the Nikodym theorem (cf. e.g. [7]) we can conclude that $f \in W_p^1(U)$.

6.3. The ordinary gradient ∇f of f considered on U is given as follows:

$$\nabla f(x) = \begin{cases} (\nabla_{x'} f(x), -1/(1 - |x'|)) & (x \in Z \backslash \{x' = 0\}), \\ 0 & (x \in U \backslash \overline{Z}), \end{cases}$$

where $\nabla_{x'} = (\partial/\partial x^1, \ldots, \partial/\partial x^{d-1})$ so that

$$\nabla_{x'} f(x) = -x^d (1 - |x'|)^{-2}(x^1/|x'|, \ldots, x^{d-1}/|x'|)$$

for $x \in Z \backslash \{x' = 0\}$. Observe that

$$\|\nabla f; L_p(U)\| = \|\nabla f; L_p(Z)\|.$$

Hence by

$$|\nabla f|^p = (|\nabla_{x'} f|^2 + |\partial f/\partial x^d|^2)^{p/2} \leq (|\nabla_{x'} f| + |\partial f/\partial x^d|)^p \leq 2^p(|\nabla_{x'} f|^p + |\partial f/\partial x^d|^p)$$

we see that

$$\|\nabla f; L_p(Z)\| \leq 2^p \int_Z (|\nabla_{x'} f(x)|^p + |\partial f(x)/\partial x^d|^p)dx.$$

The right hand side of the above inequality equals, by the Fubini theorem,

$$2^p \int_{|x'| < 1} \left(\int_0^{1-|x'|} ((x^d)^p/(1 - |x'|)^{2p} + 1/(1 - |x'|)^p)dx^d \right) dx'$$

$$= C \int_{|x'| < 1} (1 - |x'|)^{1-p} dx',$$

where $C = 2^p(1/2^{p+1}(p + 1) + 1/2)$. If we denote by $d\omega_{d-1}$ the area element on $|x'| = 1$, then we have

$$\int_{|x'|<1} (1 - |x'|)^{1-p} dx' = \int_{|x'|=1} \left(\int_0^1 (1 - r)^{1-p} r^{d-2} dr \right) d\omega_{d-1}(x')$$

$$\leqq \omega_{d-1}(\{|x'| = 1\}) \int_0^1 (1 - r)^{1-p} dr = \omega_{d-1}(\{|x'| = 1\})/(2 - p) < \infty,$$

where $\omega_{d-1}(\{|x'| = 1\})$ is the area of $\{|x'| = 1\}$ under the convention $\omega_1(\{|x'| = 1\}) = 2$ with $d\omega_1 = d\delta_e + d\delta_{-e}$ ($e = (1, 0)$) when $d = 2$. Here the assumption $1 < p < 2$ is essentially made use of. Thus $\|\nabla f; L_p(U)\| < \infty$ and we have established that $f \in W_p^1(U)$.

The proof of Lemma 6.1 is herewith complete. $\square$

7. Proof of the main theorem

7.1. We divide the assertion of the main theorem stated in the introductory part at the begining of this paper into two parts, Theorems 7.1 and 7.2 below, and we will prove them separately in this last section. The first part is formulated as follows:

THEOREM 7.1. *Suppose that $2 \leqq p < \infty$ and choose an arbitrary $\mathscr{A}$ in $\mathscr{A}_p(M)$. Any p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M can be continuously extended to $\overline{M} = M \cup \partial M$ and the extended continuous function w on $\overline{M}$ is identically zero or one on each component of ∂M.*

PROOF. Take an arbitrary p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M with exponent p in $2 \leqq p < \infty$. Since $\overline{M}$ is compact in N, the Riemannian volume $|M|$ of M is finite:

$$|M| = \int_M dV < \infty.$$

By the Hölder inequality we see that

$$|M|^{-1/2} \|w; W_2^1(M)\| \leqq |M|^{-1/p} \|w; W_p^1(M)\|.$$

Since $w \in W_p^1(M)$, the above inequality assures that $w \in W_2^1(M)$. Consider the Maz'ya decomposition (2.8), and actually the classical Weyl decomposition, of w:

(7.1) $w = u + f,$

where $u \in H_2(M) \cap W_2^1(M)$ and $f \in W_{2,0}^1(M)$. By (1.5) we see that the trace

$$(7.2) \qquad\qquad \gamma f = 0 \qquad dS\text{-a.e. on } \partial M.$$

By Proposition 3.1, we have $\gamma w \in \{0, 1\}$ dS-a.e. on ∂M. Therefore, since $\gamma u = \gamma w - \gamma f$, we conclude that

$$(7.3) \qquad\qquad \gamma u \in \{0, 1\} \qquad dS\text{-a.e. on } \partial M.$$

Again by Proposition 3.1 we deduce that u is 2-Dirichlet finite 2-harmonic measure on M.

Let $\partial M = \bigcup_{j=1}^{\ell} (\partial M)_j$ be the decomposition of ∂M into connected components $(\partial M)_j$ $(j = 1, \ldots, \ell)$. We next show that for each component $(\partial M)_j$ of ∂M $(j = 1, \ldots, \ell)$ there exists a constant c_j which is either 1 or 0 such that

$$(7.4) \qquad\qquad (\gamma u)|(\partial M)_j = c_j \qquad dS\text{-a.e.}$$

To see this choose an arbitrary point $\xi \in \partial M$ and any relatively compact parametric ball $B_\xi = B = B(\xi, 1)$ with a local parameter $x = (x^1, \ldots, x^d)$ such that $x(\xi) = 0$, $B = \{|x| < 1\}$, $B \cap M = \{|x| < 1, x^d > 0\}$ and

$$P_\xi = P := B \cap \partial M = \{|x| < 1, x^d = 0\}.$$

We denote by $d\sigma(x)$ the Euclidean area element $dx^1 \ldots dx^{d-1}$ on P. Recall that $d\sigma$ and dS are mutually absolutely continuous on P. Let

$$Q_\xi = Q := \{|x^i| < 1/4\sqrt{d} \ (i = 1, \ldots, d-1), x^d = 0\} \subset P.$$

To prove (7.4) we only have to show that

$$(7.5) \qquad\qquad (\gamma u)|Q = c \qquad d\sigma\text{-a.e.},$$

where c is either 0 or 1. By (7.3) and the Lebesgue density theorem there exists a measurable subset Q' of Q such that $\sigma(Q\backslash Q') = 0$, every point p in Q' is a Lebesgue point for γu, and $\gamma u(p) \in \{0, 1\}$ for any $p \in Q'$. To maintain (7.5) we only have to show that

$$(7.6) \qquad\qquad \gamma u(a) = \gamma u(b)$$

for every pair of points a and b in Q'.

Take an arbitrary $\delta \in (0, 1/40\sqrt{d})$ and set

$$S(a, \delta) = \{x \in B : |x^i - a^i| < \delta \ (i = 1, \ldots, d-1), x^d = 0\} \subset Q,$$

where $a = (a^1, \ldots, a^{d-1}, 0) \in Q'$. By Lemma 5.1 there exists a net $\{\alpha_\varepsilon\}_{\varepsilon \downarrow 0}$ of smooth $(d-2)$-forms α_ε in N with compact supports in B such that

$$(7.7) \qquad \int_{S(b, \delta)} (\gamma u) d\sigma - \int_{S(a, \delta)} (\gamma u) d\sigma = \lim_{\varepsilon \downarrow 0} \int_{\partial M} (\gamma u) d\alpha_\varepsilon.$$

Lemma 4.2 assures that

$$(7.8) \qquad \int_{\partial M} (\gamma u) d\alpha_\varepsilon = 0$$

for any α_ε in the net $\{\alpha_\varepsilon\}_{\varepsilon \downarrow 0}$. By $\sigma(S(a, \delta)) = \sigma(S(b, \delta))$ for any $\delta \in (0, 1/40\sqrt{d})$, (7.7) and (7.8), we conclude that

$$\frac{1}{\sigma(S(a, \delta))} \int_{S(a, \delta)} (\gamma u) d\sigma = \frac{1}{\sigma(S(b, \delta))} \int_{S(b, \delta)} (\gamma u) d\sigma.$$

On letting $\delta \downarrow 0$, the Lebesgue density theorem implies (7.6). Thus the relation (7.4) has been shown.

By (7.1), (7.2) and (7.4), we see that

$$(\gamma w)|(\partial M)_j = c_j \qquad dS\text{-a.e.}$$

In view of this, Proposition 2.2 assures that the boundary values of w on $(\partial M)_j$ is c_j ($j = 1, \ldots, \ell$), w can be continued to $\overline{M} = M \cup \partial M$ so as to be a continuous function on $\overline{M}$. $\qquad \square$

7.2. Fix an arbitrary point ξ in ∂M and any relatively compact parametric ball $B(\xi, 2)$ with a local parameter $x = (x^1, \ldots, x^d)$ such that $x(\xi) = 0$, $B(\xi, 2) = \{|x| < 2\}$, $B(\xi, 2) \cap M = \{|x| < 2, x^d > 0\}$ and

$$B(\xi, 2) \cap \partial M = \{|x| < 2, x^d = 0\}.$$

We denote by $B = B(\xi, 1) = \{|x| < 1\}$. The second of two parts into which the main theorem is divided is formulated as follows (cf. [13], [14], [4]):

THEOREM 7.2. *Suppose that $1 < p < 2$ and choose an arbitrary $\mathscr{A}$ in $\mathscr{A}_p(M)$. There exists a p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M such that the boundary values of w is one on $B \cap \partial M$ and zero on $(\partial M) \backslash \overline{B}$. In particular, the function w cannot be extended to $\overline{M} = M \cup \partial M$ so as to be a continuous function on $\overline{M}$.*

PROOF. Since $1 < p < 2$, by Lemma 6.1, we can find an f in $W_p^1(M)$ such that the boundary values of f is one on $B \cap \partial M$ and zero on $(\partial M) \backslash \overline{B}$. In view of the fact that $(\partial B) \cap \partial M$ is a C^∞ hypersurface in the boundary manifold ∂M, we see that

$$S((\partial B) \cap \partial M) = 0.$$

Hence $\gamma f \in \{0, 1\}$ dS-a.e. on ∂M. Let $w = \pi_{\mathscr{A}}^M f$ be the $\mathscr{A}$-harmonic part of f given by the Maz'ya decomposition (2.8). By Corollary 3.1, we conclude that w is a p-Dirichlet finite $\mathscr{A}$-harmonic measure on M. Since $\gamma(f - w) = 0$

dS-a.e. on ∂M by (1.5), we see that $\gamma w = 1$ dS-a.e. on $B \cap \partial M$ and $\gamma w = 0$ dS-a.e. on $(\partial M) \backslash \overline{B}$. By virtue of the fact that $B \cap \partial M$ and $(\partial M) \backslash \overline{B}$ are open subset of ∂M, Proposition 2.2 assures that the boundary values of w on $B \cap \partial M$ are identically one and those on $(\partial M) \backslash \overline{B}$ are identically zero.

Since $(\partial B) \cap \partial M$ is a nonempty subset of ∂M, we can find a point ξ in $(\partial B) \cap \partial M$ so that ξ belongs both to $\overline{B \cap \partial M}$ and $\overline{(\partial M) \backslash \overline{B}}$. By the above boundary behavior of w, we see that w cannot be continuous at ξ. Thus w cannot be continued to $\overline{M} = M \cup \partial M$ so as to be continuous on $\overline{M}$. $\square$

APPENDIX.   The case of $p > d$ on general M

We take as in 1.1, a Riemannian manifold N of class C^{∞} of dimension $d \geqq 2$, connected and orientable. We fix an *arbitrary* relatively compact subregion M of N. We do not require any sort of regularity condition whatsoever upon the relative boundary $\partial M = \overline{M} \backslash M$ of M. The number of connected components of ∂M may or may not be finite. In short, we only assume that M is a nonempty connected open subset of N and $\overline{M}$ is compact in N so that we may say that $M \cup \partial M$ is a compact bordered Riemannian manifold of class C^{∞} of dimension $d = \dim M \geqq 2$ with a general border ∂M. We can consider the class $\mathscr{A}_p(M)$ of all operators $\mathscr{A}$ on M satisfying (2.1)–(2.5). Using an $\mathscr{A} \in \mathscr{A}_p(M)$ $(1 < p < \infty)$ we can also consider $\mathscr{A}$-harmonic measures on M as in §3 and their p-Dirichlet integrals over M. We mention the following final result.

THEOREM A.   *Suppose that $d < p < \infty$ and choose an arbitrary $\mathscr{A}$ in $\mathscr{A}_p(M)$. Any p-Dirichlet finite $\mathscr{A}$-harmonic measure w on M can be continuously extended to $\overline{M} = M \cup \partial M$ and the extended continuous function w on $\overline{M}$ is identically zero or one on each connected component of ∂M.*

PROOF.   By 3.1, $w \equiv 0$ on M, $w \equiv 1$ on M, or $0 < w < 1$ on M. In the first two cases the theorem is trivially true and hence hereafter in this proof we assume that $0 < w < 1$ on M. Then, by 3.2, we see that $w(M) = (0, 1) = \{\lambda \in \mathbf{R} : 0 < \lambda < 1\}$. By the proof of Proposition 3.1 (see also [14]), we have $s := w(1 - w) \in W_{p,0}^1(M)$. We maintain that s can be continuously extended to $\overline{M}$ and vanishes on ∂M:

(A.1) $s \in C(\overline{M})$ and $s | \partial M = 0$.

If $N = \mathbf{R}^d$, this is nothing but the Sobolev imbedding theorem for $p > d$. In the case of our present general N, we take the following indirect procedure to prove (A.1). On setting $t = s$ on M and $t = 0$ on $N \backslash M$ we see that $t \in W_{p,0}^1(N)$ along with $s \in W_{p,0}^1(M)$. Let (ψ_i) be a partition of unity on N

subordinate to a locally finite covering of N by parametric balls. Suppose that the support of ψ_i is contained in a parametric ball B_i in N. Viewing $B_i \subset \mathbf{R}^d$ we see that $\psi_i t \in W^1_{p,0}(B_i)$. Since $p > d$, by the Sobolev imbedding theorem (cf. e.g. [7]), there is a $v_i \in W^1_{p,0}(B_i) \cap C(\overline{B}_i)$ such that $v_i = \psi_i t$ a.e. on B_i. Clearly we can view $v_i \in W^1_{p,0}(N) \cap C(N)$ by setting $v_i = 0$ on $N \backslash B_i$. Then, since $v_i = \psi_i t = 0$ a.e. on $N \backslash M$, $v := \sum_i v_i \in W^1_{p,0}(N) \cap C(N)$ and

$$t = \sum_i \psi_i t = \sum_i v_i = v$$

a.e. on N. This means that $s = t|M$ has a continuous extension to $\overline{M}$, i.e. $s \in W^1_{p,0}(M) \cap C(\overline{M})$. Since the $\mathscr{A}$-Perron solution H^M_s is identical with the Sobolev solution with data s on M (cf. [2]) which is identically zero by $s \in W^1_{p,0}(M)$, we conclude that $H^M_s = 0$ on M. Take any $y \in \partial M$ and observe that the p-capacity $\mathrm{cap}_p\{y\} > 0$ because $p > d$. Hence y is $\mathscr{A}$-Dirichlet regular so that

$$s(y) = \lim_{x \in M, x \to y} H^M_s(x) = 0$$

(cf. [2]). We have thus shown (A.1).

Since $\sqrt{s} = \sqrt{w(1 - w)} \leqq (w + (1 - w))/2 = 1/2$, we see that $0 \leqq s \leqq 1/4$ on M and $s = 1/4$ if and only if $w = 1/2$. We denote by

$$K = \{x \in M : s(x) = 1/4\}.$$

By virtue of (A.1), K is a compact subset of M. We fix an arbitrary relatively compact subregion X of M with the following properties: $X \supset K$; X is bounded by a finite number of disjoint closed smooth hypersurfaces; $M \backslash \overline{X}$ consists of a finite number of relatively noncompact subregions of M. Take an arbitrary boundary component γ of M, i.e. γ is a connected component of ∂M. In view of the Kerékjártó-Stoilow representation of boundary components as determining sequences (cf. e.g. [17]), there is a unique connected component Y_γ of $M \backslash \overline{X}$ with the following property: for any $y \in \gamma$ there is a small parametric ball $B(y, r)$ $(r > 0)$ such that $B(y, r) \cap M \subset Y_\gamma$ so that $\gamma \subset \overline{Y}_\gamma$. We maintain that either $w < 1/2$ on Y_γ or $w > 1/2$ on Y_γ. In fact, if there are two points x_1 and x_2 on Y_γ with $w(x_1) < 1/2$ and $w(x_2) > 1/2$, then, since Y_γ is arcwise connected, there is a polygonal line L in Y_γ connecting x_1 and x_2. The intermediate value theorem yields the existence of a $z \in L$ with $w(z) = 1/2$ so that $z \in K$, contradicting $Y_\gamma \cap K = \varnothing$. Now suppose that $w < 1/2$ on Y_γ is the case. Then $w(x)(1 - w(x)) = s(x)$ with $w(x) < 1/2$ implies that

$$w(x) = 1/2 - (\sqrt{1 - 4s(x)})/2 \qquad (x \in Y_\gamma).$$

This with (A.1) assures that

$$\lim_{x \in M, x \to y} w(x) = \lim_{x \in Y_\gamma, x \to y} w(x) = 1/2 - \left(\sqrt{1 - 4 \lim_{x \in Y_\gamma, x \to y} s(x)} \right) \Big/ 2 = 0$$

for every $y \in \gamma$. Similarly, if $w > 1/2$ on Y_γ is the case, then

$$w(x) = 1/2 + (\sqrt{1 - 4s(x)})/2 \qquad (x \in Y_\gamma),$$

and we have $\lim_{x \in M, x \to y} w(x) = 1$ for every $y \in \gamma$. This proves that w can be extended to $\overline{M}$ so as to satisfy $w \in C(\overline{M})$ and the extended $w \equiv 0$ or 1 on each connected component γ of ∂M. $\square$

References

[1] E. Gagliardo, Caractterizzazione delle traccie sulla frontiera relative ad alcune classi di funzioni in n variabili, Rend. Sem. Mat. Padova, **27** (1957), 284–305.

[2] J. Heinonen, T. Kilpeläinen and O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Claredon Press, Oxford-New york-Tokyo, 1993.

[3] M. Heins, On the Lindelöf principle, Ann. Math., **61** (1955), 440–473.

[4] D. A. Herron and P. Koskela, Continuity of Sobolev functions and Dirichlet finite harmonic measures, Potential Analysis (to appear).

[5] L. Holopainen, Nonlinear potential theory and quasiregular mappings on Riemannian manifolds, Ann. Acad. Sci. Fenn., Ser. AI Math. Dissertation, **74** (1990), 1–45.

[6] V. G. Maz'ya, On the continuity at the boundary point of solutions of quasi-linear elliptic equations, Vestnik Leningrad Univ. Math., **3** (1976), 225–242. English translation of Vestnik Leningrad. Univ., **25** (1970), 42–55 (Russian).

[7] V. G. Maz'ya, Sobolev Spaces, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[8] J. Milnor, Topology from the Differentiable Viewpoint, Univ. Press of Va., Varginia, 1965.

[9] C. Miranda, Partial Differential Equations of Elliptic Type, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[10] S. Mizohata, The Theory of Partial Differential Equations, Cambridge Univ. Press, London, 1973.

[11] M. Nakai, Dirichlet finite harmonic measures on Riemann surfaces, Complex Variables, **17** (1991), 123–131.

[12] M. Nakai, Riemannian manifolds with connected Royden harmonic boundaries, Duke Math. J., **67** (1992), 587–625.

[13] M. Nakai, Existence of Dirichlet finite harmonic measures in nonlinear potential theory, Complex Variables, **21** (1993), 107–114.

[14] M. Nakai, Existence of Dirichlet finite harmonic measures on Euclidean balls, Nagoya Math. J., **133** (1994), 85–125.

[15] M. Nakai, Existence of Dirichlet infinite harmonic measures on the unit disc, Nagoya Math. J., **138** (1995), 141–167.

[16] M. Nakai, Existence of Dirichlet infinite harmonic measures on the Euclidean unit ball, Hiroshima Math. J., **26** (1996), 605–621.

[17] B. Rodin and L. Sario, Principal Functions, Van Nostrand, Princeton, 1968.

[18] J. Serrin, Local behavior of solutions of quasilinear equations, Acta Math., **111** (1964), 247–302.

[19]   J. Serrin, Isolated singularities of solutions of quasilinear equations,   Acta Math., **113** (1965), 219–240.

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